On the eigenvalues of Laplacian ABC-matrix of graphs

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Abstract. For a simple graph G, the Laplacian ABC-matrix is defined by $\tilde{\mathrm{L}}(G) = \overline{D}(G) - \tilde{\mathrm{A}}(G)$, where $\overline{D}(G)$ is the diagonal matrix of ABC-degrees and $\tilde{\mathrm{A}}(G)$ is the ABC-matrix of G. The eigenvalues of the matrix $\tilde{\mathrm{L}}(G)$ are called the Laplacian ABC-eigenvalues of G. In this paper, we solve the problem of characterization of connected graphs having exactly three distinct Laplacian ABC-eigenvalues. We also introduce the concept of trace norm of the matrix $\tilde{\mathrm{L}}(G) - \frac{tr(\tilde{\mathrm{L}}(G))}{n} I$, called the Laplacian ABC-energy of G. We obtain some upper and lower bounds for the Laplacian ABC-energy and characterize the extremal graphs which attain these bounds.

Keywords: Adjacency matrix; ABC-matrix; Laplacian ABC-matrix; distinct eigenvalues; extremal graphs.

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1 Introduction

In this paper, we consider only connected, simple and finite graphs. A graph $G\big(V(G), E(G)\big)$ (or simply G) consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$. The number of elements in $V(G)$ is *order* n and the number of elements in $E(G)$ is *size* m of G. If u is adjacent to v, we denote it by $v \sim u$. The neighbourhood of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to $v \in V(G)$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$ (or simply d_v), is the cardinality of $N(v)$. A graph is *r-regular*, if $d_v = r$, for each $v \in V(G)$. The length of a shortest path between two vertices is known as the distance and the maximum distance between any pair of vertices is the diameter of G. For other graph theoretic notations and definitions, we follow [2].

The adjacency matrix of a graph G, denoted by $A(G)$, is defined as

$$A(G) = \begin{cases} 1 & \text{if} \quad v_i \sim v_j \\ 0 & \textit{otherwise.} \end{cases}$$

The adjacency matrix $A(G)$ is real symmetric, so its eigenvalues are real, denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and are known as the adjacency spectrum (or spectrum) of G. The energy of the adjacency matrix of G is defined by

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

The spectral parameter $\mathcal{E}(G)$ is a widely studied parameter and has its origin in theoretical chemistry, where it helps in approximating the π-electron energy of hydrocarbons. For more about the energy $\mathcal{E}(G)$ of a graph G, we refer to [19, 21]. More literature about the adjacency matrix $A(G)$ can be found in [1, 7].

The ABC-matrix of a graph G is a square matrix of order n and is defined as

$$\tilde{\mathrm{A}}(G) = (a_{ij})_{n\times n} = \begin{cases} \sqrt{\dfrac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} & \text{if} \quad v_i \sim v_j \\ 0 & \textit{otherwise.} \end{cases}$$

This matrix was introduced in [9] and is related to the topological index: atom-bond connectivity (ABC-index for short) of a graph G. The ABC-index is a degree based topological index [10] and is defined to be the sum of weights $\sqrt{\dfrac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}$ over all edges $v_i v_j$ of a graph G, that is,

$$ABC(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}.$$

In [10], the ABC-index was shown to be correlated to the heat formation of alkanes. Gutman et al. [16] proved that the ABC-index can reproduce the heat of formation with an accuracy comparable to that of high-level ab intio and DFT (MP2, B3LYP) quantum chemical calculations. More mathematical literature about ABC-index can be found in [3, 8, 12, 15].

Let $\vartheta_1 \geq \vartheta_1 \geq \cdots \geq \vartheta_n$ be the ABC-eigenvalues of G, where ϑ_1 is called the ABC-spectral radius of G. Then the ABC-energy of G is defined by

$$E_{ABC}(G) = \sum_{i=1}^{n} |\vartheta_i|.$$

The ABC-spectral parameters like energy were studied in [4], ABC-spectral radius in [14], and other spectral properties in [5, 6, 13, 18, 20].

For $v_i \in V(G)$, let $\overline{d}_{v_i} = \sum\limits_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i}d_{v_j}}}$ be the ABC-degree of a vertex v_i. From now onwards, we simply write $\overline{d}_i$ instead of $\overline{d}_{v_i}$. We observe that $\overline{d}_i$ is the same as the i-th row sum of the ABC-matrix. The Laplacian ABC-matrix of G introduced in [22] is defined as $\tilde{\mathrm{L}}(G) = \overline{D}(G) - \tilde{\mathrm{A}}(G)$, where $\overline{D}(G) = \mathrm{diag}(\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n)$ is the diagonal matrix of ABC-degrees of G. Clearly, each row sum of $\tilde{\mathrm{L}}(G)$ is zero, so 0 is its one of the eigenvalue. Besides, $\tilde{\mathrm{L}}(G)$ is a real symmetric positive semi-definite matrix, its eigenvalues are called the Laplacian ABC-eigenvalues of G, denoted by $\xi, i = 1, 2, \ldots, n$. Since each ξ_i is real so we can arrange them as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} > \xi_n = 0$, where ξ_1 is called the Laplacian ABC-spectral radius. The multiset of eigenvalues of $\tilde{\mathrm{L}}(G)$ is known as the Laplacian ABC-spectrum of G. If an eigenvalue ξ of $\tilde{\mathrm{L}}(G)$ occurs with multiplicity l, then we represent it as $\xi^{[l]}$. Yang, Deng and Li [22] studied various properties of the matrix $\tilde{\mathrm{L}}(G)$, which includes characterization of graphs with one and two distinct Laplacian ABC-eigenvalues of graphs, bounds for the largest Laplacian ABC-eigenvalue and the smallest non-zero Laplacian ABC-eigenvalue. In this paper, we carry forward the problem of characterizing graphs with three distinct Laplacian ABC-eigenvalues.

We denote the complete graph by K_n, the complete bipartite graph by $K_{a,b}$, the path graph by P_n, the cycle graph by C_n, etc. For other undefined notation and terminology from spectral graph theory, we refer to [7].

The rest of the paper is organized as follows. In Section 2, we characterize the graphs with exactly three distinct Laplacian ABC-eigenvalues. In Section 3, we introduce the concept of Laplacian ABC-energy of a graph. We obtain some upper and lower bounds for the Laplacian ABC-energy and characterize the extremal graphs for these bounds.

2 Graphs with three distinct Laplacian ABC-eigenvalues

In this section, we first mention some known results about the Laplacian ABC-eigenvalues. We obtain the Laplacian ABC-spectrum for some well-known families of graphs. Further, we completely solve the problem of characterization of graphs with three distinct Laplacian ABC-eigenvalues.

A natural problem in the spectral of theory of graph matrices is the following problem.

Problem 1 *Let G be a connected graph of order $n \geq 2$ and let $M(G)$ be a graph matrix associated to G. Let k, where $1 \leq k \leq n$, be a positive integer. Characterize the graphs having exactly k*

distinct $M(G)$-eigenvalues.

This problem has been considered for the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix, the normalized Laplacian matrix, the distance matrix, etc, for small value of k. In fact, various papers can be found in the literature regarding this problem for the mentioned matrices when $k \leq 4$. For the Laplacian ABC-matrix, this problem was considered by Yang, Deng and Li in [22] for $k = 1$ and 2 and their result is given below.

Lemma 2.1 ([22]) *Let G be a connected graph of order $n \geq 2$. Then the following holds.*

(i) *G has one distinct Laplacian ABC-eigenvalue if and only if $G \cong K_2$.*

(ii) *G has two distinct Laplacian ABC-eigenvalue if and only if $G \cong K_n$.*

In the rest of this section, we aim to solve Problem (1) for $k = 3$, and for which we need some basic properties of the Laplacian ABC-eigenvalues.

Let M be a matrix partitioned into blocks and let Q be the matrix whose entries are the average row sums (column sums) of the blocks of M. The matrix Q is known as the quotient matrix and if the row sums (columns sums) of each block in M are some constants, then the partition is regular (equitable) and we say Q is a regular (equitable) quotient matrix (see [1]). In general, the eigenvalues of M interlace the eigenvalues of Q, however for regular partitions, each eigenvalue (see [1,7]) of Q is an eigenvalue of M.

Any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as function defined on $V(G)$ which relates every v_i to x_i, that is $X(v_i) = x_i$ for all $i = 1, 2, \ldots, n$. Also, it is easy to see that

$$X^T\tilde{\mathrm{L}}(G)X = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}\,\left(x_i - x_j\right)^2 = \sum_{i=1}^{n} \overline{d}_i x_i^2 - 2\sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} x_i x_j,$$

where $\overline{d}_i = \overline{d}_{v_i} = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i}+d_{v_j}-2}{d_{v_i} d_{v_j}}}$. A real number ξ is a Laplacian ABC-eigenvalue with its associated eigenvector X if and only if $X \neq 0$ and for every $v_i \in V(G)$, we have

$$\xi X(v_i) = \sum_{v_j \in N(v_i)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}\big(X(v_i) - X(v_j)\big), \tag{2.1}$$

or equivalently

$$\xi X(v_i) - \overline{d}_i X(v_i) = -\sum_{v_j \in N(v_i)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}\; X(v_j), \tag{2.2}$$

Equations (2.1) and (2.2) are the (ξ, X)-eigenequations for the Laplacian ABC-matrix.

A subset S of the vertex set $V(G)$ is said to be an independent set if no two vertices of S are adjacent in G. It is said to be a clique if every two vertices of S are adjacent in G. The cardinality of largest possible independent set in G is called independence number of G and the cardinality of a largest possible clique in G is called clique number of G.

Next, we have a result which helps us in finding some Laplacian ABC-eigenvalues of G, provided G has some special structure.

Theorem 2.2 *Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and let $\mathcal{I} = \{v_1, v_2, \dots, v_p\}$ be a subset of G such that $N(v_i) = N(v_j)$, for all $i, j \in \{1, 2, \dots, p\}$. Then the following statements hold.*

- **(i)** *If $\mathcal{I}$ is an independent set of G, then the vertices of $\mathcal{I}$ have the same ABC-degree, say ξ and ξ is a Laplacian ABC-eigenvalue of G with multiplicity at least $p-1$.*
- **(ii)** *If $\mathcal{I}$ is a clique of G, then the vertices of $\mathcal{I}$ have the same ABC-degree, say ξ and $\xi - \frac{\sqrt{2d^*-2}}{d^*}$ is a Laplacian ABC-eigenvalue of G with multiplicity at least $p-1$, where d^* is the degree of $v_i \in \mathcal{I}$.*

Proof. We first suppose that $\mathcal{I}$ is an independent set. Since, $\mathcal{I} = \{v_1, v_2, \dots, v_p\}$ is an independent set, where each vertex sharing the same neighbourhood, therefore we have $d_1 = d_2 = \cdots = d_p$. This last equality gives us $\overline{d}_1 = \overline{d}_2 = \cdots = \overline{d}_p = \xi$. We first index the vertices in the independent set, so that the Laplacian ABC-matrix of G can be written as

$$\tilde{\mathrm{L}}(G) = \left(\begin{array}{cccc|c} \xi & 0 & \dots & 0 & \\ 0 & \xi & \dots & 0 & B_{p\times(n-p)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \xi & \\ \hline & (B_{p\times(n-p)})^T & & & C_{(n-p)\times(n-p)} \end{array}\right).$$

For $i = 2, 3, \dots, p$, let $X_{i-1} = \Big(-1, x_{i2}, x_{i3}, \dots, x_{ip}, \underbrace{0, 0, 0, \dots, 0}_{n-p}\Big)^T$ be the vector in $\mathbb{R}^n$ such that $x_{ij} = 1$ if $i = j$ and 0 otherwise. Suppose $X_1, X_2, \dots, X_{p-1}$ are linearly dependent vectors. Then there exists scalers $a_1, a_2, \dots, a_{p-1}$ not all zero, such that

$$a_1X_1 + a_2X_2 + \cdots + a_{p-1}X_{p-l} = \mathbf{0}.$$

This implies that

$$\left(-\sum_{i=1}^{p-1} a_i, a_1, a_2, \dots, a_{p-1}, 0, 0, \dots, 0\right) = \mathbf{0},$$

and it follows that $a_1 = a_2 = \cdots = a_{p-1} = 0$. Therefore, the vectors $X_1, X_2, \ldots, X_{p-1}$ cannot be linearly dependent. Noting that the rows of B are identical, we see that

$$\tilde{\mathrm{L}}(G)X_1 = \Big(-\xi, \quad \xi, \quad 0, \quad \ldots \quad ,0, \quad 0, \quad \ldots, \quad ,0\Big)^T = \xi X_1.$$

Similarly, we see that $X_2, X_3, \ldots, X_{p-1}$ are the eigenvectors of $\tilde{\mathrm{L}}(G)$ corresponding to eigenvector ξ. This completes the proof of (i).

Next, suppose that $\mathcal{I}$ is a clique in G. Let us label the vertices of G in such a way that the first p vertices are the vertices in $\mathcal{I}$. Under this labelling the Laplacian ABC-matrix of G can be written as

$$\tilde{\mathrm{L}}(G) = \left(\begin{array}{cccc|c} \xi & -\frac{\sqrt{2d^*-2}}{d^*} & \cdots & -\frac{\sqrt{2d^*-2}}{d^*} & \\ -\frac{\sqrt{2d^*-2}}{d^*} & \xi & \cdots & -\frac{\sqrt{2d^*-2}}{d^*} & B_{p\times(n-p)} \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{\sqrt{2d^*-2}}{d^*} & -\frac{\sqrt{2d^*-2}}{d^*} & \cdots & \xi & \\ \hline & & (B_{p\times(n-p)})^T & & C_{(n-p)\times(n-p)} \end{array}\right).$$

Proceeding as in (i) with the same set of eigenvectors, we can verify that $\xi - \frac{\sqrt{2d^*-2}}{d^*}$ is a Laplacian ABC-eigenvalue of G. This completes the proof. ■

Theorem 2.2 helps us to obtain the Laplacian ABC-eigenvalues of some well-known families of graphs. In the following result we mention some of these families.

Proposition 2.3 *Let G be a connected graph of order $n \geq 4$. Then the following statements hold.*

(i) *The Laplacian ABC-spectrum of $K_{1,n-1}$ is*

$$\left\{ n\sqrt{\frac{n-2}{n-1}}, \left(\sqrt{\frac{n-2}{n-1}}\right)^{[n-2]}, 0 \right\}.$$

(ii) *The Laplacian ABC-spectrum of $K_{a,b}$ is*

$$\left\{ (a+b)\sqrt{\frac{a+b-2}{ab}}, \left(b\sqrt{\frac{a+b-2}{ab}}\right)^{[a-1]}, \left(a\sqrt{\frac{a+b-2}{ab}}\right)^{[b-1]}, 0 \right\}.$$

(iii) *The Laplacian ABC-spectrum of the complete split graph $CS_{\omega,n-\omega}$, with clique number ω and independence number $n-\omega$ is*

$$\left\{ 2(n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}, \left(\omega\frac{\sqrt{2n-4}}{n-1} + (n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}\right)^{[\omega-1]}, \right.$$

$$\left((n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}\right)^{[n-\omega-1]}, 0\Big\}.$$

(iv) *The Laplacian ABC-spectrum of $K_n - e$, where e is an edge, is*

$$\left\{(n-2)\frac{\sqrt{2n-4}}{n-1}+\frac{\sqrt{2n-5}}{(n-1)(n-2)}, \left((n-2)\sqrt{\frac{2n-5}{(n-1)(n-2)}}\right)^{[n-4]},\right.$$
$$\left.\left((n-2)\frac{\sqrt{2n-4}}{n-1}+2\sqrt{\frac{2n-5}{(n-1)(n-2)}}\right)^{[n-4]}, 0\right\}.$$

(v) *The Laplacian ABC-spectrum of $K_{1,n-1}+e$ is*

$$\left\{\frac{1}{4}\left(3\sqrt{2}+(n-2)\sqrt{\frac{n-2}{n-1}}\pm\sqrt{\left(3\sqrt{2}+(n-2)\sqrt{\frac{n-2}{n-1}}\right)^2-8n\sqrt{2}\sqrt{\frac{n-2}{n-1}}}\right),\right.$$
$$\left.\left(\sqrt{\frac{n-2}{n-1}}\right)^{[n-4]}, \frac{3}{\sqrt{2}}, 0\right\}.$$

Proof. (i) is a special case of (ii), so we prove (ii). As $K_{a,b}$ has a independent vertices sharing the same neighbourhood with common ABC-degree $b\sqrt{\frac{a+b-2}{ab}}$, so by Theorem 2.2, $b\sqrt{\frac{a+b-2}{ab}}$ is a Laplacian ABC-eigenvalue of $K_{a,b}$ with multiplicity $a-1$. Likewise, b independent vertices have the common neighbourhood with each vertex having same ABC-degree $a\sqrt{\frac{a+b-2}{ab}}$. Thus, by Theorem 2.2, $a\sqrt{\frac{a+b-2}{ab}}$ is a Laplacian ABC-eigenvalue of $K_{a,b}$ with multiplicity $b-1$. Also, 0 is a simple ABC-eigenvalue of $K_{a,b}$. Using the fact that $\xi_1+\xi_2+\cdots+\xi_{n-1}=\sum_{i=1}^{n}\tilde{\mathrm{d}}_i$, we get $(a+b)\sqrt{\frac{a+b-2}{ab}}$, the remaining Laplacian ABC-eigenvalue of $K_{a,b}$.

(iv) is a special case of (iii), we proceed to prove (iii). As ω vertices of $CS_{\omega,n-\omega}$ form the clique and its each vertex shares the same neighbourhood with common ABC-degree $(\omega-1)\frac{\sqrt{2n-4}}{n-1}+(n-\omega)\sqrt{\frac{n+w-3}{\omega(n-1)}}$. So, by Theorem 2.2, it follows that $\omega\frac{\sqrt{2n-4}}{n-1}+(n-\omega)\sqrt{\frac{n+w-3}{\omega(n-1)}}$ is a Laplacian ABC-eigenvalue of $CS_{\omega,n-\omega}$ with multiplicity $\omega-1$. Again, the graph $CS_{\omega,n-\omega}$ has an independent set on $n-\omega$ vertices each sharing the same neighbourhood with common ABC-degree $(n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$ giving by Theorem 2.2 that $(n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$ is a Laplacian ABC-eigenvalue of $CS_{\omega,n-\omega}$ with multiplicity $n-\omega-1$. The other two Laplacian ABC-eigenvalues of $CS_{\omega,n-\omega}$ are 0 and $2(n-\omega)\sqrt{\frac{n+\omega-3}{\omega(n-1)}}$.

(v). As above, we can verify that $\sqrt{\frac{n-2}{n-1}}$ with multiplicity $n-4$ and $\frac{3}{\sqrt{2}}$ are Laplacian ABC-eigenvalues of $K_{1,n-1}+e$. The other three Laplacian ABC-eigenvalues of $K_{1,n-1}+e$ are the eigenvalues of the following equitable quotient matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} + (n-3)\sqrt{\frac{n-2}{n-1}} & -(n-3)\sqrt{\frac{n-2}{n-1}} \\ 0 & -\sqrt{\frac{n-2}{n-1}} & \sqrt{\frac{n-2}{n-1}} \end{pmatrix}.$$

The eigenvalues of above matrix are

$$\left\{0, \frac{1}{4}\left(3\sqrt{2} + (n-2)\sqrt{\frac{n-2}{n-1}} \pm \sqrt{\left(3\sqrt{2} + (n-2)\sqrt{\frac{n-2}{n-1}}\right)^2 - 8n\sqrt{2}\sqrt{\frac{n-2}{n-1}}}\right)\right\}.$$

This completes the proof. ■

The following theorem gives the relation between the eigenvalues of a matrix with the eigenvalues of its principal submatrices, which can be found in [17].

Theorem 2.4 (Interlacing Theorem, [17]) *Let $M \in \mathbb{M}_n$ be a real symmetric matrix. Let A be a principal submatrix of M of order m, $(m \leq n)$. Then the eigenvalues of M and A satisfy the following inequalities*

$$\lambda_{i+n-m}(M) \leq \lambda_i(A) \leq \lambda_i(M), \qquad \text{with} \;\; 1 \leq i \leq m.$$

The following is the main result of this section and gives the characterization of connected graphs with exactly three Laplacian ABC-eigenvalues.

Theorem 2.5 *Let G be a connected graph of order $n \geq 4$. Then the following statements hold.*

- **(i)** *If the diameter of G is at least 3, then there is no graph with three distinct Laplacian ABC-eigenvalues.*
- **(ii)** *If G is a bipartite graph of diameter at most 2, then G has three distinct Laplacian ABC-eigenvalues if and only if G is the star graph or the complete bipartite graph with partite sets of same cardinality.*
- **(iii)** *If G is of diameter at most 2 and multipartite, then G has three distinct Laplacian ABC-eigenvalues if and only if G is the complete t-partite graph $K_{p,p,\ldots,p}$.*
- **(iv)** *If G is unicyclic graph, then G has three distinct Laplacian ABC-eigenvalues if and only if G is either C_4 or C_5.*

Proof. (i) If G is of diameter at least 3, then the path P_4 is its induced subgraph. The principal submatrix of $\tilde{\mathrm{L}}(G)$ corresponding to the vertices v_1, v_2, v_3, v_4 in P_4 is

$$B_1 = \begin{pmatrix} d_1 & -a & -b & -c \\ -a & d_2 & -d & -e \\ -b & -d & d_3 & -f \\ -c & -e & -f & d_4 \end{pmatrix},$$

where for $i = 1, 2, 3, 4,$ d_i, a, b, c, d, e and f are non-negative real numbers. The characteristic polynomial of B_1 is $p_1(x) = x^4 - (d_1 + d_2 + d_3 + d_4)x^3 + x^2t(-a^2 - b^2 - c^2 - d^2 + d_1d_2 + d_1d_3 + d_2d_3 + d_1d_4 + d_2d_4 + d_3d_4 - e^2 - f^2) + x(a^2d_3 + a^2d_4 + 2abd + 2ace + b^2d_2 + b^2d_4 + 2bcf + c^2d_2 + c^2d_3 + d^2d_1 + d^2d_4 + d_1e^2 + d_3e^2 + 2def + d_1f^2 + d_2f^2 - d_1d_2d_3 - d_1d_2d_4 - d_1d_3d_4 - d_2d_3d_4) - a^2d_3d_4 + a^2f^2 - 2abdd_4 - 2abef - 2acd_3e - 2acdf - b^2d_2d_4 + b^2e^2 - 2bcde - 2bcd_2f + c^2d^2 - c^2d_2d_3 - d^2d_1d_4 - d_1d_3e^2 - 2dd_1ef - d_1d_2f^2 + d_1d_2d_3d_4$. It can be easily verified that the polynomial $p_1(x)$ has four distinct zeros. If $x_1 > x_2 > x_3 > x_4$ are the zeros of $p_1(x)$, then by Theorem 2.4, we get that $\xi_{n-3} \leq x_1 \leq \xi_1, \xi_{n-2} \leq x_2 \leq \xi_2, \xi_{n-1} \leq x_3 \leq \xi_3$ and $\xi_n \leq x_4 \leq \xi_4$. Using these inequalities together with the fact that x_1, x_2, x_3, x_4 are distinct we conclude that G has at least four distinct Laplacian ABC-eigenvalues.

(ii). If G is $K_{1,n-1}$, then by Proposition 2.3, it is clear that G has exactly three distinct Laplacian ABC-eigenvalues. If $G \cong K_{a,a}$ with $n = 2a$, then by (ii) of Proposition 2.3, the Laplacian ABC-spectrum of G is $\left\{0, 2\sqrt{2a-2}, \left(\sqrt{2a-2}\right)^{2a-2}\right\}$ and the result holds in this case. Conversely, assume that G is a bipartite graph of diameter at most 2 having three distinct Laplaian ABC-eigenvalues. We claim that G is either $K_{a,a}$ or $K_{1,n-1}$. Clearly, K_n is the only connected graph with diameter 1 and by Lemma 2.1, this graph has two distinct Laplacian ABC-eigenvalues. It follows that G can not be of diameter 1. Therefore, G must be of diameter 2. Let G be a bipartite graph of diameter 2. Suppose u and v are two non-adjacent vertices of G. If u has a neighbour not adjacent to v, then this neighbour along with u and v induces the path P_4, which implies that diameter of G is greater than 2, and this cannot happen. Thus any two non-adjacent vertices must share the same neighbour, so it follows that G is the complete multipartite graph. For the complete bipartite graph case, if G is either $K_{a,a}$ or $K_{1,n-1}$, then there is nothing to prove, else G can be $K_{a,n-a}, a \neq 1, n \neq 2a$ and by Proposition 2.3, it is clear that this graph has more than three distinct Laplacian ABC-eigenvalues. This completes the proof in this case.

For the complete t-partite graph with $t \geq 3$, first we assume that $G \cong K_{p,p,\dots,p}$. Then there are p independent subsets sharing the same neighbourhood such that each vertex has the same ABC-degree $\sqrt{2p(t-1)-2}$. So, by Theorem 2.2, we get a Laplacian ABC-eigenvalue

$\sqrt{2p(t-1)-2}$ with multiplicity $pt-t$. The other t Laplacian ABC-eigenvalues of $K_{p,p,\dots,p}$ are the eigenvalues of the following equitable quotient matrix

$$\begin{pmatrix} \sqrt{2p(t-1)-2} & \frac{-\sqrt{2p(t-1)-2}}{t-1} & \dots & \frac{-\sqrt{2p(t-1)-2}}{t-1} \\ \frac{-\sqrt{2p(t-1)-2}}{t-1} & \sqrt{2p(t-1)-2} & \dots & \frac{-\sqrt{2p(t-1)-2}}{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\sqrt{2p(t-1)-2}}{t-1} & \frac{-\sqrt{2p(t-1)-2}}{t-1} & \dots & \sqrt{2p(t-1)-2} \end{pmatrix}. \tag{2.3}$$

Now, it is easy to show that $\frac{t\sqrt{2p(t-1)-2}}{t-1}$ is an eigenvalue of (2.3) with multiplicity $t-1$ and 0 is always a Laplacian ABC-eigenvalue of $K_{p,p,\dots,p}$. This shows that $K_{p,p,\dots,p}$ is the candidate graph with three distinct Laplacian ABC-eigenvalues. Next, we show that $K_{p_1,p_2,\dots,p_t}$ have more than three distinct Laplacian ABC-eigenvalues. For that it is enough to prove that $K_{p,p,\dots,p,q},\ p\neq q$ has more than three distinct Laplacian ABC-eigenvalues. As above, it is easy to see that $p(t-2)\frac{\sqrt{2(p(t-1)+q)-2}}{p(t-1)+q}+q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}}$ and $p(t-1)\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}}$ are Laplacian ABC-eigenvalues of $K_{p,p,\dots,p,q}$ with multiplicities $(t-1)(p-1)$ and $q-1$, respectively. The other t Laplacian ABC-eigenvalues of $K_{p,p,\dots,p,q}$ are the eigenvalues of the following equitable quotient matrix

$$\left(\begin{array}{ccc|c} d & \dots & -\frac{p\sqrt{2(p(t-2)+q)-2}}{p(t-2)+q} & -q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \\ -\frac{p\sqrt{2(p(t-2)+q)-2}}{p(t-2)+q} & \dots & -\frac{p\sqrt{2(p(t-2)+q)-2}}{p(t-2)+q} & -q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{p\sqrt{2(p(t-2)+q)-2}}{p(t-2)+q} & \dots & d & -q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \\ \hline -p\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} & \dots & -p\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} & p(t-1)\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \end{array}\right), \tag{2.4}$$

where $d=p(t-2)\frac{\sqrt{2(p(t-1)+q)-2}}{p(t-1)+q}+q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}}$. Consider $X_{i-1}=(-1,x_{i2},x_{i3},\dots,x_{i(t-1)},0)$, where $x_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise,}\end{cases}$ for $i=2,3,\dots,t-1$. Now, we can easily verify that $X_1,\dots,X_{t-2}$ are the eigenvectors corresponding to the Laplacian ABC-eigenvalue $p(t-1)\frac{\sqrt{2(p(t-1)+q)-2}}{p(t-1)+q}+q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}}$. The other two eigenvalues of (2.3) with the given blocks are the eigenvalues of the following equitable quotient matrix

$$\begin{pmatrix} q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} & -q\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \\ -(t-1)\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} & (t-1)\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}} \end{pmatrix},$$

and its eigenvalues are 0 and $(p(t-1)+q)\sqrt{\frac{p(2t-3)+q-2}{p(t-1)(p(t-2)+q)}}$. Therefore it follows that $K_{p,p,\dots,p,q}$ has more than three distinct Laplacian ABC-eigenvalues.

Lastly, If G is a unicyclic graph, then as above the diameter of G is exactly 2. So, G must be one of the following graphs: $C_4, C_5, K_{1,n-1}+e$. By Proposition 2.3, the graph $K_{1,n-1}+e$ has more than three distinct Laplacian ABC-eigenvalues. Also, the graph C_4 is bipartite and follows by part (ii). Further, for the graph C_5, the Laplacian ABC-spectrum of C_5 is

$$\left\{(2.55834)^{[2]}, (0.977198)^{[2]}, 0\right\},$$

and so the result follows in this case. ■

Parts (iii) and (iv) of Theorem 2.5 give an insight that there can be more non-bipartite graphs with diameter 2 having three distinct Laplacian ABC-eigenvalues. Therefore, we leave the following problem.

Problem 2 *Characterize completely the non-bipartite graphs with diameter* 2 *and three distinct Laplacian ABC-eigenvalues.*

3 Laplacian ABC-energy

In this section, we introduce the concept of Laplacian ABC-energy of a graph G. We establish some tight bounds for this quantity.

For the matrix $M \in \mathbb{M}_{m\times n}(\mathbb{R})$, the positive square roots of the eigenvalues of MM^T are the *singular values* of M, denoted by $\sigma_i(M)$, (or simply by σ_i), $i = 1, 2, \dots, n$. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the singular values of M. The sum of the first k largest singular values $\|M\|_k = \sum_{i=1}^{k} \sigma_i, 1 \leq k \leq n$ is the k-norm of M. For $k = 1$, $\|M\|_1 = \sigma_1$ is the *spectral norm*, for $2 \leq k \leq n-1$, $\|M\|_k = \sum_{i=1}^{k} \sigma_i$ is known as the *Kay Fan k-norm* and for $k = n$, the norm $\|M\|_n = \sum_{i=1}^{n} \sigma_i$ is called the *trace norm* of M. In case of normal matrices and in particular for symmetric matrices, singular values are the absolute values of their eigenvalues. So for symmetric matrices, the trace norm is the sum of absolute values of the eigenvalues. Nikiforov [21], defined the energy of a symmetric matrix M as the absolute sum of values of its eigenvalues. Motivated by this, we introduce a new operator $\overline{L} = \tilde{\mathrm{L}}(G) - \overline{\xi} I_n$, where $\overline{\xi} = \sum_{i=1}^{n-1} \xi_i$ is the average of the Laplacian ABC-eigenvalues. Also, we observe that $\overline{\xi} = \sum_{i=1}^{n} \overline{d}_i = \frac{2ABC(G)}{n}$. Clearly, $\overline{L}$ is the real symmetric matrix and its eigenvalues are real, denoted by θ_i, $i = 1, 2, \dots, n$. Therefore, the Laplacian ABC-energy is defined by

$$\sum_{i=1}^{n} |\theta_i| = \sum_{i=1}^{n} \left|\xi_i - \overline{\xi}\right| = \sum_{i=1}^{n} \left|\xi_i - \frac{2ABC(G)}{n}\right| = \sum_{i=1}^{n} \sigma_i(\overline{L}). \tag{3.5}$$

Let σ be the largest positive integer such that $\xi_\sigma \geq \frac{2ABC(G)}{n}$. That is, σ is the positive integer with $\xi_\sigma \geq \frac{2ABC(G)}{n}$ and $\xi_{\sigma+1} < \frac{2ABC(G)}{n}$. It is clear that σ gives the number of Laplacian ABC-eigenvalues of G which lie in $\left[0, \frac{2ABC(G)}{n}\right]$ and the eigenvalues which lie in $\left[\frac{2ABC(G)}{n}, n\right)$. It is an interesting and hard problem in Linear Algebra to find the distribution of the eigenvalues of a given matrix. The problem of distribution of eigenvalues of a given matrix has been considered for many graph matrices and various interesting results are obtained. Like other graph matrices, the following problem can be of interest for the Laplacian ABC-matrix.

Problem 3 *Among all connected graphs G of order n with a given parameter α, like the number of edges, the independence number, the matching number, the chromatic number, the vertex covering number, the $ABC(G)$-index, etc, determine the number of Laplacian ABC-eigenvalues in the interval $[0, \alpha]$.*

The next result shows that we can express the Laplacian ABC-energy in terms of Ky Fan k-norm of the Laplacian ABC-matrix.

Theorem 3.1 *Let G be a connected graph of order $n \geq 3$ having atom-bond connectivity index $ABC(G)$. Then, the Laplacian ABC-energy of G satisfies the following relation*

$$E(\tilde{L}(G)) = 2\left(\sum_{i=1}^{\sigma} \xi_i - \frac{2\sigma ABC(G)}{n}\right) = 2 \max_{1\leq k\leq n}\left(\sum_{i=1}^{k} \xi_i - \frac{2kABC(G)}{n}\right),$$

where $\sum_{i=1}^{k} \xi_i$ is the sum of the first k largest Laplaian ABC-eigenvalues (Ky Fan k-norm) of G and σ is the number of Laplacian ABC-eigenvalues in $\left[0, \frac{2ABC(G)}{n}\right]$.

Proof. Let σ be the largest positive integer such that $\xi_\sigma \geq \frac{2ABC(G)}{n}$. Then by the definition of Laplacian ABC-energy $E(\tilde{\mathrm{L}}(G))$ and the fact $2ABC(G) = \sum_{i=1}^{n} \xi_i$, we have

$$\begin{aligned} E(\tilde{\mathrm{L}}(G)) &= \sum_{i=1}^{n}\left|\xi_i - \frac{2ABC(G)}{n}\right| = \sum_{i=1}^{\sigma}\left(\xi_i - \frac{2ABC(G)}{n}\right) + \sum_{i=\sigma+1}^{n}\left(\frac{2ABC(G)}{n} - \xi_i\right) \\ &= \sum_{i=1}^{\sigma} \xi_i - \frac{4\sigma ABC(G)}{n} + 2ABC(G) - \sum_{i=\sigma+1}^{n} \xi_i = 2\left(\sum_{i=1}^{\sigma} \xi_i - \frac{2\sigma ABC(G)}{n}\right). \end{aligned}$$

Next, we shall prove that $2\left(\sum_{i=1}^{\sigma} \xi_i - \frac{2\sigma ABC(G)}{n}\right) = 2 \max_{1\leq k\leq n}\left(\sum_{i=1}^{k} \xi_i - \frac{2kABC(G)}{n}\right)$. For $k > \sigma$, we have

$$\sum_{i=1}^{k} \xi_i - \frac{2kABC(G)}{n} = \sum_{i=1}^{\sigma} \xi_i + \sum_{i=\sigma+1}^{k} \xi_i - \frac{2kABC(G)}{n}$$

$$
\begin{aligned}
&< \sum_{i=1}^{\sigma} \xi_i + (k-\sigma)\frac{2ABC(G)}{n} - k\frac{2ABC(G)}{n} \quad \text{as } \xi_i < \frac{2ABC(G)}{n}, \text{for } i \geq \sigma+1 \\
&= \sum_{i=1}^{\sigma} \xi_i - \frac{2\sigma ABC(G)}{n}.
\end{aligned}
$$

Similarly, for $k \leq \sigma$, it can be easily verified that $\sum_{i=1}^{k} \xi_i - k\frac{2ABC(G)}{n} \leq \sum_{i=1}^{\sigma} \xi_i - \frac{2\sigma ABC(G)}{n}$, which completes the proof. ■

The following result gives a lower bound for the Laplacian ABC-energy of a graph G, in terms of the atom-bound connectivity index $ABC(G)$.

Corollary 3.2 *Let G be a connected graph of order $n \geq 3$ having atom-bound connectivity index $ABC(G)$. Then*

$$E(\tilde{L}(G)) \geq 2\Big(\xi_1 - \frac{2ABC(G)}{n}\Big),$$

with equality if and only if $\sigma = 1$; and

$$E(\tilde{L}(G)) \geq 2\Big(\frac{4ABC(G)}{n} - \xi_{n-1}\Big),$$

with equality if and only if $\sigma = n-2$

Proof. Using Theorem 3.1 and the fact that $\sum\limits_{i=1}^{k} \xi_i = 2ABC(G) - \sum\limits_{i=k+1}^{n-1} \xi_i$, the result follows. ■

From Corollary 3.2, it is clear that any lower bound for ξ_1 helps us to find a lower bound for the Laplacian ABC-energy of a graph G and any upper bound for ξ_{n-1} helps us to find a lower bound for the Laplacian ABC-energy of a graph G.

For the regular and bipartite semi-regular graphs, we have the following relation between the Laplacian ABC-energy and the corresponding Laplacian energy of a graph.

Theorem 3.3 *Let G be connected graph of order $n \geq 3$ having atom-bond connectivity index $ABC(G)$ and $LE(G)$ be its Laplacian energy. Then following statements hold.*

(i) *If G is an r-regular graph, then $E(\tilde{L}(G)) = \frac{\sqrt{2r-2}}{r} LE(G)$.*

(ii) *If G is an (r,s)-semiregular bipartite graph, then $E(\tilde{L}(G)) = \sqrt{\frac{r+s-2}{rs}} LE(G)$.*

Proof. If G is an r-regular graph, then by (1) of Theorem 3.1 in [22], we have $\xi_i = \frac{\sqrt{2r-2}}{r}\mu_i$, where μ_i is the i-th Laplacian eigenvalue of G. Also, $2ABC(G) = \xi_1 + \xi_2 + \cdots + \xi_{n-1} = \frac{\sqrt{2r-2}}{r}\Big(\mu_1 + \mu_2 + \cdots + \mu_{n-1}\Big) = \frac{\sqrt{2r-2}}{r}2m$. The first part now follows from the definition of Laplacian ABC-energy of G. Similarly, if G is an (r,s)-semiregular bipartite graph, then result follows from (2) of Theorem 3.1 in [22]. ■

A very interesting and useful lemma due to Fulton [11] is as follows.

Lemma 3.4 *Let A and B be two real symmetric matrices both of order n. If k, $1 \leq k \leq n$, is a positive integer, then*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where $\lambda_i(X)$ is the i^{th} eigenvalue of X.

Recall that $\overline{d}_i = \sum\limits_{v_j \in N(v_i)} \sqrt{\frac{d_i+d_j-2}{d_i d_j}}$ is the ABC-degree of the vertex $v_i \in V(G)$. A graph G is said to be ABC-regular if the ABC-degrees of all its vertices is the same. The following result gives an upper bound for the Laplacian ABC-energy in terms of the ABC-degrees and ABC-energy of a graph.

Theorem 3.5 *Let G be a connected graph of order $n \geq 3$ with ABC-degrees $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n$. Let σ be the number of Laplacian ABC-eigenvalues of G which are greater than or equal to $\frac{2ABC(G)}{n}$. Then*

$$E(\tilde{L}(G)) \leq E_{ABC}(G) + 2\sum_{i=1}^{\sigma}\left(\overline{d}_i - \frac{2ABC(G)}{n}\right).$$

If G is ABC-regular, then the equality occurs.

Proof. Applying Lemma 3.4 to

$$\tilde{\mathrm{L}}(G) = \overline{D}(G) - \tilde{\mathrm{A}}(G),$$

where $\overline{D}(G) = \mathrm{diag}(\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n)$ is the diagonal matrix of ABC-degrees of G, we get

$$\sum_{i=1}^{k} \xi_i(G) \leq \sum_{i=1}^{k} \overline{d}_i + \sum_{i=1}^{k} \vartheta_i, \tag{3.6}$$

where $\vartheta_i(G)$ is the i-th ABC-eigenvalue of G. Let σ be the number of Laplacian ABC-eigenvalues of G which are greater than or equal to $\frac{2ABC(G)}{n}$. Then $1 \leq \sigma \leq n-1$. From the definition of

ABC-energy, we have

$$E_{ABC}(G) = 2\max_{1\leq j\leq n}\sum_{i=1}^{k}\vartheta_i(G) \geq 2\sum_{i=1}^{\sigma}\vartheta_i(G).$$

This together with inequality 3.6 gives

$$2\sum_{i=1}^{\sigma}\xi_i(G) \leq 2\sum_{i=1}^{\sigma}\overline{d}_i + 2\sum_{i=1}^{\sigma}\vartheta_i(G),$$
$$\text{that is}, 2\sum_{i=1}^{\sigma}\xi_i - \frac{4ABC(G)\sigma}{n} \leq 2\sum_{i=1}^{\sigma}\overline{d}_i + E_{ABC}(G) - \frac{4ABC(G)\sigma}{n}.$$

Thus, using Theorem 3.1, it follows that

$$E(\tilde{\mathrm{L}}(G)) \leq E_{ABC}(G) + 2\sum_{i=1}^{\sigma}\Big(\overline{d}_i - \frac{2ABC(G)}{n}\Big).$$

If G is an ABC-regular graph, then it is clear that the equality occurs. ■

The Frobinus norm of $\tilde{\mathrm{L}}(G)$ is $\|\tilde{\mathrm{L}}(G)\|_F^2 = \sum\limits_{i=1}^{n-1}\xi_i^2$. Also, the Frobinus norm of $\overline{L}(G) = \tilde{\mathrm{L}}(G) - \frac{2ABC(G)}{n}I_n$ is

$$\|\overline{L}(G)\|_F^2 = \sum_{i=1}^{n}\theta_i^2 = \sum_{i=1}^{n}\left(\xi_i - \frac{2ABC(G)}{n}\right)^2 = \sum_{i=1}^{n}\xi_i^2 + \frac{4ABC(G)}{n^2}\sum_{i=1}^{n}.1 - \frac{4ABC(G)}{n}\sum_{i=1}^{n}\xi_i$$
$$= \sum_{i=1}^{n}\xi_i^2 - \frac{4ABC(G)^2}{n} = \|\tilde{\mathrm{L}}(G)\|_F^2 - \frac{4ABC(G)^2}{n}.$$

Next, we derive an upper bound for the Laplacian ABC-energy of a graph G, in terms of the atom-bound connectivity index, the order and the parameter $\|\tilde{\mathrm{L}}(G)\|_F^2$.

Theorem 3.6 *Let G be a connected graph of order $n \geq 3$. Then*

$$E(\tilde{L}(G)) \leq \frac{2ABC(G)}{n} + \sqrt{(n-1)\left(\|\tilde{L}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2\right)}, \qquad (3.7)$$

with equality if and only if either $G \cong K_n$ or G has three distinct Laplacian ABC-eigenvalues, which are $0, \gamma + \frac{2ABC(G)}{n}$ and $\frac{2ABC(G)}{n} - \gamma$, where $\gamma = \sqrt{\frac{\|\tilde{L}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2}{n-1}}$.

Proof. As $\xi_n = 0$, so we have

$$E(\tilde{\mathrm{L}}(G)) - \frac{2ABC(G)}{n} = \sum_{i=1}^{n-1}\left|\xi_i - \frac{2ABC(G)}{n}\right|.$$

Now, by applying the Cauchy-Schwarz inequality to the vectors

$$\left(\left|\xi_1 - \frac{2ABC(G)}{n}\right|, \left|\xi_2 - \frac{2ABC(G)}{n}\right|, \ldots, \left|\xi_{n-1} - \frac{2ABC(G)}{n}\right|\right)$$

and $(1, 1, \ldots, 1)$, we get

$$\left(\sum_{i=1}^{n-1}\left|\xi_i - \frac{2ABC(G)}{n}\right|\right)^2 \leq (n-1)\sum_{i=1}^{n-1}\left(\xi_i - \frac{2ABC(G)}{n}\right)^2 \tag{3.8}$$

Further,

$$\begin{aligned}\sum_{i=1}^{n-1}\left(\xi_i - \frac{2ABC(G)}{n}\right)^2 &= \sum_{i=1}^{n}\left(\xi_i - \frac{2ABC(G)}{n}\right)^2 - \left(\frac{2ABC(G)}{n}\right)^2\\ &= \|\tilde{\mathrm{L}}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2.\end{aligned}$$

This observation together with inequality 3.8 and the definition of Laplacian ABC-energy gives that

$$\begin{aligned}E(\tilde{\mathrm{L}}(G)) =&\frac{2ABC(G)}{n} + \sum_{i=1}^{n-1}\left|\xi_i - \frac{2ABC(G)}{n}\right|\\ \leq&\frac{2ABC(G)}{n} + \sqrt{(n-1)\left(\|\tilde{\mathrm{L}}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2\right)}.\end{aligned}$$

Suppose that Inequality (3.7) is an equality. Then equality occurs in (3.8), that is,

$$\left|\xi_1 - \frac{2ABC(G)}{n}\right| = \left|\xi_2 - \frac{2ABC(G)}{n}\right| = \cdots = \left|\xi_{n-1} - \frac{2ABC(G)}{n}\right|. \tag{3.9}$$

Since $\xi_1 - \frac{2ABC(G)}{n} > 0$ and $\xi_{n-1} - \frac{2ABC(G)}{n} \geq 0$ or $\xi_{n-1} - \frac{2ABC(G)}{n} < 0$, it follows that if $\xi_{n-1} - \frac{2ABC(G)}{n} \geq 0$, then from (3.9) we get $\xi_1 - \frac{2ABC(G)}{n} = \xi_2 - \frac{2ABC(G)}{n} = \cdots = \xi_{n-1} - \frac{2ABC(G)}{n}$, that is, $\xi_1 = \xi_2 = \cdots = \xi_{n-1}$. This shows that equality occurs in (3.7) in this case if and only if G has two distinct Laplacian ABC-eigenvalues, which is so by Lemma 2.1 if and only if $G \cong K_n$. On the other hand, if $\xi_{n-1} - \frac{2ABC(G)}{n} < 0$, then we can find a positive integer t, such that $\xi_1 - \frac{2ABC(G)}{n} = \cdots = \xi_t - \frac{2ABC(G)}{n} = \gamma$ and $\xi_{t+1} - \frac{2ABC(G)}{n} = \cdots = \xi_{n-1} - \frac{2ABC(G)}{n} = -\gamma$. This gives that $\xi_i = \gamma + \frac{2ABC(G)}{n}$, for $i = 1, 2, \ldots, t$ and $\xi_i = \frac{2ABC(G)}{n} - \gamma$, for $i = t+1, \ldots, n-1$. Since $\|\tilde{\mathrm{L}}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2 = \sum_{i=1}^{n-1}\left(\xi_i - \frac{2ABC(G)}{n}\right)^2 = \sum_{i=1}^{n-1}\left|\xi_i - \frac{2ABC(G)}{n}\right|^2 = (n-1)\left|\xi_i - \frac{2ABC(G)}{n}\right|^2$, it follows that $\gamma = \sqrt{\frac{\|\tilde{\mathrm{L}}(G)\|_F^2 - \left(\frac{2ABC(G)}{n}\right)^2}{n-1}}$. Thus, it follows that equality occurs in (3.7) in this case if and only if G has three distinct Laplacian ABC-eigenvalues, which are $0, \gamma + \frac{2ABC(G)}{n}$ and $\frac{2ABC(G)}{n} - \gamma$.

Conversely, it can be easily verified that equality holds in (3.7) for the graphs mentioned in the statement of the theorem. ■

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