Laguerre inequalities for discrete sequences

Larry X.W. Wang[1]

Center for Combinatorics, LPMC Nankai University, Tianjin, 300071, P. R. China

Eve Y.Y. Yang[2]

Department of Mathematics, Tianjin University, Tianjin, 300072, P. R. China

Abstract

The Turán inequality, the Laguerre inequality and their m-rd generalizations have been proved to be closely relative with the Laguerre-Pólya class and Riemann hypothesis. Since these two inequalities are equivalent to log-concavity of the discrete sequences, we consider whether their generalizations hold for discrete sequences. Recently, Chen, Jia and Wang proved that the partition function satisfies the Turán inequality of order 2 and thus the 3-rd Jensen polynomials associated with the partition function have only real zeros. Griffin, Ono, Rolen and Zagier proved an exciting result, that is, the n-th Jensen polynomials associated with the Maclaurin coefficients of the function in the Laguerre-Pólya class and the partition function have only real zeros except finite terms. In this paper, we show the Laguerre inequality of order 2 is true for the partition function, the overpartition function, the Bernoulli numbers, the derangement numbers, the Motzkin numbers, the Fine numbers, the Franel numbers and the Domb numbers.

Keywords: partition function, Laguerre inequality of order m, Turán inequality of order m, the Hardy-Ramanujan-Rademacher formula, Log-monotonicity

1. Introduction

The main objective of this paper is to prove some celebrated sequences satisfy the Laguerre inequality of order 2.

Email addresses: `wsw82@nankai.edu.cn` (Larry X.W. Wang[1]), `yangyaoyao@tju.edu.cn` (Eve Y.Y. Yang[2])

The Laguerre inequality [28] arises in the study of the real polynomials with only real zeros and the Laguerre-Pólya class consisting of real entire functions. Recall that a real entire function

$$\psi(x)=\sum_{k=0}^{\infty}\gamma_k\frac{x^k}{k!} \tag{1.1}$$

is said to be in the Laguerre-Pólya class, denoted $\psi(x)\in\mathcal{LP}$, if it can be represented in the form

$$\psi(x)=cx^m e^{-\alpha x^2+\beta x}\prod_{k=1}^{\infty}\left(1+x/x_k\right)e^{-x/x_k}, \tag{1.2}$$

where c, β, x_k are real numbers, $\alpha\geq 0$, m is a nonnegative integer and $\sum x_k^{-2}<\infty$. For more background on the theory of the $\mathcal{LP}$ class, we refer to [31] and [37].

One of the celebrated inequalities found in the literature of Laguerre-Pólya class is Turán inequality

$$a_k^2\geq a_{k-1}a_{k+1}.$$

Note that a sequence $\{a_k\}_{k\geq 0}$ satisfying the Turán inequality is also called log-concave sequence. Pólya and Schur [36] proved that the Maclaurin coefficients of $\psi(x)$ in the $\mathcal{LP}$ class satisfy the Turán inequality

$$\gamma_k^2-\gamma_{k-1}\gamma_{k+1}\geq 0 \tag{1.3}$$

for $k\geq 1$. Given a sequence $\{a_n\}_{n\geq 0}$, for $n\geq 1$, let

$$T_1(n)=a_n^2-a_{n-1}a_{n+1},$$

and for $k\geq 2$ and $n\geq k$, let

$$T_k(n)=T_{k-1}(n)^2-T_{k-1}(n-1)T_{k-1}(n+1).$$

A sequence $\{a_n\}_{n\geq 0}$ is said to satisfy the double Turán inequality if $T_1(n)\geq 0$ for $n\geq 1$ and $T_2(n)\geq 0$ for $n\geq 2$. In general, $\{a_n\}_{n\geq 0}$ is said to satisfy the lth order iterated Turán inequality if, for $1\leq k\leq l$ and $n\geq k$, we have $T_k(n)\geq 0$. Csordas [11] proved that the Maclaurin coefficients of $\psi(x)$ in the $\mathcal{LP}$ class satisfy the double Turán inequality. Note that the Turán inequality and the double Turán inequality are consistent with log-concavity and 2-log-concavity in combinatorics, see [1, 2].

Dimitrov [17] studied that for a real entire function $\psi(x)$ in the $\mathcal{LP}$ class, the Maclaurin coefficients satisfy the m-rd Turán inequality

$$4(\gamma_k^2-\gamma_{k-1}\gamma_{k+1})(\gamma_{k+1}^2-\gamma_k\gamma_{k+2})-(\gamma_k\gamma_{k+1}-\gamma_{k-1}\gamma_{k+2})^2\geq 0 \tag{1.4}$$

for $k \geq 1$. Note that this inequality was first observed by Pólya and Schur [36]. It is well known that the Riemann hypothesis holds if and only if the Riemann ξ-function belongs to the $\mathcal{LP}$ class. Hence, if the Riemann hypothesis is true, then the Maclaurin coefficients of the Riemann ξ-function satisfy both the Turán inequality and the Turán inequality of order 2. Csordas, Norfolk and Varga [12] proved that the coefficients of the Riemann ξ-function satisfy the Turán inequalities. And Dimitrov and Lucas [18] showed that the coefficients of the Riemann ξ-function satisfy the Turán inequalities of order 2 without the Riemann hypothesis.

Recall that if a polynomial $f(x)$ satisfies

$$f'(x)^2 - f(x)f''(x) \geq 0, \tag{1.5}$$

then it is called to satisfy Laguerre inequality. Laguerre [28] stated that if $f(x)$ is a polynomial with only real zeros, then the Laguerre inequality holds for $f(x)$. Gasper [22] used it as an important tool to deal with the positivity of special function. Laguerre stated that the Laguerre inequality is intimately relative with Riemann hypothesis. Actually, one can see that the Turán inequality of γ_k is equivalent to the Laguerre inequality of $\psi(x)$.

In 1913, Jensen [27] found a m-rd generalization of the Laguerre inequality

$$L_n(f(x)) := \frac{1}{2}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k} f^k(x) f^{2n-k}(x) \geq 0. \tag{1.6}$$

where $f^k(x)$ denotes the kth derivative of $f(x)$It yields the classical Laguerre inequality for $n = 1$. Patrick [35, 34] used (1.6) to obtain Turán-type inequalities, which hold for some fixed values of x and have essentially the same form

$$\sum_{k=0}^{2n} \frac{(-1)^{n+k}}{(2n)!}\binom{2n}{k} u_k(x) u_{2n-k}(x) \geq 0, \tag{1.7}$$

for which the sequences of functions $u_n = u_n(x)$ arise as Taylor coefficients of a function in the Laguerre-Pólya class, that is,

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = \psi(z), \tag{1.8}$$

where $\psi(z)$ is of the form (1.2).

For $k = 1$, (1.7) is the interesting and much studied Turán inequality

$$(u_{n+1}(x))^2 - u_n(x)u_{n+2}(x) \geq 0, \quad n \geq 0.$$

Skovgaard [38] showed that for such sequences $\{u_n\}_{n\geq 0}$ the Turán inequality is a consequence of the Laguerre inequality. Csordas and Escassut [11] investigated the inequalities $L_n(f(x)) \geq 0$ and related Laguerre-type inequalities. Some further generalizations and allied inequalities can be found in [3, 9, 10, 16, 21].

Let the sequence $\{\gamma(n)\}$ defined by

$$(-1+4z^2)\Lambda\left(\frac{1}{2}+z\right)=\sum_{n=0}^{\infty}\frac{\gamma(n)}{n!}z^{2n}, \tag{1.9}$$

where $\Lambda(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)=\Lambda(1-s)$. We say that a polynomial with real coefficients is hyperbolic if all of its zeros are real, and where the Jensen polynomial of degree d and shift n of an arbitrary sequence $\{\alpha(0),\alpha(1),\alpha(2),\ldots\}$ of real numbers is the polynomial

$$J_{\alpha}^{d,n}:=\sum_{j=0}^{d}\binom{d}{j}\alpha(n+j)X^j.$$

Building on work of Jensen [27], Pólya and Schur [36] showed that the Riemann hypothesis is equivalent to the function $(-1+4z^2)\Lambda\left(\frac{1}{2}+z\right)$ being in the Laguerre-Pólya class, is equivalent to $\gamma(n)$ satisfying all of the higher Turán inequalities and is equivalent to all of the associated Jensen polynomials having all real roots. Apart from this, Csordas and Varga [13] showed that the Riemann hypothesis is equivalent to the function $(-1+4z^2)\Lambda\left(\frac{1}{2}+z\right)$ satisfying all of the higher Laguerre inequalities. More properties of the Jensen polynomials can be found in [9, 12, 13].

Motivated by these results, we consider whether these inequalities hold for discrete sequences. Let us first consider the partition function. Recall that a partition of a positive integer n is a nonincreasing sequence $(\lambda_1,\lambda_2,\ldots,\lambda_r)$ of positive integers such that $\lambda_1+\lambda_2+\cdots+\lambda_r=n$. Let $p(n)$ denote the number of partitions of n. Nicolas [33], DeSalvo and Pak [15] independently showed that $p(n)$ is log-concave for $n\geq 25$. DeSalvo and Pak also proved the following two conjectures proposed by Chen [4],

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right)>\frac{p(n)}{p(n+1)}\quad\text{for } n\geq 2, \tag{1.10}$$

and

$$p(n)^2-p(n-m)p(n+m)>0\quad\text{for } n>m>1. \tag{1.11}$$

They also conjectured that for $n\geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right)>\frac{p(n)}{p(n+1)}. \tag{1.12}$$

Chen, Wang and Xie [8] showed the positivity of this conjecture and further gave the upper and lower bound for m-rd difference of $\log p(n)$.

Chen [5] proposed a bundle of conjectures on partition function $p(n)$ and the Andrews smallest parts function $spt(n)$. Soon after, Dawsey and Masri [14] proved these conjecturesby establishing an asymptotic formula with an effective bound on the error term for $spt(n)$. Further, Dawsey and Masri proved that for every $\epsilon > 0$ there was an effectively computable constant $N(\epsilon) > 0$ such that for all $n \geq N(\epsilon)$, we have

$$\frac{\sqrt{6}}{\pi}\sqrt{n}p(n) < spt(n) < \left(\frac{\sqrt{6}}{\pi} + \epsilon\right)\sqrt{n}p(n).$$

The Turán inequality of order 2 of $p(n)$ is one of the most important conjecture. Recently, Chen, Jia and Wang [7] showed that for $n \geq 95$, $p(n)$ possess the Turán inequality of order 2, i.e.,

$$4(p(n)^2-p(n-1)p(n+1))(p(n+1)^2-p(n)p(n+2))-(p(n)p(n+1)-p(n-1)p(n+2))^2 \geq 0.$$

As a corollary, the cubic polynomial

$$p(n-1) + 3p(n)x + 3p(n+1)x^2 + p(n+2)x^3$$

has three distinct real zeros for $n \geq 95$. Chen, Jia and Wang also proposed a conjecture that for any positive integer $m \geq 4$, there exists a positive integer $N(m)$ such that for any $n \geq N(m)$, the polynomial

$$\sum_{k=0}^{m} \binom{m}{k} p(n+k)x^k$$

has only real zeros. Griffin, Ono, Rolen and Zagier [23] proved this conjecture by establishing the relation between the Jensen polynomials associated with $p(n)$ and Hermite polynomials. In fact, they gave the following more generalized theorem.

Theorem 1.1 (Griffin, Ono, Rolen and Zagier)**.** *Let $\{\alpha(n)\}$, $\{A(n)\}$, and $\{\delta(n)\}$ be three sequences of positive real numbers with $\delta(n)$ tending to zero and satisfying*

$$log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2j^2 + o(\delta(n)^d) \quad \text{as } n \to \infty,$$

for some interger $d \geq 1$ and all $0 \leq j \leq d$. Then, we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_\alpha^{d,n}\left(\frac{\delta(n)X-1}{exp(A(n))}\right)\right) = H_d(X),$$

uniformly for X in any compact subset of $\mathbb{R}$, where

$$J_\alpha^{d,n} := \sum_{j=0}^{d}\binom{d}{j}\alpha(n+j)X^j,$$

and Hermite polynomials $H_d(X)$ be defined by the generating function

$$\sum_{d=0}^{\infty} H_d(X)\frac{t^d}{d!} = e^{-t^2+Xt} = 1 + Xt + \left(X^2-2\right)\frac{t^2}{2!} + \left(X^3-6X\right)\frac{t^3}{3!} + \cdots.$$

Griffin, Ono, Rolen and Zagier [23] verified both the $p(n)$ and the Maclaurin coefficients of the Riemann ξ-function have this form. As consequences, the above conjecture holds and the Jensen polynomials associated with the Riemann ξ-function have only real zeros as sufficiently large n. Based on the above work [23], Griffin, Ono, Rolen, Thorner, Tripp, and Wagner [24] made this approach effective for the Riemann ξ-function. For more log-behavior of $p(n)$, see [25, 26, 29].

Moreover, some other discrete sequences, such as the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and the Domb numbers, are log-convex, not log-concave. Thus they definitely do not satisfy the Turán inequality of order 2. Despite all this, Wang [39] proved that Turán inequality of order 2 holds for the sequences $\{a_n/n!\}_{n\geq 0}$, where a_n are the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and the Domb numbers.

In this paper, we concern with whether the discrete sequences have the similar results with Laguerre inequality of order m. We first define a sequence $\{a_n\}_{n\geq 0}$ satisfies Laguerre inequality of order m if

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}a_{n+k}a_{2m-k+n} \geq 0, \tag{1.13}$$

For $m = 1$, the above inequality reduces to

$$a_n^2 - a_{n-1}a_{n+1} > 0,$$

i.e., the log-concavity of $\{a_n\}_{n\geq 0}$. In the remaining of this paper, we will concern with the case $m = 2$ and show that the partition function, the overpartition function, the Bernoulli numbers, the derangement numbers, the Motzkin numbers, the Fine numbers, the Franel numbers and the Domb numbers possess Laguerre inequality of order 2.

Remarks. Wagner [41] recently proved that $p(n)$ satisfies all of the higher Laguerre inequalities as $n \to \infty$ and proposed a conjecture on the thresholds of the m-rd Laguerre ineequalities of $p(n)$ for $m \leq 10$. He did not mention the Laguerre inequality of order m for overpartition. Dou and Wang [19] gave an explicit bound $N(m)$ such that for $n > N(m)$ $p(n)$ satisfies the m-rd Laguerre inequalities for $3 \leq m \leq 10$. As consequences, the case $m = 3$ and 4 of Wagner's conjecture have been proved.

2. Partition function

In this section, we begin with partition function $p(n)$. We will show the Laguerre inequality of order 2 is true for $p(n)$.

Theorem 2.1. *Let $p(n)$ denote the partition function. For $n \geq 184$, we have*

$$3p(n+2)^2 - 4p(n+1)p(n+3) + p(n)p(n+4) > 0. \tag{2.1}$$

Proof. In order to prove the above theorem, let

$$u_n = \frac{p(n-1)p(n+1)}{p(n)^2}, \tag{2.2}$$

one can easily deduce that Theorem 2.1 is equivalent to that for $n > 183$,

$$3 - 4u_{n+2} + u_{n+1}u_{n+3}u_{n+2}^2 > 0. \tag{2.3}$$

For convenience, denote

$$s(n) = u_{n+1}u_{n+3}, \tag{2.4}$$

and (2.1) can be restated as

$$s(n)u_{n+2}^2 - 4u_{n+2} + 3 > 0. \tag{2.5}$$

Let

$$F(t) = s(n)t^2 - 4t + 3. \tag{2.6}$$

To prove (2.5), we need to show that

$$F(u_{n+2}) > 0. \tag{2.7}$$

Since the equation $F(t) = 0$ has two solutions

$$t_1 = \frac{2 - \sqrt{4 - 3s(n)}}{s(n)} \quad \text{and} \quad t_2 = \frac{2 + \sqrt{4 - 3s(n)}}{s(n)}.$$

From the definition of $s(n)$, it is easily seen that $0 < s(n) < 1$ for $n > 25$. Thus, $F(t)$ is positive when $t < t_1$ or $t > t_2$.

To verify (2.7), we aim to show that for $n \geq 1207$,

$$u_{n+2} < t_1 = \frac{2 - \sqrt{4 - 3s(n)}}{s(n)}. \tag{2.8}$$

For this aim, we need find a function $g(n)$ satisfying

$$u_{n+2} < g(n+2) < t_1. \tag{2.9}$$

Lehmer [30] constructed the following notation to provide an error term for the partition function, and Chen, Jia and Wang [7] adopted it to state sharper bounds for u_n.

$$\mu(n) = \frac{\pi\sqrt{24n-1}}{6}.$$

Let $r = \mu(n+3),\ j = \mu(n+4)$ and quote the equation as used in [7]:

$$x = \mu(n-1),\ y = \mu(n),\ z = \mu(n+1),\ w = \mu(n+2), \tag{2.10}$$

and

$$f(n) = e^{x-2y+z}\frac{\beta(x)y^{24}\beta(z)}{x^{12}\alpha(y)^2z^{12}}, \tag{2.11}$$

$$g(n) = e^{x-2y+z}\frac{\alpha(x)y^{24}\alpha(z)}{x^{12}\beta(y)^2z^{12}}. \tag{2.12}$$

where

$$\alpha(t) = t^{10} - t^9 + 1, \quad \beta(t) = t^{10} - t^9 - 1. \tag{2.13}$$

Notice that employing Rademacher's convergent series and Lehmer's error bound, Chen, Jia and Wang [7] proved the following inequality.

Theorem 2.2 (Chen, Jia and Wang, [7])**.** *For $n \geq 1207$,*

$$f(n) < u_n < g(n). \tag{2.14}$$

This theorem stated that

$$u_{n+2} < g(n+2),$$

i.e., the first inequality of (2.9) holds. Thus, in the remaining of this section, we will focus on the second inequality of (2.9), which can be rewritten as

$$s(n)g(n+2)^2 - 4g(n+2) + 3 > 0. \tag{2.15}$$

To verify it, we first need to give a lower bound for $s(n)$, which plays an important role in (2.5) and the proof of Theorem 2.1

Setting

$$s_1(n) = f(n+1)f(n+3), \tag{2.16}$$

from Theorem 2.2, we get the following lower bound for $s(n)$.

Corollary 2.3. *For $n \geq 1207$, we have*

$$s_1(n) < s(n). \tag{2.17}$$

To prove (2.15), it suffices to show that for $n \geq 1207$,

$$s_1(n)g(n+2)^2 - 4g(n+2) + 3 > 0. \tag{2.18}$$

From the definition (2.16) of $s_1(n)$, we can rewrite the above inequality as

$$f(n+1)f(n+3)g(n+2)^2 - 4g(n+2) + 3 > 0. \tag{2.19}$$

Substituting (2.10) and (2.13) into (2.11) and (2.12), we have

$$f(n+1) = e^{y-2z+w}\frac{\beta(y)\beta(w)z^{24}}{y^{12}w^{12}\alpha(z)^2},$$

$$f(n+3) = e^{w-2r+j}\frac{\beta(w)\beta(j)r^{24}}{w^{12}j^{12}\alpha(r)^2},$$

$$g(n+2) = e^{z-2w+r}\frac{\alpha(z)\alpha(r)w^{24}}{z^{12}r^{12}\beta(w)^2}.$$

Simplifying the left-hand side of the inequality (2.19) leads to

$$s_1(n)g(n+2)^2 - 4g(n+2) + 3 = \frac{h_1e^{y-2w+j} - 4h_2e^{z-2w+r} + 3h_3}{h_3}, \tag{2.20}$$

where

$$h_1 = \beta(y)\beta(j)w^{24}z^{12}r^{12}, \tag{2.21}$$

$$h_2 = \alpha(z)\alpha(r)w^{24}y^{12}j^{12}, \tag{2.22}$$

$$h_3 = \beta(w)^2y^{12}z^{12}r^{12}j^{12}. \tag{2.23}$$

Now we proceed to prove the numerator of (2.20) is positive for $n \geq 2$. Since h_3 is positive for all $n \geq 1$, we only need to prove

$$h_1e^{y-2w+j} - 4h_2e^{z-2w+r} + 3h_3 > 0. \tag{2.24}$$

For this aim, we need to estimate $h_1, h_2, h_3, e^{y-2w+j}$ and e^{z-2w+r}. We prefer to give the estimates of y, z, r and j by the following equalities. For $n \geq 2$,

$$y = \sqrt{w^2 - \frac{4\pi^2}{3}}, z = \sqrt{w^2 - \frac{2\pi^2}{3}}, r = \sqrt{w^2 + \frac{2\pi^2}{3}}, j = \sqrt{w^2 + \frac{4\pi^2}{3}}. \tag{2.25}$$

We can obtain the following expansions easily,

$$y = w - \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} - \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7} - \frac{28\pi^{10}}{243w^9} + O\left(\frac{1}{w^{10}}\right),$$

$$z = w - \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} - \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7} - \frac{7\pi^{10}}{1944w^9} + O\left(\frac{1}{w^{10}}\right),$$

$$r = w + \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} + \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7} + \frac{7\pi^{10}}{1944w^9} + O\left(\frac{1}{w^{10}}\right),$$

$$j = w + \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} + \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7} + \frac{28\pi^{10}}{243w^9} + O\left(\frac{1}{w^{10}}\right).$$

It can be checked that for $n \geq 17$,

$$y_1 < y < y_2, \tag{2.26}$$

$$z_1 < z < z_2, \tag{2.27}$$

$$r_1 < r < r_2, \tag{2.28}$$

$$j_1 < j < j_2, \tag{2.29}$$

where

$$y_1 = w - \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} - \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7} - \frac{29\pi^{10}}{243w^9},$$

$$y_2 = w - \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} - \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7} - \frac{28\pi^{10}}{243w^9},$$

$$z_1 = w - \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} - \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7} - \frac{8\pi^{10}}{1944w^9},$$

$$z_2 = w - \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} - \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7} - \frac{7\pi^{10}}{1944w^9},$$

$$r_1 = w + \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} + \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7},$$

$$r_2 = w + \frac{\pi^2}{3w} - \frac{\pi^4}{18w^3} + \frac{\pi^6}{54w^5} - \frac{5\pi^8}{648w^7} + \frac{7\pi^{10}}{1944w^9},$$

$$j_1 = w + \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} + \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7},$$

$$j_2 = w + \frac{2\pi^2}{3w} - \frac{2\pi^4}{9w^3} + \frac{4\pi^6}{27w^5} - \frac{10\pi^8}{81w^7} + \frac{28\pi^{10}}{243w^9}.$$

Applying (2.26), (2.27), (2.28) and (2.29) to the definition (2.13) of $\alpha(t)$

and $\beta(t)$, we obtain that for $n \geq 17$,

$$\begin{aligned}
&z^{10} - z_2 z^8 + 1 < \alpha(z) < z^{10} - z_1 z^8 + 1, \\
&r^{10} - r_2 r^8 + 1 < \alpha(r) < r^{10} - r_1 r^8 + 1, \\
&y^{10} - y_2 y^8 - 1 < \beta(y) < y^{10} - y_1 y^8 - 1, \\
&j^{10} - j_2 j^8 - 1 < \beta(j) < j^{10} - j_1 j^8 - 1.
\end{aligned} \tag{2.30}$$

It follows that

$$h_1 > \left(y^{10} - y_2 y^8 - 1\right)\left(j^{10} - j_2 j^8 - 1\right) w^{24} z^{12} r^{12}, \tag{2.31}$$

$$h_2 < \left(z^{10} - z_1 z^8 + 1\right)\left(r^{10} - r_1 r^8 + 1\right) w^{24} y^{12} j^{12}. \tag{2.32}$$

Next we turn to estimate e^{y-2w+j} and e^{z-2w+r}. By (2.26), (2.27), (2.28) and (2.29), we can see that for $n \geq 17$,

$$y_1 - 2w + j_1 < y - 2w + j < y_2 - 2w + j_2, \tag{2.33}$$

$$z_1 - 2w + r_1 < z - 2w + r < z_2 - 2w + r_2, \tag{2.34}$$

which implies that

$$e^{y_1 - 2w + j_1} < e^{y-2w+j} < e^{y_2 - 2w + j_2}, \tag{2.35}$$

$$e^{z_1 - 2w + r1} < e^{z-2w+r} < e^{z_2 - 2w + r_2}. \tag{2.36}$$

In order to give a feasible bound for e^{y-2w+j} and e^{z-2w+r}, we define

$$\Phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}, \tag{2.37}$$

$$\phi(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}. \tag{2.38}$$

It can be checked that for $t < 0$,

$$\phi(t) < e^t < \Phi(t). \tag{2.39}$$

To apply this result to (2.35) and (2.36), we have to show that both $y_2 - 2w + j_2$ and $z_2 - 2w + r_2$ are negative. By straightforward calculation, one can get that

$$y_2 - 2w + j_2 = -\frac{4\pi^4\left(5\pi^4 + 9w^4\right)}{81 w^7},$$

$$z_2 - 2w + r_2 = -\frac{\pi^4\left(5\pi^4 + 36w^4\right)}{324w^7}.$$

Obviously, for $n \geq 2$, both $y_2 - 2w + j_2$ and $z_2 - 2w + r_2$ are negative. Thus, applying (2.39) to (2.35) and (2.36), we obtain that for $n \geq 17$,

$$\phi(y_1 - 2w + j_1) < e^{y-2w+j} < \Phi(y_2 - 2w + j_2), \tag{2.40}$$

$$\phi(z_1 - 2w + r_1) < e^{z-2w+r} < \Phi(z_2 - 2w + r_2). \tag{2.41}$$

Now, we proceed to prove (2.24). For convenience, let

$$A(w) = h_1 e^{y-2w+j} - 4h_2 e^{z-2w+r} + 3h_3, \tag{2.42}$$

we need to show the positivity of $A(w)$. Making use of (2.31), (2.32), (2.40) and (2.41), we obtain that for $n \geq 17$,

$$\begin{aligned} A(w) > {} & \left(y^{10} - y_2 y^8 - 1\right)\left(j^{10} - j_2 j^8 - 1\right) w^{24} z^{12} r^{12} \phi(y_1 - 2w + j_1) \\ & - 4\left(z^{10} - z_1 z^8 + 1\right)\left(r^{10} - r_1 r^8 + 1\right) w^{24} y^{12} j^{12} \Phi(z_2 - 2w + r_2) \\ & + 3\beta(w)^2 y^{12} z^{12} r^{12} j^{12}. \end{aligned}$$

Substituting y, z, r and j with $\sqrt{w^2 - \frac{4\pi^2}{3}}$, $\sqrt{w^2 - \frac{2\pi^2}{3}}$, $\sqrt{w^2 + \frac{4\pi^2}{3}}$ and $\sqrt{w^2 + \frac{2\pi^2}{3}}$, respectively, we can rewrite the right-hand side of the above inequality as

$$\frac{\sum_{k=0}^{91} a_k w^k}{2^{15} 3^{51} 5 w^{29}}, \tag{2.43}$$

where a_k are known real numbers, and the values of $a_{91},\ a_{90},\ a_{89}$ are given below,

$$a_{91} = 2^{16} 3^{48} 5 \pi^8, \quad a_{90} = -2^{15} 3^{48} 5^3 \pi^8, \quad a_{89} = 2^{18} 3^{49} 5^2 \pi^8.$$

Thus, for $n \geq 17$, we have

$$A(w) > \frac{\sum_{k=0}^{91} a_k w^k}{2^{15} 3^{51} 5 w^{29}}. \tag{2.44}$$

As w is positive for $n \geq 1$, we have that

$$\sum_{k=0}^{91} a_k w^k > \sum_{k=0}^{90} -|a_k| w^k + a_{91} w^{91}. \tag{2.45}$$

Thus, to get (2.44), we only need to show that for $n \geq 1207$,

$$\sum_{k=0}^{90} -|a_k|w^k + a_{91}w^{91} > 0. \tag{2.46}$$

For $0 \leq k \leq 89$, we find that

$$-|a_k|w^k > -a_{89}w^{89} \tag{2.47}$$

holds for $w > \sqrt[3]{\dfrac{3625\pi^{12} - 6\pi^{10} + 5832}{1620\pi^8}} \approx 6.02$. It follows that for $w \geq 7$,

$$\begin{aligned}\sum_{k=0}^{91} a_k w^k &> \sum_{k=0}^{90} -|a_k|w^k + a_{91}w^{91} \\ &> (-90a_{89} + a_{90}w + a_{91}w^2)w^{89}.\end{aligned} \tag{2.48}$$

Combing (2.44) and (2.48), $A(w)$ is positive provided

$$-90a_{89} + a_{90}w + a_{91}w^2 > 0, \tag{2.49}$$

which is true if $w > 80$, or equivalently, if $n \geq 971$. Thus, we arrive at that (2.24) is true for $n \geq 1207$, which implies (2.18). Combining (2.18) and (2.17), we obtain that for $n \geq 1207$,

$$s(n)g(n+2)^2 - 4g(n+2) + 3 > 0. \tag{2.50}$$

This proves the second inequality of (2.9).

On the other hand, it can be checked that (2.1) is also true for $184 \leq n \leq 1206$. Thus the proof of Theorem 2.1 is completed. ∎

3. Overpartition function

In this section, we turn to consider a generalization of partition, namely, the overpartition. Recall that an overpartition of n is a ordinary partition of n with the added condition that the first occurrence of any part may be overlined or not. For example, there are eight overpartitions of 3:

$$3, \bar{3}, 2+1, \bar{2}+1, \bar{2}+\bar{1}, 1+1+1, \bar{1}+1+1$$

Engle [20] proved that for $n \geq 2$, overpartition functition $\bar{p}(n)$ satisfied the Turán inequalities, that is, $-\Delta^2 \log \bar{p}(n) \geq 0$ for $n \geq 2$. Wang, Xie and Zhang [40] proved the positivity of finite differences of the overpartition function and gave an upper bound for $\Delta^r \bar{p}(n)$ and $\Delta^r \log \bar{p}(n)$. Liu and Zhang [32] showed that the Turán inequality of order 2 is true for $\bar{p}(n)$.

Now we will prove the overpartition function satisfies the Laguerre inequality of order 2.

Theorem 3.1. *Let $\bar{p}(n)$ denote the overpartition function. For $n \geq 7$, we have*

$$3\bar{p}(n+2)^2 - 4\bar{p}(n+1)\bar{p}(n+3) + \bar{p}(n)\bar{p}(n+4) > 0. \tag{3.1}$$

Proof. In order to prove the above theorem, let

$$\bar{u}_n = \frac{\bar{p}(n-1)\bar{p}(n+1)}{\bar{p}(n)^2}, \tag{3.2}$$

one can easily deduce that Theorem 3.1 is equivalent to that for $n \geq 7$,

$$3 - 4\bar{u}_{n+2} + \bar{u}_{n+1}\bar{u}_{n+3}\bar{u}_{n+2}^2 > 0. \tag{3.3}$$

For convenience, we denote

$$\bar{s}(n) = \bar{u}_{n+1}\bar{u}_{n+3}, \tag{3.4}$$

and hence (3.1) can be rewritten as

$$\bar{s}(n)\bar{u}_{n+2}^2 - 4\bar{u}_{n+2} + 3 > 0. \tag{3.5}$$

Let

$$Q(x) = \bar{s}(n)x^2 - 4x + 3. \tag{3.6}$$

To prove (3.5), we need to show that

$$Q(\bar{u}_{n+2}) > 0. \tag{3.7}$$

Since the equation $Q(x) = 0$ has two solutions

$$x_1 = \frac{2 - \sqrt{4 - 3\bar{s}(n)}}{\bar{s}(n)} \quad \text{and} \quad x_2 = \frac{2 + \sqrt{4 - 3\bar{s}(n)}}{\bar{s}(n)}.$$

From the definition of $\bar{s}(n)$, it is easily seen that $0 < \bar{s}(n) < 1$ for $n \geq 2$. Thus, $Q(x)$ is positive when $x < x_1$ or $x > x_2$.

To verify (3.7), we aim to show that for $n \geq 55$,

$$\bar{u}_{n+2} < x_1 = \frac{2 - \sqrt{4 - 3\bar{s}(n)}}{\bar{s}(n)}. \tag{3.8}$$

Similar to the proof of Theorem 2.1, we need to find a function $\bar{g}(n)$ satisfying

$$\bar{u}_{n+2} < \bar{g}(n+2) < x_1. \tag{3.9}$$

To this aim, we adopt the following notation which constructed by Engle [20] to provide an error term for the overpartition function

$$\bar{\mu}(n) = \pi\sqrt{n}.$$

Let $\bar{r} = \bar{\mu}(n+3)$, $\bar{j} = \bar{\mu}(n+4)$ and adopt the following notation as used in [32]:

$$\bar{x} = \bar{\mu}(n-1),\ \bar{y} = \bar{\mu}(n),\ \bar{z} = \bar{\mu}(n+1),\ \bar{w} = \bar{\mu}(n+2), \tag{3.10}$$

and

$$\bar{f}(n) = e^{\bar{x}-2\bar{y}+\bar{z}}\frac{\bar{y}^{14}\bar{\beta}(x)\bar{\beta}(z)}{\bar{x}^7\bar{z}^7\bar{\alpha}(y)^2}, \tag{3.11}$$

$$\bar{g}(n) = e^{\bar{x}-2\bar{y}+\bar{z}}\frac{\bar{y}^{14}\bar{\alpha}(x)\bar{\alpha}(z)}{\bar{x}^7\bar{z}^7\bar{\beta}(y)^2}. \tag{3.12}$$

where

$$\bar{\alpha}(t) = t^5 - t^4 + 1, \quad \bar{\beta}(t) = t^5 - t^4 - 1. \tag{3.13}$$

Liu and Zhang [32] proved the following inequality.

Theorem 3.2 (Liu and Zhang [32])**.** *For $n \geq 55$,*

$$\bar{f}(n) < \bar{u}_n < \bar{g}(n). \tag{3.14}$$

Thus the first inequality of (3.9) holds. Then we shall prove the second inequality of (3.9), which can be rewritten as

$$\bar{s}(n)\bar{g}(n+2)^2 - 4\bar{g}(n+2) + 3 > 0. \tag{3.15}$$

To prove it, we need to give a lower bound for $\bar{s}(n)$. Denote

$$\bar{s}_1(n) = \bar{f}(n+1)\bar{f}(n+3), \tag{3.16}$$

from Theorem 3.2, we get the following lower bound for $\bar{s}(n)$.

Corollary 3.3. *For $n \geq 55$, we have*

$$\bar{s}_1(n) < \bar{s}(n). \tag{3.17}$$

To prove (3.15), it suffices to show that for $n \geq 55$,

$$\bar{s}_1(n)\bar{g}(n+2)^2 - 4\bar{g}(n+2) + 3 > 0. \tag{3.18}$$

From the definition (3.16) of $\bar{s}_1(n)$, we can rewrite the above inequality as

$$\bar{f}(n+1)\bar{f}(n+3)\bar{g}(n+2)^2 - 4\bar{g}(n+2) + 3 > 0. \tag{3.19}$$

Substituting (3.10) and (3.13) into (3.11) and (3.12), we have

$$\bar{f}(n+1) = e^{\bar{y}-2\bar{z}+\bar{w}} \frac{\bar{z}^{14}\bar{\beta}(\bar{y})\bar{\beta}(\bar{w})}{\bar{y}^7\bar{w}^7\bar{\alpha}(\bar{z})^2},$$

$$\bar{f}(n+3) = e^{\bar{w}-2\bar{r}+\bar{j}} \frac{\bar{r}^{14}\bar{\beta}(\bar{w})\bar{\beta}(\bar{j})}{\bar{w}^7\bar{j}^7\bar{\alpha}(\bar{r})^2},$$

$$\bar{g}(n+2) = e^{\bar{z}-2\bar{w}+\bar{r}} \frac{\bar{w}^{14}\bar{\alpha}(\bar{z})\bar{\alpha}(\bar{r})}{\bar{z}^7\bar{r}^7\bar{\beta}(\bar{w})^2}.$$

The left-hand side of the inequality (3.19) can be rewritten as

$$\bar{s}_1(n)\bar{g}(n+2)^2 - 4\bar{g}(n+2) + 3 = \frac{\bar{h}_1 e^{\bar{y}-2\bar{w}+\bar{j}} - 4\bar{h}_2 e^{\bar{z}-2\bar{w}+\bar{r}} + 3\bar{h}_3}{\bar{h}_3}, \tag{3.20}$$

where

$$\bar{h}_1 = \bar{\beta}(\bar{y})\bar{\beta}(\bar{j})\bar{w}^{14}\bar{z}^7\bar{r}^7, \tag{3.21}$$

$$\bar{h}_2 = \bar{\alpha}(\bar{z})\alpha(\bar{r})\bar{w}^{14}\bar{y}^7\bar{j}^7, \tag{3.22}$$

$$\bar{h}_3 = \bar{\beta}(\bar{w})^2\bar{y}^7\bar{z}^7\bar{r}^7\bar{j}^7. \tag{3.23}$$

Since $\bar{h}_3$ is positive for all $n \geq 1$, to prove (3.19) we have to prove

$$\bar{h}_1 e^{\bar{y}-2\bar{w}+\bar{j}} - 4\bar{h}_2 e^{\bar{z}-2\bar{w}+\bar{r}} + 3\bar{h}_3 > 0. \tag{3.24}$$

For this aim, we need to estimate $\bar{h}_1, \bar{h}_2, \bar{h}_3, e^{\bar{y}-2\bar{w}+\bar{j}}$ and $e^{\bar{z}-2\bar{w}+\bar{r}}$. We prefer to give the estimates of $\bar{y}, \bar{z}, \bar{r}$ and $\bar{j}$ by the following equalities. For $n \geq 2$,

$$\bar{y} = \sqrt{\bar{w}^2 - 2\pi^2}, \bar{z} = \sqrt{\bar{w}^2 - \pi^2}, \bar{r} = \sqrt{\bar{w}^2 + \pi^2}, \bar{j} = \sqrt{\bar{w}^2 + 2\pi^2}. \tag{3.25}$$

Expanding them leads to

$$\bar{y} = \bar{w} - \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} - \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7} - \frac{7\pi^{10}}{8\bar{w}^9} + O\left(\frac{1}{\bar{w}^{10}}\right),$$

$$\bar{z} = \bar{w} - \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} - \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7} - \frac{7\pi^{10}}{256\bar{w}^9} + O\left(\frac{1}{\bar{w}^{10}}\right),$$

$$\bar{r} = \bar{w} + \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} + \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7} + \frac{7\pi^{10}}{256\bar{w}^9} + O\left(\frac{1}{\bar{w}^{10}}\right),$$

$$\bar{j} = \bar{w} + \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} + \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7} + \frac{7\pi^{10}}{8\bar{w}^9} + O\left(\frac{1}{\bar{w}^{10}}\right).$$

It is easily seen that for $n \geq 11$,

$$\bar{y}_1 < \bar{y} < \bar{y}_2, \tag{3.26}$$

$$\bar{z}_1 < \bar{z} < \bar{z}_2, \tag{3.27}$$

$$\bar{r}_1 < \bar{r} < \bar{r}_2, \tag{3.28}$$

$$\bar{j}_1 < \bar{j} < \bar{j}_2, \tag{3.29}$$

where

$$\bar{y}_1 = \bar{w} - \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} - \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7} - \frac{\pi^{10}}{\bar{w}^9},$$

$$\bar{y}_2 = \bar{w} - \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} - \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7} - \frac{7\pi^{10}}{8\bar{w}^9},$$

$$\bar{z}_1 = \bar{w} - \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} - \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7} - \frac{5\pi^{10}}{128\bar{w}^9},$$

$$\bar{z}_2 = \bar{w} - \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} - \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7} - \frac{7\pi^{10}}{256\bar{w}^9},$$

$$\bar{r}_1 = \bar{w} + \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} + \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7},$$

$$\bar{r}_2 = \bar{w} + \frac{\pi^2}{2\bar{w}} - \frac{\pi^4}{8\bar{w}^3} + \frac{\pi^6}{16\bar{w}^5} - \frac{5\pi^8}{128\bar{w}^7} + \frac{7\pi^{10}}{256\bar{w}^9},$$

$$\bar{j}_1 = \bar{w} + \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} + \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7},$$

$$\bar{j}_2 = \bar{w} + \frac{\pi^2}{\bar{w}} - \frac{\pi^4}{2\bar{w}^3} + \frac{\pi^6}{2\bar{w}^5} - \frac{5\pi^8}{8\bar{w}^7} + \frac{7\pi^{10}}{8\bar{w}^9}.$$

Applying (3.26), (3.27), (3.28) and (3.29) to the definition (3.13) of $\bar{\alpha}(t)$ and

$\bar{\beta}(t)$, we obtain that for $n \geq 11$,

$$
\begin{aligned}
&\bar{z}_1\bar{z}^4 - \bar{z}^4 + 1 < \bar{\alpha}(\bar{z}) < \bar{z}_2\bar{z}^4 - \bar{z}^4 + 1, \\
&\bar{r}_1\bar{r}^4 - \bar{r}^4 + 1 < \bar{\alpha}(\bar{r}) < \bar{r}_2\bar{r}^4 - \bar{r}^4 + 1, \\
&\bar{y}_1\bar{y}^4 - \bar{y}^4 - 1 < \bar{\beta}(\bar{y}) < \bar{y}_2\bar{y}^4 - \bar{y}^4 - 1, \\
&\bar{j}_1\bar{j}^4 - \bar{j}^4 - 1 < \bar{\beta}(\bar{j}) < \bar{j}_2\bar{j}^4 - \bar{j}^4 - 1,
\end{aligned}
\tag{3.30}
$$

which implies that

$$\bar{h}_1 > \left(\bar{y}_1\bar{y}^4 - \bar{y}^4 - 1\right)\left(\bar{j}_1\bar{j}^4 - \bar{j}^4 - 1\right)\bar{w}^{14}\bar{z}_1\bar{z}^6\bar{r}_1\bar{r}^6, \tag{3.31}$$

$$\bar{h}_2 < \left(\bar{z}_2\bar{z}^4 - \bar{z}^4 + 1\right)\left(\bar{r}_2\bar{r}^4 - \bar{r}^4 + 1\right)\bar{w}^{14}\bar{y}_2\bar{y}^6\bar{j}_2\bar{j}^6, \tag{3.32}$$

$$\bar{h}_3 > \bar{y}_1\bar{y}^6\bar{z}_1\bar{z}^6\bar{r}_1\bar{r}^6\bar{j}_1\bar{j}^6\bar{\beta}(\bar{w})^2. \tag{3.33}$$

We proceed to estimate $e^{\bar{y}-2\bar{w}+\bar{j}}$ and $e^{\bar{z}-2\bar{w}+\bar{r}}$. Combining (3.26), (3.27), (3.28) and (3.29), we can see that for $n \geq 11$,

$$\bar{y}_1 - 2\bar{w} + \bar{j}_1 < \bar{y} - 2\bar{w} + \bar{j} < \bar{y}_2 - 2\bar{w} + \bar{j}_2, \tag{3.34}$$

$$\bar{z}_1 - 2\bar{w} + \bar{r}_1 < \bar{z} - 2\bar{w} + \bar{r} < \bar{z}_2 - 2\bar{w} + \bar{r}_2, \tag{3.35}$$

which implies that

$$e^{\bar{y}_1-2\bar{w}+\bar{j}_1} < e^{\bar{y}-2\bar{w}+\bar{j}} < e^{\bar{y}_2-2\bar{w}+\bar{j}_2}, \tag{3.36}$$

$$e^{\bar{z}_1-2\bar{w}+\bar{r}_1} < e^{\bar{z}-2\bar{w}+\bar{r}} < e^{\bar{z}_2-2\bar{w}+\bar{r}_2}. \tag{3.37}$$

In order to give a feasible bound for $e^{\bar{y}-2\bar{w}+\bar{j}}$ and $e^{\bar{z}-2\bar{w}+\bar{r}}$, we define

$$\bar{\Phi}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}, \tag{3.38}$$

$$\bar{\phi}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}. \tag{3.39}$$

It can be checked that for $t < 0$,

$$\bar{\phi}(t) < e^t < \bar{\Phi}(t). \tag{3.40}$$

To apply this result to (3.36) and (3.37), it suffices to prove that both $\bar{y}_2 - 2\bar{w} + \bar{j}_2$ and $\bar{z}_2 - 2\bar{w} + \bar{r}_2$ are negative. Straightforward calculation suggests

$$\bar{y}_2 - 2\bar{w} + \bar{j}_2 = -\frac{\pi^2\left(5\pi^6 + 4\pi^2\bar{w}^4 - 8\bar{w}^6\right)}{4\bar{w}^7},$$

$$\bar{z}_2 - 2\bar{w} + \bar{r}_2 = -\frac{\pi^4 \left(5\pi^4 + 16\bar{w}^4\right)}{64\bar{w}^7}.$$

Obviously, for $n \geq 2$, both $\bar{y}_2 - 2\bar{w} + \bar{j}_2$ and $\bar{z}_2 - 2\bar{w} + \bar{r}_2$ are negative. Thus, applying (3.40) to (3.36) and (3.37), we get that for $n \geq 11$,

$$\bar{\phi}(\bar{y}_1 - 2\bar{w} + \bar{j}_1) < e^{\bar{y} - 2\bar{w} + \bar{j}} < \bar{\Phi}(\bar{y}_2 - 2\bar{w} + \bar{j}_2), \tag{3.41}$$

$$\phi(\bar{z}_1 - 2\bar{w} + \bar{r}_1) < e^{\bar{z} - 2\bar{w} + \bar{r}} < \Phi(\bar{z}_2 - 2\bar{w} + \bar{r}_2). \tag{3.42}$$

Now, we are in the position to prove (3.24). For convenience, let

$$\bar{A}(\bar{w}) = \bar{h}_1 e^{\bar{y} - 2\bar{w} + \bar{j}} - 4\bar{h}_2 e^{\bar{z} - 2\bar{w} + \bar{r}} + 3\bar{h}_3, \tag{3.43}$$

we need to show the positivity of $\bar{A}(\bar{w})$. Making use of (3.31), (3.32), (3.33), (3.41) and (3.42), we obtain that for $n \geq 11$,

$$\begin{aligned}
\bar{A}(\bar{w}) > &\left(\bar{y}_1\bar{y}^4 - \bar{y}^4 - 1\right)\left(\bar{j}_1\bar{j}^4 - \bar{j}^4 - 1\right)\bar{w}^{14}\bar{z}_1\bar{z}^6\bar{r}_1\bar{r}^6\bar{\phi}(\bar{y}_1 - 2\bar{w} + \bar{j}_1) \\
&- 4\left(\bar{z}_2\bar{z}^4 - \bar{z}^4 + 1\right)\left(\bar{r}_2\bar{r}^4 - \bar{r}^4 + 1\right)\bar{w}^{14}\bar{y}_2\bar{y}^6\bar{j}_2\bar{j}^6\Phi(\bar{z}_2 - 2\bar{w} + \bar{r}_2) \\
&+ 3\bar{y}_1\bar{y}^6\bar{z}_1\bar{z}^6\bar{r}_1\bar{r}^6\bar{j}_1\bar{j}^6\bar{\beta}(\bar{w})^2.
\end{aligned}$$

Substituting $\bar{y}, \bar{z}, \bar{r}$ and $\bar{j}$ with $\bar{y} = \sqrt{\bar{w}^2 - 2\pi^2}, \bar{z} = \sqrt{\bar{w}^2 - \pi^2}, \bar{r} = \sqrt{\bar{w}^2 + \pi^2}$, and $\bar{j} = \sqrt{\bar{w}^2 + 2\pi^2}$, respectively, we can rewrite the right-hand side of the above inequality as

$$\frac{\sum_{k=0}^{90} a_k \bar{w}^k}{2^{47}3^1 5^1 \bar{w}^{53}}, \tag{3.44}$$

where a_k are known real numbers, and the values of $a_{90},\ a_{89},\ a_{88}$ are given below,

$$a_{90} = 2^{47}3\pi^2, \quad a_{89} = 2^{48}15\left(\pi^2 - 2\right), \quad a_{88} = 2^{48}5\pi^2\left(2\pi^4 - 6\pi^2 + 3\right).$$

Thus, for $n \geq 11$, we have

$$\bar{A}(\bar{w}) > \frac{\sum_{k=0}^{90} a_k \bar{w}^k}{2^{47}3^1 5^1 \bar{w}^{53}}. \tag{3.45}$$

As $\bar{w}$ is positive for $n \geq 1$, we have that

$$\sum_{k=0}^{90} a_k \bar{w}^k > \sum_{k=0}^{89} -|a_k|\bar{w}^k + a_{90}\bar{w}^{90}. \tag{3.46}$$

Thus, to obtain (3.45), we only need to show that for $n \geq 55$,

$$\sum_{k=0}^{89} -|a_k|\bar{w}^k + a_{90}\bar{w}^{90} > 0. \tag{3.47}$$

For $0 \leq k \leq 88$, it can be seen that

$$-|a_k|\bar{w}^k > -a_{88}\bar{w}^{88} \tag{3.48}$$

holds for $\bar{w} > \sqrt[3]{\dfrac{224\pi^{10} + 2635\pi^8 - 5520\pi^6 + 480\pi^2 - 1920}{80\pi^2\left(2\pi^4 - 6\pi^2 + 3\right)}} \approx 7.19$. It follows that for $\bar{w} \geq 7$,

$$\begin{aligned}\sum_{k=0}^{90} a_k\bar{w}^k &> \sum_{k=0}^{89} -|a_k|\bar{w}^k + a_{90}\bar{w}^{90} \\ &> (-89a_{88} + a_{89}\bar{w} + a_{90}\bar{w}^2)w^{88}.\end{aligned} \tag{3.49}$$

Combing (3.45) and (3.49), $\bar{A}(\bar{w})$ is positive provided

$$-89a_{88} + a_{89}\bar{w} + a_{90}\bar{w}^2 > 0, \tag{3.50}$$

which is true for $\bar{w} \geq 60.4$, or equivalently, for $n \geq 367$. Thus, we arrive at that (3.24) is true for $n \geq 367$, which implies (3.18). Combining (3.18) and (3.17), we obtain that for $n \geq 367$,

$$\bar{s}(n)\bar{g}(n+2)^2 - 4\bar{g}(n+2) + 3 > 0. \tag{3.51}$$

This proves the right hand side of (3.9).

On the other hand, it can be checked that (3.1) is also true for $7 \leq n \leq 366$. Hence the proof of Theorem 3.1 is completed. ∎

4. Log-monotonicity and Laguerre inequality

In this section, we will concern with some other combinatorial sequences. Wang [39] proved that for the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and the Domb numbers, the sequences $\{a_n/n!\}_{n\geq 0}$ satisfy Turán inequality of order 2 even though the sequences $\{a_n\}_{n\geq 0}$ are not log-concave. These results encourage us to consider whether the Laguerre inequality of order m holds for these sequences. In fact, we find that the Laguerre inequality of order 2 is closely related to the Log-monotonic of order 3 raised in a previous paper [6] of the first author.

In [6] an operator $\mathcal{R}$ on a sequence $\{a_n\}_{n\geq 0}$ is defined by

$$\mathcal{R}\{a_n\}_{n\geq 0} = \{b_n\}_{n\geq 0},$$

where $b_n = a_{n+1}/a_n$, and then, we say that the sequence $\{a_n\}_{n\geq 0}$ is log-monotonic of order k if for r odd and not greater than $k-1$, the sequence $\mathcal{R}^r\{a_n\}_{n\geq 0}$ is log-concave and for r even and not greater than $k-1$, the sequence $\mathcal{R}^r\{a_n\}_{n\geq 0}$ is log-convex. Now we are in a position to prove the following theorem.

Theorem 4.1. *If a sequence $\{a_n\}_{n\geq 0}$ is log-monotonic of order 3, then for $n \geq 0$, the Laguerre inequality of order 2 holds for the sequence $\{a_n\}_{n\geq 0}$, i.e.,*

$$3a_{n+2}^2 - 4a_{n+1}a_{n+3} + a_n a_{n+4} > 0. \tag{4.1}$$

Proof. To verify (4.1), we aim to show that

$$3 - 4v_{n+2} + v_{n+1}v_{n+2}^2 v_{n+3} > 0. \tag{4.2}$$

Since $\{a_n\}_{n\geq 0}$ is log-monotonic of order 3, we have $v_{n+1} = \frac{a_n a_{n+2}}{a_{n+1}^2}$ is log-convex, i.e.,

$$v_{n+2}^2 < v_{n+1}v_{n+3} \tag{4.3}$$

Thus, to prove (4.2), it suffices to show that

$$3 - 4v_{n+2} + v_{n+2}^4 > 0. \tag{4.4}$$

Since

$$3 - 4v_{n+2} + v_{n+2}^4 = (v_{n+2}-1)^2(v_{n+2}^2 + 2v_{n+2} + 3), \tag{4.5}$$

which is apparently positive for $v_{n+2} \neq 1$, we arrive at (4.4). ∎

Chen, Guo and Wang [6] showed that the Bernoulli numbers are infinitely log-monotonic. Zhu [42] proved that the sequences of the derangement numbers, the Motzkin numbers, the Fine numbers, Franel numbers and the Domb numbers are log-monotonic of order 3. Thus, by Theorem 4.1 we immediately deduce the following corollary.

Corollary 4.2. *The sequences of the Bernoulli numbers, the derangement numbers, the Motzkin numbers , the Fine numbers, the Franel numbers and the Domb numbers satisfy the Laguerre inequality of order 2.*

Acknowledgments. We thank the anonymous referee for helpful comments. This work was supported by the National Natural Science Foundation of China (grant number 12171254) and the Natural Science Foundation of Tianjin (grant number 19JCYBJC30100).

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