Weak-odd chromatic index of special digraph classes

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December 8, 2022

Abstract

Given a digraph $D = (V(D), A(D))$, let $\partial_D^+(v) = \{vw|w \in N_D^+(v)\}$ and $\partial_D^-(v) = \{uv|u \in N_D^-(v)\}$ be semi-cuts of v. A mapping $\varphi : A(D) \to [k]$ is called a *weak-odd k-edge coloring* of D if it satisfies the condition: for each $v \in V(D)$, there is at least one color with an odd number of occurrences on each non-empty semi-cut of v. We call the minimum integer k the *weak-odd chromatic index* of D. When limit to 2 colors, let $\mathrm{def}(D)$ denote the *defect* of D, i.e., the minimum number of vertices in D at which the above condition is not satisfied. In this paper, we give a descriptive characterization with respect to the weak-odd chromatic index and the defect of semicomplete digraphs and extended tournaments, which generalize results of tournaments to broader classes. In addition, we initiate the study of weak-odd edge covering on digraphs.

Keywords: weak-odd edge coloring; weak-odd edge covering; semicomplete digraph; extended tournament

1 Introduction

Throughout the paper, we follow the terminology and notion from [1, 2]. Here all digraphs considered are finite.

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Let $G = (V(G), E(G))$ be a graph. Denote by $d_G(v)$ the number of edges incident with v in G. A mapping $\varphi : E(G) \to [k]$ is called a *weak-odd k-edge coloring* of G if it satisfies the following condition:

(WO) For $v \in V(G)$ with $d_G(v) > 0$, there is at least one color $i \in [k]$ such that the number of edges incident with v colored by i is odd.

Note that this concept is a relaxation of odd edge coloring of graphs which was first introduced by Pyber in [11]. The odd edge coloring is an edge coloring such that at each non-isolated vertex every color appears an odd number of times or does not appear at all. The *weak-odd chromatic index* of G, denoted by $\chi'_{\text{wo}}(G)$, is the minimum integer k such that G admits a weak-odd k-edge coloring. This concept, motivated by [4, 5, 11], is given in [9], where Petruševski gave an intuitive characterization of graphs in terms of their weak-odd chromatic index.

Inspired by the study of the weak-odd chromatic index of graphs, Petruševski and Škrekovski [10] generalized this concept to digraphs. Given a digraph $D = (V(D), A(D))$, we use $n(D)$ to denote the number of vertices of D. Let v be a vertex of D. Let $N_D^+(v) = \{w|vw \in A(D)\}$ and $N_D^-(v) = \{w|wv \in A(D)\}$. Let $\partial_D^+(v) = \{vw|w \in N_D^+(v)\}$ and $\partial_D^-(v) = \{uv|u \in N_D^-(v)\}$ be *semi-cuts* of v. The *out-degree*(resp. *in-degree*) of v which is also called the *semi-degree* of v, denoted by $d_D^+(v)$ (resp. $d_D^-(v)$), is the cardinality of the set $\partial_D^+(v)$ (resp. $\partial_D^-(v)$). We say that a vertex $u \in V(D)$ is a *peripheral* vertex if either $d_D^+(u) = 0$ or $d_D^-(u) = 0$. Specifically, if $d_D^+(u) = 0$, then u is a *sink* of D, and if $d_D^-(u) = 0$, then u is *source*. A mapping $\varphi : A(D) \to [k]$ is said to be a *weak-odd k-edge coloring* of D if the following holds:

($\overrightarrow{\text{WO}}$) For any $v \in V(D)$, there is at least one color $i \in [k]$ such that the number of arcs in each nonempty semi-cut of v colored by i is odd.

We say that such a digraph D is *weak-odd k-edge colorable*, and call the suitable minimum integer k *weak-odd chromatic index*, denoted by $\chi'_{\text{wo}}(D)$.

In the same paper, the authors showed that $\chi'_{\text{wo}}(D) \leq 3$ and the bound is sharp. They believed that a descriptive characterization similar to graphs is impossible for all digraphs and they believed that deciding the exact value of $\chi'_{\text{wo}}(D)$ is NP-hard. In [6], the authors showed a necessary and sufficient condition for digraphs to be weak-odd 2-edge colorable, and thus $\chi'_{\text{wo}}(D)$ can be determined in polynomial time. When limit to 2 colors, let $\text{def}(D)$ denote the *defect* of D, i.e., the minimum number of vertices in D at which the condition ($\overrightarrow{\text{WO}}$) is not satisfied. Hernández-Cruz, Petruševski and Škrekovski [6] proved that $\text{def}(D)$ is related to the matching number of some graphs.

A *tournament* is an orientation of a complete graph. A digraph is called *semicomplete* if it is obtained from a complete graph by replacing each edge (u, v) with the arc uv or vu

or a pair of symmetric arcs. By *extended tournaments* we mean the digraph obtained from a tournament by blowing up some of its vertices into independent sets. Hernández-Cruz, Petruševski and Škrekovski [6] made a descriptive characterization of tournaments with respect to the weak-odd chromatic index as follows.

Theorem 1.1 ([6]). *For any tournament T, it holds that*

$$\chi'_{\mathrm{wo}}(T) = \begin{cases} 0 & \textit{if } T = K_1, \\ 1 & \textit{if } T \textit{ is nontrivial and every vertex semi-degree is odd or zero}, \\ 3 & \textit{if } T \textit{ is nontrival, of odd order, and has just one peripheral vertex}, \\ 2 & \textit{otherwise}. \end{cases}$$

And the defect of a tournament is 1 when the case $\chi'_{\mathrm{wo}}(T) = 3$.

In addition, they asked whether these results can be extended to classes of digraphs that generalize tournaments.

Problem 1.2 ([6]). *Characterize the families of semicomplete digraphs, extended tournaments and multipartite tournaments in terms of their weak-odd chromatic index.*

Problem 1.3 ([6]). *Characterize the defect in terms of the families of semicomplete digraphs, extended tournaments and multipartite tournaments when their defect is bounded.*

We give the complete characterization about the above two problems for the first two graph classes, i.e., semicomplete digraphs and extended tournaments. The results can be helpful for the remaining class. And we think the result of multipartite tournaments is also optimistic.

Hernández-Cruz, Petruševski and Škrekovski [6] also initiated the study of weak-odd edge covering and provided the weak-odd 2-edge covering conditions for graphs. Additionally, they asked about the situation for digraphs. For a digraph D, an *edge covering* with color set S is a mapping that assigns to each arc of D a nonempty subset of S. The *weak-odd edge covering* is defined as edge covering such that condition $(\overrightarrow{\mathrm{WO}})$ is satisfied.

Question 1.4 ([6]). *Does every digraph admit a weak-odd 2-edge covering?*

We give a positive answer to this question in the case of tournaments. This is of positive significance to the study of digraphs. We believe that similar research can be carried out on the simple generalization classes of tournaments.

The paper is organized as follows. In next section, we first introduce the notion and terminology that are not mentioned before, then we list some auxiliary tools that will be used in our proofs. Then, in Sections 3 and 4 we give descriptive characterizations with respect to the weak-odd chromatic index and the defect of semicomplete digraphs and extended tournaments. In the last section, we prove that every tournament admits a weak-odd 2-edge covering.

2 Preliminary

Given a digraph $D = (V(D), A(D))$, the *degree* of $v \in V(D)$, denoted by $d_D(v)$, is the total number of arcs incoming and outgoing at v, thus $d_D(v) = d_D^+(v) + d_D^-(v)$. We say that a graph or a digraph is *even* if every vertex of it has even degree. The *minimum out-degree* (*minimum in-degree*) of D is $\delta^+(D) = \min\{d_D^+(v)|v \in V(D)\}$ ($\delta^-(D) = \min\{d_D^-(v)|v \in V(D)\}$). The *minimum semi-degree* of D is $\delta^0(D) = \min\{\delta^-(D), \delta^+(D)\}$. For $X, Y \subseteq V(D)$, let $A(X,Y) = \{uv \in A(D)|u \in X, v \in Y\}$. A directed X-Y path is an (x, y)-dipath P such that $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$. The subdigraph of D induced by $X \subseteq A(D)$ is denoted by $G[X]$. A vertex u is said to *dominate* a vertex v if $v \in N_D^+(u)$.

A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, \dots, D_t$ are the strong components of D, then $V(D_i) \cap V(D_j) = \emptyset$ for every $i \neq j$ as otherwise all the vertices $V(D_i) \cup V(D_j)$ are reachable from each other. The *strong component digraph* $SC(D)$ of D is obtained by contracting the strong components of D and deleting any parallel arcs obtained in this process. The strong components of D corresponding to the vertices of $SC(D)$ of in-degree (out-degree) zero are the *initial (terminal)* strong components of D, also called the *peripheral strong components*.

We shall emphasize that when dealing with graphs, the conception *S-join* is a powerful tool. Given a graph $G = (V(G), E(G))$ and an even-sized vertex subset S, we call a spanning subgraph H is an *S-join* of G if $d_H(v)$ is odd for all $v \in S$ while $d_H(v)$ is even for all $v \in V(G) \setminus S$. It has been proved that *if G is a connected graph, then G contains an S-join for any even-sized vertex subset S* (see [12]). When turning our attention to digraphs, the problem of determining the weak-odd chromatic index of digraphs can be settled through constructing the following auxiliary graphs. Given a digraph $D = (V, A)$, its *bipartite representation* or *split* is a bipartite graph $BG(D) = (V^+, V^-, E)$ where $V^+ = \{v^+ : \ v \in V\}$, $V^- = \{v^- : \ v \in V\}$, and $(u^+, v^-) \in E$ if and only if $uv \in A$. The *partial split*, $PS(D)$, of D is a graph obtained from $BG(D)$ by re-identifying each pair $(u^+; u^-)$ for which both $d_D^+(u)$ and $d_D^-(u)$ are odd. See Figure 1.

To solve the problem whether a digraph is weak-odd 2-edge colorable, Hernández-Cruz, Petruševski and Škrekovski [6] defined a 3-partition $\{V_1; V_2; V_3\}$ of $V(PS(D))$:

- $V_1 = V(D) \cap V(PS(D))$, i.e., V_1 consists of the vertices u of D with both $d_D^+(u)$ and $d_D^-(u)$ odd.
- $V_2 = \{v \in V(PS(D)) \setminus V_1 : d_{PS(D)}(v)$ is even$\}$.

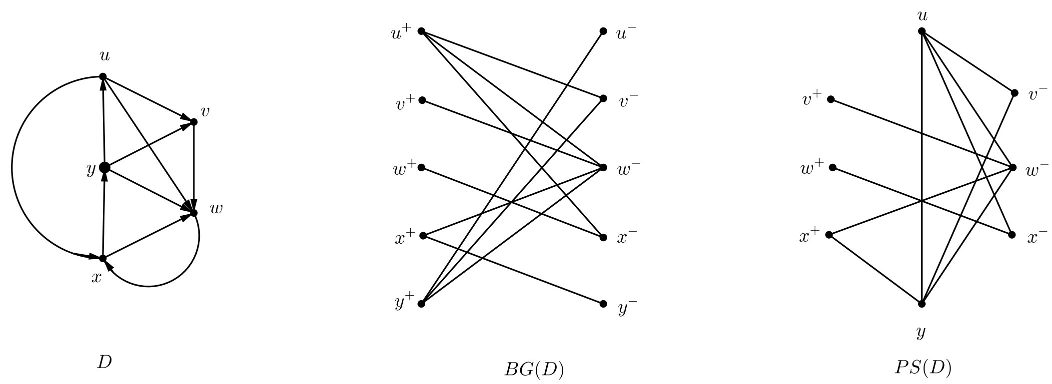

Figure 1: The split graph $BG(D)$ and partial split graph $PS(D)$ of D

- $V_3 = \{v \in V(PS(D)) \setminus V_1 : d_{PS(D)}(v) \text{ is odd}\}$.

We say that a component K of $PS(D)$ is 'bad' if $V(K) \cap V_2$ is of odd size and $V(K) \cap V_3 = \emptyset$. Let $G_D = (V_D, E_D)$ be a graph with the vertex set consisting of vertices v_K corresponding to bad components K and two distinct vertices $v_{K'}$ and $v_{K''}$ are adjacent if the respective bad components K' and K'' contain the 'halves' v^+ and v^- of some vertex $v \in V(D)$. Let α'_D be the cardinality of the maximum matching of G_D. The following results proved in [6] will be used later.

Theorem 2.1 ([6]). *A digraph D is weak-odd 2-edge colorable if and only if for every nontrivial component K of $PS(D)$ we have that $V(K) \cap V_2$ is even-sized or $V(K) \cap V_3 \neq \emptyset$.*

Proposition 2.2 ([6]). *If an even digraph D has an odd number of peripheral vertices, then $\chi'_{\mathrm{wo}}(D) = 3$.*

Theorem 2.3 ([6]). *For every digraph D, $\mathrm{def}(D) = n(G_D) - \alpha'_D$ holds.*

Finally, we give a useful statement about the weak-odd edge coloring. Let $D = (V(D), A(D))$ be a digraph with $v \in V(D)$. Let D' be the digraph obtained from D by deleting v. If D' admits a weak-odd 2-edge coloring ϕ, then we define a 2-edge coloring φ of D such that $(\overrightarrow{\mathrm{WO}})$ is satisfied for each vertex apart from v as follows.

(C) For each $u \in N_D(v)$, suppose that color i satisfies the condition $(\overrightarrow{\mathrm{WO}})$ at u for ϕ, where $i \in [2]$. If $uv \in A(D)$, then coloring uv with color $3 - i$ when $d^+_{D'}(u) > 0$ and color i when $d^+_{D'}(u) = 0$. If $vu \in A(D)$, then coloring vu with color $3 - i$ when $d^-_{D'}(u) > 0$ and color i when $d^-_{D'}(u) = 0$.

3 Semicomplete digraphs

We first state some simple properties of semicomplete digraphs, which can be found in Section 2 of [1]: (i) every semicomplete digraph has a hamiltonian dipath; (ii) every nontrivial strong semicomplete digraph contains a hamiltonian dicycle; (iii) the strong component digraph of a semicomplete digraph is an acyclic tournament and has an acyclic ordering of vertices; (iv) every semicomplete digraph has only one initial (terminal) strong component.

For simplicity of presentation, we call every nontrivial even semicomplete digraph having only one peripheral vertex 'bad' and others 'good' in the following.

Theorem 3.1. *For any semicomplete digraph D, it holds that*

$$\chi'_{\mathrm{wo}}(D) = \begin{cases} 0 & \text{if } D = K_1, \\ 1 & \text{if } D \text{ is nontrivial and every vertex semi-degree is odd or zero,} \\ 3 & \text{if } D \text{ is a nontrival even digraph with just one peripheral vertex,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 2.2 and $\chi'_{\mathrm{wo}}(D) \leq 3$, it suffices to show that every good semi-complete digraph is weak-odd 2-edge colorable.

Let D be a good semicomplete digraph. Throughout the proof, we always first find a spanning subdigraph $\hat{D}$ of D. Then we define a 2-edge coloring θ of D as the arc set of $\hat{D}$ with color 1 and $A(D) - A(\hat{D})$ with color 2. It is easy to check that $(\overrightarrow{\mathrm{WO}})$ holds for every vertex of D under θ in each case.

If D is strong, then let $\hat{D}$ be a Hamilton dicycle. If D has two trivial peripheral strong components, say x, y, then let $\hat{D}$ be a (x, y)-Hamilton dipath. If both peripheral strong components of D are nontrivial, then there exists a directed K_i-K_j path P in D that passes through every vertex $v \notin V(K_i) \cup V(K_j)$, where K_i and K_j are the initial and terminal strong components of D respectively. Let C_i and C_j, respectively, be hamiltonian dicycles in K_i and K_j. Denote by x and y, respectively, the initial and terminal vertex of P. We have that $xy \notin A(P)$ if P is of length $\ell(P) > 1$. Let $\hat{D} = D[A(C_i \cup C_j)]$ when $\ell(P) = 1$ and $\hat{D} = D[A(C_i \cup C_j \cup P) \cup \{xy\}]$ when $\ell(P) > 1$. Then color 1 meets condition $(\overrightarrow{\mathrm{WO}})$ for above cases.

We complete the proof by supposing that exactly one peripheral strong component of D, without loss of generality, the terminal one, is trivial, denoted by $\{y\}$. Then y is the sink of D. Now, there is a vertex $v \in V(D)$ such that $d_D(v)$ is odd as D is good. Let D' be the semicomplete digraph obtained from D by deleting the vertex v (note that if $d_D(y)$ is odd, then v can be the same as y). We proceed by distinguishing whether $v = y$.

Case 1. $v = y$.

First, suppose that D' does not contain peripheral strong components. Then we have $\chi'_{\mathrm{wo}}(D') \leq 2$ by the above analysis. Let ϕ be a weak-odd 2-edge coloring of D' and φ be a 2-edge coloring defined as in **(C)**. Since $d_D(v)$ is odd, color 1 or 2 satisfies the condition $(\overrightarrow{\mathrm{WO}})$ at v under φ. Hence, φ is a weak-odd 2-edge coloring of D.

Now, we may assume that there exists a sink in D', say y'. Let K be the initial strong component of D, and C be a hamiltonian dicycle in K. Take a directed K-y' path P in D' that passes through every vertex not in $V(K)$. Let x be the initial vertex of P. Then we let $\hat{D} = D[A(C \cup P) \cup \{xv, y'v\}]$. Then θ is a weak-odd 2-edge coloring of D because $d_D(v)$ is odd.

Case 2. $v \neq y$.

Recall that y is the sink of D and thus also of D' and $d^+_D(v), d^-_D(v) > 0$. First, suppose that D' has another peripheral vertex, say x. Then x is the source of D'. Obviously, vx and vy are contained in $A(D)$. Let P be a hamiltonian dipath in D'. If $d^+_D(v)$ is odd, then there is a vertex $w \in V(P)$ such that $wv \in A(D)$ and $wy \notin A(P)$, and let $\hat{D} = D[A(P) \cup \{vx, wv, wy, vy\}]$. Otherwise, let $\hat{D} = D[A(P) \cup \{vx\}]$.

Now, D' has exactly one peripheral vertex y. Suppose that $V(D') = V(K) \cup \{y\}$ where K is the initial strong component of D'. Let C be a hamiltonian dicycle in K. If $d^+_D(v)$ is odd, then there is a vertex $w \in V(C)$ such that $wv, wy \in A(D)$. Let $\hat{D} = D[A(C) \cup \{wy, wv\}]$. Otherwise, let $\hat{D} = D[A(C) \cup \{vy\}]$.

Finally, we consider the case that $V(D') \neq V(K) \cup \{y\}$. Take a directed K-y path P in D' that passes through every vertex not in $V(K)$. Let x be the initial vertex of P. By our latest assumption, the arc $xy \notin A(P)$. If $d^+_D(v)$ is even, then let $\hat{D} = D[A(C \cup P) \cup \{vy, xy\}]$. Now, assume that $d^+_D(v)$ is odd. If $xv \notin A(D)$, then there is a vertex $w \in V(D') \setminus \{x, y\}$ such that $wv \in A(D)$ and $wy \notin A(P)$. Let $\hat{D} = D[A(C \cup P) \cup \{wy, wv, xy\}]$. Otherwise, let $\hat{D} = D[A(C \cup P) \cup \{xv\}$.

It follows that θ is a weak-odd 2-edge coloring of D for Case 2. Indeed, color 2 fits the condition ($\overrightarrow{\mathrm{WO}}$) at v while color 1 works for every other vertex. $\square$

Proposition 3.2. *For any semicomplete digraph D, it holds that*

$$\operatorname{def}(D) = \begin{cases} 1 & \text{if } D \text{ is a nontrivial even digraph with just one peripheral vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1, we may assume that D is bad and has a sink y. Let D' be the digraph obtained from D by deleting the vertex y. It is not hard to find that D' is not an even semicomplete digraph. Thus, $\chi'_{\mathrm{wo}}(D') \leq 2$. Apply to $A(D)$ the particular 2-edge coloring constructed as **(C)**. The condition ($\overrightarrow{\mathrm{WO}}$) is satisfied at each vertex apart from y. $\square$

Proposition 3.3. *Every bad semicomplete digraph D admits a 2-edge coloring such that condition ($\overrightarrow{\mathrm{WO}}$) is satisfied at each vertex apart from a prescribed vertex $v \in V(D)$.*

Proof. We may assume that D has a sink y. If $v = y$, then by Proposition 3.2, we are done. Suppose that $v \neq y$. Note that $PS(D)$ has only one nontrivial component K. Observe that $V(K) \cap V_2 = V_2 \setminus \{y^+\}$ is odd-sized, and $V_3 = \emptyset$. If $v \in V_1$, then let $S = \{v\} \cup (V_2 \setminus \{y^+\})$.

If $v^+ \in V_2$, then let $S = V_2 \setminus \{v^+, y^+\}$. Take an S-join H in K, and then color $E(H)$ with color 1 and the rest of the edges of K with color 2. The obtained 2-coloring of D fits the condition. $\square$

4 Extended tournaments

In this section, we characterize the weak-odd chromatic index of extended tournaments. Let $D = (V, A)$ be a digraph with $V = \{v_1, \ldots, v_n\}$. Blow up $v_1, \ldots, v_n$ into independent sets $I_1, \ldots, I_n$ of size $s_1, \ldots, s_n$ respectively, where $s_i \geq 1, i \in [n]$. We call the resulted digraph an extended digraph of D and denote it by ED. Without loss of generality, suppose that $s_1, \ldots, s_\ell$ are odd and others are even where $\ell \leq n$. Denote by $v_i^1, \ldots, v_i^{s_i-1}$ the other $s_i - 1$ copies of v_i in ED for $i \in [n]$. Let $I_i^+ = \{v_i^+, v_i^{1+}, \ldots, v_i^{s_{i-1}+}\}$ and $I_i^- = \{v_i^-, v_i^{1-}, \ldots, v_i^{s_{i-1}-}\}$ for $i \in [n]$. Let V_1', V_2', V_3' and V_1, V_2, V_3 be the vertex partitions of $PS(D)$ and $PS(ED)$ as defined in Section 2, respectively.

Theorem 4.1. *If $\ell = n$, then $\chi'_{\mathrm{wo}}(ED) = \chi'_{\mathrm{wo}}(D)$.*

Proof. For any vertex $u \in V(ED)$, we have that $d^+_{ED}(u) \equiv d^+_D(u) \pmod 2$, $d^-_{ED}(u) \equiv d^-_D(u) \pmod 2$ since each s_j is odd for $j \in [n]$. Therefore, $V_i' \subseteq V_i$ and $|V_i| \equiv |V_i'| \pmod 2$. Thus, $|V(K) \cap V_i| \equiv |V(K') \cap V_i'| \pmod 2$ and $|V(K) \cap V_i| = 0$ if and only if $|V(K') \cap V_i'| = 0$ for $i \in \{2, 3\}$. By Theorem 2.1, $\chi'_{\mathrm{wo}}(ED) = \chi'_{\mathrm{wo}}(D)$. $\square$

In the following, let D be a tournament T with $|V(T)| = n$, $Q_1 = \bigcup_{i=1}^{\ell} I_i$ and $Q_2 = \bigcup_{i=\ell+1}^{n} I_i$. Then we have $V(ET) = Q_1 \cup Q_2$. Denote the orders of ET, Q_1, and Q_2 by q, q_1, and q_2, respectively. Obviously, we have $q = q_1 + q_2$ and $d_{ET}(u) = q - s_i$ for any $u \in I_i$.

Lemma 4.2. *Let V_2 be the vertex set of $PS(ET)$ as defined before, then the cardinality of V_2 is always even.*

Proof. First, suppose that q is even. If $u \in Q_1$, then u contributes 1 to $|V_2|$ as $d_{ET}(u)$ is odd. Otherwise, either $u \in V_1$ or u contributes 2 to $|V_2|$. Therefore, $|V_2| \equiv q_1 \pmod 2$ is even as $q_1 = q - q_2$ is even. Now, suppose that q is odd. If $u \in Q_1$, then either $u \in V_1$ or u contributes 2 to $|V_2|$ as $d_{ET}(u)$ is even. Otherwise, u contributes 1 to $|V_2|$. Therefore, $|V_2| \equiv q_2 \pmod 2$ is even as q_2 is even. $\square$

Theorem 4.3. *If $|V(T)| \leq 3$, then $\chi'_{\mathrm{wo}}(ET) \leq 2$.*

Proof. If $T = K_1$, then $\chi'_{\mathrm{wo}}(ET) = 0$. Suppose that $T = K_2 = v_1v_2$. If both v_1 and v_2 are in Q_1, then $\chi'_{\mathrm{wo}}(ET) = 1$. Otherwise, the only nontrivial component of $PS(ET)$ satisfies Theorem 2.1 and thus $\chi'_{\mathrm{wo}}(ET) \leq 2$.

Now, suppose that $|V(T)| = 3$. Suppose $\ell = 3$, i.e., s_1, s_2 and s_3 are all odd. Then $\chi'_{\text{wo}}(ET) = \chi'_{\text{wo}}(T) = 1$ by Theorem 4.1. It suffices to consider the following two cases under $\ell \le 2$.

Case 1. T is a dicycle and $A(T) = \{v_1v_2, v_2v_3, v_3v_1\}$.

Suppose $\ell = 2$. Then $PS(ET)$ has two nontrivial components K and R with $V(K) = I_1^+ \cup I_2^-$ and $V(R) = I_1^- \cup I_2^+ \cup I_3$. Suppose $\ell \le 1$. Then $PS(ET)$ has three nontrivial components K, R, S such that $V(K) = I_1^+ \cup I_2^-$, $V(R) = I_2^+ \cup I_3^-$, $V(S) = I_3^+ \cup I_1^-$. If $\ell = 2$, then $V(K) \cap V_3 \neq \emptyset$ and $V(R) \cap V_2$ is of even size. If $\ell = 1$, then $V(R) \cap V_2$ is of even size and $V(F) \cap V_3 \neq \emptyset$ for $F \in \{K, S\}$. If $\ell = 0$, then $V(F) \cap V_2$ is of even size for $F \in \{K, R, S\}$. Hence, by Theorem 2.1, $\chi'_{\text{wo}}(ET) \le 2$.

Case 2. T has two peripheral vertices.

Set $\{i, j, t\} = [3]$. Let v_i and v_j be the source and the sink of T, respectively. Then $PS(ET)$ has one nontrivial component K such that $V(K) = I_i^+ \cup I_j^- \cup I_t$ when s_i and s_j are odd, and $V(K) = I_i^+ \cup I_j^- \cup I_t^+ \cup I_t^-$ otherwise. If $s_i + s_j$ is even, then $V(K) \cap V_2$ is of even size. Otherwise, $V(K) \cap V_3 \neq \emptyset$. Hence, by Theorem 2.1, $\chi'_{\text{wo}}(ET) \le 2$. □

In the following, we consider the case when $|V(T)| > 3$. An extended tournament ET with $|V(T)| > 3$ is called 'bad' if all of the following conditions are satisfied.

(a) ET is of odd order;

(b) Exactly one independent set is even, i.e., $|I_n|$ is even;

(c) $N_T^+(v_n)$ dominates $N_T^-(v_n)$;

(d) Either $|N_T^-(v_n)| = 1$ or $|N_T^+(v_n)| = 1$.

We call every other extended tournament 'good'.

Theorem 4.4. *For $n > 3$ and $\ell < n$, $\chi'_{\text{wo}}(ET) = 3$ if and only if ET is bad.*

Proof. Recall that $|V(ET)| = q$. First, consider that ET is bad, we need to show that $\chi'_{\text{wo}}(ET) = 3$. Without loss of generality, let $|N_T^+(v_n)| = 1$ and $N_T^+(v_n) = v_i$. Then $d_{ET}^-(v_i) = s_n$ and $d_{ET}^+(v_i) = d_{ET}^-(v_n) = q - s_n - s_i$ are even, and $d_{ET}^+(v_n) = s_i$ is odd. Furthermore, for $u \in N_{ET}^-(v_n)$, $d_{ET}(u)$ is even and either $u \in V_1$ or u contributes 2 to $|V_2|$. Observe that $PS(ET)$ contains exactly two components K and R with $V(K) = I_n^+ \cup I_i^-$ and $V(R) = V(PS(ET)) \setminus V(K)$. Note that $I_i^- \subseteq V_2$, $|I_i^-|$ is odd and $V_3 = I_n^+$. Thus, we have that $|V(R) \cap V_2|$ is odd by Lemma 4.2 and $V(R) \cap V_3 = \emptyset$ as $V_3 \subseteq V(K)$. Therefore, by Theorem 2.1, $\chi'_{\text{wo}}(ET) = 3$.

Since each digraph has weak-odd chromatic index at most three, it suffices to show that if ET is good, then $\chi'_{\text{wo}}(ET) \le 2$. Now, let ET be good. We proceed our proof by considering the number of peripheral vertices in T.

First, suppose that T has a source v_i and a sink v_j. Then $PS(ET)$ contains exactly one nontrivial component K with $V(K) = V(PS(ET)) \setminus (I_i^- \cup I_j^+)$. If $v_i, v_j \in Q_1$, then $V(K) \cap V_3 \neq \emptyset$ when q is even and $V(K) \cap V_2$ is of even order or $V(K) \cap V_3 \neq \emptyset$ when q is odd. Consider, without loss of generality, that $v_i \in Q_1$ and $v_j \in Q_2$. If q is even, then $I_i^+ \subseteq V_3$, otherwise $I_j^- \subseteq V_3$. In the case when $v_i, v_j \in Q_2$, then by Lemma 4.2, we have that $V(K) \cap V_2$ is of even order. Therefore, by Theorem 2.1, $\chi'_{\rm wo}(ET) \leq 2$.

Next, suppose that T has a peripheral vertex v_j. Without loss of generality, let v_j be a sink. Then we have $V(PS(ET)) = V(K) \cup I_j^+$, where K is a nontrivial component of $PS(ET)$. Assume that $v_j \in Q_1$. If q is odd, then $\emptyset \neq V_3 \subseteq V(K)$ as $Q_2 \neq \emptyset$. Otherwise, $V(K) \cap V_3 \neq \emptyset$ as $I_j^- \subseteq (V_3 \cap V(K))$. If $v_j \in Q_2$, then K satisfies Theorem 2.1 due to Lemma 4.2. Therefore, $\chi'_{\rm wo}(ET) \leq 2$.

Finally, suppose that $\delta^0(T) \geq 1$. We choose a vertex v_i from Q_2 such that $A(N_T^-(v_i), N_T^+(v_i)) \neq \emptyset$, otherwise, let v_i be any vertex in Q_2. Now, we present a vertex partition $X \cup U \cup W$ of $PS(ET)$ with respect to v_i. Let $X = I_i$ if $v_i \in V_1$, otherwise, let $X = X_1 \cup X_2$ with $X_1 = I_i^+$ and $X_2 = I_i^-$. Define $U = U_1 \cup U_2 \cup U_3$ and $W = W_1 \cup W_2 \cup W_3$ as follows.

$$U_1 = \{u^+ : u \in N_{ET}^-(v_i) \setminus V_1\},\ U_2 = \{u^- : u \in N_{ET}^-(v_i) \setminus V_1\},\ U_3 = N_{ET}^-(v_i) \cap V_1;$$
$$W_1 = \{w^+ : w \in N_{ET}^+(v_i) \setminus V_1\},\ W_2 = \{w^- : w \in N_{ET}^+(v_i) \setminus V_1\},\ W_3 = N_{ET}^+(v_i) \cap V_1;$$

Suppose that $PS(ET)$ is a connected graph. Then V_2 is of even order or V_3 is nonempty by Lemma 4.2. Hence, by Theorem 2.1, $\chi'_{\rm wo}(ET) \leq 2$. So, in the following we consider that $PS(ET)$ is not a connected graph. If both U_1 and W_1 are empty sets, then U_3 and W_3 are nonempty. Since T is a tournament, there are edges between U_3 and W_3. Thus, $PS(ET)$ is a connected graph. Therefore, without loss of generality, we may assume that $U_1 \neq \emptyset$. Let K be the nontrivial component of $PS(ET)$ that contains U_1. It suffices to show the following two claims.

Claim 1 If v_i satisfies $A(N_T^-(v_i), N_T^+(v_i)) \neq \emptyset$, then $\chi'_{\rm wo}(ET) \leq 2$.

Proof. We have $U_1 \cup U_3 \cup X \cup W_3 \cup W_2 \subseteq V(K)$. If $U_2 \subseteq V(K)$, then $PS(ET)$ is connected because $\delta^0(ET) \geq 1$ and W_1 (if exists) is an independent set of $PS(ET)$. So, we assume that there exists a vertex $v_j^- \in U_2$ such that $v_j^- \notin V(K)$. Then v_j is a source of the subdigraph of ET induced by $N_{ET}^-(v_i)$. Since $d_T^-(v_j) > 0$, there must be a vertex $v_t^+ \in W_1$ such that $v_t^+ v_j^- \in E(PS(ET))$ and $v_t^+ \notin V(K)$. We have that v_t is a sink of subdigraph of ET induced by $N_{ET}^+(v_i)$. Thus, $PS(ET)$ has two nontrivial components K and R with $R = I_j^- \cup I_t^+$. If both s_j and s_t are even, then $V(R) \cap V_2$ is of even size and so $V(K) \cap V_2$ is of even size by Lemma 4.2. If both s_j and s_t are odd, then $V(R) \cap V_3 \neq$ and $V(K) \cap V_2$ is of even size by Lemma 4.2. If exactly one of s_i and s_j is odd, then $V(R) \cap V_3 \neq \emptyset$, and $V(K) \cap V_3 \neq \emptyset$ because $d_{ET}(v_i) = q - s_i$, s_i is even and $d_{ET}(v_j^+) = d_{ET}(v_t^-) = q - s_j - s_t$. Hence, by Theorem 2.1, $\chi'_{\rm wo}(ET) \leq 2$. ■

Claim 2 If $A(N_T^-(v), N_T^+(v)) = \emptyset$ for each $v \in Q_2$, then $\chi'_{\text{wo}}(ET) \leq 2$.

Proof. If $|N_T^+(v_i)| \geq 2$ and $|N_T^-(v_i)| \geq 2$, then $PS(ET)$ is a connected graph. So, without loss of generality we may assume that $|N_T^-(v_i)| \geq 2$ and $|N_T^+(v_i)| = 1$. If $v_i \in V_1$ or $W_3 \neq \emptyset$, then $PS(ET)$ is a connected graph. So, in the following we assume that $v_i \notin V_1$ and $W_3 = \emptyset$. Let $v_j \in N_{ET}^+(v_i)$. Thus, $PS(ET)$ has two nontrivial components K and R with $R = I_i^+ \cup I_j^-$. If s_j is even, then $V(R) \cap V_2$ is of even size and so $V(K) \cap V_2$ is of even size by Lemma 4.2. If s_j is odd, then q is odd because $v_i \notin V_1$. Since ET is good, there is a vertex $u \in Q_2$ with $u \notin I_i$. Then $V(R) \cap V_3 = I_i^+$ and $V(K) \cap V_3 \neq \emptyset$ as $d_{ET}(u)$ is odd and $\{u^+, u^-\} \subseteq V(K)$. Again by Theorem 2.1, $\chi'_{\text{wo}}(ET) \leq 2$. ■

This completes the proof of Theorem 4.4. □

Theorem 4.5. *For any extended tournament ET, it holds that*

$$\chi'_{\text{wo}}(ET) = \begin{cases} 0 & \text{if } T = K_1, \\ 1 & \text{if } \ell = n \text{ and } \chi'_{\text{wo}}(T) = 1, \\ 3 & \text{if } \ell = n \text{ and } \chi'_{\text{wo}}(T) = 3 \text{ or } ET \text{ is bad}, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.1, Theorem 4.3 and Theorem 4.4, we only need to declare the case that $\chi'_{\text{wo}}(ET) = 1$. Clearly, if there are s_i and s_j with different parity, then $\chi'_{\text{wo}}(ET) \geq 2$. By Theorem 4.1, if each s_i is odd, then $\chi'_{\text{wo}}(ET) = \chi'_{\text{wo}}(T)$. Therefore, in this case, $\chi'_{\text{wo}}(ET) = 1$ if and only if $\chi'_{\text{wo}}(T) = 1$. Now, assume that each s_i is even. Thus, we have that for each vertex $v \in ET$, both $d_{ET}^+(v)$ and $d_{ET}^-(v)$ are even, then $\chi'_{\text{wo}}(ET) \geq 2$. □

Proposition 4.6. *If $\chi'_{\text{wo}}(ET) = 3$, then ET admits a 2-edge coloring such that condition ($\overrightarrow{\text{WO}}$) is satisfied at each vertex apart from a prescribed vertex $x \in V(T)$.*

Proof. If $\ell = n$ and $\chi'_{\text{wo}}(T) = 3$, then $|V(T)|$ is odd and T has a peripheral vertex, say a sink v_i, by Theorem 1.1. Hence, $q = |V(ET)|$ is odd. Observe that $PS(ET)$ has exactly one nontrivial component K with $V(PS(ET)) = V(K) \cup I_i^-$. If $x \in V_1$, then let $F = \{x\} \cup (V_2 \setminus I_i^-)$. If $x^+ \in V_2$, then let $F = V_2 \setminus (\{x^+\} \cup I_i^-)$. Take an F-join H in K, and color $E(H)$ with color 1 and the rest of the edges of K with color 2. The obtained 2-coloring fits the condition.

Now, suppose that ET is bad. We have that $|V(ET)|$ is odd and $\delta^0(T) \geq 1$ by the definition. Then $\chi'_{\text{wo}}(T) \leq 2$ by Theorem 1.1. Let ET' be the extended tournament obtained from ET by deleting x. If $\chi'_{\text{wo}}(ET') \leq 2$, then we obtain a 2-edge coloring of ET from ET' by the coloring **(C)** such that the condition ($\overrightarrow{\text{WO}}$) is satisfied at each vertex apart from $x \in V(ET)$. Now, we only need to show that $\chi'_{\text{wo}}(ET') \leq 2$. If $x \in Q_2$, then each independent set is odd in ET'. So, $\chi'_{\text{wo}}(ET') = \chi'_{\text{wo}}(T) \leq 2$. If $x \in Q_1$, then we have that ET' is good. Therefore, by Theorem 4.5, we have $\chi'_{\text{wo}}(ET') \leq 2$. □

By Proposition 4.6, we obtain directly the following proposition.

Proposition 4.7. *For any extended tournament ET, it holds that*

$$\operatorname{def}(ET) = \begin{cases} 1 & \text{if } \ell = n \text{ and } \chi'_{\text{wo}}(T) = 3 \text{ or } ET \text{ is bad,} \\ 0 & \text{otherwise.} \end{cases}$$

5 Weak-odd edge covering of tournaments

Here we show that Question 1.4 holds for tournaments.

Proposition 5.1. *[6] Every tournament admits a 2-edge coloring such that condition $(\overrightarrow{\mathrm{WO}})$ is satisfied at each vertex apart from a prescribed vertex $v \in V(T)$.*

Theorem 5.2. *Let T be a tournament. If $\chi'_{\text{wo}}(T) = 3$, then T admits a weak-odd 2-edge covering φ such that the intersection of color classes is contained within a singleton arc in $A(T)$.*

Proof. By Theorem 1.1, we have that T is nontrivial, of odd order, and has just one peripheral vertex. Suppose that T has a sink y. By Proposition 5.1, T admits a 2-edge coloring ϕ such that $(\overrightarrow{\mathrm{WO}})$ is satisfied at each vertex apart from y. Let T'_i be the spanning subdigraph of T whose arc set is the color set $\phi^{-1}(i)$ for $i \in [2]$. Then both $d^-_{T'_1}(y)$ and $d^-_{T'_2}(y)$ are even. If there is an arc $xy \in A(T)$ such that $\phi(xy) = i$ and $(\overrightarrow{\mathrm{WO}})$ is satisfied by color i at x for $i \in [2]$, then φ can be given as follows: $\varphi(xy) = \{1, 2\}$ and $\varphi = \phi$ for other arcs. It is obvious to see that the condition $(\overrightarrow{\mathrm{WO}})$ is satisfied by color $3 - i$ at y and others are taken care of by the same color under ϕ for $i \in [2]$.

Now, suppose that for any arc $xy \in A(T)$ and $i \in [2]$, if $\phi(xy) = i$, then only color $3 - i$ satisfies the condition $(\overrightarrow{\mathrm{WO}})$ at x. Without loss of generality, assume that there is an arc $xy \in A(T)$ with $\phi(xy) = 1$, which implies that the condition $(\overrightarrow{\mathrm{WO}})$ is satisfied by color 2 at x. Then there is a vertex z such that $zx \in A(T)$ with $\phi(zx) = 2$. We can define φ as follows: $\varphi(zx) = 1$, $\varphi(zy) = \{1, 2\}$, $\varphi(xy) = 2$ and $\varphi = \phi$ for other arcs. It is easy to check that every vertex in $V(T) \setminus \{x, y, z\}$ is taken care of by the same color as in ϕ. Note that T is a tournament of odd order, thus $d^+_T(x)$ and $d^-_T(x)$ have the same parity. Combining that the condition $(\overrightarrow{\mathrm{WO}})$ is satisfied by color 2 at x, we have that both $d^+_{T'_2}(x)$ and $d^-_{T'_2}(x)$ are odd whereas $d^+_{T'_1}(x)$ and $d^-_{T'_1}(x)$ are even. Let T_i be the spanning subdigraph of T whose arc set is in $\varphi^{-1}(i)$ for $i \in [2]$. We finish the proof by the following.

Assume that the condition $(\overrightarrow{\mathrm{WO}})$ is satisfied by color i at z under ϕ for $i \in [2]$. Then we have that $\phi(zy) = 3 - i$, $d^+_{T'_i}(z)$ and $d^-_{T'_i}(z)$ are odd whereas $d^+_{T'_{3-i}}(z)$ and $d^-_{T'_{3-i}}(z)$ are even. Combining the coloring φ, we have that $d^+_{T_1}(x) = d^+_{T'_1}(x) - 1$, $d^-_{T_1}(x) = d^-_{T_1}(x) + 1$, $d^+_{T_i}(z) = d^+_{T'_i}(z) + 2(2 - i)$, $d^-_{T_i}(z) = d^-_{T'_i}(z)$ and $d^-_{T_{3-i}}(y) = d^-_{T'_{3-i}}(y) - (-1)^i$ are odd.

Hence, we are done. $\square$

Acknowledgements. This work was partially supported by the National Natural Science Foundation of China (Nos. 12001296 and 12161141006) and the Natural Science Foundation of Tianjin (Nos. 20JCJQJC00090, 20JCZDJC00840, and 21JCQNJC00060).

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