Integer Linear Programming Formulations for Double Roman Domination Problem

Qingqiong Cai[a], Neng Fan[b], Yongtang Shi[c], Shunyu Yao[c]

[a]College of Computer Science, Nankai University, Tianjin, China;
[b]Department of Systems & Industrial Engineering, University of Arizona, Tucson, AZ, USA;
[c]Center for Combinatorics and LPMC, Nankai University, Tianjin, China

ABSTRACT
For a graph $G = (V, E)$, a double Roman dominating function (DRDF) is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor u with $f(u) \geq 2$. In this paper, we consider the double Roman domination problem, which is an optimization problem of finding the DRDF f such that $\sum_{v \in V} f(v)$ is minimum. We propose five integer linear programming (ILP) formulations and one mixed integer linear programming formulation with polynomial number of constraints for this problem. Some additional valid inequalities and bounds are also proposed for some of these formulations. Further, we prove that the first four models indeed solve the double Roman domination problem, and the last two models are equivalent to the others regardless of the variable relaxation or usage of a smaller number of constraints and variables. Additionally, we use one ILP formulation to give an $H(2(\Delta + 1))$-approximation algorithm. All proposed formulations and approximation algorithm are evaluated on randomly generated graphs to compare the performance.

KEYWORDS
Double Roman domination; Integer linear programming; approximation algorithm

1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For every vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$, and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is defined as $d_G(v) = |N(v)|$. The minimum and maximum degree of a graph G are denoted by δ and Δ, respectively. The distance of two vertices is the length of a shortest path between them in a graph. The diameter d of a graph is the largest distance between vertices. The complement $\overline{G}$ of G is the graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G. The girth of G, denoted by g, is the length of a shortest cycle in G. The adjacency matrix $A(G)$ of G is a square $|V| \times |V|$ matrix such that its element a_{ij} is one when there is an edge between vertex i and vertex

j, and zero when there is no such edge. For other definitions and notations of graph theory not given here, we refer to [8].

A set $S \subseteq V$ is called a dominating set if every vertex of G is either in S or adjacent to a vertex of S. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is called the domination number. Domination of graphs has been extensively studied in the scientific literature. The variants of domination have abundant applications, including error-correcting codes constructions for digital communication and efficient data routing in wireless networks [9, 10, 12, 18, 27]. Many different kinds of domination arose, such as the connected dominating set [16], the edge dominating set [32], the total domination [23] [26], the independent domination [19], Roman domination [31], etc.

In this paper, we consider the so-called double Roman domination problem (DRDP), which was initially introduced by Beeler et al. [7]. This problem is a generalization of Roman domination problem.

A *Roman dominating function* (RDF) on $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ with the property that any vertex $v \in V$ with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The weight of an RDF is defined as $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight among all RDFs on G. An RDF on G with weight $\gamma_R(G)$ is called a γ_R-function of G. The *Roman domination problem* (RDP) is to determine the value of $\gamma_R(G)$. Roman domination was originally defined and discussed by Stewart et al. [31] , ReVelle and Rosing [30], and subsequently developed by Cockayne et al. [15]. The study of Roman domination was motivated by the defense strategies of the Roman Empire during the reign of Emperor Constantine the Great, $274 - 337$ AD. To defend the Roman Empire, the emperor decreed that for each city in the Roman Empire, at most two legions should be stationed. Moreover, a location without legion must be adjacent to at least one city at which two legions were stationed, so that if the location was attacked, then one of the two legions could be sent to defend it. So far, rich studies have been performed on various aspects of Roman domination in graphs. For more details, we refer to [2, 4, 7, 11, 13, 24].

Inspired by the concept of Roman domination, Beeler et al. [7] proposed a stronger version that doubles the protection by ensuring that any attack can be defended by at least two legions. They allowed at most three legions to be stationed at each location. This approach provides a level of defense that is both stronger and more flexible, at less than the anticipated additional cost.

Similar to Roman dominating function, a *double Roman dominating function* (DRDF) is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor u with $f(u) \geq 2$. The weight of a DRDF f is the sum $w(f) = \sum_{v \in V} f(v)$, and the *double Roman domination number* of a graph G, denoted by $\gamma_{dR}(G)$, is the minimum weight among all possible DRDFs. A DRDF on G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G. It is worth noting that Beeler et al. [7] showed that for any graph G, there exists a γ_{dR}-function of G such that no vertex needs to be assigned the value 1. The *double Roman domination problem* (DRDP) is to determine the value of $\gamma_{dR}(G)$. An example of the RDP and DRDP is illustrated in Figure 1.

The double Roman domination number has been completely determined in paths and cycles [3], and has been shown to be linear-time solvable in trees [35] and P_4-free graphs [34]. However, for a general graph, it is NP-complete to determine whether a graph has a DRDF of weight at most k [3]. Therefore, it is meaningful to obtain good upper and lower bounds for $\gamma_{dR}(G)$. Beeler et al. [7] studied the relationship between $\gamma_{dR}(G)$ and $\gamma(G)$ or $\gamma_R(G)$. They showed that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$
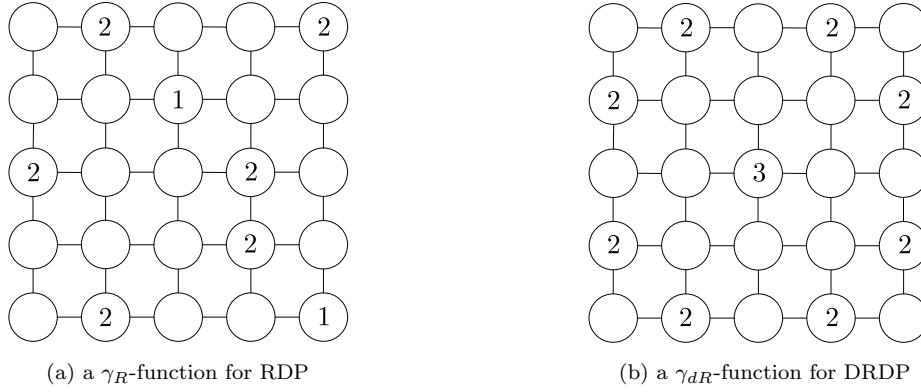
(a) a γ_R-function for RDP   (b) a γ_{dR}-function for DRDP

Figure 1. Example of RDP and DRDP on a 5×5 grid graph (The dominating function value of the unlabelled vertex is 0).

and $\gamma_R(G) < \gamma_{dR}(G) \leq 2\gamma_R(G)$. Moreover, $\gamma_{dR}(T) \geq 2\gamma(T) + 1$ for a non-trivial tree T. Later, Ahangar et al. [3] gave that a graph with maximum degree Δ satisfies $\gamma_{dR}(G) \geq \frac{2n}{\Delta} + \frac{\Delta-2}{\Delta}\gamma(G)$. Besides, there are some bounds for $\gamma_{dR}(G)$ in terms of order n, minimum degree δ, maximum degree Δ and diameter d. Beeler et al. [7] proved that $\gamma_{dR}(G) \leq 5n/4$ for a connected graph G with order $n \geq 3$, and characterized the graphs attaining this bound. In [33], Volkmann obtained the lower bound $\gamma_{dR}(G) \geq \lceil \frac{3n}{\Delta+1} \rceil$ for each nonempty graph G. Yue et al. [34] got the upper bounds $\gamma_{dR}(G) \leq 2n - 2\Delta + 1$ and $\gamma_{dR}(G) \leq \frac{3n(1+\ln\frac{2(1+\delta)}{3})}{1+\delta}$. They also showed that almost all graphs have double Roman domination number $\gamma_{dR} \leq n$. Recently, Anu et al. [6] found an upper bound with respect to order n and diameter d, that is $d + 1 \leq \gamma_{dR}(G) \leq 2n - d$. In [6], Anu et al. showed that there is no relation between the double Roman domination number of a graph and its induced subgraph. Yue et al. [34] obtained that a graph with order $n \geq 3$ satisfies $8 \leq \gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 3$ and characterized the graphs attaining these bounds. For other results on $\gamma_{dR}(G)$, we refer to [5, 21, 25, 29, 33].

However, for a specific graph, determining the exact value or designing approximation algorithms for the double Roman domination number has not been studied in the literature. For the RDP, several integer linear programming (ILP) formulations have been proposed by ReVelle and Rosing [30] and Ivanović [24]. Motivated by this, this paper will study the ILP formulations for the DRDP, and will also introduce several extra constraints to strengthen some ILP formulations. Additionally, an $H(2(\Delta+1))$-approximation algorithm for DRDP is proposed based on one ILP formulation.

The remainder of this paper is organized as follows. In Section 2, we propose two different ILP models for the DRDP. Two improved ILP formulations and two new alternative LP formulations are presented in Section 3. An approximation algorithm for DRDP is given in Section 4. Finally, we provide computational results in Section 5 and draw our conclusions in Section 6.

2. ILP models

In this section, we present two different ILP models for the DRDP. These models will be improved in Section 3 and experimentally evaluated in Section 5.

The first ILP model, called **DRDP-1**, uses three sets of binary variables. Specifi-

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cally, for each vertex $v \in V$, we define

$$x_v = \begin{cases} 1, & f(v) = 1 \\ 0, & \text{otherwise} \end{cases} \qquad y_v = \begin{cases} 1, & f(v) = 2 \\ 0, & \text{otherwise} \end{cases} \qquad z_v = \begin{cases} 1, & f(v) = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Then the DRDP can be formulated as follows:

$$[\textbf{DRDP-1}] \quad \min \quad \sum_{v \in V} x_v + 2 \sum_{v \in V} y_v + 3 \sum_{v \in V} z_v \tag{1a}$$

$$\text{s.t.} \quad x_v + y_v + z_v + \frac{1}{2} \sum_{u \in N(v)} y_u + \sum_{u \in N(v)} z_u \geq 1, \quad \forall\, v \in V \tag{1b}$$

$$\sum_{u \in N(v)} y_u + \sum_{u \in N(v)} z_u \geq x_v, \quad \forall\, v \in V \tag{1c}$$

$$x_v + y_v + z_v \leq 1, \quad \forall\, v \in V \tag{1d}$$

$$x_v, y_v, z_v \in \{0, 1\}, \quad \forall\, v \in V \tag{1e}$$

The objective function value is given by (1a). Constraints (1b) ensure that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$. Constraints (1c) ensure that if $f(v) = 1$, then vertex v must have at least one neighbor u with $f(u) \geq 2$. By constraints (1d), it establishes that every vertex is assigned with at most one value from the set $\{1, 2, 3\}$. The decision variables x_v, y_v and z_v are set to be binary by constraints (1e).

The **DRDP-1** formulation consists of $3|V|$ binary variables and $3|V|$ constraints. The following theorem shows that **DRDP-1** indeed corresponds to the mathematical formulation of the double Roman domination problem.

Theorem 2.1. *The optimal objective function value of **DRDP-1** formulation is equal to the double Roman domination number γ_{dR}.*

Proof. Let S be the set of all the possible double Roman dominating functions and T the set of all feasible solutions of **DRDP-1** formulation. First we prove that there exists a bijection between S and T.

For any double Roman dominating function $f \in S$, we construct a feasible solution $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T \subseteq \{0, 1\}^{3 \times |V|}$:

(a) if $f(v) = 0$, then set $x_v = y_v = z_v = 0$;
(b) if $f(v) = 1$, then set $x_v = 1, y_v = z_v = 0$;
(c) if $f(v) = 2$, then set $y_v = 1, x_v = z_v = 0$;
(d) if $f(v) = 3$, then set $z_v = 1, x_v = y_v = 0$.

Note that if $f(v) > 0$, then one of x_v, y_v, z_v equals one, so constraints (1b) are trivial. Otherwise, v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$, which means there exist at least two vertices $u_1, u_2 \in N(v)$ satisfying $y_{u_1} = y_{u_2} = 1$ or at least one vertex $u \in N(v)$ satisfying $z_u = 1$ from our construction. Hence, constraints (1b) are also satisfied.

Consider constraints (1c). If $f(v) \neq 1$, then $x_v = 0$, so constraints (1c) are trivial. Otherwise, v must have at least one neighbor u with $f(u) \geq 2$, which means $y_u = 1$ or $z_u = 1$ from our construction. Therefore, constraints (1c) hold as well.

Moreover, constraints (1d)–(1e) hold naturally from our construction. Consequently, we obtain an injection from S to T.

For the other direction, given a feasible solution $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in T$, we construct a function $f : V \to \{0, 1, 2, 3\}$ as follows:

(a') if $x_v = y_v = z_v = 0$, then set $f(v) = 0$;
(b') if $x_v = 1, y_v = z_v = 0$, then set $f(v) = 1$;
(c') if $y_v = 1, x_v = z_v = 0$, then set $f(v) = 2$;
(d') if $z_v = 1, x_v = y_v = 0$, then set $f(v) = 3$.

Observe that constraints (1b) ensure that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$. Constraints (1c) guarantee that if $f(v) = 1$, then vertex v must have at least one neighbor u with $f(u) \geq 2$. It follows that f is a double Roman dominating function.

We have proved that there exists a bijection between S and T. Since the objective function (1a) corresponds to the weight of a double Roman dominating function, then the optimal objective function value of **DRDP-1** formulation is equal to the minimum weight among all the double Roman dominating functions, that is γ_{dR}.

$\square$

The second ILP model, called **DRDP-2**, follows the same idea as **DRDP-1**, apart from the definition of some binary variables. Similar as above, for each vertex $v \in V$, we define

$$p_v = \begin{cases} 1, & f(v) \geq 1 \\ 0, & \text{otherwise} \end{cases} \qquad q_v = \begin{cases} 1, & f(v) \geq 2 \\ 0, & \text{otherwise} \end{cases} \qquad r_v = \begin{cases} 1, & f(v) = 3 \\ 0, & \text{otherwise} \end{cases}.$$

Then we can obtain another ILP model for DRDP:

$$[\textbf{DRDP-2}] \quad \min \quad \sum_{v \in V} p_v + \sum_{v \in V} q_v + \sum_{v \in V} r_v \tag{2a}$$

$$\text{s.t.} \quad p_v + \frac{1}{2} \sum_{u \in N(v)} q_u + \frac{1}{2} \sum_{u \in N(v)} r_u \geq 1, \quad \forall\, v \in V \tag{2b}$$

$$q_v + \sum_{u \in N(v)} q_u \geq p_v, \quad \forall\, v \in V \tag{2c}$$

$$r_v \leq q_v \leq p_v, \quad \forall\, v \in V \tag{2d}$$

$$p_v, q_v, r_v \in \{0, 1\}, \quad \forall\, v \in V \tag{2e}$$

The objective function value is given by (2a). Constraints (2b) ensure that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or at least one neighbor u with $f(u) = 3$. Constraints (2c) ensure that if $f(v) = 1$, then vertex v must have at least one neighbor u with $f(u) \geq 2$. Constraints (2d) guarantee that the binary variables are well-defined. Again, the decision variables p_v, q_v and r_v are preserved to be binary by constraints (2e).

The **DRDP-2** formulation consists of $3|V|$ binary variables and $4|V|$ constraints. Similar to the previous arguments, we have the following theorem:

Theorem 2.2. *The optimal objective function value of* **DRDP-2** *formulation is equal to the double Roman domination number* γ_{dR}.

3. Alternative ILP models

In this section, we first improve the formulations introduced in Section 2, and then propose two alternative LP formulations.

3.1. Improved ILP formulations

To begin with, we state the following result proved by Beeler et al. [7].

Theorem 3.1 ([7]). *For any graph G, there exists a γ_{dR}-function of G such that no vertex needs to be assigned the value 1.*

From Theorem 3.1, we can only consider the γ_{dR}-functions for which all vertices are assigned the value from the set $\{0, 2, 3\}$. Based on this, we substitute 0 for x_v in **DRDP-1** to obtain the new ILP model **DRDP-1$'$**.

$$[\textbf{DRDP-1}'] \quad \min \quad 2\sum_{v \in V} y_v + 3\sum_{v \in V} z_v \tag{3a}$$

$$\text{s.t.} \quad y_v + z_v + \frac{1}{2}\sum_{u \in N(v)} y_u + \sum_{u \in N(v)} z_u \geq 1, \quad \forall\, v \in V \tag{3b}$$

$$y_v + z_v \leq 1, \quad \forall\, v \in V \tag{3c}$$

$$y_v, z_v \in \{0, 1\}, \quad \forall\, v \in V \tag{3d}$$

Using the same idea as the one in Theorem 2.1, we can obtain the following statement:

Theorem 3.2. *The optimal objective function value of* **DRDP-1$'$** *formulation is equal to the double Roman domination number γ_{dR}.*

Proof. Let S be the set of all the possible double Roman dominating functions and T the set of all feasible solutions of **DRDP-1** formulation. Then let S' be the set of all the double Roman dominating functions that no vertex is assigned the value 1. Clearly, $S' \subseteq S$. Let T' be the set of all feasible solutions of **DRDP-1$'$** formulation. Note that for every feasible solution $(\mathbf{y}, \mathbf{z}) \in T'$, one can obtain a feasible solution $(\overbrace{0, \ldots, 0}^{n}, \mathbf{y}, \mathbf{z}) \in T$. Since there exists a natural bijection between T' and the set $\{(\overbrace{0, \ldots, 0}^{n}, \mathbf{y}, \mathbf{z}) \mid (\mathbf{y}, \mathbf{z}) \in T'\} \subseteq T$, $T' \subseteq T$ holds. By Theorem 3.1, S' contains at least one γ_{dR}-function. It remains to show that there exists a bijection between S' and T' and that the objective function value of **DRDP-1$'$** corresponds to the weight of a double Roman dominating function in S', whose proof is similar to that in Theorem 2.1 (by excluding the cases (b) and (b')). $\qquad \square$

Besides, we can substitute q_v for p_v in **DRDP-2** by Theorem 3.1. Then we obtain

another improved ILP model **DRDP-2$'$**.

$$[\textbf{DRDP-2}'] \quad \min \quad 2\sum_{v\in V} q_v + \sum_{v\in V} r_v \tag{4a}$$

$$\text{s.t.} \quad q_v + \frac{1}{2}\sum_{u\in N(v)} q_u + \frac{1}{2}\sum_{u\in N(v)} r_u \geq 1, \quad \forall\, v \in V \tag{4b}$$

$$r_v \leq q_v, \quad \forall\, v \in V \tag{4c}$$

$$q_v, r_v \in \{0,1\}, \quad \forall\, v \in V \tag{4d}$$

Similar to Theorem 3.2, we deduce that,

Theorem 3.3. *The optimal objective function value of **DRDP-2$'$** formulation is equal to the double Roman domination number γ_{dR}.*

Next, we explore some extra inequalities to strengthen the formulation **DRDP-1$'$**. Note that for a γ_{dR}-function f of graph G, the set $\{v \in V \mid f(v) \geq 2\}$ is a dominating set of G. Therefore, by the definition of y_v and z_v, the sum $\sum_{v\in V} y_v + \sum_{v\in V} z_v$ is equal to the cardinality of the set $\{v \in V \mid f(v) \geq 2\}$, and we can conclude that the domination number $\gamma(G)$ is bounded by $\sum_{v\in V} y_v + \sum_{v\in V} z_v$. Since $\gamma(G) \geq \lceil n/(1+\Delta)\rceil$ [22], it is valid to write

$$\sum_{v\in V} y_v + \sum_{v\in V} z_v \geq \gamma(G) \geq \lceil \frac{n}{1+\Delta}\rceil.$$

Observing that for a γ_{dR}-function f where all vertices are assigned the values from the set $\{0,2,3\}$, if we replace the function value 3 by 2 in f, the new function f' is a Roman dominating function (since f is a double Roman dominating function, we know that every vertex $v \in V$ with $f'(v) = 0$ is adjacent to at least one vertex u with $f'(u) = 2$). Thus the weight $2\sum_{v\in V} y_v + 2\sum_{v\in V} z_v$ of the function f' is an upper bound for $\gamma_R(G)$, that is

$$2\sum_{v\in V} y_v + 2\sum_{v\in V} z_v \geq \gamma_R(G).$$

There are some lower bounds for $\gamma_R(G)$ in [28]:

- $\gamma_R(G) \geq \lceil 2g/3\rceil$, when G contains a cycle;
- $\gamma_R(G) \geq \lceil (d+2)/2\rceil$, when G is connected;
- $\gamma_R(G) \geq 2\delta$, when G is connected and $g \geq 5$;
- $\gamma_R(G) \geq 4(\delta - 1)$, when G is connected, $g \geq 6$ and $\delta \geq 2$;
- $\gamma_R(G) \geq 2\Delta$, when G is connected, $g \geq 7$ and $\delta \geq 2$.

Therefore, a valid inequality for **DRDP-1$'$** is

$$2\sum_{v\in V} y_v + 2\sum_{v\in V} z_v \geq L_1, \tag{5}$$

where

$$L_1 = \max\left\{2\cdot\lceil \frac{n}{1+\Delta}\rceil, \lceil \frac{2g}{3}\rceil\right\}.$$

For convenience, we use $g \geq a$ and $\delta \geq b$ to denote $\{G \mid g(G) \geq a\}$ and $\{G \mid \delta(G) \geq b\}$, respectively. For a family $\mathcal{A}$ of graphs, we define an indicator function $\mathbf{1}_{\mathcal{A}}$ as follows:

$$\mathbf{1}_{\mathcal{A}}(G) = \begin{cases} 1 & \text{if } G \in \mathcal{A}, \\ 0 & \text{if } G \notin \mathcal{A}. \end{cases}$$

Further, for a connected graph G, we can get a stronger inequality

$$2 \sum_{v \in V} y_v + 2 \sum_{v \in V} z_v \geq L_2 \tag{6}$$

where

$$L_2 = \max \left\{ 2 \cdot \lceil \frac{n}{1+\Delta} \rceil, \lceil \frac{d+2}{2} \rceil, \lceil \frac{2g}{3} \rceil, 2\delta \cdot \mathbf{1}_{g \geq 5}, 4(\delta - 1) \cdot \mathbf{1}_{g \geq 6 \cap \delta \geq 2}, 2\Delta \cdot \mathbf{1}_{g \geq 7 \cap \delta \geq 2} \right\}.$$

Besides, there are some bounds for $\gamma_{dR}(G)$, which can also be used to derive extra inequalities. From [3, 6, 33], we know that $\gamma_{dR}(G) \geq \max\{d+1, \lceil 3n/(\Delta+1) \rceil, \lceil 2n/\Delta + (\Delta - 2)\gamma(G)/\Delta \rceil\}$. So we can get

$$2 \sum_{v \in V} y_v + 3 \sum_{v \in V} z_v \geq L_3 \tag{7}$$

where

$$L_3 = \max \left\{ d + 1, \lceil \frac{3n}{\Delta + 1} \rceil, \lceil \frac{2n}{\Delta} + \frac{\Delta - 2}{\Delta} \cdot \frac{n}{1 + \Delta} \rceil \right\}.$$

Also, from [3, 6, 7, 34], we obtain that

- $\gamma_{dR}(G) \leq \min\{2n - d, 2n - 2\Delta + 1, 3n(1 + \ln \frac{2(1+\delta)}{3})/(1 + \delta)\}$;
- $\gamma_{dR}(G) \leq n$, when $\delta \geq 3$.
- $\gamma_{dR}(G) \leq 5n/4$, when G is a connected graph with order $n \geq 3$.

These upper bounds result in the constraint

$$2 \sum_{v \in V} y_v + 3 \sum_{v \in V} z_v \leq U_1 \tag{8}$$

where

$$U_1 = \min \left\{ 2n - d, 2n - 2\Delta + 1, n \cdot \mathbf{1}_{\delta \geq 3} + 3n \cdot (1 - \mathbf{1}_{\delta \geq 3}), \lfloor \frac{3n(1 + \ln \frac{2(1+\delta)}{3})}{1 + \delta} \rfloor \right\}.$$

Further, for a connected graph G with order $n \geq 3$, the above inequality can be strengthened as

$$2 \sum_{v \in V} y_v + 3 \sum_{v \in V} z_v \leq U_2 \tag{9}$$

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where

$$U_2 = \min \left\{ 2n - d, 2n - 2\Delta + 1, n \cdot \mathbf{1}_{\delta \geq 3} + \lfloor \frac{5n}{4} \rfloor \cdot (1 - \mathbf{1}_{\delta \geq 3}), \lfloor \frac{3n(1 + \ln \frac{2(1+\delta)}{3})}{1 + \delta} \rfloor \right\}.$$

Similar as the above inequalities (5)–(7) and (8)–(9), we can also explore some extra constraints for the formulation **DRDP-2′**. Especially, the sum $2\sum_{v \in V} q_v$ can be viewed as Roman dominating function's weight, which is an upper bound for $\gamma_R(G)$. A series of extra inequalities for formulation (4a)–(4d) are listed as follows:

$$2\sum_{v \in V} q_v \geq L_1, \tag{10}$$

$$2\sum_{v \in V} q_v + \sum_{v \in V} r_v \geq L_2, \tag{11}$$

$$2\sum_{v \in V} q_v + \sum_{v \in V} r_v \leq U_1. \tag{12}$$

In addition, when G is a connected graph with order $n \geq 3$, the above constraints (10) and (12) can be strengthened as

$$2\sum_{v \in V} q_v \geq L_3, \tag{13}$$

and

$$2\sum_{v \in V} q_v + \sum_{v \in V} r_v \leq U_2. \tag{14}$$

We add the constraints (5)–(7) and (8)–(9) to **DRDP-1′** and denote the strengthened formulation by **DRDP-1′(+)**. Similarly, **DRDP-2′(+)** stands for the new formulation obtained by adding constraints (10)–(13) and (12)–(14) into **DRDP-2′**. In Section 5, we will explore the computational results for them.

3.2. Alternatives of the presented ILP formulations

In this section, we utilize the similar idea as the one in [24] to obtain two alternative LP formulations for DRDP and prove the equivalence between the new formulations and the previous ones.

The first formulation, denoted by **DRDP-1″**, is obtained from **DRDP-1′** by omitting constraints (3c).

$$[\mathbf{DRDP\text{-}1''}] \quad \min \quad 2\sum_{v \in V} y_v + 3\sum_{v \in V} z_v \tag{15a}$$

$$\text{s.t.} \quad y_v + z_v + \frac{1}{2}\sum_{u \in N(v)} y_u + \sum_{u \in N(v)} z_u \geq 1, \quad \forall\, v \in V \tag{15b}$$

$$y_v, z_v \in \{0, 1\}, \quad \forall\, v \in V \tag{15c}$$

The following theorem proves the equivalence between **DRDP-1″** and **DRDP-1′**.

Theorem 3.4. *For any graph G, optimal objective function value of formulation (15a)–(15c) is equal to that of formulation (3a)–(3d).*

Proof. Let $V = \{1, \ldots, n\}$. The optimal objective function value of **DRDP-1$'$** and **DRDP-1$''$** is denoted by $OPT_{1'}$ and $OPT_{1''}$, respectively.

First, we prove $OPT_{1''} \geq OPT_{1'}$. Note that an optimal solution to **DRDP-1$''$** can be viewed as a vector $(\tilde{y}'', \tilde{z}'') \in \{0,1\}^{2n}$, where $\tilde{y}'' = (y_1'', \ldots, y_n'')$, $\tilde{z}'' = (z_1'', \ldots, z_n'')$. Based on this, we can divide V into two disjoint subsets V_1 and V_2 such that

- $V_1 = \{\, i \mid y_i'' = 0 \text{ or } z_i'' = 0\}$,
- $V_2 = \{\, i \mid y_i'' = 1 \text{ and } z_i'' = 1\}$.

Then we define a new vector $(\tilde{y}', \tilde{z}') = (y_1', \ldots, y_n', z_1', \ldots, z_n')$ as follows:

$$
y_i' = \begin{cases} y_i'', & \forall\, i \in V_1, \\ 0, & \forall\, i \in V_2, \end{cases} \quad z_i' = z_i'', \ \forall\, i \in V
$$

Now, we prove that $(\tilde{y}', \tilde{z}')$ is a feasible solution to **DRDP-1$'$** formulation.

- For each $i \in V_1$, by the definition of variables y_i' and z_i' we know that $(y_i', z_i') = (y_i'', z_i'')$, so constraints (3b) and (3d) are satisfied. Further, from the definition of V_1, it follows that

$$
y_i' + z_i' = y_i'' + z_i'' \leq 1, \quad \forall\, i \in V_1
$$

  that is, constraints (3c) are satisfied.
- For each $i \in V_2$, by the definition of variables y_i' and z_i', we obtain $(y_i', z_i') = (0, 1)$. Then we can verify that constraints (3b)-(3d) are satisfied again:

$$
y_i' + z_i' + \frac{1}{2} \sum_{j \in N(i)} y_j' + \sum_{j \in N(i)} z_j' = 0 + 1 + \frac{1}{2} \sum_{j \in N(i)} y_j' + \sum_{j \in N(i)} z_j' \geq 1, \quad \forall\, i \in V_2
$$
$$
y_i' + z_i' = 0 + 1 \leq 1, \quad \forall\, i \in V_2
$$
$$
y_i' = 0 \in \{0,1\}, \quad z_i' = 1 \in \{0,1\}, \quad \forall\, i \in V_2
$$

Therefore, an optimal solution $(\tilde{y}'', \tilde{z}'')$ to **DRDP-1$''$** corresponds to a feasible solution $(\tilde{y}', \tilde{z}')$ to **DRDP-1$'$**. It follows that

$$
OPT_{1''} = 2 \sum_{i \in V} y_i'' + 3 \sum_{i \in V} z_i'' = 2 \sum_{i \in V_1} y_i'' + 3 \sum_{i \in V_1} z_i'' + 2 \sum_{i \in V_2} y_i'' + 3 \sum_{i \in V_2} z_i''
$$
$$
= 2 \sum_{i \in V_1} y_i'' + 3 \sum_{i \in V_1} z_i'' + 5|V_2|,
$$
$$
OPT_{1'} \leq 2 \sum_{i \in V} y_i' + 3 \sum_{i \in V} z_i' = 2 \sum_{i \in V_1} y_i' + 3 \sum_{i \in V_1} z_i' + 2 \sum_{i \in V_2} y_i' + 3 \sum_{i \in V_2} z_i'
$$
$$
= 2 \sum_{i \in V_1} y_i'' + 3 \sum_{i \in V_1} z_i'' + 3|V_2|.
$$

Hence, we get $OPT_{1''} \geq OPT_{1'}$.

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It remains to show that $OPT_{1''} \leq OPT_{1'}$. Let S', S'' be the set of feasible solutions to **DRDP-1'** and **DRDP-1''**, respectively. Note that each feasible solution to **DRDP-1'** is also a feasible solution to **DRDP-1''**. Hence $S' \subseteq S''$, which implies that

$$OPT_{1''} = \min_{S''} \left(2 \sum_{i \in V} y_i + 3 \sum_{i \in V} z_i \right) \leq \min_{S'} \left(2 \sum_{i \in V} y_i + 3 \sum_{i \in V} z_i \right) = OPT_{1'}.$$

This completes our proof.

$\square$

Next we prove that the binary variables r_v in **DRDP-2'** formulation can be relaxed to non-negative real numbers, and thus we have the following MIP formulation **DRDP-2''**.

$$[\textbf{DRDP-2''}] \quad \min \quad 2 \sum_{v \in V} q_v + \sum_{v \in V} r_v \tag{16a}$$

$$\text{s.t.} \quad q_v + \frac{1}{2} \sum_{u \in N(v)} q_u + \frac{1}{2} \sum_{u \in N(v)} r_u \geq 1, \quad \forall\, v \in V \tag{16b}$$

$$r_v \leq q_v, \quad \forall\, v \in V \tag{16c}$$

$$q_v \in \{0, 1\}, \ r_v \in [0, +\infty), \quad \forall\, v \in V \tag{16d}$$

Theorem 3.5. *For any graph G, optimal objective function value of formulation* (16a)–(16d) *is equal to that of formulation* (4a)–(4d).

Proof. Let $V = \{1, \ldots, n\}$. We denote the optimal objective function value of **DRDP-2'** and **DRDP-2''** by $OPT_{2'}$ and $OPT_{2''}$, respectively.

Since the optimal objective function value of every relaxed minimization problem is less than or equal to that of the associated original problem, hence we get $OPT_{2''} \leq OPT_{2'}$.

Therefore, we only need to prove that $OPT_{2''} \geq OPT_{2'}$. Note that an optimal solution to **DRDP-2''** can be viewed as a vector $(\tilde{q}'', \tilde{r}'') \in \{0, 1\}^{2n}$, where $\tilde{q}'' = (q_1'', \ldots, q_n'')$, $\tilde{r}'' = (r_1'', \ldots, r_n'')$. And we define a new vector $(\tilde{q}', \tilde{r}') = (q_1', \ldots, q_n', r_1', \ldots, r_n')$ as follows:

$$q_i' = q_i'', \quad \forall i \in V$$

and

$$r_i' = \begin{cases} 0, & r_i'' \in [0, 1) \\ 1, & r_i'' \in [1, +\infty) \end{cases}$$

Now we prove that $(\tilde{q}', \tilde{r}')$ is a feasible solution to **DRDP-2'** formulation. The variables q_i' and r_i' have binary values by definition, thus constraints (4d) hold. Note that

i) when $r_i'' \in [0, 1)$, $r_i' = 0 \leq q_i'$;
ii) when $r_i'' \in [1, +\infty)$, $r_i' = 1 \leq r_i'' \leq q_i'' = q_i'$, where the second inequality follows from constraints (16c).

Therefore, constraints (4c) hold as well. Next we verify the validity of constraints (4b).

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Recall that for each $i \in V$, $q_i'' + \frac{1}{2}\sum_{j \in N(i)} q_j'' + \frac{1}{2}\sum_{j \in N(i)} r_j'' \geq 1$ (constraints 16b) and $r_i'' \leq q_i''$ (constraints 16c). We consider the following three cases:

i) $q_i'' = 1$ or there exist at least two different vertices $j_1, j_2 \in N(i)$ with $q_{j_1}'' = q_{j_2}'' = 1$. Since $q_j' = q_j''$ for each $j \in V$, constraints (4b) hold naturally for q_i' and r_i'.

ii) $q_i'' = 0$ and $q_j'' = 0$ for each $j \in N(i)$. By constraints (16c), we have $r_j'' = 0$ for each $j \in N(i)$, and hence $q_i'' + \frac{1}{2}\sum_{j \in N(i)} q_j'' + \frac{1}{2}\sum_{j \in N(i)} r_j'' = 0$, which contradicts the constraints (16b). Thus this case is impossible.

iii) $q_i'' = 0$ and there exists a unique vertex $j_0 \in N(i)$ such that $q_{j_0}'' = 1$. By constraints (16c), we get $\sum_{j \in N(i) \setminus j_0} r_j'' = 0$. From constraints (16b), it follows that $r_{j_0}'' \geq 1$. So $q_{j_0}' = r_{j_0}' = 1$, then constraints (4b) are satisfied for q_i' and r_i'.

To sum up, constraints (4b)–(4d) are satisfied. Therefore, the vector $(\tilde{q}', \tilde{r}')$ is a feasible solution to **DRDP-2$'$** formulation, which implies

$$OPT_{2'} \leq 2\sum_{i \in V} q_i' + \sum_{i \in V} r_i' \leq 2\sum_{i \in V} q_i'' + \sum_{i \in V} r_i'' = OPT_{2''}.$$

The second inequality follows from the definition of $(\tilde{q}', \tilde{r}')$. Thus we complete our proof.

$\square$

Remark. Actually, from constraints (16c), we can replace constraints (16d) by

$$q_v \in \{0, 1\}, \ r_v \in [0, 1], \quad \forall \, v \in V$$

4. Approximation algorithm

In this section, we consider the approximation algorithm for the double Roman domination problem. We design a greedy algorithm based on the ILP formulation **DRDP-1$''$**. Firstly, consider the following integer programming of general covering problem:

$$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \in \{0, 1\}^{2n} \end{aligned} \tag{17}$$

Let $\mathbf{x} = (y_1, \ldots, y_n, z_1, \ldots, z_n)^T$, $\mathbf{c} = (\overbrace{2, \ldots, 2}^{n}, \overbrace{3, \ldots, 3}^{n})$, $\mathbf{b} = 2 \cdot \mathbf{1}_n$, $A = \begin{bmatrix} 2I_n + A(G) & 2(I_n + A(G)) \end{bmatrix}_{n \times 2n}$, where I_n is a unit matrix of size n. Then we transform the **DRDP-1$''$** formulation into a constrained general covering problem (17). Using the same method described in [17], we can design a greedy algorithm as follows:

Theorem 4.1 ([17]). *If OPT is the optimal value of the above covering problem (17), W_1 is the value given by the greedy algorithm, then*

$$\frac{W_1}{OPT} \leq H\left(\max_{1 \leq j \leq 2n} \sum_{i=1}^{n} a_{ij}\right),$$

Algorithm 1 Greedy
___
Input: a graph G
Output: W_1, W_2
1: $\mathbf{x} \leftarrow \mathbf{0}$, $S \leftarrow \{1, \ldots, 2n\}$
2: **while** $\mathbf{b} \neq \mathbf{0}$ **do**
3:   $k \leftarrow \arg\min_{j \in S}\{c_j / \sum_{i=1}^{n} a_{ij}\}$
4:   $x_k \leftarrow 1$
5:   $S \leftarrow S - \{k\}$
6:   **for** $i = 1$ to n **do**
7:     $b_i \leftarrow b_i - a_{ik}$
8:     **for** j in S **do**
9:       $a_{ij} \leftarrow \min\{a_{ij}, b_i\}$
10:     **end for**
11:   **end for**
12: **end while**
13: $W_1 \leftarrow \mathbf{c} \cdot \mathbf{x}$
14: **for** $i = 1$ to n **do**
15:   **if** $x_i = x_{i+n} = 1$ **then**
16:     $x_i \leftarrow 0$
17:   **end if**
18: **end for**
19: $W_2 \leftarrow \mathbf{c} \cdot \mathbf{x}$
20: **return** W_1, W_2
---

where $H(d) = \sum_{i=1}^{d} 1/i$ is the first d terms of the harmonic series.

By using Theorem 4.1, we can obtain that the greedy algorithm is an $H(2(\Delta + 1))$-approximation algorithm for double Roman domination problem.

Corollary 4.2. *There exists an $H(2(\Delta+1))$-approximation algorithm for double Roman domination problem.*

Proof. Let OPT be the optimal objective function value of the formulation **DRDP-1''**. By applying algorithm 1, we have a feasible solution $\mathbf{x}$ satisfying the formulation **DRDP-1''**. Since $W_2 \leq W_1$ (step 13–19), with theorem 4.1, we have

$$\frac{W_2}{OPT} \leq \frac{W_1}{OPT} \leq H\left(\max_{1 \leq j \leq 2n}\sum_{i=1}^{n} a_{ij}\right) = H(2(\Delta + 1)),$$

Thus we complete our proof. $\qquad\square$

Note that $H(d) \leq 1 + \ln d$, hence the approximation factor is bound by $O(\ln(2(\Delta + 1)))$. Next we will show this approximation factor is best possible in some sense. Recall that $2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G)$. We get the following result:

Lemma 4.3. (1) *If minimum dominating set problem can be approximated up to a factor of α in some graph class, then DRDP can be approximated up to a factor of $\frac{3}{2}\alpha$ in the same graph class.*
  (2) *If DRDP can be approximated up to a factor of α in some graph class, then*

minimum dominating set problem can be approximated up to a factor of $\frac{3}{2}\alpha$ in the same graph class.

Proof. (1) Let S be the solution returned by some α-approximation algorithm for minimum dominating set problem on input $G = (V, E)$. Then $|S| \leq \alpha\gamma(G)$. Let f be the function such that $f(u) = 3$ for each $u \in S$ and $f(u) = 0$ for each $u \in V \setminus S$. It is easy to see that f is a double Roman dominating function with weight $3|S|$. Thus $3|S| \leq 3\alpha\gamma(G) \leq \frac{3}{2}\alpha\gamma_{dR}(G)$.

(2) Let f be the double Roman dominating function with (V_0, V_1, V_2, V_3) returned by some α-approximation algorithm for DRDP on input $G = (V, E)$, where $V_i = \{u \in V \mid f(u) = i\}$. Then $|V_1| + 2|V_2| + 3|V_3| \leq \alpha\gamma_{dR}(G)$. Obviously, $V_2 \cup V_3$ forms a dominating set of G. So we get $|V_2| + |V_3| \leq \frac{1}{2}(|V_1| + 2|V_2| + 3|V_3|) \leq \frac{1}{2}\alpha\gamma_{dR}(G) \leq \frac{3}{2}\alpha\gamma(G)$.

$\square$

Lemma 4.4 ([14]). *Minimum dominating set problem cannot be approximated to within a factor of $(1 - \varepsilon)\ln(n)$ in polynomial time for any constant $\varepsilon > 0$ unless $NP \subseteq DTIME\left(n^{O(\log(\log(n)))}\right)$. The same results hold also in bipartite and split graphs (hence in chordal graphs, and in complements of chordal graphs as well).*

Combining the above two lemmas, we can conclude that

Theorem 4.5. *DRDP cannot be approximated to within a factor of $\frac{2}{3}(1 - \varepsilon)\ln(n)$ in polynomial time for any constant $\varepsilon > 0$ unless $NP \subseteq DTIME\left(n^{O(\log(\log(n)))}\right)$. The same results hold also in bipartite and split graphs (hence in chordal graphs, and in complements of chordal graphs as well).* $\square$

Remark. For Roman domination problem, if we use the new ILP formulation presented in [24] and set $\mathbf{c} = (\overbrace{1, \ldots, 1}^{n}, \overbrace{2, \ldots, 2}^{n})$, $A = \left[\begin{array}{cc} I_n & I_n + A(G) \end{array}\right]_{n \times 2n}$, $\mathbf{b} = \mathbf{1}_n$ in previous constrained general covering problem (17), then we can also apply the greedy algorithm 1 to Roman domination problem to obtain the approximate result. Hence we get

Corollary 4.6. *There exists an $H(\Delta + 1)$-approximation algorithm for Roman domination problem.*

5. Computational results

In this section, computational results which show effectiveness of the proposed DRDP formulations are summarized. All DRDP formulations were solved using CPLEX 12.8 optimization solvers. We use a Python package — "NetworkX"[36] to generate all graphs of different types and sizes and the greedy algorithm was carried out by Python 3.7.0. All computational experiments were performed on Intel(R) Core(TM) i5-4200U CPU @ 1.60GHz 2.30GHz with 8GB RAM under Windows 10 operating system.

For all computational experiments, we used grid graphs, random graphs generated by Erdős-Rényi $G(n, p)$ model and random trees which were chosen uniformly at random from the set of all trees on n nodes. All the formulations posed above were applied exactly once to each problem instance. The computation time limit for the applications

of CPLEX to each model was 3600 seconds for each graph. The results are presented in numerical form in three tables: Table 2 contains the results for all grid graphs. The results for all random graphs and random trees are presented in Table 1 and Table 3, respectively. The first fourth columns of Table 2 indicate the name of the instances, the number of vertices ($|V|$), the number of edges ($|E|$) and the best solution value. For the Table 1, the first fourth columns are the number of vertices ($|V|$), the number of edges ($|E|$), the edge probability (p) to generate random graphs and the best solution value. The first three columns of Table 3 indicate the name of the instances, the number of vertices ($|V|$) and the optimal solution value. Each random graph and random tree with the same number of vertices has three different instances. These entries of three tables are followed by the results obtained using all the above formulations and approximation algorithm. For each formulation, the computation time required to prove optimality is given in seconds together with the number of nodes visited in the search tree or the number of iterations. For the greedy algorithm, results and time were presented in the table. More specifically, all formulations and the greedy algorithm are compared with each other and the best method of DRDP formulations (the least computation time) is marked in bold in each row. We use the sign "–" to show the instances where the optimal solution was not reached within 3600 seconds. For that in Table 1 and Table 2, the best results are shown for the corresponding case when 3600-second limit reaches.

Table 1 presents the computational results on random graphs. Considering the instances in Table 1, it can be seen that all six DRDP formulations perform better in dense graphs than in sparse graphs. And we can conclude that the computation time of the six models is closely related to the number of edges in a graph. Besides, comparing the performance of the six models, **DRDP-1′** and **DRDP-1″** have better performance than others in general. Although we have added some extra constraints to **DRDP-1′** and **DRDP-2′**, these new formulations **DRDP-1′(+)** and **DRDP-2′(+)** do not perform very well as we expect.

From Table 2, we observe that, on the whole, **DRDP-1′**, **DRDP-1′(+)** and **DRDP-1″** perform better than the other formulations in grid graphs. In particular, for all instances with less than 200 vertices, **DRDP-1″** outperforms **DRDP-1′**. This was to be expected, as the only difference in the two models is the elimination of a set of constraints. And **DRDP-1′(+)** perform better than **DRDP-1′** in some degree. It can also be observed that **DRDP-2′** and **DRDP-2″** are competitive with **DRDP-1′** and **DRDP-1″** in the context of grid graphs with less than 200 vertices.

When considering all instances together in Table 3, **DRDP-1″** is the best-performing model on random trees, which is followed by **DRDP-2′**. Note that the DRDP problem of tree is much easier than gird or random graph, since the six models can even handle tens of thousands of nodes within the time limit of 3600 seconds. Perhaps it is relevant to the complexity of the problem (One can construct a linear-time algorithm to compute the value of $\gamma_{dR}(T)$ for any tree T [35]).

Furthermore, we can notice that the greedy algorithm was very competitive on grid and random graphs. However, this method performed much worse than any formulation in the case of random trees, especially when the tree's order is very large. The main reason is probably that for random trees, this approximation algorithm does not sufficiently utilize the sparsity of matrix A.

6. Conclusions

In this paper, we investigated the double Roman domination problem. We first proposed two ILP formulations **DRDP-1** and **DRDP-2**, and proved these models indeed correspond to the mathematical formulation of the double Roman domination problem. By Theorem 3.1, we can restrict our attention to the functions with no vertex assigned value 1, which leads to two improved models **DRDP-1$'$** and **DRDP-2$'$**. Next, we explore some extra inequalities to strengthen the formulations **DRDP-1$'$** and **DRDP-2$'$**. Furthermore, it was shown that constraints (3c) could be excluded from **DRDP-1$'$** and variables z_v in **DRDP-2$'$** can be relaxed to non-negative real numbers. Thus we obtained two alternative formulations **DRDP-1$''$** and **DRDP-2$''$**. Finally, we use one ILP formulation **DRDP-1$''$** to give an $H(2(\Delta+1))$-approximation algorithm and provide an inapproximability result for this problem.

Our computational experiments were carried out on grid graphs and random graphs with up to 370 vertices, and random trees up to 30000 vertices. The results showed that **DRDP-1$'$** and **DRDP-1$''$** outperformed the other formulations in most situations. Especially, **DRDP-1$''$** is the best-performing model on random trees and the formulations with extra constraints did not perform very well like we expect. We also notice that the greedy algorithm was very competitive on grid and random graphs. We made several improvements (decrease some variables or add some extra constraints) in some formulations, but significant improvements in computational effort for strengthening those formulations were still expected. Given that optimization solver was not efficient in solving large sparse graphs, designing new exact methods or other approximation algorithms for DRDP is an interesting direction for future work.

Table 1. Numerical results for grid graphs

Instance				DRDP-1			DRDP-1′			DRDP-1′(+)			DRDP-1″			DRDP-2			DRDP-2′			DRDP-2′(+)			DRDP-2″			Greedy	
name	|V|	|E|	best	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time
Grid05×10	50	85	38	38	0.63	79	**38**	**0.20**	**0**	38	0.31	0	38	0.41	13	38	0.81	81	38	0.31	0	38	0.41	0	38	0.63	163	54	0.39
Grid05×15	75	130	56	56	4.06	2528	56	1.20	630	56	1.73	550	**56**	**0.72**	**201**	56	1.7	2236	56	0.81	169	56	1.80	232	56	1.89	2821	80	1.07
Grid05×20	100	175	74	74	29.44	15505	74	4.86	3348	74	3.41	2105	74	3.31	2576	74	38.48	56261	**74**	**2.67**	**2773**	74	3.02	2607	74	6.14	7440	105	2.19
Grid05×25	125	220	92	92	200.39	87272	92	15.69	9631	92	14.50	5501	**92**	**10.22**	**5210**	92	173.11	89529	92	17.88	15051	92	32.27	29349	92	30.20	19124	134	4.21
Grid05×30	150	265	110	110	–	–	110	235.53	73607	110	212.36	108141	**110**	**135.11**	**84417**	111	–	–	110	259.31	124201	110	220.97	115958	110	293.45	147695	160	7.35
Grid05×35	175	310	128	129	–	–	129	507.59	142150	**128**	**317.27**	**112420**	128	353.78	92031	129	–	–	128	583.75	141225	128	537.42	178804	128	894.69	362089	185	12.33
Grid10×10	100	180	72	72	22.94	18817	72	2.45	1418	72	2.03	796	**72**	**1.25**	**461**	72	16.19	14135	72	2.28	2335	72	2.56	2965	72	6.00	5554	90	1.78
Grid10×15	150	275	106	106	738.36	140939	106	89.03	31426	106	76.48	32497	**106**	**21.44**	**9335**	106	493.26	160496	106	65.92	26816	106	180.01	144437	106	97.27	40352	138	6.10
Grid10×20	200	370	140	141	–	–	**140**	**1180.09**	**215452**	140	2014.41	325629	140	3362.61	619175	140	–	–	141	–	–	140	2744.03	513429	139.99	–	–	182	13.98
Grid10×25	250	465	174	175	–	–	174	–	–	174	–	–	175	–	–	174	–	–	174	–	–	175	–	–	175	–	–	220	25.90
Grid10×30	300	560	208	209	–	–	211	–	–	209	–	–	208	–	–	208	–	–	209	–	–	210	–	–	210	–	–	268	44.43
Grid15×15	225	420	155	155	–	–	155	1373.06	215668	**155**	**1205.52**	**168916**	155	1355.20	192572	156	–	–	155	1709.80	225916	155	2843.16	341775	155	1905.75	392388	194	19.21
Grid15×20	300	565	204	205	–	–	204	–	–	204	–	–	204	–	–	205	–	–	205	–	–	205	–	–	204.99	–	–	258	42.19
Grid15×25	375	710	254	254	–	–	254	–	–	259	–	–	254	–	–	253	–	–	254	–	–	254	–	–	254	–	–	323	79.26

Table 2. Numerical results for random graphs

Instance				DRDP-1			DRDP-1′			DRDP-1′(+)			DRDP-1″			DRDP-2			DRDP-2′			DRDP-2′(+)			DRDP-2″			Greedy	
|V|	|E|	p	best	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time	nodes	result	time
100	994	0.2	21	21	3.80	5724	**21**	**2.22**	**5162**	21	3.30	4827	21	2.42	5427	21	15.38	16799	21	6.08	8283	21	3.05	4994	21	19.52	32246	23	0.40
	2484	0.5	11	11	7.23	7519	**11**	**4.45**	**8344**	11	5.58	8356	11	4.53	8676	11	30.38	41450	11	6.70	8778	11	6.48	9196	11	256.86	302139	12	0.26
	3949	0.8	6	6	0.55	197	**6**	**0.53**	**0**	6	1.16	185	6	0.75	181	6	0.97	201	6	1.09	168	6	1.17	189	6	4.30	8249	6	0.17
150	2239	0.2	24	24	535.69	550441	24	1745.03	1987548	24	429.39	477868	24	3506.45	3619108	26	–	–	24	788.81	1176644	**24**	**402.88**	**474078**	23.99	–	–	27	0.89
	5596	0.5	11	11	73.67	54703	11	45.56	57470	11	31.25	31104	11	39.91	56138	11	422.19	338528	11	69.92	64322	**11**	**29.16**	**18769**	11	1523.83	2121825	12	0.45
	8903	0.8	6	6	1.97	298	6	1.56	302	6	2.19	291	**6**	**1.44**	**298**	6	2.83	221	6	2.39	267	6	2.44	212	6	16.48	12621	6	0.25
200	0	0.2	27	27	–	–	27	–	–	27	–	–	27	–	–	27	–	–	27	–	–	27	–	–	26.99	–	–	29	1.56
	9941	0.5	12	12	219.22	100591	12	139.25	93823	12	216.39	123961	**12**	**122.53**	**100557**	12	2159.20	986301	12	225.19	135567	12	239.94	107938	12	–	–	14	0.86
	15850	0.8	6	6	3.95	387	**6**	**3.16**	**397**	6	5.47	427	6	3.84	395	6	8.23	312	6	5.44	381	6	5.16	287	6	46.02	20269	9	0.58
250	6205	0.2	29	29	–	–	29	–	–	30	–	–	29	–	–	29	–	–	30	–	–	29	–	–	29	–	–	31	2.81
	15527	0.5	12	12	479.44	127117	12	294.11	122411	12	295.64	125937	**12**	**253.73**	**127877**	12	–	–	12	551.56	174352	12	500.33	143796	12	–	–	12	1.07
	24869	0.8	6	6	31.13	1906	6	9.83	2729	6	14.86	1743	**6**	**7.08**	**995**	6	70.42	3704	6	12.45	493	6	46.98	3059	6	142.36	37039	8	0.80
300	8948	0.2	30	30	–	–	30	–	–	30	–	–	30	–	–	30	–	–	30	–	–	30	–	–	30	–	–	30	4.51
	22393	0.5	12	12	1499.45	244786	**12**	**802.33**	**217587**	12	1319.98	349369	14	–	–	14	–	–	12	1470.81	252751	12	–	–	12	–	–	15	1.97
	35902	0.8	8	8	270.17	57352.00	8	113.47	37007	8	148.28	34773	**8**	**99.38**	**37041**	8	1289.20	128757	8	180.73	34821	8	223.06	31940	8	–	–	10	1.46

Table 3. Numerical results for random trees

name	$|V|$	value	DRDP-1 time	iterations	DRDP-1' time	iterations	DRDP-1'(+) time	iterations	DRDP-1'' time	iterations	DRDP-2 time	iterations	DRDP-2' time	iterations	DRDP-2'(+) time	iterations	DRDP-2'' time	iterations	Greedy time	result
Tree-100-1		102	0.22	371	0.14	185	0.08	198	**0.03**	**17**	0.41	371	0.05	182	0.30	195	0.06	165	1.78	113
Tree-100-2	100	100	0.19	347	0.17	179	0.27	195	**0.05**	**5**	0.55	374	0.09	167	0.25	207	0.08	166	1.92	114
Tree-100-3		98	0.16	269	0.06	152	0.11	167	**0.02**	**3**	0.33	280	0.08	136	0.05	187	0.05	132	1.74	110
Tree-1000-1		998	2.20	3258	0.52	1778	0.75	1926	**0.11**	**20**	1.55	3480	0.42	1654	0.47	1898	0.61	1576	1681.76	1149
Tree-1000-2	1000	964	1.11	2993	0.78	1673	0.86	1774	**0.08**	**16**	1.16	3132	0.50	903	0.63	1706	0.88	1167	1523.83	1092
Tree-1000-3		982	1.30	3224	0.33	1719	0.80	1779	**0.25**	**44**	1.44	3318	0.53	1395	0.97	1763	0.73	1342	1628.42	1136
Tree-10000-1		9838	15.84	32244	5.17	18151	7.25	1913	**0.66**	**29**	73.94	66361	3.17	14733	4.45	710	5.11	15042	–	–
Tree-10000-2	10000	9833	13.73	32843	10.25	18082	7.44	1982	**2.14**	**60**	82.91	63080	3.52	15539	4.59	665	6.73	15183	–	–
Tree-10000-3		9868	26.83	32579	7.78	18064	7.09	2169	**2.39**	**59**	101.69	67826	5.28	15391	7.13	663	5.64	15166	–	–
Tree-20000-1		19624	33.16	22924	7.81	4063	11.58	3807	**1.16**	**123**	97.83	70320	5.75	1317	10.23	1325	7.91	29821	–	–
Tree-20000-2	20000	19622	30.64	22741	10.03	4205	13.83	3975	**1.7**	**61**	103.05	71581	5.61	1398	10.63	1510	8.03	30133	–	–
Tree-20000-3		19655	34.63	22315	8.08	4101	18.59	3924	**3.19**	**828**	95.89	70016	5.86	1352	13.05	1425	9.25	30120	–	–
Tree-30000-1		29410	47.25	32731	12.08	5894	23.11	5817	**2.25**	**393**	161.22	103441	9.00	1996	19.27	2116	13.27	5500	–	–
Tree-30000-2	30000	29309	44.33	31951	13.47	5816	21.70	5575	**2.09**	**58**	161.00	103398	9.58	2123	18.97	2225	13.52	5302	–	–
Tree-30000-3		29423	44.56	32812	19.19	5961	21.98	6098	**2.27**	**458**	149.20	102554	9.77	2024	17.56	2093	13.51	5532	–	–

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