Note on rainbow cycles in edge-colored graphs*

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Abstract

Let G be a graph of order n with an edge-coloring c, and let $\delta^c(G)$ denote the minimum color-degree of G. A subgraph F of G is called rainbow if all edges of F have pairwise distinct colors. There have been a lot of results on rainbow cycles of edge-colored graphs. In this paper, we show that (i) if $\delta^c(G) > \frac{2n-1}{3}$, then every vertex of G is contained in a rainbow triangle; (ii) if $\delta^c(G) > \frac{2n-1}{3}$ and $n \geq 13$, then every vertex of G is contained in a rainbow C_4; (iii) if G is complete, $n \geq 7k - 17$ and $\delta^c(G) > \frac{n-1}{2} + k$, then G contains a rainbow cycle of length at least k, where $k \geq 5$.

Keywords: edge-coloring; edge-colored graph; rainbow cycle; color-degree condition

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1 Introduction

We consider finite simple undirected graphs. An edge-coloring of a graph G is a mapping $c : E(G) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. A graph G is called an *edge-colored graph* if G is assigned an edge-coloring. The color of an edge e of G and the set of colors assigned to $E(G)$ are denoted by $c(e)$ and $C(G)$, respectively. For $V_1, V_2 \subset V(G)$ and $V_1 \cap V_2 = \emptyset$, we set $E(V_1, V_2) = \{xy \in E(G), x \in V_1, y \in V_2\}$, and when $V_1 = \{u\}$, we write $E(u, V_2)$ for $E(\{u\}, V_2)$. The set of colors appearing on the edges between V_1 and V_2 in G is denoted by $C(V_1, V_2)$. When $V_1 = \{v\}$, we use $C(v, V_2)$

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instead of $C(\{v\}, V_2)$. For a subgraph T of G, the set of colors appearing on $E(T)$ is denoted by $C(T)$, and we use T^C to denote $G - T$. A subset F of edges of G is called *rainbow* if no distinct edges in F receive the same color, and a graph is called *rainbow* if its edge-set is rainbow. Specially, a path P is *rainbow* if no distinct edges in $E(P)$ are assigned the same color. The length of a path $P = u_1u_2 \cdots u_p$ is the number of edges in $E(P)$, denoted by $\ell(P)$. We use u_iPu_j to denote the segment between u_i and u_j on P. If $i < j$, then $u_iPu_j = u_iu_{i+1} \cdots u_j$; if $i > j$, then $u_iPu_j = u_iu_{i-1} \cdots u_j$. For a vertex $v \in V(G)$, the *color-degree* of v in G is the number of distinct colors assigned to the edges incident to v, denoted by $d^c_G(v)$. We use $\delta^c(G) = \min\{d^c_G(v) : v \in V(G)\}$ to denote the *minimum color-degree* of G. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. Let $N^c(v)$ be a subset of $N_G(v)$ such that $|N^c(v)| = d^c_G(v)$ and each color in $C(v, N_G(v))$ appears in $E(v, N^c(v))$ exactly once. For each vertex $v \in V(G)$ and a color subset $C' = \{c_1, c_2, \cdots, c_k\}$ of $C(G)$, let $N_{C'}(v) = \{u \mid u \in N^c(v), c(uv) \in C'\}$. For a subset S of V, we denote $N_{C'}(v) \cap S$ by $N_{C'}(v, S)$. When $S = V(P)$, we use $N_{C'}(v, P)$ instead of $N_{C'}(v, V(P))$. For other notation and terminology not defined here, we refer to [1].

The existence of rainbow substructures in edge-colored graphs has been widely studied in literature. We mention here only those known results that are related to our paper. For short rainbow cycles, a minimum color-degree condition for the existence of a rainbow triangle was given by Li [5] in 2013.

Theorem 1.1 ([5])**.** *Let G be an edge-colored graph of order $n \geq 3$. If $\delta^c(G) > \frac{n}{2}$, then G has a rainbow triangle.*

In 2014, Li et al. [7] improved Theorem 1.2 and got the following result.

Theorem 1.2 ([7])**.** *Let G be an edge-colored graph of order $n \geq 3$ satisfying one of the following conditions:*

(1) $\sum_{u \in V(G)} d^c(u) \geq \frac{n(n+1)}{2}$,

(2) $\delta^c(G) \geq \frac{n}{2}$ and $G \notin \{K_{\frac{n}{2},\frac{n}{2}}, K_4, K_4 - e\}$.

Then G contains a rainbow triangle.

There is also a result about the rainbow triangles in an edge-colored complete graph which was proved by Fujita and Magnant in 2011.

Theorem 1.3 ([8])**.** *Let G be an edge-colored complete graph of order $n \geq 3$. If $\delta^c(G) \geq \frac{n+1}{2}$, then each vertex of G is contained in a rainbow triangle.*

In this paper, we get the the following result.

Theorem 1.4. *Let G be an edge-colored graph of order $n \geq 3$. If $\delta^c(G) > \frac{2n-1}{3}$, then each vertex of G is contained in a rainbow triangle and the bound is sharp.*

Next, for the rainbow C_4, Čada et al. in [2] obtained the following result.

Theorem 1.5 ([2]). *Let G be an edge-colored graph of order n. If G is triangle-free and $\delta^c(G) > \frac{n}{3} + 1$, then G contains a rainbow C_4.*

In this paper, we also get a result as follows.

Theorem 1.6. *Let G be an edge-colored graph of order $n \geq 13$. If $\delta^c(v) > \frac{2n-1}{3}$, then each vertex of G is contained in a rainbow C_4.*

Remark 1.7. We think that the lower bound in Theorem 1.4 is sharp and that in Theorem 1.6 is not sharp, since in the end of the proof of Theorem 1.4, (in Section 2), we construct an edge-colored graph G with $\delta^c(G) = \frac{2n-1}{3}$ in which there exists a vertex such that no rainbow triangle contains it, while each vertex is contained in a rainbow C_4.

Finally, for long rainbow cycles, Li and Wang in [6] got the following result.

Theorem 1.8 ([6]). *Let G be an edge-colored graph of order $n \geq 8$. If for each vertex v of G, $d^c(v) \geq d \geq \frac{3n}{4} + 1$, then G has a rainbow cycle of length at least $d - \frac{3n}{4} + 2$.*

In 2016, Čada et al. in [2] obtained a result on rainbow cycles of length at least four.

Theorem 1.9 ([2]). *Let G be an edge-colored graph of order n. If for each vertex v of G, $d^c(v) > \frac{n}{2} + 2$, then G contains a rainbow cycle of length at least four.*

At the end of their paper [2], they raised the following conjecture.

Conjecture 1.10 ([2]). *Let G be an edge-colored graph of order n and k be a positive integer. If for each vertex v of G, $d^c(v) > \frac{n+k}{2}$, then G contains a rainbow cycle of length at least k.*

Inspired by Theorem 1.9, Tangjai in [9] proved the following result.

Theorem 1.11 ([9]). *Let G be an edge-colored graph of order n and k be a positive integer. If G has no rainbow cycle of length 4 and $\delta^c(G) \geq \frac{n+3k-2}{2}$, then G contains a rainbow cycle of length at least k, where $k \geq 5$.*

Recently, the authors in [4] proved that G has a rainbow cycle of length ℓ when the order of G is large enough, depending on ℓ.

Theorem 1.12 ([4])**.** *For every integer* $\ell \geq 3$*, every edge-colored graph* G *of order* $n \geq n_0(\ell)$ *with* $\delta^c(G) \geq \frac{n+1}{2}$ *admits a rainbow* ℓ*-cycle* C_ℓ*, where* $n_0(\ell) \leq 432\ell$*.*

We will show the following result.

Theorem 1.13. *Let* G *be an edge-colored complete graph of order* n *and* k *be a positive integer at least 5. If* $n \geq 7k - 17$ *and* $\delta^c(G) > \frac{n-1}{2} + k$*, then* G *contains a rainbow cycle of length at least* k*.*

In order to prove our main result, we need the following result of Chen and Li in [3] on the existence of long rainbow paths.

Theorem 1.14 ([3])**.** *Let* G *be an edge-colored graph, where* $\delta^c(G) \geq t \geq 7$*. Then the maximum length of rainbow paths in* G *is at least* $\lceil \frac{2t}{3} \rceil + 1$*.*

In the following sections, we will give the proofs of our three results, Theorems 1.4, 1.6 and 1.13.

2 Proofs of Theorems 1.4 and 1.6

To present the proof of Theorems 1.4 and 1.6, we need some auxiliary lemmas. Let G be an edge-colored graph and v be a vertex of G. A subset A of $N_G(v)$ is said to have the *dependence property* with respect to a vertex $v \notin A$, denoted by DP_v, if $c(aa') \in \{c(va), c(va')\}$ for all $aa' \in E(G[A])$.

Lemma 2.1. *If a subset* A *of vertices in an edge-colored graph* G *has the* DP_v*, then there exists a vertex* $x_0 \in A$ *such that the number of colors different from* $c(vx_0)$ *on the edges incident with* x_0 *in* $G[A]$ *is at most* $\frac{|A|-1}{2}$*.*

Proof. We prove this lemma by constructing an oriented graph. Orient the edges in $E(G[A])$ by applying the following rule: for an edge xy, if $c(xy) = c(vx)$, then the orientation of xy is from y to x; otherwise, the orientation of xy is from x to y. Thus, we get an oriented graph $D(A)$. Evidently, the arcs with colors different from $c(vx)$ are out-arcs from x. Let x_0 be a vertex in $D(A)$ with minimum out-degree. Clearly, $d^+_{D(A)}(x_0) \leq \frac{|A|-1}{2}$. Thus, the number of colors different from $c(vx_0)$ on the edges incident with x_0 in $G[A]$ is at most $\frac{|A|-1}{2}$. □

Proof of Theorem 1.4: Let G be a graph satisfying the assumptions of Theorem 1.4 and suppose, to the contrary, that there exists a vertex v such that no rainbow triangle contains it. For any edge $e = xy \in E(G[N^c(v)])$, since $c(vx) \neq c(vy)$, we have $c(xy) \in \{c(vx), c(vy)\}$; otherwise $vxyv$ is a rainbow triangle. Thus, $N^c(v)$ has the dependence property with respect to v. According to Lemma 2.1, there is a vertex $x_0 \in N^c(v)$ such that the number of colors different from $c(vx_0)$ on the edges incident with x_0 in $G[N^c(v)]$ is at most $\frac{|N^c(v)|-1}{2}$. Then, we have $|N^c(x_0) \cap (N^c(v) \cup \{v\})| \leq \frac{d^c(v)-1}{2} + 1$. Thus, $|N^c(x_0) \cap (V(G) \setminus (N^c(v) \cup \{v\}))| \geq \delta^c(G) - (\frac{d^c(v)-1}{2} + 1)$. So, we have

$$\begin{aligned} n &\geq |N^c(v)| + |V(G) \setminus (N^c(v) \cup \{v\})| + |\{v\}| \\ &\geq d^c(v) + |N^c(x_0) \cap (V(G) \setminus (N^c(v) \cup \{v\}))| + 1 \\ &\geq d^c(v) + \delta^c(G) - (\tfrac{d^c(v)-1}{2} + 1) + 1 \\ &\geq \tfrac{3}{2}\delta^c(G) + \tfrac{1}{2}. \end{aligned}$$

Hence, we have $\delta^c(G) \leq \frac{2n-1}{3}$, a contraction.

$c(u_iu_j)$ ($u_j \in V_2$ / $u_i \in V_1$)	u_{2n+2}	u_{2n+3}	$\cdots$	u_{3n}	u_{3n+1}
u_1	c_{n+2}	c_{n+3}	$\cdots$	c_{2n}	c_{2n+1}
u_2	c_{n+3}	c_{n+4}	$\cdots$	c_{2n+1}	c_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_n	c_{2n+1}	c_1	$\cdots$	c_{n-2}	c_{n-1}
u_{n+1}	c_1	c_2	$\cdots$	c_{n-1}	c_n
u_{n+2}	c_2	c_3	$\cdots$	c_n	c_{n+1}
u_{n+3}	c_3	c_4	$\cdots$	c_{n+1}	c_{n+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{2n+1}	c_{n+1}	c_{n+2}	$\cdots$	c_{2n-1}	c_{2n}

Table 1: An edge-coloring of $E[V_1, V_2]$

Now we show that the bound on $\delta^c(G)$ is tight. Consider an edge-colored graph G of order $3n + 2$ with $C(G) = \{c_1, c_2, \cdots, c_{2n+1}\}$. Let v be a vertex of G and the vertex-set of G is $\{v\} \cup V_1 \cup V_2$ satisfying that $|V_1| = 2n + 1$ and $|V_2| = n$. We label the vertices in V_1 by $\{u_1, u_2, \cdots, u_{2n+1}\}$ and those in V_2 by $\{u_{2n+2}, u_{2n+3}, \cdots, u_{3n+1}\}$. Let $G[\{v\} \cup V_1]$ be a complete graph and $G[\{u_i\}, V_2]$ be a complete bipartite graph for all $u_i \in V_1$. The edge-set of G is $E(G) = \bigcup_{1 \leq i \leq 2n+1} E[\{u_i\}, V_2]) \cup E(G[\{v\} \cup V_1])$.

The edge-coloring of $E(G)$ satisfies the following three conditions:

(1) $c(vu_i) = c_i$ for $1 \leq i \leq 2n + 1$;

(2) for any two vertices u_i and u_j in V_1 (w.l.o.g., $i \geq j$):

$$c(u_iu_j) = \begin{cases} c_i, & \text{if } i-j \leq n, \\ c_j, & \text{if } i-j \geq n+1; \end{cases}$$

(3) for any two vertices $u_i \in V_1$ and $u_j \in V_2$, the color of u_iu_j follows Table 1.

It is easy to verify that $d^c(u_k) = 2n+1$ for $u_k \in \{v\} \cup V_2$. We then discuss the color-degrees of the vertices u_k in V_1. By (2) we know that if $k \leq n+1$, then

$$c(u_ku_i) = \begin{cases} c_k, & \text{if } i \in [1, k-1] \cup [n+k+1, 2n+1] \\ c_i, & \text{if } i \in [k+1, n+k] \end{cases}$$

and $C(u_k, V_1) = \{c_k, c_{k+1}, \cdots, c_{n+k}\}$. If $k \geq n+2$, then

$$c(u_ku_i) = \begin{cases} c_k, & \text{if } i \in [k-n, k-1] \\ c_i, & \text{if } i \in [1, k-(n+1)] \cup [k+1, 2n+1] \end{cases}$$

and $C(u_k, V_1) = \{c_1, c_2, \cdots, c_{k-(n+1)}, c_k, c_{k+1}, \cdots, c_{2n+1}\}$. Thus, $d^c(u_k) = 2n+1$ for $u_k \in V_1$. Note that v is not contained in any rainbow triangle in G. Therefore, the bound of $\delta^c(G)$ in Theorem 1.4 is sharp. $\square$

Proof of Theorem 1.6: Let G be a graph satisfying the assumptions of Theorem 1.6 and suppose, to the contrary, that there exists a vertex x such that no rainbow C_4 contains it. If x is not contained in a rainbow triangle either, then by Theorem 1.4, we have $\delta^c(G) \leq \frac{2n-1}{3}$, a contradiction. Thus, we assume that x is contained in a rainbow triangle $T = xyzx$ in G. Let $N^c(x)$ be a rainbow neighbor-set of x containing y and z. At first, we use the following algorithm to output a family $\mathcal{F}$ of q disjoint subsets $S_0, S_1, \cdots, S_q$ of $N^c(x)$ which will be useful for our proof.

Algorithm 1

Input: G.

Output: A family $\mathcal{F}$ of disjoint subsets.

1: **Set** $S_0 = \{z\}$.
2: Let $S_z = \{y\}$.
3: **Set** $S_1 = S_z$.
4: Let $S_y = \{u \in N_{C(G)\setminus C(T)}(x) \cap N_G(y) : c(yu) \notin \{c(xy), c(zy)\}\}$.
5: **Set** $S_2 = S_y$.
6: **Set** $\mathcal{F} = \{S_0, S_1, S_2\}$.
7: **Set** $i = 2$.
8: **for** $S_i \neq \emptyset$ **do**
9: for $v \in S_i$, use v^- to denote one of the predecessors of v such that $v \in S_{v^-}$,
10: and let $S_v = \{u \in N^c(x) \cap N_G(v) \setminus \bigcup_{k=0}^{i} S_k : c(uv) \notin \{c(xv), c(vv^-)\}\}$.
11: **Set** $S_{i+1} = \bigcup_{v \in S_i} S_v$.
12: **Set** $i = i + 1$.
13: **Set** $\mathcal{F} = \mathcal{F} \bigcup \{S_i\}$.
14: **if** $S_i = \emptyset$ **then**
15: **Set** $q = i$ and return $\mathcal{F}$.

The following claim states that steps 9 and 10 in Algorithm 1 can be executed.

Claim 1. (1) If $u \in S_v$ for $v \in \cup_{i=1}^{q-1} S_i$, then $c(uv) \in \{c(xu), c(xv^-)\}$.

(2) For $u \in S_i$ and $i \geq 3$, if there exist two distinct vertices $v, w \in S_{i-1}$ such that $u \in S_v \cap S_w$, then $c(uv) = c(uw)$, that is, $|C(u, S_u^-)| = 1$ where $S_u^- = \{v \in S_{i-1} | u \in S_v\}$.

Proof. (1) Suppose $v \in S_i$ and $1 \leq i \leq q-1$. Since $u \in S_v$, we have $c(uv) \notin \{c(xv), c(vv^-)\}$. We prove this claim by induction on i. When $i = 1$, we have $v = y$ and $v^- = y^- = z$. Since $u \in N_{C(G)\setminus C(T)}(x) \cap N_G(y)$, we know that $c(xu), c(xz), c(zy)$ are three distinct colors. Since $xzyux$ is not a rainbow C_4, we have $c(yu) \in \{c(xz), c(xu)\} = \{c(xy^-), c(xu)\}$. When $i \geq 2$, by the induction hypothesis, we have $c(vv^-) \in \{c(xv), c(xv^{--})\}$. Then $c(xu), c(vv^-), c(xv^-)$ are three distinct colors. Since xv^-vux is not a rainbow C_4, we have $c(uv) \in \{c(xu), c(xv^-)\}$. The result thus follows.

(2) According to (1), we have $c(uv) \in \{c(xu), c(xv^-)\}$ and $c(uw) \in \{c(xu), c(xw^-)\}$. Since $v, w \in S_{i-1}$, we have $v^- \neq w$ and $w^- \neq v$. Thus, $\{c(uv), c(uw)\} \cap \{c(xv), c(xw)\} = \emptyset$. Hence, $c(uv) = c(uw)$; otherwise, $xvuwx$ is a rainbow C_4, a contradiction. □

Since $N^c(x)$ is a finite set, Algorithm 1 can return $\{S_0, S_1, \cdots, S_q\}$ within finite steps.

Apparently, $\{S_0, S_1, \cdots, S_q\}$ is a family of disjoint subsets of $N^c(x)$. Let $S = \bigcup_{i=2}^{q} S_i$. Then we get the following Claims 2 through 5, which are crucial steps for completing the proof of Theorem 1.6.

Claim 2. If $vu \in E(G[S])$ with $v \in S_i$ and $u \in S_j$ (w.l.o.g., assume that $2 \le i \le j \le q$), then $c(vu) \in \{c(xv), c(xu), c(xv^-), c(xv^{--})\}$. Furthermore, we can get the following specific results:

(1) If $c(vu) = c(xv)$, then $\begin{cases} v \neq u^-, c(vv^-) = c(xv), & \text{if } i < j \\ c(vv^-) = c(xv), & \text{if } i = j. \end{cases}$

(2) If $c(vu) = c(xu)$, then $\begin{cases} v = u^-, & \text{if } i < j \\ c(uu^-) = c(xu), & \text{if } i = j. \end{cases}$

(3) If $c(vu) = c(xv^-)$, then $u^- = \begin{cases} v, & \text{if } i < j \\ v^-, & \text{if } i = j. \end{cases}$

(4) If $c(vu) = c(xv^{--})$, then

$$\begin{cases} u^{--} = v, c(vv^-) = c(xv^{--}), c(uu^-) = c(xv), c(vu^-) = c(xu^-), & \text{if } i < j \\ u^{--} = v^{--}, c(vv^-) = c(uu^-) = c(xu^{--}) = c(xv^{--}), & \text{if } i = j. \end{cases}$$

Then, if $i < j$, we have $c(vu) \in \{c(xv), c(xu), c(xu^{--}), c(xu^{----})\}$ as well.

Proof. According to (1) of Claim 1, we know $c(vv^-) \in \{c(xv), c(xv^{--})\}$. Since xv^-vux is not a rainbow C_4, we have $c(vu) \in \{c(xv^-), c(vv^-), c(xu)\} \subseteq \{c(xv), c(xu), c(xv^-), c(xv^{--})\}$.

We distinguish the following two cases.

Case 1. $i < j$.

If $v = u^-$, then from (1) of Claim 1 we have $c(vu) \in \{c(xu), c(xv^-)\}$. Thus we assume $v \neq u^-$. Then we have $c(vu) \in \{c(xv), c(vv^-)\}$ by steps 4 and 10 of Algorithm 1. Note that $\{c(xv), c(vv^-)\} \cap \{c(xu), c(xv^-)\} = \emptyset$.

If $c(vu) = c(xv)$, then since xv^-vux is not a rainbow C_4, we have $c(vv^-) = c(vu)$. Since $c(vv^-) \in \{c(xv), c(xv^{--})\}$ by (1) of Claim 1, we have $c(vv^-) = c(vu) = c(xv)$, and so the "if $i < j$" case of (1) follows.

If $c(vu) = c(vv^-)$, then since $c(vu) \in \{c(xv), c(xv^{--})\}$ by (1) of Claim 1, we assume $c(vu) = c(xv^{--})$, which means that $c(vu) = c(vv^-) = c(xv^{--})$. Note that $v^{--} \in S_{i-1}$. Then $v^{--} \neq u^-$. Therefore, $c(xv), c(xu^-), c(vu)$ are three distinct colors. Since $xvuu^-x$ is not a rainbow C_4, by (1) of Claim 1 we have $c(uu^-) \in \{c(xv), c(xu^-), c(xv^{--})\} \cap \{c(xu^{--}), c(xu)\}$. Hence, $c(uu^-) = c(xu^{--}) = c(xv)$, which implies that $u^{--} = v$. Then,

$u^- \in S_v$. Hence, $c(vu^-) \neq c(vv^-) = c(vu)$ by steps 4 and 10 of Algorithm 1. Since xu^-vux is not a rainbow C_4, we have $c(vu^-) \in \{c(xu), c(xu^-), c(vu)\} \cap \{c(xu^-), c(xv^-)\}$ by (1) of Claim 1. Apparently $c(vu^-) = c(xu^-)$, i.e., the "if $i < j$" case of (4) follows.

Combining with the first sentence of this case, the "if $i < j$" cases of (2) and (3) follow.

Case 2. $i = j$.

Then by Claim 1 we have that $c(vv^-) \notin \{c(xu), c(xv^-)\}$ and $c(uu^-) \notin \{c(xv), c(xu^-)\}$. Since neither xu^-uvx nor xv^-vux is rainbow, we have

$$c(vu) \in \{c(xv), c(xu^-), c(uu^-)\} \cap \{c(xu), c(xv^-), c(vv^-)\}.$$

By (1) of Claim 1 again, we have

$$c(vu) \in \{c(xv), c(xu), c(xu^-), c(xu^{--})\} \cap \{c(xu), c(xv), c(xv^-), c(xv^{--})\}.$$

If $c(vu) = c(xv)$, then since $c(xv) \notin \{c(xu), c(xv^-), c(xv^{--})\}$, we have $c(vv^-) = c(xv)$. If $c(vu) = c(xu)$, then $c(uu^-) = c(xu)$ similarly. If $c(vu) = c(xv^-)$, then since $c(xv^-) \notin \{c(xu), c(xv), c(xu^{--})\}$, we have $c(xu^-) = c(xv^-)$, which implies $v^- = u^-$. If $c(vu) = c(xv^{--})$, then since $c(xv^{--}) \notin \{c(xu), c(xv), c(xu^-)\}$, we have $c(vu) = c(vv^-) = c(uu^-) = c(xu^{--}) = c(xv^{--})$, which implies $v^{--} = u^{--}$. Then the "if $i = j$" cases of (1)-(4) follow. □

Claim 3. For each vertex $v \in S_i$ with $2 \le i \le q$, if $c(xv)$ or $c(xv^-)$ is incident with v in $G[S]$, then $c(xv^{----})$ is not incident with v in $G[S]$.

Proof. Suppose to the contrary that there exists a vertex $u \in S$ such that $c(uv) = c(xv^{----})$. By Claim 2, we can get that $i > j$, and then $c(uv) \in \{c(xv), c(xu), c(xu^-), c(xu^{--})\}$. If $c(uv) = c(xu^-)$, then by (3) of Claim 2, we have $v^- = u$, which gives $c(vu) = c(xv^{--})$, a contradiction. Hence from (4) of Claim 2, we have $c(uv) = c(xu^{--})$ and $u = v^{--}$. Thus $u \in S_{i-2}$.

(i) if there exists a vertex $w \in S$ such that $c(wv) = c(xv^-)$, then by (3) of Claim 2, we have $v = w^-$ or $v^- = w^-$, which implies that $w \in S_i \cup S_{i+1}$. It is easy to verify that $xuvwx$ is a rainbow C_4, a contradiction.

(ii) if there exists a vertex $w \in S$ such that $c(wv) = c(xv)$, then $w \neq v^{----}$ clearly. It is easy to verify that $xuvwx$ is a rainbow C_4, a contradiction. □

Now we define a directed graph D on $S = \bigcup_{k=2}^{q} S_k$. For any two distinct vertices u, v in S, if $uv \in E(G)$, we define the arcs joint them as follows:

(a) If u, v are in the same set of S_k for $2 \leq k \leq q$, then by Claim 2, $c(uv) \in \{c(xv), c(xu), c(xv^-), c(xv^{--})\}$. Hence, $\overrightarrow{uv}$ exists if $c(uv) \in \{c(xv), c(xv^-), c(xv^{--})\}$; otherwise $\overleftarrow{uv}$ exists.

(b) If u, v are in distinct sets of S_k for $2 \leq k \leq q$, say $v \in S_i$ and $u \in S_j$, then $\overrightarrow{uv}$ exists if $v \in S_u$ and $\overrightarrow{vu}$ exists if $u \in S_v$; otherwise, $\overrightarrow{uv}$ exists if $i < j$ and $\overrightarrow{vu}$ exists if $i > j$.

Note that $u^- = v^-$ if $c(uv) = c(xv^-)$, and $u^{--} = v^{--}$ if $c(uv) = c(xv^{--})$, when $i = j$ by (3) and (4) of Claim 2. Hence, both $\overrightarrow{uv}$ and $\overleftarrow{uv}$ exist if $c(uv) \in \{c(xv^-), c(xv^{--})\}$ in (a). We obtain $A(D)$ by deleting one in each pair of oppositely oriented arcs with the same ends. Hence, we get an oriented graph $D = D(S, A(D))$. According to Algorithm 1 and Claims 1 and 2, it is easy to verify that for each vertex $v \in S$,

$$\text{each in-arc } \overrightarrow{uv} \text{ from } v \text{ in } D \text{ satisfies } c(uv) \in \{c(xv), c(xv^-), c(xv^{--})\}. \qquad (*1)$$

We analyze the color of each out-arc $\overleftarrow{uv}$ from v in D depending on the way of orientation:

(c) uv is oriented by the method (a) above, then $c(uv) \in \{c(xu), c(xu^-), c(xu^{--})\} = \{c(xu), c(xv^-), c(xv^{--})\}$;

(d) uv is oriented by the method (b) above, then if $u \in S_v$, then $c(uv) \in \{c(xu), c(xv^-)\}$ by (1) of Claim 1; if $u \notin S_v$, then $j < i$ and $c(uv) \in \{c(xu), c(uu^-)\}$ by steps 4 and 10 of Algorithm 1. Note that $c(uu^-) \in \{c(xu), c(xu^{--})\}$. If $c(uv) = c(xu^{--})$, then by the "if $i < j$" case of (4) in Claim 2, we have $v^{--} = u$. Hence, $c(uv) = c(xv^{----})$.

Thus, by Claim 3 we have

$$\begin{aligned} &\text{each out-arc } \overleftarrow{uv} \text{ from } v \text{ in } D \text{ satisfies } c(uv) \in \{c(xu), c(xv^-), c(xv^{--})\} \qquad (*2)\\ &\text{or } c(uv) \in \{c(xu), c(xv^{--}), c(xv^{----})\}. \end{aligned}$$

According to Algorithm 1, for $v \in S$ we have

$$C(v, N^c(x) \setminus (S \cup \{y, z\})) \subseteq \{c(xv), c(vv^-)\} \subseteq C(v, S \cup \{x, y, z\}).$$

Hence,

$$C(v, N^c(x) \cup \{x\}) = C(v, N^+_D(v)) \cup C(v, N^-_D(v)) \cup C(v, \{x, y, z\}),$$

and then

$$d^c_{G[N^c(x)\cup\{x\}]}(v) \leq d^+_D(v) + |C(v, N^-_D(v) \cup \{x, y, z\})|.$$

Claim 4. For $v \in S$ with $c(xv) \neq c(zy)$, we have

(i) $|C(v, N^-_D(v) \cup \{x, y, z\})| \leq 3$, and the equality holds if and only if $C(v, N^-_D(v) \cup \{x, y, z\}) = \{c(xv), c(xv^-), c(xv^{--})\}$;

(ii) if the color $c(xv)$ is on an edge incident with v in $G[S]$, then $|C(v, N^-_D(v) \cup \{x, y, z\})| \leq 2$, and the equality holds if and only if $C(v, N^-_D(v) \cup \{x, y, z\}) = \{c(xv), c(xv^-)\}$.

Proof. Since neither $xzvv^-x$ nor $xyzvx$ is a rainbow C_4 (note that $y = v^-$ if $v \in S_2$), we have

$$c(vz) \in \{c(xz), c(xv^-), c(vv^-)\} \cap \{c(xv), c(xy), c(zy)\} \text{ if } vz \text{ exists.}$$

Since $c(xz) \notin \{c(xv), c(xy), c(zy)\}$, we have

$$c(vz) \in \{c(xv^-), c(vv^-)\} \cap \{c(xv), c(xy), c(zy)\} \text{ if } vz \text{ exists.} \quad (*4)$$

If $v \notin S_2$, since neither $xzyvx$ nor $xyvv^-x$ is a rainbow C_4 and $c(vy) \in \{c(xy), c(zy)\}$ by step 4 of Algorithm 1, we have

$$c(vy) \in \{c(xv), c(xz), c(zy)\} \cap \{c(xy), c(xv^-), c(vv^-)\} \cap \{c(xy), c(zy)\} \text{ if } vy \text{ exists.}$$

Since $c(xy) \notin \{c(xv), c(xz), c(zy)\}$ and $c(zy) \notin \{c(xy), c(xz), c(xv)\}$, we have

$$c(vy) \in \{c(xv^-), c(vv^-)\} \cap \{c(zy)\} \text{ if } vy \text{ exists.} \quad (*5)$$

If $v \in S_2$, by (1) of Claim 1, we have

$$c(vy) \in \{c(xv), c(xv^{--})\}. \quad (*6)$$

Above all, by $(*1)$ and (1) of Claim 1 we have

$$C(v, N_D^-(v) \cup \{x, y, z\}) \subseteq \{c(xv), c(xv^-), c(xv^{--})\}.$$

Moreover, if the color $c(xv)$ is on an edge incident with v in $G[S]$, by (1) and (2) of Claim 2 we have $c(vv^-) = c(xv)$.

According to $(*4)$-$(*6)$, if $c(xv^{--}) \in \{c(vy), c(vz)\}$, then $c(vv^-) = c(xv^{--})$, a contradiction. Hence, $C(v, \{x, y, z\}) \subseteq \{c(xv), c(xv^-)\}$.

Suppose there exists an arc $\overrightarrow{wv} \in A(D)$ with $c(wv) = c(xv^{--})$. By the definition of D, we have $w \neq v^{--}$; otherwise, the direction of wv is from v to w depending on (b). Then xv^-vwx is rainbow C_4, a contradiction. Hence, there exists no in-arc from v assigned $c(xv^{--})$.

Therefore, $c(xv^{--}) \notin C(v, N_D^-(v) \cup \{x, y, z\})$. The proof is thus complete. □

Claim 5. If $S \neq \emptyset$, then there exists a vertex v_0 in $N^c(x)$ such that $d^c_{G[N^c(x) \cup \{x\}]}(v_0) \leq \frac{d^c(x)+1}{2}$.

Proof. Suppose to the contrary that D contains no such vertex. Since $n \geq 13$, one has $d^c(x) \geq 9$. Since $S \neq \emptyset$, we assume that $\frac{d^c(x)-4}{2} \leq |S_2| \leq d^c(x) - 2$. Otherwise,

$d^c_{G[N^c(x)\cup\{x\}]}(y) \le |S_2| + 3 \le \frac{d^c(x)+1}{2}$, and thus the result follows. Since D is an oriented graph, we have $\delta^+(D) \le \frac{|S|-1}{2}$. We distinguish the following cases.

Case 1. $\delta^+(D) = \frac{|S|-1}{2}$.

In this case, each vertex in D has an out-degree $\frac{|S|-1}{2}$. At first we assert that for any vertex $v \in S$ with $c(xv) \ne c(zy)$, we have that $c(xv)$ is not incident with it in $G[S]$, since otherwise, by (2) of Claim 4 we have $d^c_{G[N^c(x)\cup\{x\}]}(v) \le \frac{|S|-1}{2} + 2 \le \frac{d^c(x)+1}{2}$, a contradiction. If it exists, let a_0 be such a vertex in S that $c(xa_0) = c(zy)$. Hence, for any edge $uv \in E[S \setminus \{a_0\}]$, $c(uv) \notin \{c(xu), c(xv)\}$.

Since $|S_2| \ge 2$, by Claim 2 we have $C(v, S\setminus\{a_0\}) \in \{c(xv^-), c(xv^{--})\}$ for $v \in S_2\setminus\{a_0\}$. Thus, by Claim 4 we have $C(v, N^c(x) \cup \{x\}) \subseteq \{c(xv), c(xa_0), c(xv^-), c(xv^{--})\}$. Hence, $d^c_{G[N^c(x)\cup\{x\}]}(v) \le 4 \le \frac{d^c(x)+1}{2}$, a contradiction.

Case 2. $\delta^+(D) \le \frac{|S|-2}{2}$.

Set $A = \{a \in S \mid d^+_D(a) \le \frac{|S|-1}{2}\}$. We first assert that $|A| \ge \frac{|S|}{2} - 1$. By Claim 4, we have that for any vertex $a \in A$ with $c(xa) \ne c(zy)$, $d^+_D(a) \ge \frac{|S|-2}{2}$, and for $a \in A$ with $c(xa) = c(zy)$, $d^+_D(a) \ge \frac{|S|-6}{2}$ by $(*1)$. If $d^+_D(S) \le \frac{|S|-3}{2}$, then $|S| \le 3$ and the assertion follows. If $d^+_D(S) \le \frac{|S|-1}{2}$, then there are at most $\frac{|S|}{2} + 1$ vertices with out-degree more than $\frac{|S|-1}{2}$.

If it exists, let a_0 be such a vertex in S that $c(xa_0) = c(zy)$. By the same argument as in Case 1, we have that for $a \in A$ with $c(xa) \ne c(zy)$, $c(xa)$ is not incident with a in $G[S]$. Let $B = S \setminus (A \cup \{a_0\})$. W.l.o.g., assume that $B \ne \emptyset$, and $|B| \le \frac{|S|}{2} + 1$.

Subcase 2.1. $\delta^+(D[B]) = \frac{|B|-1}{2}$.

In this case, each vertex in B has an out-degree $\frac{|B|-1}{2}$. There exists a vertex $b \in B$ with $c(ba_0) \ne c(xa_0)$; otherwise, by Claims 2 and 3, $C(a_0, N^c(x)\cup\{x\}) \subseteq \{c(xa_0), c(xa_0^-), c(xa_0^{--}), c(za_0), c(ya_0)\}$, a contradiction. If $c(xb)$ is incident with b in $G[S]$, then by Claim 3, $c(xb^{----})$ is not incident with b in $G[S]$. Hence from $(*2)$,

$$C(b, N^+_D(b) \cap (A \cup \{a_0\})) \subseteq \{c(xb^-), c(xb^{--})\}.$$

Thus,

$$\begin{aligned} C(b, N^c(x) \cup \{x\}) &\subseteq C(b, N^+_D(b) \cap B) \cup C(b, N^+_D(b) \cap (A \cup \{a_0\})) \cup C(b, N^-_D(b) \cup \{x, y, z\}) \\ &\subseteq C(b, N^+_{D[B]}(b)) \cup \{c(xb), c(xb^-), c(xb^{--})\}. \end{aligned}$$

Therefore,

$$\begin{aligned} d^c_{G[N^c(x)\cup\{x\}]}(b) &\le d^+_{D[B]}(b) + 3 \\ &\le \lfloor \tfrac{|B|-1}{2} \rfloor + 3 \\ &\le \tfrac{d^c(x)+1}{2}, \end{aligned}$$

a contradiction.

Let $B_1 = \{b \in B \mid c(ba_0) \neq c(xa_0)\}$ and $B_2 = \{b \in B \mid c(ba_0) = c(xa_0)\}$. Then for each $b \in B_1$, $c(xb)$ is not incident with b in $G[S]$. For any $b \in B_2$, by (∗2) we have

$$C(b, N_D^+(b) \cap (A \cup B_1 \cup \{a_0\}) \subseteq \{c(xa_0), c(xb^-), c(xb^{--})\}$$
$$\text{or } \subseteq \{c(xa_0), c(xb^{--}), c(xb^{----})\}.$$

If $c(xb^{----}) \in C(b, N_D^+(b))$, then by Claim 3, $c(xb^-) \notin C(b, S)$. If we suppose $c(xb^-) \in C(b, \{x, y, z\})$, then by (∗4)-(∗6), we have $c(zy) = c(xb^-)$ or $c(xy) = c(xb^-)$. Note that $c(zy) = c(xa_0)$ and if $c(xy) = c(xb^-)$, the vertex b^{----} does not exist. Thus,

$$\begin{aligned} & C(b, N_D^+(b_0) \cap (A \cup B_1 \cup \{a_0\})) \cup C(b_0, N_D^-(b_0) \cup \{x, y, z\}) \\ \subseteq\ & \{c(xa_0), c(xb), c(xb^-), c(xb^{--})\} \\ & \text{or} \\ \subseteq\ & \{c(xa_0), c(xb), c(xb^{--}), c(xb^{----})\}. \end{aligned}$$

Since $D[B_2]$ is an oriented graph, there is a vertex $b_0 \in B_2$ with $d^+_{D[B_2]}(b_0) \leq \frac{|B_2|-1}{2}$. Note that $B_1 \neq \emptyset$. Hence,

$$\begin{aligned} d^c_{G[N^c(x)\cup\{x\}]}(b_0) &\leq |C(b, N_D^+(b) \cap B_2)| \\ &\quad + |C(b_0, N_D^-(b_0) \cup \{x, y, z\}) \cup C(b, N_D^+(b_0) \cap (A \cup B_1 \cup \{a_0\}))| \\ &\leq \lfloor \tfrac{|B|-2}{2} \rfloor + 4 \\ &\leq \tfrac{d^c(x)+1}{2}. \end{aligned}$$

Subcase 2.2. $\delta^+(D[B]) \leq \frac{|B|-2}{2}$.

Let b_0 be a vertex in B such that $d^+_{D[B]}(b_0) \leq \frac{|B|-2}{2}$. By the same argument as for B_2 in Subcase 2.1, we can easily get

$$|C(b_0, N_D^-(b_0) \cup \{x, y, z\}) \cup C(b, N_D^+(b_0) \cap (A \cup \{a_0\}))| \leq 4.$$

Then, $d^c_{G[N^c(x)\cup\{x\}]}(b_0) \leq \frac{d^c(x)+1}{2}$. The claim is thus proved. □

Now it is time to give the proof of Theorem 1.6. We distinguish two cases.

Case I. $S = \emptyset$.

Since $S = \emptyset$ implies that

$$N_{C(G)\setminus C(T)}(x) \cap N_{C(G)\setminus\{c(xy),c(zy)\}}(y) = \emptyset,$$

we have $N_{C(G)\setminus\{c(xy),c(zy)\}}(y) \subseteq V(G) \setminus N_{C(G)\setminus C(T)}(x)$. Hence,

$$\begin{aligned} n &\geq |N^c(x) \setminus \{u \mid c(xu) = c(zy)\}| + |N_{C(G)\setminus\{c(xy),c(zy)\}}(y)| + |\{x\}| \\ &\geq 2\delta^c(G) - 3 + 1. \end{aligned}$$

Then, $\delta^c(G) \le \frac{n+2}{2} \le \frac{2n-1}{3}$, a contradiction.

Case II. $S \neq \emptyset$.

According to Claim 5, there exists a vertex v_0 with $|N^c(v_0) \cap (V(G) \setminus (N^c(x) \cup \{x\}))| \ge \delta^c(G) - \frac{d^c(x)+1}{2} + 1$. So, we have

$$\begin{aligned} n &\ge |N^c(x)| + |V(G) \setminus (N^c(x) \cup \{x\})| + |\{x\}| \\ &\ge |N^c(x)| + |N^c(v_0) \cap (V(G) \setminus (N^c(x) \cup \{x\}))| + 1 \\ &\ge d^c(x) + \delta^c(G) - (\tfrac{d^c(x)+1}{2}) + 1 \\ &\ge \tfrac{3}{2}\delta^c(G) + \tfrac{1}{2}. \end{aligned}$$

Hence, we have $\delta^c(G) \le \frac{2n-1}{3}$, a contraction. □

3 Proof of Theorem 1.13

To present the proof of Theorem 1.13, we need some auxiliary theorems and lemmas. Lemmas 3.1 and 3.3 are used to prove Theorem 1.9. We will use them to prove our theorem.

Lemma 3.1 ([2])**.** *Let G be an edge-colored graph of order n and $P = u_1u_2\cdots u_p$ be a rainbow path in G. If G contains no rainbow cycle of length at least k, where $k \le p$, then for any color $a \in C(u_1, u_kPu_p)$ and vertex $u_i \in V(u_kPu_p)$, where $c(u_1u_i) = a$, there is an edge $e \in E(u_1Pu_i)$ such that $c(e) = a$.*

Similarly, we have the following lemmas.

Lemma 3.2. *Let G be an edge-colored graph of order n and $P = u_1u_2\cdots u_p$ be a rainbow path in G. If G contains no rainbow cycle of length at least k, where $k \le p$, then for any positive integers s, t with $t \ge s + (k-1)$, we have $u_su_t \notin E(G)$ or $c(u_su_t) \in C(u_sPu_t)$.*

Lemma 3.3 (Lemma 4 in [2])**.** *Let G be an edge-colored graph of order n and $P = u_1u_2\cdots u_p$ be a longest rainbow path in G. If G contains no rainbow cycle of length at least k, where $k \le p$, then for any positive integers s, t such that $s + t = k$, we have $|C(u_1, u_kPu_{p-(t-1)}) \cap C(u_p, u_sPu_{p-(k-1)})| \le 1$.*

Lemma 3.4. *Let G be an edge-colored complete graph of order n and $P = u_1u_2\cdots u_p$ be a longest rainbow path in G. If G contains no rainbow cycle of length at least k, where $2k-1 \le p$, then one of the following statements holds:*

(1) $C(u_1, u_kPu_{p-(k-2)}) = \{c(u_1u_p)\}$;

(2) $C(u_p, u_{k-1}Pu_{p-(k-1)}) = \{c(u_1u_p)\}$;

(3) There exists an edge $u_qu_{q+1} \in E(u_kPu_{p-(k-1)})$ *with* $c(u_qu_{q+1}) = c(u_1u_p)$ *such that* $C(u_1, u_{q+1}Pu_{p-(k-2)}) = \{c(u_1u_p)\}$ *and* $C(u_p, u_{k-1}Pu_q) = \{c(u_1u_p)\}$.

Proof. From Lemma 3.1, we have $c(u_1u_k) \in C(u_1Pu_k)$ and $c(u_pu_{p-(k-1)}) \in C(u_{p-(k-1)}Pu_p)$. Then we have

$$c(u_1u_p) = c(u_1u_k) \text{ or } c(u_1u_p) \in C(u_kPu_p);$$

otherwise, $u_1u_kPu_pu_1$ is a rainbow cycle of length at least k. Similarly, we have

$$c(u_1u_p) = c(u_pu_{p-(k-1)}) \text{ or } c(u_1u_p) \in C(u_1Pu_{p-(k-1)}).$$

Since $p \geq 2k-1$, we have $c(u_1u_k) \neq c(u_pu_{p-(k-1)})$. Thus, one of the following statements holds:

$$c(u_1u_p) = c(u_1u_k) \in C(u_1Pu_k), \tag{1}$$

$$c(u_1u_p) = c(u_pu_{p-(k-1)}) \in C(u_{p-(k-1)}Pu_p). \tag{2}$$

or

$$c(u_1u_p) \in C(u_kPu_{p-(k-1)}). \tag{3}$$

If (1) holds, then for any vertex $u_i \in V(u_kPu_{p-(k-2)})$, we have $c(u_1u_i) = c(u_1u_p)$; otherwise, $u_1u_iPu_pu_1$ is a rainbow cycle of length at least k. By the symmetry, if (2) holds, then we have $c(u_pu_i) = c(u_1u_p)$ for $u_i \in V(u_{k-1}Pu_{p-(k-1)})$.

If (3) holds, suppose $c(u_1u_p) = c(u_qu_{q+1})$. By Lemma 3.1, we know that $c(u_1u_i) \in C(u_1Pu_i)$ for $i \in [q+1, p-(k-2)]$. Hence, $C(u_1, u_{q+1}Pu_{p-(k-2)}) = \{c(u_1u_p)\}$, since otherwise, $u_1u_iPu_pu_1$ is a requested rainbow cycle. Similarly, $C(u_p, u_{k-1}Pu_q) = \{c(u_1u_p)\}$ holds. □

Proof of Theorem 1.13: Let G be a graph satisfying the assumptions of Theorem 1.13. Suppose to the contrary that G contains no rainbow cycle of length at least k. Let $P = u_1u_2\cdots u_p$ be a longest rainbow path in G. Since $n \geq 7k-17$, from Theorem 1.14 it follows that $p \geq 3k-5$. From Lemma 3.4, by symmetry we assume that (1) or (3) holds.

For convenience, we label some sets of colors as follows:

$A_1 = C(u_1, u_kPu_{p-1})$, $A_2 = C(u_1, u_2Pu_{k-1})$,

$B_1 = C(u_p, u_{k-1}Pu_{p-(k-1)})$, $B_2 = C(u_p, u_2Pu_{k-2})$, $B_3 = C(u_p, u_{p-(k-2)}Pu_{p-1})$,

and

$C_0 = (C(u_1, P^C) \setminus C(u_1, P)) \cap (C(u_p, P^C) \setminus C(u_p, P))$,

$C_1 = C(u_1, P^C) \setminus (C_0 \cup C(u_1, P))$, $C_2 = C(u_p, P^C) \setminus (C_0 \cup C(u_p, P))$.

At first, we give some useful claims.

Claim 1. *Let $u_s, u_t \in V(P)$ with $s < t$ and $\ell(u_sPu_t) \geq 2k-3$. Then for any pair of vertices u_a and u_b with $s < a < b < t$, if $k-1 \leq \ell(u_aPu_b) \leq \ell(u_sPu_t) - (k-2)$ and $c(u_su_t) \in C(u_aPu_b)$, then we have $c(u_au_b) = c(u_su_t)$.*

Proof. Since $\ell(u_aPu_b) \geq k-1$, from Lemma 3.2 we have $c(u_au_b) \in C(u_aPu_b)$. If $c(u_au_b) \neq c(u_su_t)$, then $u_sPu_au_bPu_tu_s$ is a rainbow cycle of length at least k, a contradiction. □

From Lemma 3.3, setting $t = 2$ and $s = k-2$, we can get the following claim.

Claim 2. $|A_1 \cap B_1| \leq 1$.

Claim 3. $|A_1 \cap (B_2 \setminus (B_1 \cup \{c(u_1u_p)\}))| \leq 1$.

Proof. Suppose to the contrary that there are at least two distinct colors m and m' in $A_1 \cap (B_2 \setminus (B_1 \cup \{c(u_1u_p)\}))$. Assume that there exist two vertices u_s and u_t where $s \in [2, k-2]$ and $t \in [k, p-1]$, such that

$$c(u_1u_t) = c(u_pu_s) = m \in A_1 \cap (B_2 \setminus (B_1 \cup \{c(u_1u_p)\})).$$

Case 1. (1) of Lemma 3.4 holds.

Since $C(u_1, u_kPu_{p-(k-2)}) = \{c(u_1u_p)\}$ and $p \geq 3k-5$, we have $t \in [p-(k-3), p-1]$. Note that $c(u_pu_{s+(k-2)}) \in B_1$ and $c(u_1u_{t-(k-2)}) = c(u_1u_p)$. We have $m \notin \{c(u_pu_{s+(k-2)}), c(u_1u_{t-(k-2)})\}$. By Lemma 3.1, $c(u_pu_{s+(k-2)}) \in C(u_pPu_{s+(k-2)})$ and $c(u_1u_{t-(k-2)}) \in C(u_1Pu_{t-(k-2)})$. To avoid $u_sPu_{s+(k-2)}u_pu_s$ and $u_1u_{t-(k-2)}Pu_tu_1$ being rainbow cycles of length k, we have $m \in C(u_sPu_{s+(k-2)}) \cap C(u_{t-(k-2)}Pu_t)$, which guarantees that $k-2 \leq t-(k-2) < s+(k-2) \leq p-(k-2)$, and

$$m \in C(u_{t-(k-2)}Pu_{s+k-2}) \subseteq C(u_{k-2}Pu_{p-(k-2)}).$$

Since $p \geq 3k-5$, $\ell(u_{k-2}Pu_{p-(k-2)}) \geq k-1$ and $\ell(u_1Pu_t) \geq 2k-3$. Hence, from Lemma 3.2 and Claim 1, we have $c(u_{k-2}u_{p-(k-2)}) = c(u_1u_t) = m$; see Figure 1. Then by a same argument, we can get that $c(u_{k-2}u_{p-(k-2)}) = m'$, a contradiction.

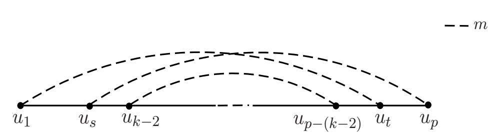

Figure 1: For the proof of Claim 3

Case 2. (3) of Lemma 3.4 holds.

By Lemma 3.1 we have $m \in C(u_sPu_t)$. If $m \in C(u_{q+1}Pu_{p-1}) \cap C(u_sPu_t)$, then $u_1u_tPu_pu_qPu_1$ is a required cycle, a contradiction. If $m \in C(u_sPu_q) \cap C(u_sPu_t)$, then $u_1u_{q+1}Pu_pu_su_1$ is a required cycle, a contradiction. Hence, $A_1 \cap (B_2 \setminus (B_1 \cup \{c(u_1u_p)\})) = \emptyset$, a contradiction. □

Claim 4. *Let* $D = \{x \in V(P^C) : x \in N_{C_1 \cup C_0}(u_1, P^C) \cap N_{C_2 \cup C_0}(u_p, P^C)$ *and* $c(u_1 x) \neq c(u_p x)\}$*. Then* $|D| \leq 2$*.*

Proof. Suppose to the contrary that $|D| \geq 3$. Since the colors in $C(u_1, D)$ are distinct, there exists a vertex $x \in D$ such that

$$c(u_1 x) \notin \{c(u_p u_{k-1}), c(u_p u_{p-(k-1)})\}. \tag{4}$$

Then, $c(u_p u_{k-1}) \notin \{c(u_1 x), c(u_p x)\}$. By Lemma 3.1, $c(u_p u_{k-1}) \in C(u_{k-1} P u_p)$. Thus,

$$c(u_1 x) \in C(u_2 P u_{k-1}) \text{ or } c(u_p x) \in c(u_1 P u_{k-1}). \tag{5}$$

Otherwise, $u_1 P u_{k-1} u_p x u_1$ is a rainbow cycle of length at least k; see (a) in Figure 2, a contradiction. According to Lemma 3.4, we have $c(u_1 u_{p-(k-2)}) = c(u_1 u_p)$, and then $c(u_1 u_{p-(k-2)}) \notin C(u_1, D) \cup C(u_p, D)$. Thus,

$$c(u_1 x) \in C(u_{p-(k-2)} P u_p) \text{ or } c(u_p x) \in C(u_{p-(k-2)} P u_{p-1}). \tag{6}$$

Otherwise, $u_1 u_{p-(k-2)} P u_p x u_1$ is a rainbow cycle of length at least k; see (b) in Figure 2, a contradiction.

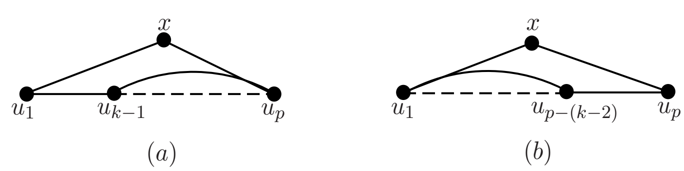

Figure 2: Two rainbow cycles of length at least k in Claim 4

Since P is a rainbow path, combining (5) and (6), we have

$$c(u_1 x) \in C(u_2 P u_{k-1}) \text{ and } c(u_p x) \in C(u_{p-(k-2)} P u_{p-1}), \tag{7}$$

or

$$c(u_1 x) \in C(u_{p-(k-2)} P u_p) \text{ and } c(u_p x) \in C(u_1 P u_{k-1}). \tag{8}$$

W.l.o.g., assume that (7) holds. By Lemma 3.1, we have $c(u_1 u_k) \in C(u_1 P u_k)$ and

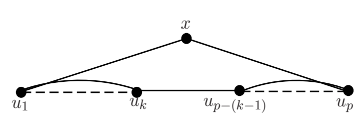

Figure 3: A rainbow cycle $u_1 u_k P u_{p-(k-1)} u_p x_0 u_1$

$c(u_p u_{p-(k-1)}) \in C(u_{p-(k-1)} P u_p)$. From these together with (4), we get that $u_1 u_k P u_{p-(k-1)} u_p x u_1$ is a rainbow cycle of length at least k; see Figure 3, a contradiction. $\square$

By Lemma 3.1, we can get that $B_1 \cup B_2 \subseteq C(u_2Pu_p)$ and $A_1 \subseteq C(u_1Pu_{p-1})$. Thus, $c(u_1u_2) \notin B_1 \cup B_2$ and $c(u_pu_{p-1}) \notin A_1$. However, possibly we have $c(u_1u_2) \in A_1$ or $c(u_pu_{p-1}) \in B_1 \cup B_2$. Thus, we set

$$\varepsilon_1 = \begin{cases} 1, & \text{if } c(u_1u_2) \notin A_1 \\ 0, & \text{if } c(u_1u_2) \in A_1, \end{cases}$$

$$\varepsilon_2 = \begin{cases} 1, & \text{if } c(u_pu_{p-1}) \notin B_1 \cup B_2 \\ 0, & \text{if } c(u_pu_{p-1}) \in B_1 \cup B_2. \end{cases}$$

By Lemma 3.4, we know $c(u_1u_p) \in A_1$, and then by Lemma 3.1 we have $c(u_1u_p) \in C(u_1Pu_k)$. Thus, we set

$$\varepsilon_3 = \begin{cases} 1, & \text{if } c(u_1u_p) \notin B_1 \\ 0, & \text{if } c(u_1u_p) \in B_1. \end{cases}$$

Then,

$$\begin{aligned} d^c(u_1) &\leq |C(u_1, P) \cup C_0 \cup C_1| \\ &\leq \varepsilon_1 + |A_2 \setminus \{c(u_1u_2)\}| + |A_1| + |C_0| + |C_1|, \end{aligned}$$

$$\begin{aligned} d^c(u_p) &\leq |C(u_p, P) \cup C_0 \cup C_2| \\ &\leq \varepsilon_2 + \varepsilon_3 + |B_1| + |B_2 \setminus (B_1 \cup \{c(u_1u_p)\})| \\ &\quad + |B_3 \setminus \{c(u_pu_{p-1})\}| + |C_0| + |C_2|. \end{aligned}$$

Since $|A_2 \setminus \{c(u_1u_2)\}| \leq k-3$ and $|B_3 \setminus \{c(u_pu_{p-1})\}| \leq k-3$, we have

$$|A_1| + |C_0| + |C_1| + \varepsilon_1 \geq \delta^c(G) - (k-3), \tag{9}$$

$$|B_1| + |B_2 \setminus (B_1 \cup \{c(u_1u_p)\})| + |C_0| + |C_2| + \varepsilon_2 + \varepsilon_3 \geq \delta^c(G) - (k-3). \tag{10}$$

Since $c(u_1x) \neq c(u_px)$ for each $x \in N_{C_1}(u_1, P^C) \cap N_{C_2}(u_p, P^C)$, we have $x \in D$. By Claim 4, we can get that $|N_{C_1}(u_1, P^C) \cap N_{C_2}(u_p, P^C)| \leq 2$. Note that $C_0 = (C(u_1, P^C) \setminus C(u_1, P)) \cap (C(u_p, P^C) \setminus C(u_p, P))$. If there is a vertex $x \in N_{C_2}(u_p, P^C)$ with $c(u_1x) = c_0 \in C_0$, then $x \in D$ and there is another distinct vertex y such that $c(u_py) = c_0$. If $c(u_1y) \in C_1$, then y is also contained in D. Since $|D| \leq 2$, for all $z \in N_{C_0 \setminus \{c_0\}}(u_1, P^C)$, we have $z \notin N_{C_2}(u_p, P^C)$, which implies that $|N_{C_0}(u_1, P^C) \cup N_{C_0}(u_p, P^C) \setminus (N_{C_1}(u_1, P^C) \cup N_{C_2}(u_p, P^C))| \geq |C_0| - 1$. Therefore,

$$\begin{aligned} |V(P^C)| &\geq |N_{C_1}(u_1, P^C) \cup N_{C_2}(u_p, P^C)| + \\ &\quad |N_{C_0}(u_1, P^C) \cup N_{C_0}(u_p, P^C) \setminus (N_{C_1}(u_1, P^C) \cup N_{C_2}(u_p, P^C))| \\ &\geq |C_1| + |C_2| - |D| + |C_0| - 1 \\ &\geq |C_1| + |C_2| + |C_0| - 3. \end{aligned} \tag{11}$$

Note that for any color $a \in C_0$, there is an edge in P whose color is a, since otherwise, P is not a longest rainbow path. Then together with Claims 2 and 3, we have

$$\begin{aligned}|V(P)| &= |E(P)|+1\\ &\geq |e\in E(P), c(e)\in A_1\cup B_1\cup B_2| + |e\in E(P), c(e)\in C_0|\\ &\quad +\varepsilon_1+\varepsilon_2+1\\ &\geq |A_1|+|B_1|+|B_2\setminus(B_1\cup c(u_1u_p))|+|C_0|+\varepsilon_1+\varepsilon_2-1.\end{aligned} \tag{12}$$

By Inequalities (11) and (12), we have

$$\begin{aligned}n &\geq |V(P)|+|V(P^C)|\\ &\geq (|A_1|+|C_0|+|C_1|+\varepsilon_1)+(|B_1|+|B_2\setminus(B_1\cup c(u_1u_p))|+|C_0|+|C_2|+\varepsilon_2)-4,\end{aligned} \tag{13}$$

Combining with Inequalities (9) and (10), we get that

$$\begin{aligned}n &\geq 2(\delta^c(G)-(k-3))-\varepsilon_3-4\\ &\geq 2\delta^c(G)-2k+1.\end{aligned} \tag{14}$$

Then, $\delta^c(G) \leq \frac{n-1}{2}+k$, a contradiction. The proof is thus complete. □

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