More on the rainbow disconnection in graphs*

Xuqing Bai, Renying Chang, Zhong Huang, Xueliang Li

Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

Email: baixuqing0@163.com, changrysd@163.com

Abstract

Let G be a nontrivial edge-colored connected graph. An edge-cut R of G is called a rainbow-cut if no two edges of it are colored the same. An edge-colored graph G is rainbow disconnected if for every two vertices u and v of G, there exists a u-v-rainbow-cut separating them. For a connected graph G, the rainbow disconnection number of G, denoted by $\mathrm{rd}(G)$, is defined as the smallest number of colors that are needed in order to make G rainbow disconnected. In this paper, we first determine the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ for any given integers k and n with $1 \leq k \leq n-1$, which solves a conjecture posed only for the case that n is odd in [Chartrand et al., Rainbow disconnection in graphs, Discuss. Math. Graph Theory 38(4)(2018), 1007–1021]. From this result and a result in their paper, we obtain Erdős-Gallai-type results for $\mathrm{rd}(G)$. Secondly, we discuss bounds on $\mathrm{rd}(G)$ for complete multipartite graphs, critical graphs with respect to the chromatic number, minimal graphs with respect to the chromatic index, and regular graphs, and we also give the values of $\mathrm{rd}(G)$ for several special graphs. Thirdly, we get Nordhaus-Gaddum-type bounds for $\mathrm{rd}(G)$, and examples are given to show that the upper and lower bounds are sharp. Finally, we show that for a connected graph G, to compute $\mathrm{rd}(G)$ is NP-hard. In particular, we show that it is already NP-complete to decide if $\mathrm{rd}(G) = 3$ for a connected cubic graph. Moreover, we show that for a given edge-colored (with an unbounded number of colors) connected graph G it is NP-complete to decide whether G is rainbow disconnected.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the *degree* and the *neighborhood* of v in G (or simply $d(v)$ and $N(v)$ respectively, when the graph G is clear from the context). We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of G, respectively. The notion $\overline{G}$ denotes the *complement* of G. For any notation or terminology not defined here, we follow those used in [4].

Throughout this paper, we use P_n, C_n, K_n to denote the path, cycle and complete graph of order n, respectively. Given two disjoint graphs G and H, the *join* of G and H, denoted by $G \vee H$, is obtained from the vertex-disjoint copies of G and H by adding all edges between the vertices in $V(G)$ and the vertices in $V(H)$.

Throughout the paper, $[k]$ denotes the set $\{1, 2, ..., k\}$ of integers. Let G be a graph with an *edge-coloring* c: $E(G) \to [k]$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. When adjacent edges of G receive different colors by c, the edge-coloring c is called *proper*. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper edge-coloring of G. By a famous theorem of Vizing [22], one has that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph G. If $\chi'(G) = \Delta(G)$, then G is said to be in *Class* 1; if $\chi'(G) = \Delta(G) + 1$, then G is said to be in *Class* 2.

A path is called *rainbow* if no two edges of the path are colored the same. An edge-colored graph G is called *rainbow connected* if every two vertices of G are connected by a rainbow path in G. An edge-coloring under which G is rainbow connected is called a *rainbow connection coloring*. Clearly, if a graph is rainbow connected, it must be connected. For a connected graph G, the *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. The concept of rainbow connection was introduced by Chartrand et al. [7] in 2008. For more details on rainbow connection, we refer the reader to a book [18] and two survey papers [17, 19].

In this paper, we investigate a new concept introduced by Chartrand et al. in [6] that is somehow reverse to the rainbow connection.

An *edge-cut* of a connected graph G is a set F of edges such that $G-F$ is disconnected. The minimum number of edges in an edge-cut of G is the *edge-connectivity* of G, denoted by $\lambda(G)$. We have the well-known inequality $\lambda(G) \leq \delta(G)$. For two vertices u and v of G, let $\lambda_G(u,v)$ (or simply $\lambda(u,v)$ when the graph G is clear from the context), denote the minimum number of edges in an edge-cut F such that u and v lie in different components of $G-F$. A *u-v-path* is a path with ends u and v. The following proposition presents an alternate interpretation of $\lambda(u,v)$ (see [12], [13]).

Proposition 1.1 *For every two vertices u and v in a graph G, $\lambda(u,v)$ is equal to the maximum number of pairwise edge-disjoint u-v-paths in G.*

An edge-cut R of an edge-colored connected graph G is called a *rainbow-cut* if no two edges in R are colored the same. A rainbow-cut R of G is said to *separate two vertices u and v* of G if u and v belong to different components of $G-R$. Such a rainbow-cut is called a *u-v-rainbow-cut*. An edge-colored graph G is called *rainbow disconnected* if for every two vertices u and v of G, there exists a u-v-rainbow-cut in G separating them. In this case, the edge-coloring is called a *rainbow disconnection coloring* of G. For a connected graph G, we similarly define the *rainbow disconnection number* (or rd-*number* for short) of G, denoted by $\mathrm{rd}(G)$, as the smallest number of colors that are needed in order to make G rainbow disconnected. A rainbow disconnection coloring with $\mathrm{rd}(G)$ colors is called an rd-*coloring* of G.

One of the many interesting problems in extremal graph theory is Erdős-Gallai-type problem which is used to determine the maximum or minimum size of a graph with a given value of a graph parameter. We will obtain Erdős-Gallai-type results for the graph parameter $\mathrm{rd}(G)$.

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus-Gaddum-type" is given because Nordhaus and Gaddum are the first to establish [21] the following type of inequalities for the chromatic number in 1956. They proved that if G and $\overline{G}$ are complementary graphs on n vertices whose chromatic numbers are $\chi(G)$ and $\chi(\overline{G})$, respectively, then

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1,$$

$$n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2.$$

For more results of Nordhaus-Gaddum-type, we refer to papers [8, 14, 15] and a survey paper [2].

The remainder of this paper will be organized as follows. In Section 2, we determine the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ for given integers k and n with $1 \leq k \leq n-1$. This solves a conjecture posed only for n odd by Chartrand et al. in [6]. From this and a result in [6], we obtain Erdős-Gallai-type results for $\mathrm{rd}(G)$. In Section 3, we discuss bounds on the rainbow disconnection number of graphs depending on some parameters, and we also give the values of $\mathrm{rd}(G)$ for some well-known special graphs. In Section 4, we obtain Nordhaus-Gaddum-type bounds for $\mathrm{rd}(G)$ and show that these bounds are sharp. In Section 5, we show that to compute $\mathrm{rd}(G)$ for a connected graph G is NP-hard. In particular, we show that it is already NP-complete to decide if $\mathrm{rd}(G) = 3$ for a connected cubic graph G. Moreover, we show that for a given edge-colored (with an unbounded number of colors) connected graph G, it is NP-complete to decide whether G is rainbow disconnected under the given edge-coloring.

2 Erdős-Gallai-type results

In this section, we consider two kinds of Erdős-Gallai-type problems for $\mathrm{rd}(G)$.

Problem A. Given two positive integers n and k with $1 \leq k \leq n-1$, compute the maximum integer $g(n,k)$ such that for any graph G of order n, if $|E(G)| \leq g(n,k)$, then $\mathrm{rd}(G) \leq k$.

Problem B. Given two positive integers n and k with $1 \leq k \leq n-1$, compute the minimum integer $f(n,k)$ such that for any graph G of order n, if $|E(G)| \geq f(n,k)$, then $\mathrm{rd}(G) \geq k$.

It is worth mentioning that the two parameters $f(n,k)$ and $g(n,k)$ are equivalent to the following two parameters. Let $t(n,k) = \min\{|E(G)| : G$ is a connected graph with $|V(G)| = n$ and $\mathrm{rd}(G) \geq k\}$ and $s(n,k) = \max\{|E(G)| : G$ is a connected graph with $|V(G)| = n$ and $\mathrm{rd}(G) \leq k\}$. It is easy to see that $g(n,k) = t(n,k+1) - 1$ for $1 \leq k \leq n-2$ and $f(n,k) = s(n,k-1) + 1$ for $2 \leq k \leq n-1$.

To solve Problems A and B, the following results will be used.

For given integers k and n with $1 \leq k \leq n-1$, the authors in [6] determined the minimum size of a connected graph G of order n with $\mathrm{rd}(G) = k$.

Lemma 2.1 [6] *For integers k and n with $1 \leq k \leq n-1$, the minimum size of a connected graph of order n with* $\mathrm{rd}(G) = k$ *is $n + k - 2$.*

For the maximum size, they posed the following conjecture only for n odd.

Conjecture 2.2 *Let k and n be integers with $1 \leq k \leq n-1$ and $n \geq 5$ is odd. Then the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ is $\frac{(k+1)(n-1)}{2}$.*

We will show the following result for the maximum size, regardless of whether n is odd or even.

Theorem 2.3 *Let k and n be integers with $1 \leq k \leq n-1$. Then the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ is $\left\lfloor \frac{(k+1)(n-1)}{2} \right\rfloor$.*

Before we give the proof of Theorem 2.3, some auxiliary lemmas are stated as follows.

Lemma 2.4 [6] *If G is a nontrivial connected graph, then*

$$\lambda(G) \leq \lambda^+(G) \leq \mathrm{rd}(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where the upper edge-connectivity $\lambda^+(G)$ is defined by $\lambda^+(G) = \max\{\lambda(u, v) : u, v \in V(G)\}$.

Lemma 2.5 [6] *Let G be a nontrivial connected graph. Then $\mathrm{rd}(G) = 1$ if and only if G is a tree.*

Lemma 2.6 [6] *If G is a cycle of order $n \geq 3$, then $\mathrm{rd}(G) = 2$.*

Lemma 2.7 [6] *For each integer $n \geq 2$, $\mathrm{rd}(K_n) = n - 1$.*

Lemma 2.8 [6] *Let G be a connected graph of order $n \geq 2$. Then $\mathrm{rd}(G) = n - 1$ if and only if G contains at least two vertices of degree $n - 1$.*

Lemma 2.9 [20] *Let G be a graph of order n $(n \geq k + 2 \geq 3)$. If $|E(G)| > \frac{k+1}{2}(n-1) - \frac{1}{2}\sigma_k(G)$, where $\sigma_k(G) = \sum\limits_{\substack{x \in V(G) \\ d(x) \leq k}} (k - d(x))$, then $\lambda^+(G) \geq k + 1$.*

We give an observation before the proof of Lemma 2.11.

Observation 2.10 *Let G be a graph and u be a vertex of G. If G admits an edge-coloring c with k colors such that the set E_x of edges incident with x is rainbow for every vertex x in $V(G - u)$, then $\mathrm{rd}(G) \leq k$.*

Lemma 2.11 *For a graph G, the following results hold.*

(i) *For any vertex u of G, let $H = G - u$. Then* $\mathrm{rd}(G) \leq \Delta(H) + 1$.

(ii) *If there exists a vertex u of G such that $H = G - u$ is in Class 1 and $d_H(x) \leq \Delta(H) - 1$ for any $x \in N_G(u)$, then* $\mathrm{rd}(G) \leq \Delta(H)$.

(iii) *Let uv be an edge of G and $H = G - uv$. If $\chi'(H) = \Delta(H) = \Delta(G)$, then* $\mathrm{rd}(G) \leq \Delta(G)$.

Proof. (i) Let $H = G - u$. Then we obtain a proper edge-coloring c_0 of H using colors from the set $[\Delta(H) + 1]$. For each vertex $x \in V(H)$, since $d_H(x) \leq \Delta(H)$, there is an $a_x \in [\Delta(H) + 1]$ such that a_x is not assigned to any edge incident with x in H. Since $E(G) = E(H) \cup \{ux \mid x \in N_G(u)\}$, we now extend the edge-coloring c_0 of H to an edge-coloring c of G by assigning $c(ux) = a_x$ for any vertex $x \in N_G(u)$. Note that the set E_x of edges incident with x is a rainbow set for each vertex $x \in V(H)$. Hence, $\mathrm{rd}(G) \leq \Delta(H) + 1$ by Observation 2.10.

(ii) Since H is in Class 1, we have $\chi'(H) = \Delta(H)$. Then we obtain a proper edge-coloring c_0 of H using colors from $[\Delta(H)]$. For each vertex $x \in N_G(u)$, since $d_H(x) \leq \Delta(H) - 1$, there is an $a_x \in [\Delta(H)]$ such that a_x is not assigned to any edge incident with x in H. Since $E(G) = E(H) \cup \{ux \mid x \in N_G(u)\}$, we now extend the edge-coloring c_0 of H to an edge-coloring c of G by assigning $c(ux) = a_x$ for any vertex $x \in N_G(u)$. Note that the set E_x of edges incident with x is a rainbow set for each vertex $x \in V(H)$. Hence, $\mathrm{rd}(G) \leq \Delta(H)$ by Observation 2.10.

(iii) Since $\chi'(H) = \Delta(H) = \Delta(G)$, we obtain a proper edge-coloring c_0 of H using colors from $[\Delta(G)]$. Since $\Delta(H) = \Delta(G)$, we have $d_H(u) < \Delta(G)$, and thus there is an $a_u \in [\Delta(G)]$ such that a_u is not assigned to any edge incident with u in H. Now we extend c_0 to an edge-coloring c of G by defining $c(uv) = a_u$. Note that the set E_x of edges incident with x in G is a rainbow set for each vertex $x \in V(G) \setminus v$. Hence, $\mathrm{rd}(G) \leq \Delta(G)$ by Observation 2.10. $\square$

Proof of Theorem 2.3. If $k = n - 1$, the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = n - 1$ is $\frac{n(n-1)}{2}$ since $\mathrm{rd}(K_n) = n - 1$ by Lemma 2.7, and thus the result is true. Now we consider k with $1 \leq k \leq n - 2$. Suppose that $|E(G)| > \frac{(k+1)(n-1)}{2} - \frac{1}{2}\sigma_k(G)$. Then $\mathrm{rd}(G) \geq \lambda^+(G) \geq k + 1$ by Lemmas 2.4 and 2.9. Therefore, if $\mathrm{rd}(G) = k$, then $|E(G)| \leq \frac{(k+1)(n-1)}{2} - \frac{1}{2}\sigma_k(G) \leq \frac{(k+1)(n-1)}{2}$ since $\sigma_k(G)$ is nonnegative.

It remains to show that for each pair of integers k and n with $1 \leq k \leq n - 2$, there exists a connected graph G_k with order n and size $\left\lfloor \frac{(k+1)(n-1)}{2} \right\rfloor$ such that $\mathrm{rd}(G_k) = k$. We distinguish the following two cases.

Case 1. n is odd.

For $n = 3$, it is easy to verify that the result is true for $G_k = P_3$. For $n \geq 5$, the construction of the graph G_k was already given in [6], where the inequality $\mathrm{rd}(G_k) \leq k$ was proved. Here we restate it as follows. Set $G_k = H_k \vee K_1$, where H_k is a $(k-1)$-regular graph of order $n-1$ and $K_1 = \{u\}$. Since $n-1$ is even, such graphs H_k exist. Then G_k is a connected graph of order n having one vertex u of degree $n-1$ and $n-1$ vertices of degree k, and the size of G_k is $\frac{(k+1)(n-1)}{2}$.

Since $\Delta(H_k) = k-1$, we obtain that $\mathrm{rd}(G_k) \leq \Delta(H_k) + 1 = k$ by Lemma 2.11(i). Note that $|E(G_k)| = \frac{(k+1)(n-1)}{2} > \frac{k(n-1)}{2} \geq \frac{k(n-1)}{2} - \frac{1}{2}\sigma_{k-1}(G_k)$ since $\sigma_{k-1}(G_k)$ is nonnegative. Thus, $\lambda^+(G_k) \geq k$ by Lemma 2.9. Combining with Lemma 2.4, we have $\mathrm{rd}(G_k) \geq k$. Therefore, the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ is $\left\lfloor \frac{(k+1)(n-1)}{2} \right\rfloor$ when $1 \leq k \leq n-2$ and n is odd.

Case 2. n is even.

For $n = 2t \geq 4$, we construct a graph G_k as follows. Let $G = K_{2t}$ and $V(G) = \{u, v_0, v_1, \cdots, v_{2t-2}\}$. Arrange $v_0, v_1, \cdots, v_{2t-2}$ in the order on the vertices of a regular $(2t-1)$ polygon, and let u be the center of the regular $(2t-1)$ polygon. Figure 1 shows the vertex order of a regular $(2t-1)$ polygon with $t = 15$. For $0 \leq i \leq 2t-2$, let $E_i = \{uv_i\} \cup \{e | e \text{ is perpendicular to the line containing } uv_i,\ e \in E(K_{2t})\}$. In Figure 1, the edges of the sets E_0 and E_2 are drawn for $t = 15$. Obviously, each $G[E_i]$ forms a 1-factor of K_{2t}, and $E_0,\ E_1,\ \cdots,\ E_{2t-2}$ are edge-disjoint. Let $H_{k-1} = G[E_1 \cup E_2 \cup \cdots \cup E_{k-1}]$ where $2 \leq k \leq 2t-1$. In particular, H_0 is an empty graph. It follows that H_{k-1} is $(k-1)$-regular and H_{k-1} is 1-factorable, that is, $\chi'(H_{k-1}) = k-1$.

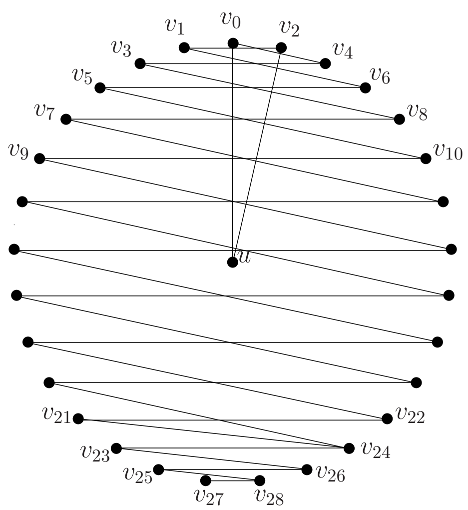

Figure 1: Graph for the proof of Theorem 2.3.

By the previous constructions, $E(H_{k-1})$, $\{uv_0\}$, $\bigcup\limits_{i=k}^{2t-2}\{uv_i\}$, and the edge set $\{v_1v_2, v_3v_4, \cdots, v_{2\lfloor\frac{k-1}{2}\rfloor-1}v_{2\lfloor\frac{k-1}{2}\rfloor}\}$ are edge-disjoint. Let $G_k = H_{k-1} + \{uv_0\} + \bigcup\limits_{i=k}^{2t-2}\{uv_i\} + \{v_1v_2, v_3v_4, \cdots, v_{2\lfloor\frac{k-1}{2}\rfloor-1}v_{2\lfloor\frac{k-1}{2}\rfloor}\}$. Then G_k is a graph of order n with $|E(G_k)| = \frac{(k-1)n}{2} + 1 + (n-k-1) + \lfloor\frac{k-1}{2}\rfloor = \left\lfloor\frac{(k+1)(n-1)}{2}\right\rfloor$. Since $\chi'(H_{k-1}) = k-1$, we obtain a proper edge-coloring c_0 of H_{k-1} using colors from $[k-1]$. We can extend c_0 to an edge-coloring c of G_k by assigning a new color k to all newly added edges in H_{k-1}. Note that the set E_x of edges incident with x in G_k is a rainbow set for each vertex $x \in V(G_k) \setminus u$. Therefore, $\mathrm{rd}(G_k) \leq k$ by Observation 2.10. On the other hand, $E(G_k) = \left\lfloor\frac{(k+1)(n-1)}{2}\right\rfloor > \frac{k(n-1)}{2}$ since $n \geq 4$. It follows from Lemmas 2.4 and 2.9 that $\mathrm{rd}(G_k) \geq k$. Therefore, the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ is $\left\lfloor\frac{(k+1)(n-1)}{2}\right\rfloor$ when $1 \leq k \leq n-2$ and n is even. $\square$

We are now in the position to solve Problem A by giving the exact value of $g(n,k)$, using Lemma 2.1.

Theorem 2.12 *For integers k and n with $1 \leq k \leq n-1$,*

$$g(n,k) = \begin{cases} \frac{n(n-1)}{2}, & \text{if } k = n-1, \\ n+k-2, & \text{if } 1 \leq k \leq n-2. \end{cases}$$

Proof. First, since $\mathrm{rd}(K_n) = n-1$, we get $g(n, n-1) = \frac{n(n-1)}{2}$. Next, it follows from Lemma 2.1 that $t(n,k) = n+k-2$ for $1 \leq k \leq n-1$. Thus, $g(n,k) = t(n,k+1)-1 = n+k-2$ for $1 \leq k \leq n-2$. $\square$

Now we solve Problem B by giving the exact value of $f(n,k)$.

Theorem 2.13 *For integers k and n with $1 \leq k \leq n-1$,*

$$f(n,k) = \begin{cases} n-1, & \text{if } k = 1, \\ \left\lfloor\frac{k(n-1)}{2}\right\rfloor + 1, & \text{if } 2 \leq k \leq n-1. \end{cases}$$

Proof. First, let T be a nontrivial tree of order n. Since $\mathrm{rd}(T) = 1$ by Lemma 2.5, we get $f(n,1) = n-1$. Next, it follows from Theorem 2.3 that $s(n,k) = \frac{(k+1)(n-1)}{2}$ for $1 \leq k \leq n-1$. Thus, $f(n,k) = s(n,k-1)+1 = \left\lfloor\frac{k(n-1)}{2}\right\rfloor + 1$ for $2 \leq k \leq n-1$. $\square$

3 The rd-numbers of some classes of graphs

In this section, we investigate the rainbow disconnection numbers of complete multipartite graphs, critical graphs with respect to the chromatic number, minimal graphs with respect to the chromatic index, and regular graphs.

At first, we give the rainbow disconnection numbers of complete multipartite graphs.

Theorem 3.1 *If $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph with order n where $k \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_k$, then*

$$\mathrm{rd}(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} n - n_2, & \text{if } n_1 = 1, \\ n - n_1, & \text{if } n_1 \geq 2. \end{cases}$$

To prove Theorem 3.1 we need a lemma below. Let G_Δ denote the *core* of G, that is, the subgraph of G induced by the vertices of maximum degree $\Delta(G)$.

Lemma 3.2 [1] *Let G be a connected graph. If every connected component of G_Δ is a unicyclic graph or a tree, and G_Δ is not a disjoint union of cycles, then G is in Class 1.*

Proof of Theorem 3.1. Let $V_1, V_2, \ldots V_k$ be the k-partition of the vertices of G with $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ for every i, $1 \leq i \leq k$. We distinguish the following two cases.

Case 1. $n_1 = 1$.

First, we have $V_1 = \{v_{1,1}\}$ and $d(v_{1,1}) = n - 1$. Let $H = G - \{v_{1,1}\}$. Then $\Delta(H) = n - n_2 - 1$. By Lemma 2.11(i), we obtain $\mathrm{rd}(G) \leq \Delta(H) + 1 = n - n_2$.

If $n_2 = 1$, then $\mathrm{rd}(G) = n-1$ by Lemma 2.8, and thus the result is true. Otherwise, for any two vertices u and v of V_2, since they are adjacent with all the vertices of $V(G) \setminus V_2$, we get $\lambda(u, v) \geq n - n_2$. It follows from Lemma 2.4 that $\mathrm{rd}(G) \geq n - n_2$. Hence, $\mathrm{rd}(G) = n - n_2$.

Case 2. $n_1 \geq 2$.

Pick a vertex u of V_1 and let $F = G - u$. Then $\Delta(F) = n - n_1$ since $n_1 \geq 2$ and $F_\Delta = G[V_1 - u]$. It follows from Lemma 3.2 that F is in Class 1. For each vertex $x \in N_G(u)$, since $d_F(x) \leq \Delta(F) - 1 = n - n_1 - 1$, we have $\mathrm{rd}(G) \leq n - n_1$ by Lemma 2.11(ii).

For any two vertices of V_1, since all vertices of $V(G) \setminus V_1$ are their common neighbors, we get $\lambda^+(G) \geq n - n_1$. It follows from Lemma 2.4 that $\mathrm{rd}(G) \geq n - n_1$. Hence, $\mathrm{rd}(G) = n - n_1$. □

A graph G is said to be *color-critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. The study of critical k-chromatic graphs was initiated by Dirac ([10], [11]).

Here, for simplicity, we abbreviate the term "color-critical" to "critical". A *k-critical* graph is one that is k-chromatic and critical. We get a lower bound of the rainbow disconnection number for $(k+1)$-critical graphs.

Theorem 3.3 *If G is a connected $(k+1)$-critical graph, then* $\mathrm{rd}(G) \geq k$.

Our proof will follow from the next two lemmas. First, we give a lower bound on the rainbow disconnection number of a graph depending on its average degree.

Lemma 3.4 *If G is a connected graph of order n with average degree d, then* $\mathrm{rd}(G) \geq \lfloor d \rfloor$.

Proof. If G is a tree, then $1 \leq d < 2$ since $d = \frac{2(n-1)}{n}$. By Lemma 2.5 we have $\mathrm{rd}(G) = 1$. Obviously $\mathrm{rd}(G) = 1 \geq \lfloor d \rfloor$ and the result is true. If G is not a tree, then $d \geq 2$ since $\frac{2|E(G)|}{n} \geq \frac{2n}{n} = 2$. We have $|E(G)| = \frac{1}{2}dn \geq \frac{1}{2}\lfloor d \rfloor n > \frac{1}{2}\lfloor d \rfloor (n-1)$. So $\lambda^+(G) \geq \lfloor d \rfloor$ by Lemma 2.9. Therefore, $\mathrm{rd}(G) \geq \lfloor d \rfloor$ by Lemma 2.4. $\square$

Lemma 3.5 [10] *If G is a connected $(k+1)$-critical graph, then $\delta(G) \geq k$.*

Proof of Theorem 3.3: Let G be a $(k+1)$-critical graph with average degree d. We know that $\delta(G) \geq k$ by Lemma 3.5. Obviously, $d \geq \delta(G) \geq k$. Therefore, it follows from Lemma 3.4 that $\mathrm{rd}(G) \geq \lfloor d \rfloor \geq k$ since k is an integer. $\square$

A graph G with at least two edges is called *minimal with respect to the chromatic index* if $\chi'(G-e) < \chi'(G)$ for any edge e of G, i.e., $\chi'(G-e) = \chi'(G) - 1$ for any edge e of G. We show that the rainbow disconnection number of a connected minimal graph G with respect to the chromatic index is no more than the maximum degree of G.

Theorem 3.6 *If G is a connected minimal graph with respect to the chromatic index, then* $\mathrm{rd}(G) \leq \Delta(G)$.

In order to prove Theorem 3.6, we need the next two lemmas.

Lemma 3.7 [22] *Let G be a connected graph of Class 2 that is minimal with respect to the chromatic index. Then every vertex of G is adjacent to at least two vertices of degree $\Delta(G)$. In particular, G contains at least three vertices of degree $\Delta(G)$.*

Lemma 3.8 [3] *Let G be a connected graph with $\Delta(G) \geq 2$. Then G is minimal with respect to the chromatic index if and only if either*

(i) *G is in Class 1 and $G = K_{1,\Delta(G)}$, or*

(ii) *G is in Class 2 and $G - e$ is in Class 1 for every edge e of G.*

Proof. Here we restate the proof. Assume first that $G = K_{1,\Delta(G)}$. Then $\chi'(G) = \Delta(G) \geq 2$ and $\chi'(G-e) = \Delta(G) - 1$ for every edge e of G. Since G is in Class 1, $\chi'(G-e) = \chi'(G) - 1$. Next suppose that G is in Class 2 and $G - e$ is in Class 1 for every edge e of G. Then for any edge e of G, we have $\chi'(G-e) = \Delta(G-e) < \Delta(G) + 1 = \chi'(G)$. Therefore, $\chi'(G-e) = \chi'(G) - 1$.

Conversely, assume that $\chi'(G-e) < \chi'(G)$ for every edge e of G. If G is in Class 1, then $\Delta(G) \leq \Delta(G-e) + 1 \leq \chi'(G-e) + 1 = \chi'(G) = \Delta(G)$. Therefore, $\Delta(G-e) = \Delta(G) - 1$ for every edge e of G, which implies that $G = K_{1,\Delta(G)}$. If G is in Class 2, then $\chi'(G-e) + 1 = \chi'(G) = \Delta(G) + 1$, i.e., $\chi'(G-e) = \Delta(G)$ for every edge e of G. Suppose that G contains an edge e_1 such that $G - e_1$ is in Class 2. Then $\chi'(G-e_1) = \Delta(G-e_1) + 1$. Thus, $\Delta(G) = \Delta(G-e_1) + 1$, which implies that G has at most two vertices of degree $\Delta(G)$, which contradicts Lemma 3.7. $\square$

Proof of Theorem 3.6. Let G be a minimal connected graph with respect to the chromatic index. We distinguish the following two cases according to Lemma 3.8.

Case 1. G is in Class 1 and $G = K_{1,d}$ with $d \geq 2$. It follows that $\mathrm{rd}(G) = 1$ from Lemma 2.5. Obviously, $\mathrm{rd}(G) < d = \Delta(G)$.

Case 2. G is in Class 2 and for any edge $e \in E(G)$, $\chi'(G-e) = \Delta(G-e)$. We pick a vertex $v \in V(G)$ such that $d_G(v) = \Delta(G)$. Let $H = G - uv$ for some vertex $u \in N_G(v)$. Then $\chi'(H) = \Delta(H)$ and $\chi'(H) = \chi'(G) - 1 = \Delta(G)$ since G is minimal with respect to the chromatic index and G is in Class 2. Thus, it implies that $\chi'(H) = \Delta(H) = \Delta(G)$. Therefore, we have $\mathrm{rd}(G) \leq \Delta(G)$ by Lemma 2.11(iii). $\square$

For regular graphs, we know that not all k-regular graph have $\mathrm{rd}(G) = k$. For example, we know from [6] that the Petersen graph P is a 3-regular graph but $\mathrm{rd}(P) = 4$. The following results give some regular graphs with $\mathrm{rd}(G) = k$.

Theorem 3.9 *If G is a connected k-regular graph of even order satisfying $k \geq \frac{6}{7}|V(G)|$, then $\mathrm{rd}(G) = k$.*

Theorem 3.10 *If G is a connected k-regular bipartite graph, then $\mathrm{rd}(G) = k$.*

Theorem 3.11 *If G is a connected $(n-k)$-regular graph of order n, where $1 \leq k \leq 4$, then $\mathrm{rd}(G) = n - k$.*

To prove these results, we need the following lemmas.

Lemma 3.12 [9] *Let G be a regular graph of even order n and degree $d(G)$ equal to $n-3$, $n-4$, or $n-5$. Let $d(G) \geq 2\left\lfloor \frac{1}{2}(\frac{n}{2}+1)\right\rfloor - 1$. Then G is in Class 1.*

Lemma 3.13 [9] *Let G be a regular graph of even order n whose degree $d(G)$ satisfies $d(G) \geq \frac{6}{7}n$. Then G is in Class 1.*

For regular graphs, we can easily get the following result.

Lemma 3.14 *If G is a connected k-regular graph, then $k \leq \mathrm{rd}(G) \leq k+1$.*

Proof. Since the average degree of a k-regular graph G is k, it follows from Lemma 3.4 that $\mathrm{rd}(G) \geq k$. On the other hand, it follows from Lemma 2.4 that $\mathrm{rd}(G) \leq \chi'(G) \leq \Delta + 1 = k+1$. $\square$

Proof of Theorem 3.9: Let G be a connected k-regular graph of even order n satisfying $k \geq \frac{6}{7}n$. We have that G is in Class 1 by Lemma 3.13. Thus $\chi'(G) = k$. The result then follows from Lemmas 2.4 and 3.14. $\square$

Proof of Theorem 3.10: Since G is a bipartite graph, $\chi'(G) = \Delta(G) = k$ (see [4]). The result then follows from Lemmas 2.4 and 3.14. $\square$

Proof of Theorem 3.11. We distinguish the following three cases.

Case 1. $k = 1$. We have $G = K_n$. Hence, the result is true by Lemma 2.7.

Case 2. $k = 2$ or 3. Let $u \in V(G)$ and consider the graph $H = G - u$. Then $\Delta(H) = n-k$ and the number of vertices of H with maximum degree is 1 or 2. So each component of H_Δ is a tree. Therefore, it follows from Lemma 3.2 that H is in Class 1 and for each vertex $x \in N_G(u)$, $d_H(x) \leq \Delta(H) - 1 = n-k-1$. By Lemma 2.11(ii), $\mathrm{rd}(G) \leq n-k$. On the other hand, by Lemma 3.14, $\mathrm{rd}(G) \geq n-k$. Thus, $\mathrm{rd}(G) = n-k$.

Case 3. $k = 4$. Let G be an $(n-4)$-regular graph of order n, where $n \geq 5$. Then we know that n must be even since $2|E(G)| = n(n-4)$. First, we consider $n \geq 8$. It is easy to verify that $d(G) = n-4 \geq 2\left\lfloor \frac{1}{2}(\frac{n}{2}+1)\right\rfloor - 1$. It follows from Lemma 3.12 that G is in Class 1. So, $\chi'(G) = n-4$. Furthermore, we get $\mathrm{rd}(G) = n-4$ by Lemmas 2.4 and 3.14. Secondly, it remains to consider the case $n = 6$. In this case, we have $G = C_6$. By Lemma 2.6, we obtain $\mathrm{rd}(G) = 2 = n-4$. $\square$

4 Nordhaus-Gaddum-type results

In this section, we consider Nordhaus-Gaddum-type results for the rainbow disconnection number of graphs. We know that if G is a connected graph with n vertices, then the number of edges in G is at least $n-1$. Since $2(n-1) \leq |E(G)| + |E(\overline{G})| = |E(K_n)| = \frac{n(n-1)}{2}$, if both G and $\overline{G}$ are connected, then n is at least 4.

In the rest of this section, we always assume that all graphs have at least four vertices, and that both G and $\overline{G}$ are connected. For any vertex $u \in V(G)$, let $\bar{u}$ denote the vertex in $\overline{G}$ corresponding to the vertex u. Now we give a Nordhaus-Gaddum-type result for the rainbow disconnection number.

Theorem 4.1 *If G is a connected graph such that $\overline{G}$ is also connected, then $n-2 \leq \mathrm{rd}(G) + \mathrm{rd}(\overline{G}) \leq 2n-5$ and $n-3 \leq \mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) \leq (n-2)(n-3)$. Furthermore, these bounds are sharp.*

For the proof of Theorem 4.1, we need the following four lemmas.

Lemma 4.2 [6] *If H is a connected subgraph of a graph G, then $\mathrm{rd}(H) \leq \mathrm{rd}(G)$.*

Lemma 4.3 [6] *Let G be a connected graph, and let B be a block of G such that $\mathrm{rd}(B)$ is maximum among all blocks of G. Then $\mathrm{rd}(G) = \mathrm{rd}(B)$.*

Lemma 4.4 *Let G be a connected graph of order $n \geq 4$. If G has at least two vertices of degree 1, then $\mathrm{rd}(G) \leq n-3$.*

Proof. Let B be a block of G such that $\mathrm{rd}(B)$ is maximum among all blocks of G. Then $|V(B)| \leq n-2$ since G has at least two vertices of degree 1. It follows from Lemmas 2.7 and 4.2 that $\mathrm{rd}(B) \leq \mathrm{rd}(K_{n-2}) = n-3$. On the other hand, $\mathrm{rd}(G) = \mathrm{rd}(B) \leq n-3$ by Lemma 4.3. □

Lemma 4.5 *If G is a connected graph of order n which contains at most one vertex of degree at least $n-2$, then $\mathrm{rd}(G) \leq n-3$.*

Proof. We distinguish the following three cases.

Case 1. There exists exactly one vertex, say u, of degree $n-1$.

Let $F = G - u$. We have $\Delta(F) \leq n-4$ since $d_G(u) = n-1$ and $d_G(v) \leq n-3$ for any vertex $v \in V(G) \setminus u$. Therefore, $\mathrm{rd}(G) \leq \Delta(F) + 1 \leq n-3$ by Lemma 2.11(i).

Case 2. There exists exactly one vertex, say u, of degree $n-2$.

Let $F = G - u$. If $\Delta(F) \leq n-4$, as discussed in Case 1, we obtain $\mathrm{rd}(G) \leq \Delta(F)+1 \leq n-3$. Otherwise, if $\Delta(F) = n-3$, then there exists exactly one vertex, say v, with degree $n-3$ in F. Then F is in Class 1 by Lemma 3.2. Since $v \notin N_G(u)$, for each vertex $x \in N_G(u)$, $d_F(x) \leq \Delta(F) - 1 = n-4$, and so $\mathrm{rd}(G) \leq \Delta(F) = n-3$ by Lemma 2.11(ii).

Case 3. $\Delta(G) \leq n-3$.

If $\Delta(G) \leq n-4$, then $\mathrm{rd}(G) \leq \chi'(G) \leq n-3$ by Lemma 2.4. Thus, we may assume that $\Delta(G) = n-3$. Let $d(u) = n-3$ and $F = G-u$. If $\Delta(F) \leq n-4$, then $\mathrm{rd}(G) \leq \Delta(F)+1 \leq n-3$ by Lemma 2.11(i). If $\Delta(F) = n-3$, then there exist at most two vertices of degree $n-3$ in F. So, each component of F_Δ is a tree. It follows from Lemma 3.2 that F is in Class 1. Since $\Delta(G) \leq n-3$, for each vertex $x \in N_G(u)$, we have $d_F(x) \leq \Delta(F)-1 = n-4$. It follows that $\mathrm{rd}(G) \leq \Delta(F) = n-3$ from Lemma 2.11(ii). □

By the above Lemma 4.5, we can immediately get the following result.

Corollary 4.6 *Let G be a connected graph with order n. If $\mathrm{rd}(G) \geq n-2$, then there are at least two vertices of degree at least $n-2$.*

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let d and $\bar{d}$ be the average degree of G and $\overline{G}$, respectively. Then $\mathrm{rd}(G) \geq \lfloor d \rfloor$ and $\mathrm{rd}(\overline{G}) \geq \lfloor \bar{d} \rfloor$ by Lemma 3.4. Thus,

$$\begin{aligned}
\mathrm{rd}(G) + \mathrm{rd}(\overline{G}) &\geq \lfloor d \rfloor + \lfloor \bar{d} \rfloor \\
&\geq \lfloor d + \bar{d} \rfloor - 1 \\
&= \left\lfloor \frac{2|E(G)|}{n} + \frac{2|E(\overline{G})|}{n} \right\rfloor - 1 \\
&= \left\lfloor \frac{2}{n} \cdot \frac{n(n-1)}{2} \right\rfloor - 1 \\
&= n-2.
\end{aligned}$$

One can see that the minimum value $n-2$ of $\mathrm{rd}(G) + \mathrm{rd}(\overline{G})$ can be reached if $\mathrm{rd}(G) = 1$ and $\mathrm{rd}(\overline{G}) = n-3$, or $\mathrm{rd}(\overline{G}) = 1$ and $\mathrm{rd}(G) = n-3$. Furthermore, Since both G and $\overline{G}$ are connected, it follows that both $\Delta(G)$ and $\Delta(\overline{G})$ are at most $n-2$. Thus, both $\mathrm{rd}(G)$ and $\mathrm{rd}(\overline{G})$ are at most $n-2$ by Lemma 2.8. Therefore, $n-2 \leq \mathrm{rd}(G) + \mathrm{rd}(\overline{G}) \leq 2n-4$ and $n-3 \leq \mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) \leq (n-2)^2$. Now we claim that for a graph G we cannot have both $\mathrm{rd}(G) = n-2$ and $\mathrm{rd}(\overline{G}) = n-2$. Assume that $\mathrm{rd}(G) = \mathrm{rd}(\overline{G}) = n-2$. Then G has at least two vertices of degree $n-2$ by Corollary 4.6, which implies that $\overline{G}$ has at least two vertices of degree 1. It follows from Lemma 4.4 that $\mathrm{rd}(\overline{G}) \leq n-3$, which contradicts that $\mathrm{rd}(\overline{G}) = n-2$. Finally, we get that $n-2 \leq \mathrm{rd}(G)+\mathrm{rd}(\overline{G}) \leq 2n-5$ and $n-3 \leq \mathrm{rd}(G)\cdot\mathrm{rd}(\overline{G}) \leq (n-2)(n-3)$.

Next we will show that the four bounds are sharp. First, for the lower bound, let $G = P_4$. We then have $\overline{G} = P_4$. Since $\mathrm{rd}(P_4) = 1$, we get $\mathrm{rd}(G) + \mathrm{rd}(\overline{G}) = 2 = n-2$, and $\mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) = 1 = n-3$. Second, for the upper bound, we construct a graph G of order n, where $n \geq 6$, satisfying $\mathrm{rd}(G)$+$\mathrm{rd}(\overline{G}) = 2n-5$ and $\mathrm{rd}(G)\cdot$ $\mathrm{rd}(\overline{G}) = (n-2)(n-3)$ as follows. Let G be a graph of order $n \geq 6$ constructed as follows. Let $u, v, w, x \in V(G)$. We then set $E(G) = \{uv, wx\} \cup \{uy, vy | y \in V(G) \setminus \{u, v, w\}\}$. Obviously, G and $\overline{G}$ are both connected. Now we claim that $\mathrm{rd}(G)$+$\mathrm{rd}(\overline{G}) = 2n-5$ and $\mathrm{rd}(G)\cdot\mathrm{rd}(\overline{G}) = (n-2)(n-3)$. We only need to show that $\mathrm{rd}(G) + \mathrm{rd}(\overline{G}) \geq 2n-5$ and $\mathrm{rd}(G)\cdot\mathrm{rd}(\overline{G}) \geq (n-2)(n-3)$. First, we have $\lambda(u, v) = n-2$ by the construction of G, and so $\mathrm{rd}(G) \geq n-2$ by Lemma 2.4. Next, for any two vertices $p, q \in V(\overline{G}) \setminus \{\bar{u}, \bar{v}, \bar{w}, \bar{x}\}$, we have $\lambda(p, q) = n-3$ since y is a common neighbor of p and q for each vertex $y \in V(\overline{G}) \setminus \{\bar{u}, \bar{v}, p, q\}$ and pq is an edge in $\overline{G}$. So, $\mathrm{rd}(\overline{G}) \geq n-3$ by Lemma 2.4. Hence, $\mathrm{rd}(G)$+$\mathrm{rd}(\overline{G}) \geq 2n-5$ and $\mathrm{rd}(G)\cdot\mathrm{rd}(\overline{G}) \geq (n-2)(n-3)$. □

5 Hardness results

The following result is due to Holyer [16].

Theorem 5.1 [16] *It is NP-complete to determine whether the chromatic index of a cubic graph is* 3 *or* 4.

At first we show that our problem is in NP for any fixed integer k.

Lemma 5.2 *For a fixed positive integer k, given a k-edge-colored graph G, deciding whether G is rainbow disconnected under this coloring is in P.*

Proof. Let n and m be the number of vertices and edges of G, respectively. Let s and t be two vertices of G. Since G is k-edge-colored, any rainbow-cut S contains at most k edges, and so, we have no more than $\binom{m}{k}$ choices for S. Given a set S of edges, it is polynomial-time checkable to decide whether s and t lie in different components of $G \setminus S$. There are at most $\binom{n}{2}$ pairs of vertices in G. Then, we can deduce that deciding whether G is rainbow disconnected can be checked in polynomial-time. □

Let G be a graph and let X be a proper subset of V. To *shrink* X is to delete all the edges between the vertices of X and then identify the vertices of X into a single vertex. We denote the resulting graph by G/X. The next lemma is crucial for the proof of our result.

Lemma 5.3 *Let G be a 3-edge-connected cubic graph. Then $\chi'(G) = 3$ if and only if* $\mathrm{rd}(G) = 3$.

Proof. Assume that $\chi'(G) = 3$, and let us show that $\mathrm{rd}(G) = 3$. Noticing that G is 3-edge-connected, we have $\mathrm{rd}(G) \geq 3$. Since $\mathrm{rd}(G) \leq \chi'(G)$ by Lemma 2.4, we then have $\mathrm{rd}(G) = 3$.

Assume that $\mathrm{rd}(G) = 3$ with an associated rainbow disconnection coloring f. We say that a graph G has Property 1, if G has a rainbow 3-edge-cut S such that $G \setminus S$ has two non-trivial components C_1 and C_2, i.e., no component is a singleton. We do an operation, introduced in the following, on G when graph G has Property 1. If the three edges of S share a common vertex, then one of C_1 and C_2 is a singleton, a contradiction. If two edges of S are adjacent, say e_1, e_2, let e_3 be the third edge adjacent to e_1, e_2, then $S \cup \{e_3\} \setminus \{e_1, e_2\}$ is a 2-edge-cut of G, a contradiction. Hence, we have that none of the edges in S are adjacent. Then we shrink the vertices of component C_1 to a vertex x_1. If there exists a 2-edge-cut of $G/V(C_1)$, then it is also a 2-edge-cut of G, a contradiction. So, we have that $G/V(C_1)$ is a 3-edge connected cubic graph.

Claim 1. f maintains a rainbow disconnection coloring on $G/V(C_1)$.

Proof of Claim 1: Let S' be a rainbow 3-edge-cut of G between two vertices $s, t \in V(C_2)$. Since C_2 is 2-edge connected, then at least two edges of S' are in $E(C_2)$. Suppose that the third edge e of S' is in $E(C_1)$. As C_1 is 2-edge connected, the graph F induced by edge set $E(C_1) \cup S \setminus \{e\}$ is connected. Then F is a subgraph of one component of $G \setminus S'$. As a result, the two end points of e lie in one component of $G \setminus S'$. So, $S' \setminus e$ is a 2-edge-cut of G, a contradiction. Hence, we have $S' \subseteq E(C_2) \cup S$. Then, S' is also a rainbow 3-edge-cut of $G/V(C_1)$ between s and t. As S is a rainbow 3-edge-cut, the three edges adjacent to x_1 are properly colored. Thus, the coloring f maintains a rainbow disconnection coloring on $G/V(C_1)$, and similar thing is true for $G/V(C_2)$. ∎

After this shrinking operation, we get two graphs $G/V(C_1)$ and $G/V(C_2)$. Since the choice of the rainbow 3-edge-cut S is arbitrary, then we can give an order to the edges of graph G to fix the choice of S. Let p be a positive integer, and each G_i ($i \in [p]$) be a 3-edge connected cubic graph with an associated rainbow disconnection coloring. Then we define the operation functions o and O as follows:

$$o(\{G\}) = \begin{cases} \{G/V(C_1), G/V(C_2)\}, & \text{if a graph } G \text{ has Property 1,} \\ \{G\}, & \text{otherwise.} \end{cases}$$

$$O(\{G_1, G_2, \cdots, G_p\}) = \cup_{i=1}^{p} o(\{G_i\}).$$

Since the graph is split into two pieces when we do the operation, then the operation cannot last endlessly. Hence, there exists a integer r such that $O^r(\{G\}) = O^{r+1}(\{G\})$. Finally, we get a finite sequence of edge-colored cubic graphs $O^r(\{G\}) = \{H_1, H_2, \cdots, H_q\}$, where q is a positive integer. We say that a vertex is proper, if the three edges incident with this vertex is properly colored.

Claim 2. Every vertex of H_j is proper, for $j \in [q]$.

Proof of Claim 2: Suppose that there exists two vertices of H_j which are not proper, for a $j \in [q]$. Since there exists a rainbow 3-edge-cut between these two vertices by Claim 1, then the rainbow 3-edge-cut separates a non-trivial component and a singleton by the definition of H_j. Therfore, one of these two vertices is proper, a contradiction. Then we deduce that every vertex of H_j is proper except for one, say s_0. Let H_{12} be the subgraph of H_j induced by the set of edges with color 1 or 2. Then we have that the degree of vertex $v \in V(H_{12})$ equals 2 except for s_0. Let k_i denote the number of edges incident with s_0 with color i. Since the degree sum of H_{12} is an even number, then we have $k_1 + k_2 + 2(n(H_{12}) - 1) \equiv 0 \pmod 2$, which gives $k_1 \equiv k_2 \pmod 2$. Similarly, $k_2 \equiv k_3 \pmod 2$. As $k_1 + k_2 + k_3 = 3$, we have that $k_1 = k_2 = k_3 = 1$. Then s_0 is also proper. As a result, every vertex of H_j is proper, for $j \in [q]$. ∎

Let u be a vertex of the graph G. Then u is also a vertex of some H_j, which gives that u is proper in H_j, for $j \in [q]$. Since the operation maintains the coloring, then u is also proper in G. Thus, the coloring f is a proper edge-coloring of G. Hence, we have $\chi'(G) = 3$. □

Corollary 5.4 *It is NP-complete to determine whether the rainbow disconnection number of a cubic graph is* 3 *or* 4.

Proof. The problem is in NP by Lemma 5.2. Notice that the graph G_ϕ in [16] is 3-edge connected. Then the result is a direct corollary from Theorem 5.1 and Lemma 5.3. □

Lemma 5.2 tells us that deciding whether a given k-edge-colored graph G is rainbow disconnected for a fixed integer k is in P. However, it is NP-complete to decide whether a given edge-colored (with an unbounded number of colors) graph is rainbow disconnected. The proof of the following result uses a technique similar to the one used in [5].

Theorem 5.5 *Given an edge-colored graph G and two vertices s, t of G, deciding whether there is a rainbow-cut between s and t is NP-complete.*

Proof. Clearly, the problem is in NP, since checking whether a given edge set is a rainbow edge-cut can be done in polynomial-time. We now show that the problem is NP-complete by giving a polynomial reduction from 3-SAT to our problem. Given a 3CNF formula $\phi = \wedge_{i=1}^{m} c_i$ over n variables $x_1, x_2, \cdots, x_n$, we construct a graph G_ϕ with two special vertices s, t and an edge-coloring f such that there is a rainbow-cut between s and t in G_ϕ if and only if ϕ is satisfiable.

We define G_ϕ as follows:

$$V(G_\phi) = \{c_i^0, c_i^1, c_i^2, c_i^3 : i \in [m]\} \cup \{x_j^0, x_j^1 : j \in [n]\} \cup \{s, t\}$$

$$\begin{aligned} E(G_\phi) = \quad & \big\{x_j^0 c_i^0, x_j^1 c_i^k : \text{If variable } x_j \text{ is positive in the } k\text{-th literature of clause } c_i, \\ & i \in [m], j \in [n], k \in \{1,2,3\}\big\} \\ \cup \quad & \big\{x_j^1 c_i^0, x_j^0 c_i^k : \text{If variable } x_j \text{ is negative in the } k\text{-th literature of clause } c_i, \\ & i \in [m], j \in [n], k \in \{1,2,3\}\big\} \\ \cup \quad & \big\{c_i^k c_i^0 : i \in [m], k \in \{1,2,3\}\big\} \\ \cup \quad & \{s x_j^0, s x_j^1 : j \in [n]\} \\ \cup \quad & \{t c_i^0 : i \in [m]\} \\ \cup \quad & \{st\} \end{aligned}$$

The edge-coloring f is defined as follows (see Figure 2):

- the edges $\{st, tc_i^0 : i \in [m]\}$ are colored with a special color r_0^0;
- the edges $\{sx_j^0, sx_j^1 : j \in [n]\}$ are colored with a special color r_j^0, $j \in [n]$;
- the edge $x_j^0 c_i^0$ or $x_j^1 c_i^0$ is colored with a special color r_i^k, $i \in [m], j \in [n], k \in \{1,2,3\}$;
- the edge $c_i^k x_j^0$ or $c_i^k x_j^1$ is colored with a special color r_i^4, $i \in [m], j \in [n], k \in \{1,2,3\}$;
- the edge $c_i^k c_i^0$ is colored with a special color r_i^5, $i \in [m], k \in \{1,2,3\}$.

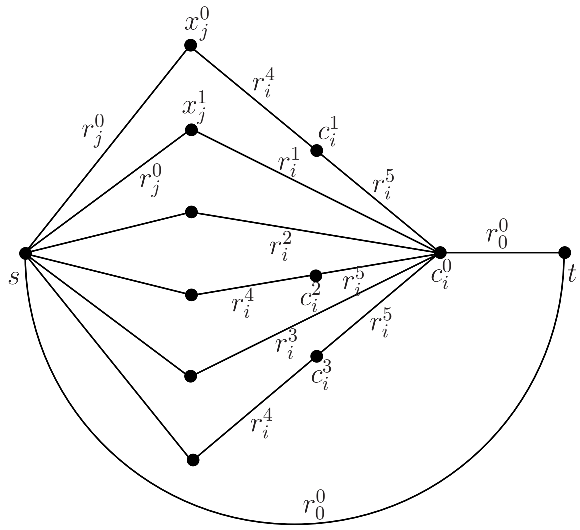

Figure 2: Variable x_j is negative in the first literature of clause c_i.

We now claim that there is a rainbow-cut between s and t in G_ϕ if and only if ϕ is satisfiable.

Assume that there is a rainbow edge-cut S between s and t in G_ϕ under f, and let us show that ϕ is satisfiable. First, we consider the color r_0^0. Since s and t are adjacent in G_ϕ, then the edge st is in S. Next, the color r_j^0 appears twice in G_ϕ. If $sx_j^0 \in S$, then we set $x_j = 0$. If $sx_j^1 \in S$, then we set $x_j = 1$. Finally, the color r_i^4 (r_i^5) appears three times in G_ϕ. If the literature associated with x_j in clause c_i is false, then at least one edge colored with r_i^4 or r_i^5 is in S. Suppose that the three literatures of c_i are false. Then there are three edges colored with r_i^4 or r_i^5 in S. So, S cannot be a rainbow edge-cut, a contradiction. Hence, ϕ is satisfiable.

Assume that ϕ is satisfiable, and let us construct a rainbow edge-cut S between s and t in G_ϕ under f. Clearly, edge st is in S. Suppose $x_j = 0$. The edge sx_j^0 is in S for $j \in [n]$. If the vertex x_j^0 is adjacent to c_i^0, then one edge of $c_i^k x_j^1, c_i^k c_i^0$ is in S for $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex x_j^0 is adjacent to c_i^k, then the edge $x_j^1 c_i^0$ is in S for $i \in [m], j \in [n], k \in \{1, 2, 3\}$. Suppose $x_j = 1$. The edge sx_j^1 is in S for $j \in [n]$. If the vertex x_j^1 is adjacent to c_i^0, then one edge of $c_i^k x_j^0, c_i^k c_i^0$ is in S for $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex x_j^1 is adjacent to c_i^k, then the edge $x_j^0 c_i^0$ is in S for $i \in [m], j \in [n], k \in \{1, 2, 3\}$. Now we verify that S is indeed a rainbow edge-cut. In fact, if a literature of c_i is false, then one edge colored with r_i^4 or r_i^5 is in S. Since the three literatures of c_i cannot be false at the same time, then we can find a rainbow edge-cut S between s and t in G_ϕ under f.

The proof is thus complete. $\square$

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