Rainbow edge-pancyclicity of strongly edge-colored graphs[1]

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Abstract

In an edge-colored graph G, a subgraph is *rainbow* if all its edges have different colors. A *strongly edge-colored graph* is an edge-colored graph in which each path of length three is rainbow. A cycle C in an edge-colored graph G of order n is called a *rainbow Hamiltonian cycle* if the cycle is rainbow and its length is n. An edge-colored graph G of order n is *rainbow vertex(edge)-pancyclic* if each vertex(edge) of G is contained in a rainbow cycle of length k for each k with $3 \leq k \leq n$. Cheng et al. in 2019 showed that every strongly edge-colored graph G of order n with minimum degree $\delta \geq \frac{2n}{3}$ contains a rainbow Hamiltonian cycle. Later in 2021, Wang et al. extended this result and showed that every strongly edge-colored graph G of order n with minimum degree $\delta \geq \frac{2n}{3}$ is rainbow vertex-pancyclic. In this paper, we further show that every strongly edge-colored graph G of order n with minimum degree $\delta \geq \frac{2n+1}{3}$ is rainbow edge-pancyclic. Moreover, from the proof we get a polynomial time algorithm for every given edge e in any such graph G to find a rainbow l-cycle for each $3 \leq l \leq n$ that contains the edge e.

Keywords: Rainbow; Strongly edge-colored graph; Rainbow Hamiltonian cycle; Rainbow vertex(edge)-pancyclic.

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1 Introduction

In this paper, we only consider finite, undirected and simple graphs. Let G be a graph consisting of a vertex set $V(G)$ and an edge set $E(G)$. We use $d(v)$ to denote the number

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of edges incident to a vertex v in G, called the *degree* of v in G. Furthermore, we use $\delta(G)$ to denote the minimum value of $d(v)$ over all vertices v in G, called the *minimum degree* of G. The *length* of a path or a cycle is the number of edges on it. We call a cycle(path) of length k a k-cycle(path). For a vertex subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. For two distinct vertex subsets X and Y of G, we use $E(X,Y)$ to denote the edge subset of G such that one end of each edge of $E(X,Y)$ in X and the other end in Y. An *edge-coloring* of G is a mapping $c: E(G) \to \mathbb{N}$, where $\mathbb{N}$ is a color set. An *edge-colored graph* is a graph with an edge-coloring. In an edge-colored graph G, we use $c(e)$ to denote the color of an edge e of G and $c(G)$ to denote the set of colors assigned to the edges of G. A subgraph is *rainbow* (*properly colored*) in an edge-colored graph G if any two (adjacent) edges of the subgraph have different colors. A *strongly edge-coloring* of G is an edge-coloring such that every path of length three in G is rainbow. A cycle C in an edge-colored graph G of order n is called a *rainbow Hamiltonian cycle* if C is rainbow and its length is n. An edge-colored graph of order n is *rainbow vertex(edge)-pancyclic* if each vertex(edge) of G is contained in a rainbow cycle of length k for each k with $3 \leq k \leq n$. Let $d^c(v)$ denote the number of different colors on the edges incident with a vertex v in an edge-colored graph G, called the *color-degree* of v in G, and let $\delta^c(G)$ denote the minimum value of $d^c(v)$ over all vertices v in G, called the *minimum color-degree* of G. For notation and terminology not defined here, we refer the reader to [2].

The classical Dirac's theorem states that every graph G of order n with minimum degree $\delta \geq \frac{n}{2}$ contains a Hamiltonian cycle. Inspired by this famous theorem, Hendry in [7] showed that every graph G of order n with minimum degree $\delta \geq \frac{n+1}{2}$ is vertex-pancyclic. Recently, the problems on the existences of properly colored cycles and rainbow cycles in an edge-colored graph attracted much attention, and thus a lot of work have been done extensively. For more details, the reader can find results on the properly colored cycles in [6, 9, 11, 12, 14] and on the rainbow cycles in [4, 5, 8]. For the edge-colored version of Dirac's problem, Lo in [10] proved the the following asymptotic result by probabilistic method.

Theorem 1.1. *[10] For any $\epsilon > 0$, there is an integer n_0 such that every edge-colored graph G with $n \geq n_0$ vertices and $\delta^c(G) \geq (\frac{2}{3} + \epsilon)n$ contains a properly edge-colored cycle of length k for all $3 \leq k \leq n$.*

In 2019, Cheng et al. in [3] considered the problem of the existence of rainbow Hamiltonian cycles in strongly edge-colored graphs and proposed the following conjecture.

Conjecture 1.2. *[3] Every strongly edge-colored graph G with n vertices and minimum degree $\delta \geq \frac{n+1}{2}$ has a rainbow Hamiltonian cycle.*

They constructed a class of graphs in [3] to show that the lower bound of Conjecture

1.2 is tight if it was true. To support the correctness of Conjecture 1.2, they proved the following result.

Theorem 1.3. *[3] Let G be a strongly edge-colored graph G with n vertices and minimum degree δ. If $\delta \geq \frac{2n}{3}$, then G has a rainbow Hamiltonian cycle.*

In fact, Bondy in [1] stated a significant conjecture that almost any condition that implies a graph being Hamiltonian will imply the graph being pancyclic, possibly with a well defined class of exceptional graphs. Hence, Wang et al. in [13] considered the rainbow vertex-pancyclicity of strongly edge-colored graphs under the condition of Theorem 1.3, and they got the following result.

Theorem 1.4. *[13] Let G be a strongly edge-colored graph G with n vertices and minimum degree δ. If $\delta \geq \frac{2n}{3}$, then G is rainbow vertex-pancyclic.*

Inspired by the above results, we consider the rainbow edge-pancyclicity of strongly edge-colored graphs further in this paper, and obtain the following result.

Theorem 1.5. *Let G be a strongly edge-colored graph G of order n with minimum degree δ. If $\delta \geq \frac{2n+1}{3}$, then G is rainbow edge-pancyclic. Furthermore, for every edge e of G, one can find a rainbow l-cycle containing e for each l with $3 \leq l \leq n$ in polynomial time.*

As one can see that for the results in both [3] and [13] the authors did not discuss the sharpness of the lower bounds for the minimum degree. This is so because it could be very difficult to get the best possible lower bounds. For us, the same problem exists, too. So, we also cannot construct examples to show the sharpness of our result, at least at the moment.

From the proof of Theorem 1.4 in [13], we can also get a polynomial time algorithm to find a rainbow l-cycle containing v for each l with $3 \leq l \leq n$.

Corollary 1.6. *Let G be a strongly edge-colored graph G of order n with minimum degree $\delta \geq \frac{2n}{3}$. Then for every vertex v of G, one can find a rainbow l-cycle containing v for each l with $3 \leq l \leq n$ in polynomial time.*

The rest of the paper is to give a proof of our this result.

2 Proof of Theorem 1.5

Before proving Theorem 1.5, we first introduce more useful notation and lemmas.

Let G be a strongly edge-colored graph with an edge-coloring c. For every vertex v of G, the *color neighborhood* of v in G, denoted by $CN_G(v)$, is defined as the set of colors assigned to the edges that are incident to v. When there is no confusion, we write $CN(v)$ instead of $CN_G(v)$. For a subgraph H of G, $c(H)$ denotes the set of colors used on the edges of H. Let $C = v_1v_2\cdots v_lv_1$ be a rainbow cycle in G. We call a color f a C-color ($\widetilde{C}$-color) if $f \in c(C)$ ($f \notin c(C)$). An edge e is called a C-color edge ($\widetilde{C}$-color edge) if $c(e) \in c(C)$ ($c(e) \notin c(C)$). For any two vertices x and y, we say that x and y are C-adjacent ($\widetilde{C}$-adjacent) if $c(xy) \in c(C)$ ($c(xy) \notin c(C)$). For any two disjoint vertex subsets V_1 and V_2, we use $E_C(V_1, V_2)$ and $E_{\widetilde{C}}(V_1, V_2)$ to denote the C-color edges and $\widetilde{C}$-color edges of $E(V_1, V_2)$, respectively. Similarly, for two subgraphs H_1 and H_2 of G, we use $E_C(H_1, H_2)$ and $E_{\widetilde{C}}(H_1, H_2)$ to denote $E_C(V(H_1), V(H_2))$ and $E_{\widetilde{C}}(V(H_1), V(H_2))$, respectively. Choose two vertices v_i and v_j in C, we use v_iCv_j to denote the rainbow path $v_iv_{i+1}\cdots v_{j-1}v_j$ on C, where all the subscripts of vertices are taken by mod l.

Lemma 2.1. *Let G be a strongly edge-colored graph of order $n \geq 5$ with minimum degree $\delta \geq \frac{2n+1}{3}$. Then each edge of G is contained in rainbow cycles of lengths 3, 4 and 5, respectively.*

Proof. It is straightforward to deduce that each edge of G is contained in a rainbow triangle. If $n \leq 6$, then G is a rainbow complete graph and it is not difficult to check that G is rainbow edge-pancyclic. Next, we consider $n \geq 7$. Choose an arbitrary edge xy, and let $xyzx$ be a rainbow triangle containing xy. Note that $d(x)+d(z)-n \geq \frac{4n+2}{3}-n = \frac{n+2}{3} \geq 3$. Then there is a vertex w such that $w \in N(x) \cap N(z) - \{y\}$ in G. Clearly, $xyzwx$ is a rainbow cycle of length four in G.

Finally, we show that xy is contained in a rainbow cycle of length five. Since $d(x) + d(w) - n \geq \frac{4n+2}{3} - n = \frac{n+2}{3} \geq 3$, there is a vertex u such that $u \in N(x) \cap N(w) - \{y, z\}$ in G. Hence, $xyzwux$ is a rainbow cycle of length five in G, and the result thus follows. □

Let G be an edge-colored graph and $C = v_1v_2\cdots v_tv_1$ be a rainbow cycle in G. A vertex $u \in V(G) - V(C)$ is called an *extendable vertex* respect to v_1v_2 if u is $\widetilde{C}$-adjacent to two successive vertices v_i and v_{i+1} in $\{v_2, v_3, ..., v_t, v_1 = v_{t+1}\}$. It is clear that $v_iuv_{i+1}Cv_i$ is a rainbow $(t+1)$-cycle containing v_1v_2 in G if u is an extendable vertex respect to v_1v_2.

Lemma 2.2. *Let G be a strongly edge-colored graph of order n with minimum degree $\delta \geq \frac{2n+1}{3}$. Then each edge of G is contained in a rainbow cycle of length l for each l with $3 \leq l \leq \frac{n+8}{3}$. Furthermore, one can find these rainbow cycles for each fixed edge of G in polynomial time.*

Proof. In fact, choose an arbitrary edge v_1v_2. To prove that each edge of G is contained in a rainbow cycle of length l for each l with $3 \leq l \leq \frac{n+8}{3}$. We only need to show that v_1v_2 is contained in a rainbow $(t+1)$-cycle if v_1v_2 is contained in a rainbow t-cycle for all $3 \leq t \leq \frac{n+5}{3}$ in G. Let $C = v_1v_2\cdots v_tv_1$ be a rainbow cycle in G. From Lemma 2.1, we consider $t \geq 5$. Suppose that c is the strong edge-coloring of G and $c(v_iv_{i+1}) = i$, where

$1 \leq i \leq t$ and $v_{t+1} = v_1$. Next, we find a rainbow $(t+1)$-cycle containing the edge v_1v_2 in G.

We assert that there is an extendable vertex respect to v_1v_2 in $V(G) - V(C)$. If not, assume that N_i is the set of vertices on C which are adjacent to v_i and M_i is the set of vertices in $V(G)-V(C)$ which are $\widetilde{C}$-adjacent to v_i. Note that the color j is not in $CN(v_i)$ if $v_j \in N_i$, where $1 \leq j \leq i-2$ or $i+1 \leq j \leq t$. This implies that there are at least $(|N_i|-1)$ C-colors that do not appear in $CN(v_i)$. Then the number of C-colors included in $CN(v_i)$ is at most $t-|N_i|+1$. Combining the fact that $c(v_iv_{i+1}) = i$ and $c(v_{i-1}v_i) = i-1$ belong to $CN(v_i)$, we can get that $|E_C(v_i, V(G)-V(C))| \leq t-|N_i|+1-2 = t-|N_i|-1$. Since $|E(v_i, V(G) - V(C))| \geq \delta - |N_i|$, we have

$$|M_i| = |E_{\tilde{C}}(v_i, V(G) - V(C))| = |E(v_i, V(G) - V(C))| - |E_C(v_i, V(G) - V(C))|,$$

where $1 \leq i \leq t$. Then,

$$|M_i| \geq \delta - |N_i| - (t - |N_i| - 1) = \delta - t + 1.$$

The hypothesis that G contains no extendable vertex implies that $M_2 \cap M_3 = \emptyset$. Since $|M_2| + |M_3| + t \leq n$, we have $t \geq 2\delta + 2 - n \geq \frac{n+8}{3}$, a contradiction. Then $M_2 \cap M_3 \neq \emptyset$. Clearly, each vertex of $M_2 \cap M_3$ is an extendable vertex and we can find an extendable vertex $u \in V(G) - V(C)$ in polynomial time. Hence, $v_2uv_3Cv_2$ is a rainbow $(t+1)$-cycle containing v_1v_2 in G, the result thus follows. $\square$

Lemma 2.3. *Let G be a strongly edge-colored graph of order n with $\delta \geq \frac{2n+1}{3}$. If an edge e of G is contained in a rainbow t-cycle in G, then e is also contained in a rainbow $(t+1)$-cycle in G, where $t \in \{n-2, n-1\}$. Furthermore, one can find the rainbow $(t+1)$-cycle in polynomial time.*

Proof. Assume that c is the strong edge-coloring of G, $C = v_1v_2 \cdots v_{n-2}v_1$ is a rainbow cycle, and $c(v_iv_{i+1}) = i$, where $1 \leq i \leq t$ and $v_{t+1} = v_1$. Without loss of generality, suppose $e = v_1v_2$. Next, we need to show that there is a rainbow $(n-1)$-cycle containing v_1v_2 in G.

We assert that there is an extendable vertex respect to v_1v_2 in $V(G) - V(C)$. If not, choose an arbitrary vertex $u \in V(G) - V(C)$. Let $p = |E_C(u, V(C))|$ and $q = |E_{\widetilde{C}}(u, V(C))|$. Then,

$$p + q = |E(u, V(C))| \geq \delta - 1 \geq \frac{2n-2}{3}.$$

For each edge uv_i, note that the colors $i-1$ and i cannot appear in $CN(u)$. Hence, there are at least $(p+q)$ C-color edges that are not incident to u, which means that $p \leq n-2-(p+q)$. Then,

$$2p + q \leq n - 2. \tag{1}$$

Assume that $v_{i_1}, v_{i_2}, ..., v_{i_q}$ are all the vertices in C that are $\widetilde{C}$-adjacent to u, where $1 \leq i_1 < i_2 \cdots < i_q \leq t$. Then the hypothesis that there is no extendable vertex respect to v_1v_2 in $V(G) - V(C)$ implies that u is not $\widetilde{C}$-adjacent to two successive vertices v_i and v_{i+1} in C for all $2 \leq i \leq t$. Therefore, we have $|i_{j+1} - i_j| \geq 2$ for all $j \in \{2, 3, ..., q-1\}$. Then $i_1 - 1, i_1, i_2, i_3 - 1, i_3, ..., i_q - 1, i_q$ are pairwise distinct. Furthermore, the definition of a strong edge-coloring implies that

$$\{i_1 - 1, i_1, i_2, i_3 - 1, i_3, ..., i_q - 1, i_q\} \cap CN(u) = \emptyset.$$

Hence, we have $2q - 1 + p \leq n - 2$. Then $2q + p \leq n - 1$. From Inequality (1), we know that $3|E(u, V(C))| = 3p + 3q \leq 2n - 3$. Then, $|E(u, V(C))| \leq \frac{2n-3}{3}$. However, $\frac{2n+1}{3} \leq d(u) \leq |E(u, V(C))| + 1 \leq \frac{2n}{3}$, a contradiction. Suppose that $u \in V(G) - V(C)$ is an extendable vertex respect to v_1v_2 and u is $\widetilde{C}$-adjacent to two successive vertices v_i and v_{i+1} in C, where $2 \leq i \leq t$. Then $v_iuv_{i+1}Cv_i$ is a rainbow $(n-1)$-cycle containing v_1v_2 in G. Clearly, we can find the above rainbow in polynomial time. Similarly, we can show that for the case $t = n - 1$, and the result also follows. $\square$

Now we are ready to give a proof of our Theorem 1.5.

Proof of Theorem 1.5: Choose an arbitrary edge $e^* = v_1v_2$ in G. In fact, To prove that G is rainbow edge-pancyclic, we only need to prove that v_1v_2 is contained in a rainbow $(t+1)$-cycle if v_1v_2 is contained in a t-cycle in G for each $3 \leq t \leq n-1$. Let $C = v_1v_2 \cdots v_tv_1$ be a rainbow t-cycle in G. From Lemmas 2.2 and 2.3, we only need to consider the case that $\frac{n+8}{3} \leq t \leq n - 3$. Without loss of generality, suppose that c is the strong edge-coloring of G and $c(v_iv_{i+1}) = i$, where $1 \leq i \leq t$ and $v_{t+1} = v_1$.

If there is an extendable vertex respect to v_1v_2 in $V(G) - V(C)$, by a similar argument to the proofs of Lemmas 2.2 and 2.3, the result follows. Hence, we suppose there is no extendable vertex respect to v_1v_2 in $V(G) - V(C)$ in the following.

Let $H = K_m$ be a maximal complete subgraph in $G[V(G) \setminus V(C)]$ such that each edge of H is a $\widetilde{C}$-color edge and let F be the induced subgraph of the vertex subset $V(G) \setminus V(C) \setminus V(H)$. Choose an arbitrary vertex $u \in H$, and set $p_u = |E_{\widetilde{C}}(u, C)|$, $q_u = |E_C(u, C)|$, $r_u = |E_{\widetilde{C}}(u, F)|$ and $s_u = |E_C(u, F)|$. Then,

$$d(u) = p_u + q_u + r_u + s_u + (m - 1) \geq \delta. \tag{2}$$

For each edge uv_i, note that the colors $i - 1$ and i cannot appear in $CN(u)$. Hence, there are at least $(p_u + q_u)$ C-color edges that are not incident to u, which means that $q_u + s_u \leq t - (p_u + q_u)$. Then,

$$2q_u + s_u + p_u \leq t. \tag{3}$$

Assume that $v_{i_1}, v_{i_2}, ..., v_{i_{p_u}}$ are all the vertices in C that are $\widetilde{C}$-adjacent to u, where $1 \leq i_1 < i_2 \cdots < i_{p_u} \leq t$. Note that if u is $\widetilde{C}$-adjacent to v_1 and v_2, then $v_1uv_2Cv_1$ is a rainbow $(t+1)$-cycle that does not contain the edge v_1v_2. The fact that u is not an extendable vertex respect to v_1v_2 implies that $|i_{j+1}-i_j| \geq 2$ for all $j \in \{2,3,...,p_u-1\}$. Then $i_1-1, i_1, i_2, i_3-1, i_3, ..., i_{p_u}-1, i_{p_u}$ are pairwise distinct. Furthermore, the definition of a strong edge-coloring implies that

$$\{i_1-1, i_1, i_2, i_3-1, i_3, ..., i_{p_u}-1, i_{p_u}\} \cap CN(u) = \emptyset.$$

Hence, we have

$$2p_u + q_u + s_u \leq t+1. \tag{4}$$

Noticing that $V(F) = V(G) \setminus V(C) \setminus V(H)$, we have $r_u + s_u \leq n - t - m$. Combining with Inequalities (3) and (4), we can get

$$3q_u + 3p_u + 3s_u + r_u \leq n + t + 1 - m.$$

Next, set $P = \sum_{u\in H} p_u$, $Q = \sum_{u\in H} q_u$, $R = \sum_{u\in H} r_u$ and $S = \sum_{u\in H} s_u$. Then, we have

$$3Q + 3P + 3S + R \leq m(n+t+1-m). \tag{5}$$

The maximality of H implies that each vertex of F is $\widetilde{C}$-adjacent to at most $m-1$ vertices of H. Then, we have

$$R \leq (m-1)(n-t-m). \tag{6}$$

Note that

$$m\delta \leq P + Q + R + S + m(m-1). \tag{7}$$

Hence, using Inequalities (2), (5) and (6), we have

$$\begin{aligned} 3m\delta &\leq 3P + 3Q + 3R + 3S + 3m(m-1) \\ &\leq m(n+t+1-m) + 2(m-1)(n-t-m) + 3m(m-1) \\ &\leq n(3m-2) + t(2-m). \end{aligned}$$

We can see that if $m=1$, then $t \geq n$, a contradiction. If $m=2$, then $\delta \leq \frac{2n}{3}$, a contradiction. In conclusion, we have $m \geq 3$. Let H^* be an induced subgraph by all the $\widetilde{C}$-color edges of $G[V(G) \setminus V(C)]$. Since we can get H^* by visiting all edges of $G[V(G) \setminus V(C)]$, and then we can easily find a maximal clique in a graph in polynomial time. Hence, we can construct H from H^* in polynomial time.

Since H is complete, the definition of a strong edge-coloring implies that $E(H) \cup E(H, F \cup C)$ is rainbow. We can see that

$$Q + S \leq t. \tag{8}$$

We assert that

$$P \geq t. \tag{9}$$

If not, we suppose $P \le t-1$. From Inequalities (6), (7) and (8), we have

$$m\delta \le P+Q+S+R+m(m-1) \le 2t-1+(m-1)(n-t),$$

which means that $m(n-t-\delta) \ge n+1-3t$. If $n-t-\delta < 0$, recalling that $m \ge 3$, then $3(n-t-\delta) \ge m(n-t-\delta) \ge n+1-3t$. This can deduce that $\delta \le \frac{2n-1}{3}$, a contradiction. If $n-t-\delta \ge 0$, then $t \le \frac{n}{3}$, which contradicts Lemma 2.2.

Algorithm 1 Find a sequence of l disjoint paths $P_1, P_2, ..., P_l$ respect to a rainbow cycle $C = v_1v_2 \cdots v_tv_1$ and a rainbow clique K_m in a strongly edge-colored graph G.

Input: A strongly edge-colored graph G, a rainbow cycle $C = v_1v_2 \cdots v_tv_1$ and a rainbow clique $H = K_m$ such that C and H are disjoint.

Output: A sequence of l disjoint paths $P_1, P_2, ..., P_l$ such that P_i is a subgraph of C.

1: **Set** a_1 be the smallest subscript such that $E_{\widetilde{C}}(v_{a_1}, H) \neq \emptyset$ for $1 \le a_1 \le t$.
2: **if** $a_1 = 1$ **then**
3:     **Set** $v_k = v_{t+2-k}$ for all $2 \le k \le t$, $a_1 = 2$ and go to 4.
4: **if** $a_1 \ge 2$ **then**
5:     **Set** $j = 1$.
6:     **if** $a_j + m < t+1$ **then**
7:         **Set** $P_j = v_{a_j}v_{a_j+1} \cdots v_{a_j+m}$
8:         **if** $E_{\widetilde{C}}(v, H) = \emptyset$ for all $v \in \{v_{a_j+m+1}, ..., v_t, v_1\}$ **then**
9:             **Set** $l = j$.
10:             **return** $P_1, P_2, ..., P_l$.
11:         **else** Set $j = j+1$, a_j be the smallest subscript such that $E_{\widetilde{C}}(v_{a_j}, H) \neq \emptyset$ for $a_{j-1}+m+1 \le a_j \le t+1$ and go to 6.
12:     **else** Set $P_j = v_{a_j}v_{a_j+1} \cdots v_tv_1$ and $l = j$.
13: **return** $P_1, ..., P_l$.

By Algorithm 1, we can construct a sequence of disjoint paths $P_1, P_2, ..., P_l$ respect to C and H. Firstly, we prove the correctness of Algorithm 1. Clearly, Inequality (9) implies that such an a_1 exists. Since $|V(C)|$ is finite and $P_1, P_2, ..., P_l$ are pairwise disjoint, Algorithm 1 will stop and output a result in polynomial time. Secondly, from Algorithm 1 we can see that

$$P_i = \begin{cases} v_{a_i}v_{a_i+1} \cdots v_{a_i+m}, & 1 \le i \le l-1; \\ v_{a_i}v_{a_i+1} \cdots v_tv_1, & i = l. \end{cases}$$

Note that $e^* \notin E(P_i)$ for all $1 \le i \le l$.

For any two vertices $u, w \in H$, we call (u, w) an *extendable pair* respect to v_1v_2 if u and w are $\widetilde{C}$-adjacent to two vertices v_j and v_k of C such that $2 \le k-j \le m+1$ and v_jCv_k contains no v_1v_2. Note that if we can find an extendable pair (u, w) respect to v_1v_2 in H, without loss of generality, assume that $uv_j, wv_k \in E_{\widetilde{C}}(H, \{v_j, v_k\})$ and $2 \le k-j \le m+1$. Clearly, $v_j uQwv_k Cv_j$ is a rainbow $(t+1)$-cycle containing v_1v_2, where Q is a rainbow path of length $(k-j-1)$ in H. Hence, we only need to show that there is an extendable pair respect to v_1v_2 in H in the following. If not, recalling that there is no extendable vertex in H, we show the following claim.

Claim 1. (1) $|E_{\widetilde{C}}(P_i, H)| \le m$ if $|V(P_i)| \le 2$ for all $1 \le i \le l$;

(2) $|E_{\widetilde{C}}(P_i, H)| \le m+1$ for all $1 \le i \le l$. In particular, $E_{\widetilde{C}}(v_{a_i+m+1}, H) = \emptyset$ if $|E_{\widetilde{C}}(P_i, H)| = m+1$ for all $1 \le i \le l-1$.

Proof. It is clear that $|E_{\widetilde{C}}(P_i, H)| \le m$ if $|V(P_i)| = 1$. Suppose that $|V(P_i)| = 2$ and $|E_{\widetilde{C}}(P_i, H)| \ge m+1$. Then there is at least one vertex $w \in H$ such that $|E_{\widetilde{C}}(\{v_{a_i}, v_{a_i+1}\}, w)| = 2$, which means that w is an extendable vertex respect to v_1v_2, a contradiction.

Next, we prove statement (2). In fact, we only need to show it for the case $i = 1$, i.e., $|E_{\widetilde{C}}(P_1, H)| \le m+1$.

Firstly, we assume that $a_1 + m < t+1$ and $|E_{\widetilde{C}}(v_{a_1}, H)| \ge 2$. Let h_1 and h_2 be two vertices of H such that $v_{a_1}h_1, v_{a_1}h_2 \in E_{\widetilde{C}}(v_{a_1}, H)$.

We assert that $E_{\widetilde{C}}(v_{a_1+i}, H) = \emptyset$ for each vertex v_{a_1+i} with $2 \le i \le m$. If not, then there exists a vertex w in H such that $v_{a_1+i}w \in E_{\widetilde{C}}(P_1, H)$. Without loss of generality, say $h_1 \ne w$. Then (w, h_1) is an extendable pair respect to v_1v_2, a contradiction. Hence, $E_{\widetilde{C}}(P_1, H) = E_{\widetilde{C}}(\{v_{a_1}, v_{a_1+1}\}, H)$. If $|E_{\widetilde{C}}(\{v_{a_1}, v_{a_1+1}\}, H)| \ge m+1$, then there is at least one vertex $w \in H$ such that $|E_{\widetilde{C}}(\{v_{a_1}, v_{a_1+1}\}, w)| = 2$. Then w is an extendable vertex respect to v_1v_2, a contradiction. Thus, $|E_{\widetilde{C}}(P_1, H)| = |E_{\widetilde{C}}(\{v_{a_1}, v_{a_1+1}\}, H)| \le m$.

Secondly, we suppose that $a_1+m < t+1$ and $|E_{\widetilde{C}}(v_{a_1}, H)| = 1$, say $v_{a_1}h_0 \in E_{\widetilde{C}}(v_{a_1}, H)$.

If $|E_{\widetilde{C}}(v_{a_1+1}, H)| = 0$, then $|E_{\widetilde{C}}(v_{a_1+i}, H)| \le 1$ for all $2 \le i \le m$. Otherwise, suppose that there is an integer $2 \le i \le m$ such that $|E_{\widetilde{C}}(v_{a_1+i}, H)| \ge 2$. By a similar discussion, we can get a contradiction. Then $|E_{\widetilde{C}}(P_1, H)| \le m$.

If $|E_{\widetilde{C}}(v_{a_1+1}, H)| = 1$, say $v_{a_1+1}h_1 \in E_{\widetilde{C}}(v_{a_1+1}, H)$. Clearly, $h_1 \ne h_0$. For each vertex v_{a_1+i} with $2 \le i \le m$, if $E_{\widetilde{C}}(v_{a_1+i}, H) \ne \emptyset$, say $v_{a_1+i}w \in E_{\widetilde{C}}(v_{a_1+i}, H)$ and $w \ne h_1$. Consequently, (w, h_1) is an extendable pair respect to v_1v_2, a contradiction. Thus, $E_{\widetilde{C}}(v_{a_1+i}, H) = \emptyset$ for $2 \le i \le m$. Then, $|E_{\widetilde{C}}(P_1, H)| \le 2$.

If $|E_{\widetilde{C}}(v_{a_1+1}, H)| \ge 2$, then by a similar discussion, we have $|E_{\widetilde{C}}(P_1 \setminus v_{a_1}, H)| \le m$. Thus, $|E_{\widetilde{C}}(P_1, H)| \le m+1$. Similarly, we can show that $|E_{\widetilde{C}}(P_1, H)| \le m+1$ when $a_1 + m \ge t+1$. In conclusion, we have $|E_{\widetilde{C}}(P_i, F)| \le m+1$ for all $1 \le i \le l$.

Finally, we prove the second half of statement (2). It is clear that $l \ge 2$. From the above discussion, we can find that $|E_{\widetilde{C}}(v_{a_1+1}, H)| \ge 2$ if $|E_{\widetilde{C}}(P_1, H)| = m+1$. Let w_1 and w_2 be two vertices of H such that $v_{a_1+1}w_1, v_{a_1+1}w_2 \in E_{\widetilde{C}}(v_{a_1+1}, H)$. If $E_{\widetilde{C}}(v_{a_1+m+1}, H) \ne \emptyset$, then there is at least one vertex $w \in H$ such that $v_{a_1+m+1}w \in E_{\widetilde{C}}(v_{a_1+m+1}, H)$. Without loss of generality, set $w \neq w_1$. Then (w, w_1) is an extendable pair respect to v_1v_2, a contradiction. The claim thus follows. $\square$

If $l = 1$, then $P = |E_{\widetilde{C}}(P_1, H)| \leq m + 1$ from Claim 1. Combining with Inequalities (6), (7) and (8), we can get that

$$\begin{aligned} m\delta &\leq P + Q + R + S + m(m-1) \\ &\leq m(n-t+1) + 2t - n + 1. \end{aligned}$$

This means that $m(\delta + t - n - 1) \leq 2t - n + 1$. From Lemma 2.2, we know that $t \geq \frac{n+8}{3}$. Then $\delta + t - n - 1 > 0$. Recalling that $m \geq 3$, we have

$$2t - n + 1 \geq m(\delta + t - n - 1) \geq 3(\delta + t - n - 1).$$

From Lemma 2.1, we have $t \geq 5$. Consequently, we can conclude that $\delta \leq \frac{2n+4-t}{3} \leq \frac{2n-1}{3}$, a contradiction.

Next, assume $l \geq 2$. From the definition of the sequence $P_1, P_2, ..., P_l$, we can see that $|V(P_i)| = m + 1$ for all $1 \leq i \leq l - 1$ and $1 \leq |V(P_l)| \leq m + 1$. Then, from Claim 1, we have

$$\begin{aligned} P = |E_{\widetilde{C}}(C, H)| &= \sum_{i=1}^{l-1} |E_{\widetilde{C}}(P_i, H)| + |E_{\widetilde{C}}(P_l, H)| \\ &\leq \sum_{i=1}^{l-1} |V(P_i)| + |E_{\widetilde{C}}(P_l, H)| \\ &\leq t - |V(P_l)| - (a_1 - 2) - (l - 1) + |E_{\widetilde{C}}(P_l, H)| \\ &\leq t - a_1 - l + 3 - |V(P_l)| + |E_{\widetilde{C}}(P_l, H)|. \end{aligned}$$

Now we consider the following two cases.

Case 1. $|V(P_l)| \geq 2$.

From Claim 1, we know that $|E_{\widetilde{C}}(P_l, H)| \leq m + 1$ if $|V(P_l)| \geq 3$ and $|E_{\widetilde{C}}(P_l, H)| \leq m$ if $|V(P_l)| = 2$. Then $-|V(P_l)| + |E_{\widetilde{C}}(P_l, H)| \leq m - 2$. Recalling that $l \geq 2$ and $a_1 \geq 2$, we have $P \leq t + m - 3$. Using inequalities (6), (7) and (8), we can get that

$$\begin{aligned} m\delta &\leq P + Q + R + S + m(m-1) \\ &\leq t + m - 3 + t + (m-1)(n-t-m) + m(m-1) \\ &\leq 3t - n - 3 + m(n-t+1). \end{aligned}$$

Then we have $m(\delta - n + t - 1) \leq 3t - n - 3$. Recalling that $t \geq \frac{n+8}{3}$ and $\delta \geq \frac{2n+1}{3}$, we can conclude that $\delta - n + t - 1 \geq 2$. The result $m \geq 3$ implies that

$$3(\delta - n + t - 1) \leq m(\delta - n + t - 1) \leq 3t - n - 3.$$

Then $\delta \le \frac{2n}{3}$, which contradicts the assumption that $\delta \ge \frac{2n+1}{3}$.

Case 2. $|V(P_l)| = 1$.

From Claim 1, we know that $|E_{\widetilde{C}}(P_l, H)| \le m$. Recalling that $a_1 \ge 2$ and $l \ge 2$, we have $P \le t + m - 2$. Using Inequalities (6), (7) and (8), we can get that

$$\begin{aligned} m\delta &\le P + Q + R + S + m(m-1) \\ &\le t + m - 2 + t + (m-1)(n-t-m) + m(m-1) \\ &\le 3t - n - 2 + m(n - t + 1). \end{aligned}$$

Then we have

$$m(\delta - n + t - 1) \le 3t - n - 2.$$

Recalling that $t \ge \frac{n+8}{3}$ and $\delta \ge \frac{2n+1}{3}$, we can conclude that $\delta - n + t - 1 \ge 2$. If $m \ge 4$, then $3t - n - 2 \ge 4(\delta - n + t - 1)$. Hence, $\delta \le \frac{3n-t+2}{4}$. Using the result $t \ge \frac{n+8}{3}$, we can get $\delta \le \frac{4n-1}{6} < \frac{2n+1}{3}$, a contradiction. Next, we suppose $m = 3$. Thus, we can conclude that $\delta \le \frac{2n+1}{3}$.

The condition $\delta \ge \frac{2n+1}{3}$ implies that $\delta = \frac{2n+1}{3}$, which means that $a_1 = 2$, $l = 2$, $P_1 = v_2v_3v_4v_5$, $P_2 = v_1$ and $|E_{\widetilde{C}}(P_2, H)| = |E_{\widetilde{C}}(v_1, H)| = m = 3$. Then, $P \le t - a_1 - l + 3 - |V(P_2)| + |E_{\widetilde{C}}(P_2, H)| = t + 1$. Combining Inequality (9), we have $t \le P \le t + 1$.

We assert that $|E_{\widetilde{C}}(P_1, H)| \ge 3$. If not, then $P \le |E_{\widetilde{C}}(P_1, H)| + |E_{\widetilde{C}}(P_2, H)| \le 5$. Hence, using Inequalities (6), (7) and (8), we can get that

$$\begin{aligned} 3\delta &\le P + Q + R + S + m(m-1) \\ &\le 5 + t + 2(n - t - 3) + 6 \\ &\le 2n - t + 5. \end{aligned}$$

Since $t \ge 5$, we have $\delta \le \frac{2n}{3}$, a contradiction. Then, from Claim 1, we have $3 \le |E_{\widetilde{C}}(P_1, H)| \le 4$.

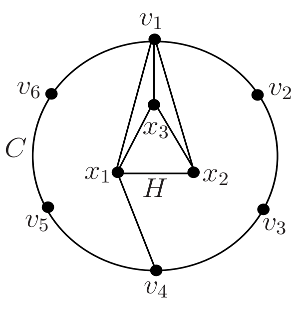

Figure 1: $n = 9$, $|E_{\widetilde{C}}(v_1, H)| = 3$ and $t = 6$.

Now we assume that $V(H) = \{x_1, x_2, x_3\}$. If $|E_{\widetilde{C}}(P_1, H)| = 3$, then $P = 6$ and $t = 5$ or 6. If $P = 6$ and $t = 5$, then $n \ge t + m = 8$. Recalling that $t \ge \frac{n+8}{3}$, we have $n \le 7$, a contradiction. If $P = t = 6$, then $n \ge t + m = 9$. Combining the fact that $t \ge \frac{n+8}{3}$, we have $9 \leq n \leq 10$. If $n = 9$, then $d(v_4) \geq 7$. Since $(V(C), V(H), V(F))$ is a vertex partition of G, we can obverse that $V(F) = \emptyset$. Thus, $E(v_4, H) \neq \emptyset$, say $v_4x_1 \in E(v_4, H)$; see Figure 1. The definition of a strong edge-coloring and the fact that $|E_{\widetilde{C}}(v_1, H)| = 3$ imply that (x_1, x_3) is an extendable pair respect to v_1v_2 and $v_1v_2v_3v_4x_1x_2x_3v_1$ is a rainbow 7-cycle contains v_1v_2 in G, a contradiction. Similarly, we can show it for the case $n = 10$.

If $|E_{\widetilde{C}}(P_1, H)| = 4$, then $P = 7$ and $t = 6$ or 7. If $P = 7$ and $t = 6$, by a similar argument for the case that $P = t = 6$, we can find a rainbow cycle of length seven containing e^* in G, a contradiction. If $P = t = 7$, using Inequalities (6), (7) and (8), we can get that $3\delta \leq P + Q + R + S + m(m-1) \leq 7 + t + 2(n - 7 - 3) + 6 = 2n$, which means that $\delta \leq \frac{2n}{3}$, a contradiction.

Combining with the above two cases, we prove that there is an extendable pair respect to v_1v_2 in H. Clearly, we can find an extendable pair by visiting all the vertex pairs of H. The operation can be finished in polynomial time, and then we can construct a rainbow $(t+1)$-cycle containing v_1v_2 in G in polynomial time, and the result thus follows. $\square$

To end of this section, we give an algorithm for finding the $(n-2)$ rainbow cycles of different lengths containing a given edge in a strongly edge-colored graph of order n. The correctness of the algorithm is contained in the proof of Theorem 1.5.

Algorithm 2 Find $(n-2)$ rainbow cycles of different lengths containing a given edge in a strongly edge-colored graph of order n.

Input: A strongly edge-colored graph G of order n with minimum degree $\delta \geq \frac{2n+1}{3}$ and a given edge $e^* = v_1v_2 \in E(G)$.

Output: $(n-2)$ rainbow cycles C^i $(3 \leq i \leq n)$ such that C^i contains v_1v_2 and the length of C^i is i in G.

1: **Set** $i = 3$.
2: Choose a vertex v_3 such that $v_3 \in N(v_1) \cap N(v_2)$.
3: **Set** $C^i = v_1v_2 \cdots v_iv_1$ and $e^* = v_1v_2$.
4: **if** there is an extendable vertex $v \in V(G) - V(C^i)$ respect to v_1v_2 **then**
5: Choose an extendable vertex $v \in V(G) - V(C^i)$ respect to v_1v_2 such that v is $\widetilde{C^i}$-adjacent to v_j and v_{j+1} for $2 \leq j \leq t$.
6: **Set** $C^{i+1} = v_jvv_{j+1}Cv_j$, $i = i+1$ and go to 3.
7: **else**
8: **if** $\frac{n+8}{3} \leq i \leq n-2$ **then**
9: Choose a maximal complete subgraph $H = K_m$ in $G[V(G) \setminus V(C^i)]$ such that each edge of H is a $\widetilde{C}$-color edge and $m \geq 3$.
10: Get a sequence of disjoint paths $\{P_1, P_2, ..., P_l\}$ respect to C^i and H in G by Algorithm 1.
11: **for** $1 \leq j \leq l$ **do**
12: **if** there is an extendable pair respect to v_1v_2 in H. **then**
13: Choose an extendable pair (u_α, u_β) respect to v_1v_2 in H and two vertices $v_\alpha, v_\beta \in V(C^i)$ such that $v_\alpha u_\alpha, v_\beta u_\beta \in E_{\widetilde{C^i}}(\{v_\alpha, v_\beta,\}H)$ and $|\beta - \alpha| \geq 2$.
14: Choose a $(\beta - \alpha - 1)$-path P^* from u_α to u_β in H.
15: **Set** $C^{i+1} = v_\alpha u_\alpha P^* u_\beta v_\beta C v_\alpha$, $i = i+1$ and go to 3.
16: **return** $C^3, C^4, ..., C^n$.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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