PENTAVALENT SEMISYMMETRIC GRAPHS OF SQUARE-FREE ORDER

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Abstract. A regular graph is semisymmetric if its automorphism group acts transitively on the edge set but not on the vertex set. In this paper, we give a complete list of connected semisymmetric graphs of square-free order and valency 5. The list consists of a single graph, the incidence graph of a generalized hexagon of order $(4,4)$, and an infinite family arising from some groups with cyclic Fitting subgroup.

Keywords. Edge-transitive graph, vertex-transitive graph, semisymmetric graph, simple group.

1. introduction

In this paper we consider only finite and simple graphs.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. The size $|V|$ is called the order of Γ. Let $\mathsf{Aut}\Gamma$ be the automorphism group of Γ, that is, the group consisting of all permutations on V which preserve the adjacency of Γ. Then the action of $\mathsf{Aut}\Gamma$ on V induces a natural action on the edge set E by

$$\{u, w\}^g = \{u^g, w^g\}, \forall \{u, w\} \in E, g \in \mathsf{Aut}\Gamma.$$

The graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on V or E, respectively. If Γ is regular, edge-transitive but not vertex-transitive, then Γ is called a *semisymmetric* graph. It is well-known that a semisymmetric graph is bipartite with two parts being the orbits of its automorphism group on the vertices.

Folkman [9] started the study of semisymmetric graphs and posed eight open problems. Folkman's problems stimulated a wide interest in constructing or classifying semisymmetric graphs, see [1, 3, 6, 7, 8, 11, 13, 14, 18, 19, 20, 21] for example.

In this paper, we make an attempt towards Folkman's problems (4.1) and (4.8), which ask for which pairs (n, k) there are connected semisymmetric graphs of order $2n$ and valency k. By giving a classification result on semisymmetric graphs, we prove that there are connected semisymmetric graphs of valency 5 and order $10p_1p_2 \cdots p_r$, where $r \geq 2$, and p_i are distinct primes with every $p_i - 1$ divisible by 5. The main result of this paper is stated as follows.

Theorem 1.1. *Let Γ be a connected edge-transitive graph of valency 5 and square-free order. Then Γ is semisymmetric if and only if Γ is isomorphic to one of the following graphs: the incidence graph of the generalized hexagon associated with the simple group $\mathrm{G}_2(4)$, and the graphs given in Construction 3.2.*

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2. Preliminaries

Let $\Gamma = (V, E)$ be a graph of valency $k \geq 3$, and $G \leq \mathsf{Aut}\Gamma$. For $v \in V$, set $G_v = \{g \in G \mid v^g = v\}$ and $\Gamma(v) = \{u \in V \mid \{u, v\} \in E\}$, called the stabilizer and the neighbourhood of v in G and in Γ, respectively. Then G_α induces a permutation group on $\Gamma(v)$, denoted by $G_v^{\Gamma(v)}$. Letting $G_v^{[1]}$ be the kernel of G_v acting on $\Gamma(v)$, we have $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$.

Let p be a prime with $p \geq k$, and P a Sylow p-subgroup of $G_{vu} := G_v \cap G_u$, where $\{v, u\} \in E$. Then P fixes both $\Gamma(v)$ and $\Gamma(u)$ point-wise. It follows that for every path from u to some vertex v' of Γ, the subgroup P fixes every vertex on this path. Thus, if Γ is connected, then P fixes V point-wise, and so $P = 1$. Then the next lemma follows.

Lemma 2.1. *Assume that $\Gamma = (V, E)$ is a connected graph of valency k, and $v \in V$. If p is a divisor of $|G_v|$ with $p \geq k$, then $p = k$, G_v is transitive on $\Gamma(v)$ and $|G_v|$ is not divisible by p^2.*

Assume further that G acts transitively on the edge set E of Γ. It is well-known and easily shown that either G is transitive on V, or Γ is a bipartite graph two parts being the orbits of G on V. For the latter case, we call Γ a G-semisymmetric graph. Clearly, if Γ is G-semisymmetric then for an edge $\{u, w\}$ of Γ, the stabilizers G_u and G_w have the same order, and they are transitive on $\Gamma(u)$ and $\Gamma(w)$, respectively. Moreover, we have the following simple fact, refer to [8, Lemma 2.3].

Lemma 2.2. *Assume that $\Gamma = (V, E)$ is a G-semisymmetric graph, and $\{u, w\} \in E$. Then Γ is connected if and only if $G = \langle G_u, G_w\rangle$. In particular, if Γ is connected, then G_u and G_w have no non-trivial normal subgroup in common.*

Lemma 2.3. *Assume that $\Gamma = (V, E)$ is a connected G-semisymmetric graph of valency k, and $\{u, w\} \in E$. If p is a divisor of $|G_u|$ then p is a divisor of $|G_u^{\Gamma(u)}|$ or $|G_w^{\Gamma(w)}|$.*

Proof. Suppose that p is a divisor of $|G_u|$, and that both $G_u^{\Gamma(u)}$ and $G_w^{\Gamma(w)}$ are p'-groups. Then every Sylow p-subgroup of G_u is contained in $G_u^{[1]}$, and every Sylow p-subgroup of G_w is contained in $G_w^{[1]}$. For $v \in \{u, w\}$, let K_v be the subgroup generated by all Sylow p-subgroups of G_v. Then K_v is normal in G_v. By the choice of p, we have $K_v \neq 1$.

Let P be an arbitray Sylow p-subgroup of G_u. Then $P \leq G_u^{[1]} \leq G_{uw} \leq G_w$. Since $|G_u| = |G_w|$, we know that P is also a Sylow p-subgroup of G_w, and then $P \leq K_w$. It follows that $K_u \leq K_w$. Similarly, $K_w \leq K_u$. Then $K_u = K_w$. Since Γ is connected, by Lemma 2.2, $K_u = K_w = 1$, a contradiction. Thus this lemma follows. □

Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph of valency k with bipartition $V = U \cup W$. Suppose that G has a normal subgroup N which is intransitive on both U and W. For $v \in V$, we denote by $\bar{v}$ the N-orbit containing v, and let $\overline{V} = \{\bar{v} \mid v \in V\}$, $\overline{U} = \{\bar{u} \mid u \in U\}$ and $\overline{W} = \{\bar{w} \mid w \in W\}$. Let $\overline{G}$ be the permutation group induced by G on $\overline{V}$. Define a graph Γ_N on $\overline{V}$ such that $\{\bar{u}, \bar{w}\}$ is an edge if and only if $\{u, w\} \in E$. Then Γ_N is well-defined, and $\overline{G}$ acts transitively on the edge set of Γ_N.

In general, Γ_N is not necessarily a regular graph. For the case where Γ is G-locally primitive, that is, $G_v^{\Gamma(v)}$ is a primitive permutation group for each $v \in V$, one can easily prove that Γ_N is $\overline{G}$-locally primitive and of valency k, refer to [10]. In particular, we have the following fact.

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph of prime valency k with bipartition $V = U \cup W$. Suppose that N is a normal subgroup of G such that N is intransitive on both U and W. Then N is semiregular on V, $\overline{G} \cong G/N$, and Γ_N is $\overline{G}$-semisymmetric and of valency k.*

3. A family of semisymmetric graphs

Let G be a finite group, and H, $K \leq G$ with $G = \langle H, K\rangle$. Assume that $H \cap K$ contains no normal subgroup of G other than 1. Let $[G : H]$ and $[G : K]$ be the sets of right cosets of H and K in G, respectively. Then G acts faithfully on $[G : H] \cup [G : K]$ by

$$g : Hx \mapsto Hxg,\ Ky \mapsto Kyg;\ g, x, y \in G.$$

Define a bipartite graph $\mathrm{B}(G, H, K)$ with two parts $[G : H]$ and $[G : K]$ such that $\{Hx, Ky\}$ is an edge if and only if $yx^{-1} \in KH$. Then $\mathrm{B}(G, H, K)$ is well-defined and connected, and G acts transitively on its edge set. Thus $\mathrm{B}(G, H, K)$ is G-semisymmetric if and only if $|H| = |K|$.

Assume that $\Gamma = (V, E)$ is a connected G-semisymmetric graph, and let $\{u, w\} \in E$. Then $u^x \mapsto G_u x$, $w^y \mapsto G_w y$ gives a bijection between V and the vertex set of $\mathrm{B}(G, H, K)$. It is easily shown that this bijection is in fact an isomorphism form Γ to $\mathrm{B}(G, G_u, G_w)$.

Lemma 3.1. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph and $\{u, w\} \in E$. Then $\Gamma \cong \mathrm{B}(G, G_u, G_w)$.*

Construction 3.2. Take r distinct primes p_1, $p_2, \ldots, p_r$ with $r \geq 2$ and every $p_i - 1$ divisible by 5. Let $F = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$, and identify $\mathsf{Aut}(F)$ with $\mathbb{Z}^*_{p_1} \times \cdots \times \mathbb{Z}^*_{p_r}$, where the action of $\mathsf{Aut}(F)$ on F is given by

$$(a_1, a_2, \ldots, a_r)^{(l_1,l_2,\cdots,l_r)} = (l_1a_1, l_2a_2, \ldots, l_ra_r).$$

Fix two elements $\sigma = (m_1, m_2, \ldots, m_r)$ and $\tau = (n_1, n_2, \ldots, n_r)$ of $\mathsf{Aut}(F)$ satisfying the following conditions:

(C1) $m_i \neq 1$, $n_i \neq 1$, $m_i^5 = 1 = n_i^5$, where $1 \leq i \leq r$;
(C2) $(m_1^l, m_2^l, \ldots, m_r^l) \neq (n_1, n_2, \ldots, n_r)$ for $1 \leq l \leq 4$.

Let $G = F{:}\langle \sigma, \tau\rangle$, the semi-direct product of groups F and $\langle \sigma, \tau\rangle$. Then

$$G = \{x\delta \mid x \in F, \delta \in \langle \sigma, \tau\rangle\}$$

with the product given by

$$x_1\delta_1x_2\delta_2 = (x_1 + x_2^{\delta_1^{-1}})(\delta_1\delta_2).$$

Let $a = (1, 1, \cdots, 1) \in F$, $H = \langle \sigma\rangle$ and $K = \langle \tau^a\rangle$. Define

$$\Gamma(p_1, p_2, \cdots, p_r; \sigma, \tau) = \mathrm{B}(G, H, K).$$

□

Recall that an arc in a graph Γ is an ordered pair of adjacent vertices. A graph Γ is said to be *symmetric* (or *arc-transitive*) if $\mathsf{Aut}\Gamma$ is transitive on the vertex set and the set of arcs. It is easily shown that a connected G-semisymmetric graph Γ with bipartition

$U \cup W$ is symmetric if and only if there is some $\sigma \in \mathsf{Aut}\Gamma$ such that $U^\sigma = W$. For an integer $n \geq 2$, denote by D_{2n} the dihedral group of order $2n$.

Lemma 3.3. *Let $\Gamma = \Gamma(p_1, p_2, \cdots, p_r; \sigma, \tau)$ be as in Construction 3.2. Then Γ is connected and semisymmetric.*

Proof. By the choices of σ and τ, we have that $\mathbf{C}_G(F) = F$, and the only element in F fixed by σ or τ^{-1} is the zero of F. It follows that $F = \langle x^\sigma - x\rangle = \langle x^{\tau^{-1}} - x\rangle$ provided that $F = \langle x\rangle$. Note that $\tau^a = -a\tau a = -a(\tau a\tau^{-1})\tau = (-a + a^{\tau^{-1}})\tau$. Let $x = -a + a^{\tau^{-1}}$. Then $\langle x\rangle = F$, and so

$$\langle H, K\rangle = \langle \sigma, \tau^a\rangle = \langle \sigma, (\tau^a)^\sigma(\tau^a)^{-1}, \tau^a\rangle = \langle \sigma, x^\sigma - x, \tau^a\rangle = F\langle \sigma, \tau\rangle = G.$$

Thus Γ is connected. By the construction of Γ, we know that Γ is G-semisymmetric.

Suppose that Γ is vertex-transitive. By [16], checking the symmetric graphs of square-free order and valency 5, we conclude that $\mathsf{Aut}\Gamma \cong \mathrm{D}_{10p_1p_2\cdots p_r}{:}\mathbb{Z}_5$, which has order $2|G|$. It follows that $G \cong \mathbb{Z}_{5p_1p_2\cdots p_r}{:}\mathbb{Z}_5$. Thus $|\mathbf{C}_G(F)| \geq 5p_1p_2\cdots p_r > |F|$, a contradiction. Then the lemma holds. □

4. Graphs associated with soluble groups

Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph of valency 5 with bipartition $V = U \cup W$. By Lemma 2.1, for $v \in V$, we have that

$$|G_v| = 2^s \cdot 3^t \cdot 5, \ \ |G| = 2^s \cdot 3^t \cdot 5|U| \tag{4.a}$$

for some integers s, $t \geq 0$. Since $G_v^{\Gamma(v)}$ is a transitive permutation group of degree 5, we have

$$G_v^{\Gamma(v)} \cong \mathbb{Z}_5,\ \mathbb{Z}_5{:}\mathbb{Z}_2,\ \mathbb{Z}_5{:}\mathbb{Z}_4, \mathrm{A}_5, \text{ or } \mathrm{S}_5. \tag{4.b}$$

If G acts unfaithfully on one of U and W, then it is easily shown that Γ is the complete bipartite graph $\mathsf{K}_{5,5}$. Thus we assume next that G is faithful on both U and W. Then $\Gamma \not\cong \mathsf{K}_{5,5}$; otherwise G has a subgroup isomorphic to $\mathbb{Z}_5^2$ which is neither faithful on U nor faithful on W, a contradiction. In particular,

$$|U| = |W| > 5.$$

Lemma 4.1. *Assume that $|V|$ is square-free, and G is faithful on both U and W. Let N be a normal subgroup of G which is intransitive on both U and W. Suppose that $\Gamma_N \cong \mathsf{K}_{5,5}$. Then 5 is the smallest prime divisor of $|U|$, N is a Hall subgroup of G, and G is soluble.*

Proof. We continue the notation at the end of Section 2. Note that $\overline{G} \leq \mathrm{S}_5 \times \mathrm{S}_5$ and Γ_N is $\overline{G}$-semisymmetric. Then $\overline{G}$ contains a subgroup isomorphic to $\mathbb{Z}_5^2$, which is transitive on the edge set of Γ_N. Let $X \leq G$ with $N \leq X$ and $X/N \cong \mathbb{Z}_5^2$. Then Γ is X-semisymmetric, and X is soluble and contains a normal subgroup R which is regular on both U and W.

Let p is the smallest prime divisor of $|U|$. Since $|R| = |U|$ is square-free, R has a unique p'-Hall subgroup L. Then L is characteristic in H, and hence L is normal in X. Clearly, L is intransitive on both U and W. Applying Lemma 2.4 to the pair (X, L), we have $p \geq 5$, and thus $p = 5$. Since $|U| = 5|N|$ is square-free and $|G/N|$ has no prime divisor greater than 5, N is a Hall subgroup of G.

Noting that N is a normal Hall subgroup of G, there is a subgroup H of G with $G = NH$ and $N \cap H = 1$. Then $H \cong G/N \cong \overline{G} \leq \mathrm{S}_5 \times \mathrm{S}_5$. Since $|U| = |W|$ has prime divisor greater than 5, we know that the actions of H on U and W are not transitive. Denote by $\mathbf{C}_H(N)$ the centralizer of N in H. Then $\mathbf{C}_H(N)$ is normal in G. Applying Lemma 2.4 to the pair $(G, \mathbf{C}_H(N))$, we have that $\mathbf{C}_H(N)$ is semiregular on U, and so $|\mathbf{C}_H(N)|$ is square-free. In particular, $\mathbf{C}_H(N)$ is soluble. Considering the conjugation of H on N, we conclude that H induces a subgroup of $\mathsf{Aut}(N)$ with kernel $\mathbf{C}_H(N)$. Since N has square-free order, $\mathsf{Aut}(N)$ is soluble. Thus $H/\mathbf{C}_H(N)$ is soluble, and so H is soluble. It follows that G is soluble. □

Theorem 4.2. *Assume that $|V|$ is square-free, and that G is soluble and faithful on both U and W. Then Γ is either a symmetric graph, or semisymmetric and isomorphic to a graph given by Construction* 3.2.

Proof. Let F be the Fitting subgroup, the maximal nilpotent normal subgroup, of G. Let $\mathbf{C}_G(F)$ be the centralizer of F in G. Then $\mathbf{C}_G(F)$ is normal in G and, since G is soluble, we have $\mathbf{C}_G(F) \leq F$. Consider the conjugation of G on F. Then G induces a subgroup of the automorphism group $\mathsf{Aut}(F)$ with kernel $\mathbf{C}_G(F)$. Thus $G/\mathbf{C}_G(F)$ is isomorphic to a subgroup of $\mathsf{Aut}(F)$.

Case 1. Assume that F is transitive on one of U and W, without loss of generality, say U. Note that every Sylow subgroup Q of F and every maximal subgroup of Q are normal in F. Since $|U|$ is an odd square-free number and F is faithful on U, it follows that $|F| = |U|$, and then F is regular on U. In particular, F is cyclic, and so $\mathbf{C}_X(F) = F$ and $\mathsf{Aut}(F)$ is abelian. Note that every subgroup of F is normal in G, and thus has orbits of the same size on W. Since G is faithful on W, we conclude that F is transitive and hence regular on W.

Let $\{u, w\} \in E$. Write $U = \{u^x \mid x \in F\}$, $W = \{w^y \mid y \in F\}$ and $D = \{z \in F \mid w^z \in \Gamma(u)\}$. Then $\{u^x, w^y\} \in E$ if and only if $yx^{-1} \in D$. Define θ by

$$\theta : V \to V;\ u^x \mapsto w^{x^{-1}},\ w^y \mapsto u^{y^{-1}},\ x, y \in F.$$

Since F is abelian, it is easy to check that θ is an automorphism of Γ, which interchanges U and W. Thus Γ is vertex-transitive, and so Γ is symmetric.

Case 2. Assume that F is intransitive on both U and W. Then F is semiregular on V by Lemma 2.4, Thus F has square-free order, and F is cyclic, and then $\mathbf{C}_G(F) = F$ and $\mathsf{Aut}(F)$ is abelian. Then G/F is abelian. By Lemma 2.4, Γ_F admits an abelian group acting transitively on the edge set but not on the vertex set. The only possibility is that $\Gamma_F \cong \mathsf{K}_{5,5}$ and $G/F \cong \mathbb{Z}_5^2$. By Lemma 4.1, F is a normal Hall subgroup of G, and each prime divisor of $|F|$ is greater than 5.

Let P be a Sylow 5-subgroup of G. Then $P \cong \mathbb{Z}_5^2$ and $G = F{:}P$. By the choice of F, we may identify P with a subgroup of $\mathsf{Aut}(F)$. Then $G = \{ax \mid a \in F, x \in \mathsf{Aut}(F)\}$ with the product given by

$$axby = ab^{x^{-1}}(xy).$$

We write F as the additive abelian group $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$, and identify $\mathsf{Aut}(F)$ with $\mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_r}^*$, where the action of $\mathsf{Aut}(F)$ on F is given by

$$(a_1, a_2, \ldots, a_r)^{(l_1, l_2, \cdots, l_r)} = (l_1 a_1, l_2 a_2, \ldots, l_r a_r).$$

Fix an edge $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Then $|U||G_u| = |G| = |W||G_w|$, yielding $|G_u| = |G_w| = 5$. Set $G_u = \langle \sigma \rangle$ and $G_w = \langle \delta \rangle$. Since Γ is connected, $G = \langle \sigma, \delta \rangle$ by Lemma 2.2. Without loss of generality, we let $\sigma \in P$. Take a Sylow 5-subgroup Q of G with $\delta \in Q$. Then $P \neq Q$, and thus there is $1 \neq a \in F$ such that $P^a = Q$. Choose $\tau \in P$ with $\tau^a = \delta$. Then $G = \langle \sigma, \tau^a \rangle \leq \langle \sigma, \tau, a \rangle = \langle a \rangle{:}\langle \sigma, \tau \rangle \leq FP$, yielding $F = \langle a \rangle$ and $P = \langle \sigma, \tau \rangle$. It is easily shown that any two generators of F are conjugate under the action of $\mathsf{Aut}(F)$. Replacing P by a suitable conjugation under $\mathsf{Aut}(F)$, we may choose $a = (1, 1, \ldots, 1) \in F$.

Note that every subgroup of F is a normal Hall subgroup of F. Then $F = \mathbf{C}_F(\tau){:}F_1$ for some $F_1 \leq F$. Thus $a = cb$ for $b \in F_1$ and $c \in \mathbf{C}_F(\tau)$. We have $G = \langle \sigma, \tau^a \rangle = \langle \sigma, \tau^b \rangle \leq \langle \sigma, \tau, b \rangle \leq F_1{:}P$, yielding $F = F_1 = \langle b \rangle$. It follows that $\mathbf{C}_F(\tau) = 1$. Thus $F{:}\langle \tau \rangle$ is a Frobinus group with Frobenius kernel F. In particular, 5 is a divisor of $p_i - 1$ for each p_i. Noting that $F = \langle -a \rangle$ and $G = \langle \sigma, \tau^a \rangle = \langle \sigma^{-a}, \tau \rangle \leq \langle \sigma, \tau, -a \rangle \leq FP$, a similar argument leads to $\mathbf{C}_F(\sigma) = 1$. Recalling that $\mathbf{C}_G(F) = F$, it follows that $\sigma = (m_1, m_2, \ldots, m_r)$ and $\tau = (n_1, n_2, \ldots, n_r)$ satisfy the conditions (C1) and (C2) listed in Construction 3.2. Then this theorem follows from Lemmas 3.1 and 3.3. □

5. Graphs arising from almsot simple groups

In this section, we assume that G is an almost simple group with socle $\mathrm{soc}(G) = T$, and $\Gamma = (V, E)$ is a connected G-semisymmetric graph of square-free order and valency 5 with bipartition $V = U \cup W$. Clearly, T fixes both U and W set-wise.

Note that $|U| = |W|$ is odd and square-free. Since $|T|$ has even order, T is not semiregular on V. By Lemma 2.4, T is transitive on at least one of U and W. It is easily shown that $\mathsf{Aut}(\mathsf{K}_{5,5})$ contains no almost simple subgroup which acts transitively on the edge set. Thus we have $|U| = |W| > 5$. By [10, Lemma 5.5], the following lemma holds.

Lemma 5.1. *One of the following holds.*

(1) *Γ is T-semisymmetric, and T_v is transitive on $\Gamma(v)$ for every $v \in V$;*
(2) *T is transitive on one of U and W and has 5 orbits on the other one; in particular, $|G : T|$ is divisible by 5, and T_v is transitive on $\Gamma(v)$ for some $v \in V$.*

By Lemmas 2.1 and 5.1, we have the following observation, which is useful for us to determine T.

(5.a) $$5 \mid |T|, 5^3 \nmid |T|, r^2 \nmid |T|,$$

where r is a prime not less than 7.

By Lemma 5.1, there exists $v \in V$ such that T_v is transitive on $\Gamma(v)$. Take a subgroup M of T with $T_v \leq M$. Noting that $|T : T_v|$ is a divisor of $|U|$,

(5.b) $5 \mid |T_v|$, $5 \mid |M|$, $|T : M|$ and $|M : T_v|$ are odd, square-free and coprime.

Lemma 5.2. *Let N be a normal $\{2, 3\}$-subgroup of M. Then one of the following holds.*

(1) *M is a $\{2, 3, 5\}$-group and M/N has a subgroup of index 5;*

(2) *M/N has a maximal subgroup with order divisible by 5 and index odd, square-free and coprime to $|T : M|$.*

Proof. Noting that $NT_v/N \cong T_v/(N \cap T_v)$, it follows that NT_v/N has order divisible by 5. Since T_v is a transitive on $\Gamma(v)$, we know that T_v has a subgroup of index 5, and thus NT_v/N has a subgroup of index 5. If $M = NT_v$ then $M/N = NT_v/N$, and part (1) of the lemma follows. Assume that $M \neq NT_v$. Then NT_v/N is a proper subgroup of M/N. Since $|M/N : (NT_v/N)| = |M : (NT_v)|$ and $|T : T_v| = |T : M||M : T_v|$, we know that $|M/N : (NT_v/N)|$ is odd, square-free and coprime to $|T : M|$. Considering the maximal subgroups of M/N which contain NT_v/N, we get part (2) of this lemma. □

In the following, we always choose $v \in V$ and $T_v \leq M < T$ such that T_v is transitive on $\Gamma(v)$ and M is maximal in T. Using (5.a), (5.b) and Lemma 5.2, we shall read out the pair (T, M) from [17, Tables 1-4], and then determine all possible candidates for T_v.

5.1. In this subsection, we deal with the alternating groups and sporadic simple groups.

Lemma 5.3. *$T \not\cong \mathrm{A}_n$ for all $n \geq 5$.*

Proof. Suppose that $T \cong \mathrm{A}_n$ for some $n \geq 5$. Then $5 \leq n \leq 13$ as $|T|$ is indivisible by 7^2. Choose $v \in V$ such that T_v is transitive on $\Gamma(v)$, and take a maximal subgroup of T with $T_v \leq M$. Then $|T : M|$ is odd and square-free, and $|M|$ is divisible by 5. Checking [17, Table 1], we conclude that M is the stabilizer of some k-subset under the natural action of A_n on $\Omega = \{1, 2, 3, \ldots, n\}$, where (k, n) is one of $(1, 7)$, $(2, 7)$, $(1, 11)$, $(2, 11)$, $(3, 11)$, $(1, 13)$ and $(4, 13)$. Let N be the normal subgroup of M with $N \cong \mathrm{A}_k$, where $\mathrm{A}_k = 1$ for $k \in \{1, 2\}$. Then $M/N \cong \mathrm{A}_{n-k}$ or S_{n-k}. Clearly, $n - k \geq 5$.

Assume that $n - k > 5$. By Lemma 5.2, either A_{n-k} or S_{n-k} has a maximal subgroup with odd square-free index and order divisible by 5. It follows from [17, Table 1] that $n - k \in \{7, 11, 13\}$, which is impossible.

Assume that $n - k = 5$. Then $n = 7$ and $k = 2$. Clearly, $|G : T| \leq 2$. By Lemma 5.1, Γ is T-semisymmetric. In particular, $|U| = |W| = |T : M||M : T_v| = 21|M : T_v|$. Since $|U|$ is an odd square-free number and $|T_v|$ is divisible by 5, we conclude that $T_v = M$. Thus the actions of T on U and W are equivalent to the action of T on the 2-subsets of $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. Then two vertices $u \in U$ and $w \in W$ are adjacent if and only if, as 2-subsets of Ω, the intersection $u \cap w$ is empty, or if and only if $|u \cap w| = 1$. It follows that Γ has valency 10, a contradiction. □

Lemma 5.4. *T is not a sporadic simple group.*

Proof. Suppose that T is one of the 26 sporadic simple groups. Choose $v \in V$ such that T_v is transitive on $\Gamma(v)$, and take a maximal subgroup M of T with $T_v \leq M$. Then $|M|$ is divisible by 5, and $|T : M|$ is odd and square-free. By [17, Table 2], up to isomorphism, T and M are listed as follows:

T	M_{11}	M_{22}	M_{22}	M_{23}	M_{23}	M_{23}	M_{23}	M_{24}	M_{24}	J_1
M	M_{10}	$2^4{:}\mathrm{A}_6$	$2^4{:}\mathrm{S}_5$	M_{22}	$\mathrm{PSL}(3,4){:}2$	$2^4{:}\mathrm{A}_7$	$2^4{:}(3 \times \mathrm{A}_5){:}2$	$2^6{:}\mathrm{A}_8$	$2^6{:}3.\mathrm{S}_6$	$2 \times \mathrm{A}_5$
$\|T : M\|$	11	$7 \cdot 11$	$3 \cdot 7 \cdot 11$	23	$11 \cdot 23$	$11 \cdot 23$	$7 \cdot 11 \cdot 23$	$3 \cdot 11 \cdot 23$	$7 \cdot 11 \cdot 23$	$7 \cdot 11 \cdot 19$

Noting that $|G : T| \leq 2$ for each T listed above, by Lemma 5.1, Γ is T-semisymmetric.

By the Atlas [4], A_6, M_{10}, S_6 and A_8 have no maximal subgroup of odd square-free index and order divisible by 5. Thus the pairs $(\mathrm{M}_{11}, \mathrm{M}_{10})$, $(\mathrm{M}_{22}, 2^4{:}\mathrm{A}_6)$, $(\mathrm{M}_{24}, 2^6{:}\mathrm{A}_8)$ and $(\mathrm{M}_{24}, 2^6{:}3.\mathrm{S}_6)$ are excluded by Lemma 5.2.

Case 1. Suppose that (T, M) is one of $(\mathrm{M}_{22}, 2^4{:}\mathrm{S}_5)$ and $(\mathrm{J}_1, 2 \times \mathrm{A}_5)$. Recall that the indices $|T : M|$ and $|M : T_v|$ are odd, square-free and coprime. Checking the subgroups of M of square-free index, we get $T_v = M$; in particular, T is primitive on both U and W. By the information given in the Atlas [4], all subgroups isomorphic to M are conjugate in T. Then there are $u \in U$ and $w \in W$ with $T_u = T_w$. Thus $\Gamma(u)$ is a T_w-orbit on W of length 5, which is impossible. (Confirmed by GAP [23], see also the Web-version of [4]).

Case 2. Suppose that (T, M) is one of $(\mathrm{M}_{23}, 2^4{:}(3 \times \mathrm{A}_5){:}2)$, $(\mathrm{M}_{23}, \mathrm{PSL}(3, 4).2)$, $(\mathrm{M}_{23}, 2^4{:}\mathrm{A}_7)$ and $(\mathrm{M}_{23}, \mathrm{M}_{22})$. If $T_v = M$ then $(T, M) = (\mathrm{M}_{23}, 2^4{:}(3 \times \mathrm{A}_5){:}2)$, and then a similar argument as in Case 1 gives a contradiction. Next let $T_v \neq M$.

Recall that $T = \mathrm{M}_{23}$ is the automorphism group of the unique $S(4, 7, 23)$ Steiner system. Let Ω and $\mathcal{B}$ be the point set and block set of the $S(4, 7, 23)$ Steiner system.

Check the subgroups of M which have odd and square-free index and order divisible by 5. For $M \cong 2^4{:}(3 \times \mathrm{A}_5){:}2$, we have that $T_v \cong 2^4{:}\mathrm{S}_5$, M is the stabilizer of a 3-subset $\{\alpha, \beta, \delta\}$ of Ω, and T_v is one of the point-stabilizers of M acting on $\{\alpha, \beta, \delta\}$. For $M \cong \mathrm{PSL}(3, 4).2$, we have that $T_v \cong 2^4{:}\mathrm{S}_5$, M is the stabilizer of a 2-subset $\{\alpha, \beta\}$ of Ω, and T_v is the stabilizer of a block containing $\{\alpha, \beta\}$. For $M \cong 2^4{:}\mathrm{A}_7$, we have that $T_v \cong 2^4{:}\mathrm{S}_5$, M is the stabilizer of a block $B \in \mathcal{B}$, and T_v is one of the point-stabilizers of M acting on B. For $M \cong \mathrm{M}_{22}$, we have that $T_v \cong 2^4{:}\mathrm{S}_5$, M is the stabilizer of some $\alpha \in \Omega$, and T_v is the stabilizer of some 2-subset $\{\beta, \delta\}$ of $\Omega \setminus \{\alpha\}$. All in all, $T_v \cong 2^4{:}\mathrm{S}_5$, and one of the following cases occurs:

(i) $T_v = T_{\{\alpha,\beta\}} \cap T_B$ for some $B \in \mathcal{B}$ and some 2-subset $\{\alpha, \beta\} \subset \Omega$;
(ii) $T_v = T_{\{\alpha,\beta\}} \cap T_{\{\alpha,\beta,\delta\}}$ for some 2-subset $\{\alpha, \beta\} \subset \Omega$ and $\delta \in \Omega \setminus \{\alpha, \beta\}$.

Fix an edge $\{u, w\}$ of Γ. Next we deduce the contradiction in three cases.

(1). Suppose that both T_u and T_w satisfy (i), say $T_u = T_{\{\alpha,\beta\}} \cap T_B$ and $T_w = T_{\{\alpha',\beta'\}} \cap T_{B'}$. Since Γ is connected, $\langle T_u, T_w\rangle = T$ by Lemma 2.2. Then $\{\alpha, \beta\} \neq \{\alpha', \beta'\}$ and $B \neq B'$. Considering the action of T on $\mathcal{B}$, we know that T_B has 2 orbits on $\mathcal{B} \setminus \{B\}$, which have sizes 112 and 140, refer to the Web-version of the Atlas [4]. This implies that $|T_B \cap T_{B'}| = 360$ or 288, respectively. Noting that $T_u \cap T_w \leq T_B \cap T_{B'}$ and $|\Gamma(u)| = |T_u : (T_u \cap T_w)|$, we have $5 = |\Gamma(u)| \geq \frac{|T_u|}{360} > 5$, a contradiction.

(2). Suppose that both T_u and T_w satisfy (ii), say $T_u = T_{\{\alpha,\beta\}} \cap T_{\{\alpha,\beta,\delta\}}$ and $T_w = T_{\{\alpha',\beta'\}} \cap T_{\{\alpha',\beta',\delta'\}}$. Then $\{\alpha, \beta\} \neq \{\alpha', \beta'\}$ and $\{\alpha, \beta, \delta\} \neq \{\alpha', \beta', \delta'\}$ as $\langle T_u, T_w\rangle = G$. Considering the action of T on the 3-subsets of Ω, we conclude that $T_{\{\alpha,\beta,\delta\}}$ has 7 orbits with length not equal to 1, and the minimum length is 20, refer to the Web-version of the Atlas [4]. Thus $|T_u \cap T_w| \leq |T_{\{\alpha,\beta,\delta\}} \cap T_{\{\alpha',\beta',\delta'\}}| \leq \frac{|T_{\{\alpha,\beta,\delta\}}|}{20} = 288$. Then $5 = |\Gamma(u)| = |T_u : (T_u \cap T_w)| \geq \frac{|T_u|}{288} > 5$, a contradiction.

(3). Finally, let $T_u = T_{\{\alpha,\beta\}} \cap T_B$ and $T_w = T_{\{\alpha',\beta'\}} \cap T_{\{\alpha',\beta',\delta'\}}$ be as in (i) and (ii), respectively. Then $\{\alpha, \beta\} \neq \{\alpha', \beta'\}$. Note that $T_{\{\alpha,\beta\}}$ has three orbits on the 2-subsets of Ω, say $\{\{\alpha, \beta\}\}$, $\{\{\alpha, \delta\} \mid \delta \in \Omega \setminus \{\alpha, \beta\}\} \cup \{\{\beta, \delta\} \mid \delta \in \Omega \setminus \{\alpha, \beta\}\}$ and $\{\{\delta, \eta\} \mid \{\delta, \eta\} \cap \{\alpha, \beta\} = \emptyset\}$. If $\{\alpha', \beta'\} \cap \{\alpha, \beta\} = \emptyset$. Then $|T_u \cap T_w| \leq |T_{\{\alpha,\beta\}} \cap T_{\{\alpha',\beta'\}}| = \frac{|T_{\{\alpha,\beta\}}|}{210} = 192$, and hence $5 = |\Gamma(u)| = |T_u : (T_u \cap T_w)| \geq \frac{|T_u|}{192} = 10$, a contradiction. Thus we may assume that $\beta' = \beta$. Then $T_u \cap T_w \leq T_{\{\alpha,\beta\}} \cap T_{\{\alpha',\beta\}} = T_{\alpha\beta\alpha'} \cong 2^4{:}\mathrm{A}_5$; in particular, $|T_u \cap T_w|$ is not divisible by 2^7. Since $T_u \cong 2^4{:}\mathrm{S}_5$ has order divisible by 2^7, we have that $|\Gamma(u)| = |T_u : (T_u \cap T_w)|$ is even, again a contradiction. □

5.2. Now we deal with the simple groups of Lie type. For a power $q = p^f$ of some prime p, we denote by $\mathbb{F}_q$ the finite field of order q.

Let $t \geq 2$ be an integer. A prime r is a primitive divisor of $p^t - 1$ if r is a divisor of $p^t - 1$ but not a divisor of $p^s - 1$ for any $1 \leq s < t$. If r is a primitive divisor of $p^t - 1$, then p has order t modulo r, and thus t is a divisor of $r - 1$; in particular, $r \geq t + 1$.

Lemma 5.5. *Let p be a prime and t be a positive integer.*

(1) *If $p^t - 1$ has no prime divisor greater than 5 then $t \leq 4$.*
(2) *If $p^t + 1$ has no prime divisor greater than 5 then either $t \leq 2$ or $p^t = 8$.*

Proof. Suppose that $t > 4$ and $p^t - 1$ has no prime divisor greater than 5. Then $p^t - 1$ has no primitive prime divisor. By Zsigmondy's Theorem (see [24]), $t = 6$ and $p = 2$, and then $p^t - 1$ is divisible by 7, a contradiction. Thus part (1) of this lemma holds.

Assume that $p^t + 1$ has no prime divisor greater than 5. If $p^{2t} - 1$ has a primitive prime divisor r then $r \geq 2t + 1$ and r is a divisor of $p^t + 1$, and so $t \leq 2$. Suppose that $p^{2t} - 1$ has no primitive prime divisor. By Zsigmondy's Theorem, either $2t = 2$ or $(2t, p) = (6, 2)$. Then part (2) of this lemma follows. □

Lemma 5.6. *If T is a exceptional simple group of Lie type, then $T = \mathrm{G}_2(4)$, $\mathsf{Aut}\Gamma = \mathrm{G}_2(4).2$, Γ is semsymmetric and isomorphic to the incidence graph of the generalized hexagon associated with $\mathrm{G}_2(4)$.*

Proof. Assume that T is a exceptional simple group defined over $\mathbb{F}_q$, where $q = p^f$ for a prime p. Choose $v \in V$ such that T_v is transitive on $\Gamma(v)$, and take a maximal subgroup M of T with $T_v \leq M$. Then $|T : M|$ is odd and square-free, and $|M|$ is divisible by 5.

It follows from [17, Table 4] that q is even and T is one of ${}^2\mathrm{B}_2(q)$, $\mathrm{G}_2(q)$, ${}^3\mathrm{D}_4(q)$, $\mathrm{F}_4(q)$, ${}^2\mathrm{E}_6(q)$ and $\mathrm{E}_7(q)$. Suppose that $T = {}^2\mathrm{B}_2(q)$. Then $|M| = q^2(q-1)$. In this case, q is an odd power of 2, and then $q \equiv \pm 2 \,(\mathsf{mod}\ 5)$. It follows that $|M|$ is indivisible by 5, a contradiction. Thus $T \neq {}^2\mathrm{B}_2(q)$, and then $|T|$ has a divisor $(q^2 - 1)^2$. Recall that $|T|$ has no divisor r^2 for any prime $r > 5$, see the observation (5.a). Then $q^2 - 1$ has no prime divisor greater than 5. By Lemma 5.5, since q is even, we have $q \in \{2, 4\}$. If T is one of ${}^3\mathrm{D}_4(q)$, $\mathrm{F}_4(q)$, ${}^2\mathrm{E}_6(q)$ and $\mathrm{E}_7(q)$, then calculation shows that $|T|$ has a divisor 7^2, a contradiction. Thus we have $T = \mathrm{G}_2(4)$.

Clearly, $|G : T| \leq 2$. By Lemma 5.1, Γ is T-semisymmetric. In particular, $|U| = |W| = |T : T_v| = |T : M||M : T_v|$. Checking the maximal subgroups of T in the Atlas [4], we conclude that $M \cong 2^{2+8}{:}(3 \times \mathrm{A}_5)$ or $2^{4+6}{:}(3 \times \mathrm{A}_5)$, and $|T : M| = 3 \cdot 5 \cdot 7 \cdot 13$. It follows that $T_v = M$. If there are $u \in U$ and $w \in W$ with $T_u = T_w$, then $\Gamma(u)$ is a T_w-orbit on W of size 5, which is impossible, see the Web-version of [4]. Thus the actions of T on U and W are not equivalent. Then the lemma follows from the information for $\mathrm{G}_2(4)$ given in the Atlas [4]. □

Lemma 5.7. *Assume that T is a classical simple group. Then Γ is T-semisymmetric, and T is isomorphic to one of $\mathrm{PSL}(2,p)$, $\mathrm{PSL}(2,25)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(3,16)$, $\mathrm{PSL}(4,4)$, $\mathrm{PSL}(5,4)$, $\mathrm{PSU}(3,4)$ and $\mathrm{PSp}(4,4)$, where p is a prime.*

Proof. Assume that T is an n-dimensional classical simple group defined over $\mathbb{F}_q$, where $q = p^f$ for a prime p. Then either $T = \mathrm{PSL}(2,q)$ or $|T|$ has a divisor q^3. Recall that $|T|$ has no divisor 5^3 or r^2, where r is a prime not less than 7. If $p \geq 5$ then $T = \mathrm{PSL}(2,q)$, yielding $T = \mathrm{PSL}(2,p)$ or $\mathrm{PSL}(2,25)$. Thus we assume that $p \in \{2, 3\}$ in the following.

Choose $v \in V$ such that T_v is transitive on $\Gamma(v)$, and take a maximal subgroup M of T with $T_v \leq M$. Then $|T:M|$ is odd and square-free, and $|M|$ is divisible by 5.

Assume that $T = \mathrm{PSL}(2,q)$. It follows from [17, Table 3] that $|T:M| = p^f + 1$ and $M \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{q-1}{(2,q-1)}}$; in particular, $p = 2$. Since $|M:T_v|$ is odd (see (5.b)) and T_v is transitive on $\Gamma(v)$, we have $T_v \cong \mathbb{Z}_2^f{:}\mathbb{Z}_l$ and $T_v^{\Gamma(v)} \cong \mathbb{Z}_5$, where l is a divisor of $2^f - 1$ and divisible by 5. Let P be the unique Sylow 2-subgroup of T_v. Then $P \leq T_v^{[1]} \leq G_v^{[1]}$, and P is normal in G_v as T_v is normal in G_v. Let $w \in \Gamma(v)$. Then $P \leq T_w$, and so P is a Sylow 2-subgroup of T_w. Check the subgroups of T, refer to [12, II.8.27]. Noting that $|T:T_w|$ is odd, we conclude that P is normal in T_w, and so P is characteristic in T_w. Then P is normal in G_w as T_w is normal in G_w. Thus P is normal in both G_v and G_w, which contradicts Lemma 2.2.

Assume that $T = \mathrm{PSL}(3,q)$. Then $|T|$ has a divisor $(p^f-1)^2$ or $\frac{(p^f-1)^2}{3}$, and so $p^f - 1$ has no prime divisor greater than 5. By Lemma 5.5, $f \leq 4$. Recalling that $p \in \{2,3\}$, we know that $p^3 - 1$ has a divisor 7 or 13, and so $f \neq 3$. Thus $q = p^f \in \{2,3,2^2,3^2,2^4,3^4\}$. Since $|T|$ has a divisor 5, we have $q \notin \{2,3\}$. Suppose that $q = 3^2$ or 3^4. By [17, Table 3], $|T:M| = q^2 + q + 1$ and $M = N{:}X$, where N is a 3-group and $X \cong \mathrm{GL}(2,9)$ or $\mathrm{GL}(2,81)$. It is easily shown that $\mathrm{GL}(2,9)$ has neither subgroup of index 5 nor maximal subgroup with odd and square-free index and order divisible by 5. Then, by Lemma 5.2, we conclude that $q = 3^4$ and $X \cong \mathrm{GL}(2,81)$. Noting $NT_v = NT_v \cap M = N(T_v \cap X)$, we have $|M:(NT_v)| = |X:(T_v \cap X)|$. Since $|M:T_v| = |M:(NT_v)||NT_v:T_v|$ and $|M:T_v|$ is odd and square-free, $|X:(T_v \cap X)|$ is odd and square-free. Then $(T_v \cap X)Z/Z$ is a subgroup of X/Z with odd and square-free index, where Z is the center of X. By [17, Table 3], $\mathrm{PGL}(2,81)$ has no maximal subgroup with odd and square-free index. It follows that $(T_v \cap X)Z/Z = X/Z \cong \mathrm{PGL}(2,81)$. Then $|T_v|$ has a divisor 41, which contradicts Lemma 2.1. Thus $q \in \{4,16\}$, and so $T = \mathrm{PSL}(3,4)$ or $\mathrm{PSL}(3,16)$.

Assume that $T = \mathrm{PSU}(3,q)$. Then $|T|$ has a divisor $(p^f+1)^2$ or $\frac{(p^f+1)^2}{3}$, and so $p^f + 1$ has no prime divisor greater than 5. Recalling $p \in \{2,3\}$, by Lemma 5.5, we have $q \in \{2,3,4,8,9\}$. By [17, Table 3], $|T:M| = q^3 + 1$. Since $|T:M|$ is odd and square-free, we have $q = 4$, and thus $T = \mathrm{PSU}(3,4)$.

Assume next that $n \geq 4$. Then $|T|$ has a divisor $(q^2-1)^2 = (p^{2f}-1)^2$. Then $p^{2f} - 1$ has no prime divisor greater than 5. By Lemma 5.5, $2f \leq 4$, and then $q \in \{2,3,4,9\}$.

Suppose that $T = \mathrm{PSL}(n,q)$. Then $n \leq 5$; otherwise, $|T|$ has a divisor $(q^3-1)^2$, and so $|T|$ is divisible by 7^2 or 13^2, a contradiction. By [17, Table 3], $|T:M| = \frac{\prod_{i=0}^{k-1}(q^{n-i}-1)}{\prod_{i=1}^{k}(q^i-1)}$ for some $1 \leq k < n$. Since $|T:M|$ is odd and square-free, calculation shows that $(n,q) = (4,2)$, $(4,4)$, $(5,2)$ or $(5,4)$. Noting $\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, by Lemma 5.3, we have $(n,q) \neq (4,2)$. Assume that $T = \mathrm{PSL}(5,2)$. Then $M \cong 2^4{:}\mathrm{PSL}(4,2)$ or $2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}(3,2))$. Since $|M|$ is divisible by 5, we have $M \cong 2^4{:}\mathrm{PSL}(4,2)$. By Lemma 5.2, $\mathrm{PSL}(4,2)$ has a maximal subgroup with odd and square-free index and order divisible by 5, which is impossible. Then we get $T = \mathrm{PSL}(4,4)$ or $\mathrm{PSL}(5,4)$.

Suppose that $T = \mathrm{PSU}(n,q)$. By [17, Table 3], $|T:M| = \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1}$. In particular, $|T:M|$ is divisible by $q-1$ or $q+1$. Since $|T:M|$ is odd, q is even, and so $q = 2$ or 4. If $n \geq 10$ then $|T|$ has a divisor $(q^5+1)^2$, and so $|T|$ has a divisor 11^2 or 41^2, a contradiction. If $q = 4$ and $n \geq 6$ then $|T|$ has a divisor $(q^3+1)^2$, and so $|T|$ has a divisor 13^2, a contradiction. Thus $4 \leq n \leq 9$ for $q = 2$, and $4 \leq n \leq 5$

for $q = 4$. Then $(n, q) = (4, 4)$, $(5, 2)$ or $(8, 2)$. Assume that $(n, q) = (8, 2)$. Then $M \cong 3^{1+12}{:}\mathrm{SU}(6, 2){:}3$, which is the stabilizer of some totally isotropic 1-subspace. By Lemma 5.2, $\mathrm{PGU}(6, 2)$ has a maximal subgroup of odd and square-free index say m. By [17, Table 3], $m = \frac{(2^6-1)(2^5+1)}{2^2-1} = 3^2 \cdot 7 \cdot 11$, a contradiction. If $(n, q) = (4, 4)$ or $(5, 2)$ then either $|T|$ is divisible by 5^3 or M is a $\{2, 3\}$-group, which contradicts (5.a) or (5.b).

Suppose that T is one of $\mathrm{PSp}(n, q)$ (with n even and $(n, q) \neq (4, 2)$), $\mathrm{PSp}(4, 2)'$ or $\Omega(n, q)$ (with nq odd). Since $\mathrm{PSp}(4, 2)' \cong \mathrm{A}_6$, we have $T \neq \mathrm{PSp}(4, 2)'$ by Lemma 5.3. By [17, Table 3], $|T : M|$ has a divisor $q + 1$. Since $|T : M|$ is odd, q is even, and hence $q = 2$ or 4. Then $T = \mathrm{PSp}(n, q)$. If $nf \geq 12$ then $|T|$ has a divisor $(2^6 - 1)^2$, and so $|T|$ has a divisor 7^2, a contradiction. It follows that either $q = 2$ and $n \in \{6, 8, 10\}$ or $(n, q) = (4, 4)$. By [17, Table 3], $|T : M| = \frac{q^n-1}{q-1}$ or $\frac{(q^n-1)(q^{n-2}-1)}{(q^2-1)(q-1)}$, and M is the stabilizer of some totally isotropic 1-subspace or 2-subspace, respectively. For $(n, q) = (6, 2)$, we have $|T : M| = 3^2 \cdot 7$ or $3^2 \cdot 5 \cdot 7$, which is not square-free. Assume that $(n, q) = (8, 2)$. Then $|T : M| = 3 \cdot 5 \cdot 17$ and $M \cong 2^{1+6}{:}\mathrm{PSp}(6, 2)$. By Lemma 5.2, $\mathrm{PSp}(6, 2)$ has a maximal subgroup of odd and square-free index, which is impossible. Assume that $(n, q) = (10, 2)$. Then $M \cong 2^9{:}\mathrm{PSp}(8, 2)$ or $2^{3+12}{:}(\mathrm{S}_3 \times \mathrm{PSp}(6, 2))$. Again by Lemma 5.2, $\mathrm{PSp}(8, 2)$ or $\mathrm{PSp}(6, 2)$ has a maximal subgroup of odd and square-free index, which is impossible. Thus we have $T = \mathrm{PSp}(4, 4)$.

Suppose that $T = \mathrm{P}\Omega^{\pm}(2m, q)$, where $n = 2m \geq 8$. By [17, Table 3], $|T : M|$ has a divisor $q^{m-i} + 1$, where $i \in \{0, 1, 2, 3\}$. Since $|T : M|$ is odd, q is even, and hence $q = 2$ or 4. If $m \geq 5$ then $|T|$ has a divisor $(q^4 - 1)^2$, and so $q = 2$; otherwise, $|T|$ is divisible by 17^2, a contradiction. If $q = 2$ and $m > 6$ then $|T|$ is divisible by $(q^6 - 1)^2 = 3^4 \cdot 7^2$, a contradiction. It follows that (m, q) is one of $(4, 4)$, $(4, 2)$, $(5, 2)$ and $(6, 2)$. Calculation shows that $\mathrm{P}\Omega^{\pm}(8, 4)$ and $\mathrm{P}\Omega^{-}(12, 2)$ have order divisible by 5^3, and $\mathrm{P}\Omega^{+}(12, 2)$ has order divisible by 7^2. By the observation (5.a), we conclude that T is one of $\mathrm{P}\Omega^{\pm}(8, 2)$ and $\mathrm{P}\Omega^{\pm}(10, 2)$. Checking the maximal subgroups of T in the Atlas [4], since $|T : M|$ is odd and square-free, one of the following occurs: $T = \mathrm{P}\Omega^{-}(8, 2)$ and $M \cong 2^6{:}\mathrm{PSU}(4, 2)$, $T = \mathrm{P}\Omega^{-}(10, 2)$ and $M \cong 2^{1+12}{:}(\mathrm{S}_3 \times \mathrm{PSU}(4, 2))$, $T = \mathrm{P}\Omega^{+}(10, 2)$ and $M \cong 2^8{:}\mathrm{P}\Omega^{+}(8, 2)$. Then, by Lemma 5.2, we conclude that either $\mathrm{PSU}(4, 2)$ or $\mathrm{P}\Omega^{+}(8, 2)$ has a maximal subgroup of odd and square-free index, which is impossible.

By the above argument, all possible candidates for T are desired as in this lemma. In particular, $|G : T|$ is indivisible by 5. By Lemma 5.1, Γ is T-semisymmetric, and the lemma follows. □

Lemma 5.8. *Let $T = \mathrm{PSL}(2, p)$ for a prime p. Then Γ is symmetric, $11 \leq p \equiv \pm 3 \pmod 8$ and $p \equiv \pm 1 \pmod 5$, and $T_v \cong \mathrm{D}_{20}$ or A_5.*

Proof. By Lemma 5.3, $T \not\cong \mathrm{A}_5$. Since $|T|$ is divisible by 5, we have $p \geq 11$ and $p \equiv \pm 1 \pmod 5$. For $v \in V$, since $|T : T_v|$ is odd, $|T_v|$ is divisible by 20. Check the subgroups of $\mathrm{PSL}(2, p)$, refer to [12, II.8.27]. We conclude that either $T_v \cong \mathrm{A}_5$, and so $p \equiv \pm 3 \pmod 8$, or T_v is contained in a maximal subgroup isomorphic to $\mathrm{D}_{p+\varepsilon}$, where $\varepsilon = \pm 1$ such that $p + \varepsilon$ is divisible by 10. Let $\{u, w\}$ be an edge of Γ.

Assume that one of T_u and T_w, say T_u, is soluble. Then T_u is a dihedral group. Suppose that $T_w \cong \mathrm{A}_5$. Since Γ is T-semisymmetric, $T_{uw} = T_u \cap T_w$ has index 5 in both T_u and T_w. It follows that $T_u \cap T_w \cong \mathrm{A}_4$; however T_u has no subgroup isomorphic to A_4, a contradiction. Thus T_w is also soluble. Take a positive integer t such that $p+\varepsilon$ is divisible by 2^t but not by 2^{t+1}. Then $t \geq 2$ and, by Lemma 2.3 and (4.b), $T_u \cong T_w \cong \mathrm{D}_{2^t \cdot 5}$. Thus T_{uw} is a Sylow 2-subgroup of both T_u and T_w, which is isomorphic to D_{2^t}. If $t \geq 3$ then T_u, T_w and T_{uw} have the same center isomorphic to $\mathbb{Z}_2$, which contradicts Lemma 2.2. Thus $t = 2$, and then $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$. In particular, a Sylow 2-subgroup of T has order 4. Enumerating the Sylow 2-subgroups of T, we conclude a Sylow 2-subgroup of T is exactly contained in three distinct subgroups isomorphic to D_{20}, say T_u, T_w and H. Let $N = \mathbf{N}_{\mathrm{PGL}(2,p)}(T_{uw})$. Then N has an action on $\{T_u, T_w, H\}$ by conjugation, where the kernel say K contains T_{uw}. Noting that $N \cong \mathrm{S}_4$ and T_u is self-normalized in T, it follows that $K = T_{uw}$. Then, noting that $\mathbf{N}_T(T_{uw}) \cong \mathrm{A}_4$, we may choose an involution $\sigma \in N \setminus T$ such that $T_u^\sigma = T_w$. Define

$$\theta : V \to V,\, u^x \mapsto w^{x^\sigma},\, w^x \mapsto u^{x^\sigma}.$$

It is easily shown that θ is an automorphism of Γ, and θ interchanges U and W. Then Γ is vertex-transitive (see also [8, Lemma 2.6]), and so Γ is symmetric.

Assume that $T_u \cong T_w \cong \mathrm{A}_5$. Then $T_{uw} \cong \mathrm{A}_4$. Note that all subgroups isomorphic to A_4 are conjugate in T. (In fact, each A_4 is the normalizer of some Sylow 2-subgroup of T.) Enumerating the subgroups isomorphic to A_4, we conclude two conjugations of A_5 under T can not intersect at a subgroup of order 12, and each subgroup A_4 is exactly contained in two subgroups A_5. It follows that T_u and T_w are not conjugate in T. Noting that $\mathbf{N}_{\mathrm{PGL}(2,p)}(T_{uw}) \cong \mathrm{S}_4$, we conclude that $T_u^\sigma = T_w$ for some involution $\sigma \in \mathbf{N}_{\mathrm{PGL}(2,p)}(T_{uw}) \setminus T$. Similarly as above, there is an automorphism of Γ interchanging U and W. Thus Γ is symmetric, and the lemma follows. □

Lemma 5.9. *$T \neq \mathrm{PSL}(2, 25)$.*

Proof. Suppose that $T = \mathrm{PSL}(2, 25)$. Let $\{u, w\}$ be an edge of Γ. Since Γ is T-semisymmetric, $|T_u| = |T_w|$. Checking the subgroups of $\mathrm{PSL}(2, 25)$, since $|T : T_u|$ is odd, we conclude that $T_u \cong T_w \cong \mathrm{S}_5$, and then $T_u \cap T_w \cong \mathrm{S}_4$. In $\mathrm{PSL}(2, 25)$, there are two conjugacy classes of subgroups isomorphic to S_5 and two conjugacy classes of subgroups isomorphic to S_4. It follows that two distinct subgroups S_5 can not intersect at a subgroup S_4, a contradiction. □

Lemma 5.10. *$T \neq \mathrm{PSU}(3, 4)$ or $\mathrm{PSL}(3, 16)$.*

Proof. Let $v \in V$, and take a maximal subgroup M of T with $T_v \leq M$.

Suppose that $T = \mathrm{PSU}(3, 4)$. Checking the maximal subgroups of T in the Atlas [4], since $|T : M|$ is odd and square-free, we have $M \cong 2^{2+4}{:}\mathbb{Z}_{15}$ and $|T : M| = 65$. It follows that T_v has a unique Sylow 2-subgroup. Since Γ is T-semisymmetric, for an edge $\{u, w\}$ of Γ, we have $5 = |T_u : T_{uw}| = |T_w : T_{uw}|$. Thus T_{uw} contains the unique Sylow 2-subgroup of T_u and the unique Sylow 2-subgroup of T_w. Then T_u and T_w have a nontrivial normal subgroup in common, which contradicts Lemma 2.2.

Suppose that $T = \mathrm{PSL}(3, 16)$. Then $|T| = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$, and hence $|T : T_v| = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ or $5 \cdot 7 \cdot 13 \cdot 17$. By [17, Table 3], we have $|T : M| = 3 \cdot 7 \cdot 13$ and $M \cong 2^8{:}(5 \times \mathrm{PSL}(2, 16))$. It follows that $|T : T_v| = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$, and T_v has index 85 in M. Then $T_v \cong 2^8{:}(5 \times 2^4{:}3)$ or $2^8{:}(2^4{:}15)$. In particular, T_v has a unique Sylow 2-subgroup. Then we have a similar contradiction as above. □

Let $\mathbb{F}_q^n$ be the vector space over $\mathbb{F}_q$ with dimension n. An (l,m)-flag in $\mathbb{F}_q$ is an ordered pair $(\mathbf{u},\mathbf{v})$ of subspaces with $1 \le l = \dim(\mathbf{u}) < \dim(\mathbf{w}) = m < n$ and $\mathbf{u} \subset \mathbf{v}$.

Lemma 5.11. *$T \ne \mathrm{PSL}(4,4)$ or $\mathrm{PSL}(5,4)$.*

Proof. Let $T = \mathrm{PSL}(n,q)$ with $q = 4$ and $n \in \{4,5\}$. Let $v \in V$. Since $|T : T_v|$ is odd and square-free, considering the maximal subgroups of T which contain T_v, it follows from [17, Table 3] that T_v is contained in the stabilizer of some subspace of $\mathbb{F}_q^n$. Thus, for convenience, we use boldface $\mathbf{v}$ to denote a subspace of $\mathbb{F}_q^n$ with $T_v \le T_{\mathbf{v}}$.

Case 1. Suppose that $T = \mathrm{PSL}(4,4)$. Then $|T| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$, and so $|T : T_v| = 3 \cdot 5 \cdot 7 \cdot 17$ or $5 \cdot 7 \cdot 17$. By [17, Table 3] and [2, Table 8.8], we have $\dim(\mathbf{v}) \in \{1,2,3\}$, and $T_{\mathbf{v}} \cong 2^6{:}\mathrm{GL}(3,4)$ or $2^8{:}(\mathrm{SL}(2,4) \times \mathrm{SL}(2,4)){:}3$.

Assume that $\dim(\mathbf{v}) = 1$ or 3. Consider the action of $T_{\mathbf{v}}$ on the quotient space $\mathbb{F}_4^4/\mathbf{v}$ or $\mathbf{v}$, respectively. Then we have a surjective homomorphism $\phi : T_{\mathbf{v}} \to \mathrm{GL}_3(4)$, and $\ker\phi \cong \mathbb{Z}_2^6$. Since T_v is a $\{2,3,5\}$-group and $|T_{\mathbf{v}} : T_v|$ is odd and square-free, $\phi(T_v)$ is a $\{2,3,5\}$-subgroup of $\mathrm{GL}_3(4)$ with odd and square-free index. Computation using GAP shows that $\phi(T_v) \lesssim 3 \times 2^4{:}\mathrm{GL}(2,4)$, that is, $\phi(T_v)$ is contained in the stabilizer in $\mathrm{GL}(3,4)$ of some 1 or 2-dimensional subspace. It follows that T_v is contained in the stabilizer in T of some $(1,2)$, $(1,3)$ or $(2,3)$-flag. It is easily shown that the numbers of $(1,2)$, $(1,3)$ and $(2,3)$-flags are all equal to $3 \cdot 5 \cdot 7 \cdot 17$. Thus $|T : T_v|$ is divisible by $3 \cdot 5 \cdot 7 \cdot 17$, yielding $|T : T_v| = 3 \cdot 5 \cdot 7 \cdot 17$. Then the action of T on the T-orbit containing v is equivalent to the action of T on the set of $(1,2)$, $(1,3)$ or $(2,3)$-flags.

Assume that $\dim(\mathbf{v}) = 2$. Then $T_{\mathbf{v}} \cong 2^8{:}(\mathrm{SL}(2,4) \times \mathrm{SL}_2(4)){:}3$. Note that $|T_v|$ is indivisible by 5^2, in particular, $T_{\mathbf{v}} \ne T_v$. Considering the action of $T_{\mathbf{v}}$ on $\mathbb{F}_4^4/\mathbf{v}$ or $\mathbf{v}$, we conclude that T_v is contained in the stabilizer in T of some $(1,2)$ or $(2,3)$-flag. It follows that $|T : T_v| = 3 \cdot 5 \cdot 7 \cdot 17$, and the action of T on the T-orbit containing v is equivalent to the action of T on the set of $(1,2)$ or $(2,3)$-flags.

By the above argument, we may let U the set of (i,j)-flags and W be the set of (i',j')-flags, where $1 \le i < j < 4$ and $1 \le i' < j' < 4$. Let $u \in U$ and $w \in W$ with $\{u,w\} \in E$. Suppose that $i = i' = 1$. Then we may choose 1-dimensional subspaces $\mathbf{u}$ and $\mathbf{w}$ of $\mathbb{F}_4^4$ with $T_u \le T_{\mathbf{u}}$ and $T_w \le T_{\mathbf{w}}$. Then $T_{uw} \le T_{\mathbf{u}} \cap T_{\mathbf{w}}$ and, since $T = \langle T_u, T_w\rangle$, we have $\mathbf{u} \ne \mathbf{w}$. Noting T acts 2-transitively on the set of 1-dimensional subspaces, we have $|T_{\mathbf{u}} : (T_{\mathbf{u}} \cap T_{\mathbf{w}})| = \frac{q^n-1}{q-1} - 1 = 84$. It follows that $|T_{\mathbf{u}} : T_{uw}|$ is even. Noting that $|T_{\mathbf{u}} : T_{uw}| = |T_{\mathbf{u}} : T_u||T_u : T_{uw}| = 5|T_{\mathbf{u}} : T_u|$, we know that $|T_{\mathbf{u}} : T_u|$ is even, a contradiction. For $j = j' = 3$, we get a similar contradiction. Thus, without of generality, we let u be a $(1,2)$-flag and w be a $(2,3)$-flag. Then we may choose 2-dimensional subspaces $\mathbf{u}$ and $\mathbf{w}$ of $\mathbb{F}_4^4$ with $T_u \le T_{\mathbf{u}}$ and $T_w \le T_{\mathbf{w}}$. It is easily shown that $T_{\mathbf{u}}$ has 3-orbits on the set of 2-dimensional subspaces, which have length 1, 100 and 256. Note that $\mathbf{u} \ne \mathbf{w}$, for otherwise, $T = \langle T_u, T_w\rangle \le T_{\mathbf{u}} < T$. It follows that $|T_{\mathbf{u}} : (T_{\mathbf{u}} \cap T_{\mathbf{w}})|$ is even, which yields a similar contradiction as above.

Case 2. Suppose that $T = \mathrm{PSL}(5,4)$. By [17, Table 3] and [2, Table 8.18], we have $\dim(\mathbf{v}) \in \{1,2,3,4\}$, and $T_{\mathbf{v}} \cong 2^8{:}\mathrm{GL}(4,4)$ or $2^{12}{:}(\mathrm{SL}(2,4) \times \mathrm{SL}(3,4)){:}3$. Let N be a normal subgroup of $T_{\mathbf{v}}$ with $N \cong 2^8{:}\mathrm{SL}(4,4)$ or $2^{12}{:}(\mathrm{SL}(2,4) \times \mathrm{SL}(3,4))$, respectively. Then $|(NT_v) : T_v|$ is a divisor of $|T_{\mathbf{v}} : T_v|$. Noting that $|N||T_v| = |NT_v||N \cap T_v|$, we have $|N : (N \cap T_v)| = |(NT_v) : T_v|$, and so $|N : (N \cap T_v)|$ is a divisor of $|T_{\mathbf{v}} : T_v|$. In particular, $|N : (N \cap T_v)|$ is odd and square-free. Note that every Sylow 5-subgroup of $T_{\mathbf{v}}$ is contained in N, and each Sylow 5-subgroup of T_v is contained in some Sylow

5-subgroup of $T_{\mathbf{v}}$. It follows that $|N\cap T_v|$ is divisible by 5. Clearly, $|N\cap T_v|$ is indivisible by 5^2, see Lemma 2.1.

Let K be the maximal soluble normal subgroup of N. Then $(N\cap T_v)K/K$ is a subgroup of N/K with odd and square-free index. Considering the order of N/K, we have $|(N\cap T_v)K/K| = 2^l\cdot 3^m\cdot 5$ for some positive integers l and m. Assume that $N/K\cong \mathrm{PSL}(4,4)$. Then, by the argument in Case 1, $(N\cap T_v)K/K$ is isomorphic to the stabilizer in $\mathrm{PSL}(4,4)$ of some $(1,2)$, $(1,3)$ or $(2,3)$-flag. In particular, in this case, $(N\cap T_v)K/K$ has a composition factor A_5. Assume that $N/K\cong \mathrm{PSL}(2,4)\times\mathrm{PSL}(3,4)$. Using GAP program, we search the subgroups of $\mathrm{PSL}(2,4)\times\mathrm{PSL}(3,4)$ with odd and square-free index. It follows that $(N\cap T_v)K/K$ has a composition factor A_5.

Noting that $N\cap T_v$ is normal in T_v, by the above argument, T_v has a composition factor A_5. Recall that Γ is T-semisymmetric. By Lemma 2.1, $T_v^{[1]}$ is a $\{2,3\}$-group, and then $T_v^{\Gamma(v)}$ has a composition factor A_5. Thus $T_v^{\Gamma(v)}\cong \mathrm{A}_5$ or S_5. Let P is a Sylow 2-subgroup of T_v. It follows from [22, Theorem 2] that either $|P|\le 2^{18}$ or $|P|\ge 2^{24}$, which is impossible as $|P|=2^{20}$. This completes the proof. □

Theorem 5.12. *Let $\Gamma=(V,E)$ be a connected G-semisymmetric graph of square-free order and valency 5. Assume that G is an almost simple group with socle $\mathrm{soc}(G)=T$. Then Γ is T-semisymmetric, and one of the following holds.*

(1) *Γ is the incidence graph of the generalized hexagon associated with $\mathrm{G}_2(4)$;*
(2) *Γ is symmetric and isomorphic to the incidence graph of the projective plane $\mathrm{PG}(2,4)$ over $\mathbb{F}_4$;*
(3) *Γ is symmetric and isomorphic the incidence graph of the generalized quadrangule associated with $\mathrm{PSp}(4,4)$;*
(4) *$T=\mathrm{PSL}(2,p)$ and Γ is symmetric, where p is a prime with $11\le p\equiv \pm 3 \pmod 8$ and $p\equiv\pm 1 \pmod 5$.*

Proof. By Lemmas 5.3 to 5.11, Γ is T-semisymmetric, and either one of (1) and (4) of this theorem holds or T is one of $\mathrm{PSL}(3,4)$ and $\mathrm{PSp}(4,4)$. Let $v\in V$, and take a maximal subgroup M of T with $T_v\le M$.

Let $T=\mathrm{PSL}(3,4)$. Checking the maximal subgroups of $\mathrm{PSL}(3,4)$ in the Atlas [4], we have $M\cong 2^4{:}\mathrm{A}_5$, which is the stabilizer of a point or a line of the projective plane $\mathrm{PG}(2,4)$. It is easily shown that M has no subgroup with odd index and order divisible by 5, and then $T_v=M$. Since Γ has valency 5, it is the incidence graph of $\mathrm{PG}(2,4)$. The inverse transpose automorphism of $\mathrm{PSL}(3,4)$ induces an automorphism of Γ which interchanges U and W, and so Γ is symmetric. Then part (2) follows.

Let $T=\mathrm{PSp}(4,4)$. Then, by the Atlas [4], $M\cong 2^6{:}(3\times\mathrm{A}_5)$, which is the stabilizer of a point or a line of the self-dual generalized quadrangle $\mathrm{GQ}(4)$. Noting that $|G:M|=85$, it follows from (5.b) that $T_v=M$ or $T_v\cong 2^6{:}\mathrm{A}_5$. Confirmed by GAP, in the group $\mathrm{PSp}(4,4)$, any two subgroups isomorphic to $2^6{:}\mathrm{A}_5$ do not intersect at a subgroup of index 5, and any two conjugate subgroups isomorphic to $2^6{:}(3\times\mathrm{A}_5)$ do not intersect at a subgroup of index 5. Therefore, $T_v=M$, and then Γ is the incidence graph of $\mathrm{GQ}(4)$. Moreover, the duality automorphism of $\mathrm{PSp}(4,4)$ induces an automorphism of Γ which interchanges U and W. Thus part (3) of this theorem follows. □

6. The proof of Theorem 1.1

Let $\mathit{\Gamma} = (V, E)$ be a connected semisymmetric graph of square-free order and valency 5 with bipartition $V = U \cup W$. Noting that $\mathsf{K}_{5,5}$ is symmetric, we have $\mathit{\Gamma} \ncong \mathsf{K}_{5,5}$. Thus $G := \mathsf{Aut}\mathit{\Gamma}$ is faithful on both U and W. If G is soluble then, by Theorem 4.2, $\mathit{\Gamma}$ is a graph described as in Construction 3.2.

Assume that G is insoluble. Let N be a normal subgroup of G, which is maximal among the normal subgroups of G intransitive on both U and W. We consider the quotient $\mathit{\Gamma}_N$, and use the notation given at the end of Section 2. By [10, Theorem 1.1], N is semiregular on V, $\overline{G} \cong G/N$, $\mathit{\Gamma}_N$ is $\overline{G}$-semisymmetric and of valency 5, and either

(i) $\mathit{\Gamma}_N \cong \mathsf{K}_{5,5}$, and $|U| = 5|N| > 5$; or
(ii) $|\overline{U}| = |\overline{W}| > 5$, $\overline{G}$ is faithful on both $\overline{U}$ and $\overline{W}$, and $\overline{G}$ is quasiprimitive on at least one of $\overline{U}$ and $\overline{W}$.

Since G is insoluble, by Lemma 4.1, only (ii) occurs.

Without loss of generality, we let $\overline{G}$ be quasiprimitive on $\overline{U}$, that is, every non-trivial normal subgroup of $\overline{G}$ is transitive on $\overline{U}$. Take a maximal $\overline{G}$-invariant partition $\mathcal{B}$ of $\overline{U}$. Then $|\mathcal{B}|$ is square-free, and $\overline{G}$ acts faithfully and primitively on $\mathcal{B}$. It follows from [17] that $\overline{G}$ is an almost simple group. If $N = 1$ then our theorem follows from Lemma 5.6 and Theorem 5.12. Thus, to complete the proof, we next show $N = 1$.

By Theorem 5.12, $\mathit{\Gamma}_N$ is $\mathrm{soc}(\overline{G})$-semisymmetric, and $\mathrm{soc}(\overline{G}) \cong \mathrm{G}_2(4)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(4,4)$ or $\mathrm{PSL}(2,p)$, where p is a prime with $11 \le p \equiv \pm 3 \pmod 8$ and $p \equiv \pm 1 \pmod 5$. Let $N \le X \le G$ with $X/N \cong \mathrm{soc}(\overline{G})$. Then $\mathit{\Gamma}$ is X-semisymmetric, and $X = N \times T$ by [15, Theorem 30], where T is a simple subgroup of X. In particular, $T \cong \mathrm{soc}(\overline{G})$. For $v \in V$, noting that $X_{\bar{v}} = NT \cap X_{\bar{v}} = NT_{\bar{v}}$, we have $\mathrm{soc}(\overline{G})_{\bar{v}} \cong X_{\bar{v}}/N = NT_{\bar{v}}/N \cong T_{\bar{v}}$.

Since X is normal in G and T is characteristic in X, we know that T is normal in G. Since T has even order, T is not semiregular on V. By Lemma 2.4, T is transitive on one of U and W, say on U without loss of generality. Let $u \in U$. Then $T_{\bar{u}}$ is transitive on the N-orbit $\bar{u}$ on U. Since N centralizes $T_{\bar{u}}$, it implies that $T_{\bar{u}}$ has a normal subgroup of index $|N|$, refer to [5, Theorem 4.2A]. Since N is semiregular on U, the order of N is odd and square-free. If $T \cong \mathrm{PSL}(2,p)$ then $T_{\bar{u}} \cong \mathrm{A}_5$ or D_{20} by Lemma 5.8, and so $T_{\bar{u}}$ has no proper normal subgroup of odd index, yielding $N = 1$, as desired.

Assume next that $T \cong \mathrm{soc}(\overline{G}) \cong \mathrm{G}_2(4)$, $\mathrm{PSL}(3,4)$ or $\mathrm{PSp}(4,4)$. Applying Lemma 5.6 and Theorem 5.12 to the pair $(\mathrm{soc}(\overline{G}), \mathit{\Gamma}_N)$, one of the following cases occurs:

(1) $T \cong \mathrm{G}_2(4)$ and $T_{\bar{u}} \cong 2^{2+8}{:}(3 \times \mathrm{A}_5)$ or $2^{4+6}{:}(3 \times \mathrm{A}_5)$;
(2) $T \cong \mathrm{PSL}(3,4)$ and $T_{\bar{u}} \cong 2^4{:}\mathrm{A}_5$;
(3) $T \cong \mathrm{PSp}(4,4)$ and $T_{\bar{u}} \cong 2^6{:}(3 \times \mathrm{A}_5)$.

For each of these cases, we have $|N| = 1$ or 3. Then T has at most three orbits on W as T is transitive on $\overline{W}$. Thus T is transitive on W by [10, Lemma 5.5], and so $\mathit{\Gamma}$ is T-semisymmetric. Applying Lemma 5.6 and Theorem 5.12 to the pair $(T, \mathit{\Gamma})$, we conclude that $T_u \cong T_{\bar{u}}$. Noting that $|T : T_u| = |U| = |N||T : T_{\bar{u}}|$, we have $N = 1$, as desired. This completes the proof of Theorem 1.1.

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