The proper vertex-disconnection of graphs*

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Abstract

Let G be a vertex-colored connected graph. A subset X of the vertex-set of G is called *proper* if any two adjacent vertices in X have distinct colors. The graph G is called *proper vertex-disconnected* if for any two vertices x and y of G, there exists a vertex subset S of G such that when x and y are nonadjacent, S is proper and x and y belong to different components of $G - S$; whereas when x and y are adjacent, $S + x$ or $S + y$ is proper and x and y belong to different components of $(G - xy) - S$. For a connected graph G, the *proper vertex-disconnection number* of G, denoted by $pvd(G)$, is the minimum number of colors that are needed to make G proper vertex-disconnected.

In this paper, we firstly characterize the graphs of order n with proper vertex-disconnection number k for $k \in \{1, n-2, n-1, n\}$. Secondly, we give some sufficient conditions for a graph G such that $pvd(G) = \chi(G)$, and show that almost all graphs G have $pvd(G) = \chi(G)$ and $pvd(\overline{G}) = \chi(\overline{G})$. We also give an equivalent statement of the famous Four Color Theorem. Furthermore, we study the relationship between the proper disconnection number $pd(G)$ of G and the proper vertex-disconnection number $pvd(L(G))$ of the line graph $L(G)$ of G. Finally, we show that it is NP-complete to decide whether a given vertex-colored graph is proper vertex-disconnected, and it is NP-hard to decide for a fixed integer $k \geq 3$, whether the pvd-number of a graph G is no more than k, even if $k = 3$ and G is a planar graph with $\Delta(G) = 12$. We also show that it is solvable in polynomial time to determine the proper vertex-disconnection number for a graph with maximum degree less than four.

Keywords: Vertex-coloring, Vertex-cut, Proper vertex-disconnection, Chromatic number, NP-complete (hard), polynomial time algorithm

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1 Introduction

All graphs considered in this paper are finite, simple, undirected and nontrivial. Let $G = (V(G), E(G))$ be a connected graph with vertex-set $V(G)$ and edge-set $E(G)$. For a vertex $v \in V$, the *open neighborhood* of v in G is the set $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and the *degree* of v is $d(v) = |N_G(v)|$, and the *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. If there is no confusion, we use $N(x)$ and $N[x]$ to denote them. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex v and vertex subset S of $V(G)$, we simply let $S + v = S \cup \{v\}$. A *clique* of G is a set Q of vertices of G such that every pair of vertices of Q are adjacent. The maximum size of a clique of G is called the *clique number* of G, denoted by $w(G)$. For any notation and terminology not defined here, we follow those used in [8].

For a graph G and a positive integer k, let $c : V(G) \to [k]$ be a *vertex-coloring*, or simply a *coloring*, of G, where and in what follows $[k]$ denotes the set $\{1, 2, ..., k\}$ of integers. Similarly, an edge-coloring of a graph G is an assignment of colors to the edges of G. A coloring of G is *proper* if no two adjacent vertices in G are assigned the same color. For a coloring c, let $\Gamma(c)$ be the set of colors used in c and $|\Gamma(c)|$ be the number of colors of c. A coloring c is a *k-coloring* if $|\Gamma(c)| = k$. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k for which the graph G is k-colorable. If $\chi(G) = k$, the graph G is said to be *k-chromatic*. In a vertex-colored graph G, a set of vertices of G is *rainbow* if any two vertices of the set have different colors and *proper* if any two adjacent vertices of the set have different colors. In an edge-colored graph G, a set of edges of G is *rainbow* if any two edges of the set have different colors and *proper* if any two adjacent edges of the set have different colors.

For a connected graph G, let x and y be any two vertices of G. If x and y are nonadjacent, then an *x-y vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $G - S$. If x and y are adjacent, then an *x-y vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $(G - xy) - S$. An *edge-cut* of a graph G is a set R of edges of G such that $G - R$ is disconnected.

From Menger's Theorem, we learn that paths are in the same position as cuts in the study of graph connectivity. Chartrand et al. in [12] introduced the concept of rainbow connection of graphs. Ten years later, Chartrand et al. [11] introduced the concept of rainbow disconnection of graphs, which is a dual concept of the rainbow connection of graphs. An edge-coloring is called a *rainbow disconnection coloring* of G if for every two distinct vertices of G, there exists a rainbow edge-cut in G separating them. For a connected graph G, the *rainbow disconnection number* of G, denoted by $rd(G)$, is the smallest number of colors required for a rainbow disconnection coloring of G. For more results about the rainbow disconnection of graphs, we refer to [2, 5, 6, 7].

Inspired by the concept of rainbow disconnection of graphs, the authors of [4] introduced the concept of rainbow vertex-disconnection of graphs. For a vertex-colored connected graph G, let x and y be two vertices of G. An *x-y rainbow vertex-cut* is an x-y vertex-cut S such that if x and y are nonadjacent, then S is rainbow; if x and y are adjacent, then $S + x$ or $S + y$ is rainbow. A vertex-colored graph G is called *rainbow vertex-disconnected* if for any two distinct vertices x and y of G, there exists an x-y rainbow vertex-cut. In this case, the vertex-coloring is called a *rainbow vertex-disconnection coloring* of G. For a connected graph G, the *rainbow vertex-disconnection number* of G, denoted by $rvd(G)$, is the minimum number of colors that are needed to make G rainbow vertex-disconnected. We refer the reader to [13, 22] for more relevant results.

Andrews et al. [1] and Borozan et al. [9] independently introduced the concept of proper connection of graphs. Inspired by the concept of rainbow disconnection and proper connection of graphs, the authors of [3] and [13] introduced the concept of proper disconnection of graphs. An edge-colored graph is called *proper disconnected* if for each pair of distinct vertices of G, there exists a proper edge-cut separating them. At this time, the edge-coloring is called a *proper disconnection coloring* of G. For a connected graph G, the *proper disconnection number* of G, denoted by $pd(G)$, is defined as the minimum number of colors that are needed to make G proper disconnected. In addition, monochromatic versions of above concepts have also been introduced, and we refer reader to [19, 20, 21].

As shown above, in the studies of rainbow (vertex-)disconnection and proper disconnection, people studied the colored connectivity of graphs by rainbow edge(vertex)-cut and proper edge-cut. All these parameters can be regarded as some kinds of new chromatic numbers, compared to the classical chromatic number. In order to strength the research of colored connectivity of graphs, as well as try to explore the difference between the new chromatic numbers and the classical chromatic number, we naturally turn our attention to proper vertex-cut and introduce the concept of proper vertex-disconnection in this paper.

For a vertex-colored connected graph G, let x and y be two vertices of G. An *x-y proper vertex-cut* is an x-y vertex-cut S such that if x and y are nonadjacent, then S is proper; if x and y are adjacent, then $S+x$ or $S+y$ is proper. A vertex-colored graph G is called *proper vertex-disconnected* if for any two vertices x and y of G, there exists an x-y proper vertex-cut. In this case, the vertex-coloring is called a *proper vertex-disconnection coloring* of G, denoted by *pvd-coloring* for short. For a connected graph G, the *proper vertex-disconnection number* (abbreviated as *pvd-number*) of G, denoted by $pvd(G)$, is the minimum number of colors that are needed to make G proper vertex-disconnected. A graph G is called *k-proper vertex-disconnection colorable* if $pvd(G) \leq k$. A pvd-coloring of G with $pvd(G)$ colors is called an *optimal pvd-coloring* of G.

This paper is organized as follows. In Section 2, we provide some fundamental results that will be used in later discussion, and also characterize the graphs of order n with pvd-number k for $k \in \{1, n-2, n-1, n\}$. In Section 3, we give some sufficient conditions for a graph G such that $pvd(G) = \chi(G)$ and prove that almost all graphs G have $pvd(G) = \chi(G)$ and $pvd(\overline{G}) = \chi(\overline{G})$. Furthermore, we present an equivalent assertion of the Four Color Theorem. In Section 4, we study the relationship between $pd(G)$ and $pvd(L(G))$. In Section 5, we show that it is NP-complete to decide whether a given vertex-colored graph is proper vertex-disconnected. We also show that it is NP-hard to decide for a fixed integer $k \geq 3$, whether the pvd-number of a graph G is no more than k, even if $k = 3$ and the graph G is a planar graph with $\Delta(G) = 12$. In addition, we show that it is solvable in polynomial time to determine the pvd-number for a graph with maximum degree less than four , and we also give a polynomial time algorithm for pvd-coloring a graph with maximum degree less than four.

2 Preliminaries

At the very beginning, we state some fundamental results about the pvd-numbers of graphs, which will be used frequently in the sequel.

Lemma 2.1 *If G is a connected graph and H is a connected subgraph of G, then $pvd(H) \leq pvd(G)$.*

Proof. Suppose that c is an optimal pvd-coloring of G and H is a connected subgraph of G. Let c' be a coloring that is obtained by restricting c to H. Let x and y be two vertices of H and S be an x-y proper vertex-cut of G. Then $S' = S \cap V(H)$ is an x-y proper vertex-cut in H; otherwise, if there exists an x-y path P with length at least 2 in $H - S'$, then P is also in $G - S$, a contradiction. Thus, c' is a pvd-coloring of H and then $pvd(H) \leq pvd(G)$. $\square$

A *block* of a graph G is a maximal connected subgraph of G that contains no cut vertices. So, a block of G is a cut edge of G or a 2-connected subgraph of G with at least three vertices. The *block decomposition* of G is the set of blocks of G. Next we will show that $pvd(G)$ is equal to maximal pvd-number $pvd(B)$ among the blocks B of G. The proof method is similar to the one used in [4].

Lemma 2.2 *Let G be a connected graph, and let B be a block of G such that $pvd(B)$ is maximum among all blocks of G. Then $pvd(G) = pvd(B)$.*

Proof. Suppose that $\{B_1, B_2, \cdots, B_t\}$ is the block decomposition of G. Let $k = \max\{\text{pvd}(B_i)|i \in [t]\}$. If G has no cut vertex, then $G = B_1$ and the result follows. Next, we assume that G has at least one cut vertex. Since each block is a connected subgraph of G, we have $pvd(G) \geq k$ by Lemma 2.1.

Let c_i be an optimal pvd-coloring of B_i, where $i \in [t]$. Let H be a connected graph consisting of some blocks of G and c' be an optimal pvd-coloring of H. Let B_i ($i \in [t]$) be the block having a common vertex with H, where B_i is the subgraph of G but not of H. Suppose that v is the common vertex of B_i and H. We define a *color exchange operation* on B_i as follows: If $c'(v) = c_i(v)$, we do nothing. If $c'(v) \neq c_i(v)$, we may assume that $c'(v) = a$ and $c_i(v) = b$. Assign color a to the vertices of B_i with color b, and color b to the vertices of B_i with color a (if such vertices exist) . By doing the color exchange operation on B_i, we can separately obtain a coloring of $H \cup B_i$ with $\max\{|\Gamma(c')|, |\Gamma(c_i)|\}$ colors, and an optimal pvd-coloring, denoted by c_i^*, of B_i. Note that c_i^* may be the same as c_i.

Firstly, we take a block, say B_1, and let $G_1 = B_1$. Then we find a block $B(\in \{B_2, \cdots, B_t\})$ which has a common vertex with the graph G_i $(1 \leq i \leq t-1)$. Let $G_{i+1} = G_i \cup B$. By doing the color exchange operation on B, we can obtain a vertex-coloring of G_{i+1} and an optimal pvd-coloring of B. Repeatedly, we have $G_t = G$, and can separately obtain a vertex-coloring c of G with k colors and an optimal pvd-coloring c_i^* of B_i for each $i \in [t]$.

Let x and y be two vertices of G. If there exists a block, say B_i, which contains both x and y, then any x-y proper vertex-cut in B_i under the coloring c_i^* is an x-y proper vertex-cut in G. If x and y are in different blocks, then there is exactly one x-y internally disjoint path, say P, in G and the path P contains at least one cut vertex, say w. Then $\{w\}$ is an x-y proper vertex-cut in G. Hence, $pvd(G) \leq k$. □

As a consequence of Lemma 2.2, the study of proper vertex-disconnection can be restricted to 2-connected graph. We now present an upper bound for the pvd-number of a graph.

Theorem 2.3 *If G is a connected graph, then $pvd(G) \leq \chi(G)$.*

Proof. Let c be a proper coloring of G with $|\Gamma(c)| = \chi(G)$. Note that $N(x)$ is proper for each vertex v of G. For any two vertices x, y, if x, y are nonadjacent, then $N(x)$ is an x-y proper vertex-cut. If x, y are adjacent, consider the set $F = N(x) \setminus \{y\}$. Since F is an x-y vertex-cut and $F \cup \{y\} = N(x)$ is proper, F is an x-y proper vertex-cut. Then c is also a pvd-coloring of G, and so $pvd(G) \leq \chi(G)$. □

Theorem 2.4 *[10] Brooks' Theorem*
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

By above results, we have the following consequence: if G is a connected graph, and neither an odd cycle nor a complete graph, then

$$1 \leq pvd(G) \leq \chi(G) \leq \Delta(G) \leq n-1. \tag{1}$$

Next we give two useful lemmas, which will be used frequently in the following.

Lemma 2.5 *Let G be a connected graph, and let x, y be two adjacent vertices of G having at least two common neighbors. Then x and y receive different colors in any optimal pvd-coloring.*

Proof. Let c be any optimal pvd-coloring of G. Suppose vertices v_1, v_2 are two common neighbors of x and y. Then for any v_1-v_2 proper vertex-cut S of G under the coloring c, there must be $\{x, y\} \subseteq S$. Since x, y are adjacent, we have $c(x) \neq c(y)$. □

Lemma 2.6 *Let G be a connected graph. If any two adjacent vertices of G have at least two common neighbors, then $pvd(G) = \chi(G)$.*

Proof. Let c be an optimal pvd-coloring of G. If any two adjacent vertices, say x, y, of G have at least two common neighbors, then $c(x) \neq c(y)$ by Lemma 2.5. Thus c is also a proper coloring of G. Then $\chi(G) \leq pvd(G)$. By Theorem 2.3, we have $pvd(G) = \chi(G)$. □

For a graph G of order n, since its pvd-coloring is a vertex-coloring, there is naturally an inequality $1 \leq pvd(G) \leq n$. We next characterize the connected graphs of order n with pvd-number k for $k \in \{1, n-2, n-1, n\}$. Firstly, we give the pvd-number of a triangle.

Lemma 2.7 *If G is a triangle, then $pvd(G) = 2$.*

Proof. Let $V(G) = \{v_1, v_2, v_3\}$. For any optimal pvd-coloring c of G, consider the v_1-v_3 proper vertex-cut S. Note that $S = \{v_2\}$. Since v_1, v_3 are adjacent, the set $\{v_1, v_2\}$ or $\{v_2, v_3\}$ is proper. Since v_2 is adjacent to both v_1 and v_3, we have $c(v_1) \neq c(v_2)$ or $c(v_2) \neq c(v_3)$. Then $|\Gamma(c)| \geq 2$ and $pvd(G) \geq 2$. Define a coloring c' of G such that $c'(v_1) = c'(v_2) = 1$ and $c'(v_3) = 2$. Obviously, the set $\{v_1\}$ is a v_2-v_3 vertex-cut and $\{v_1, v_3\}$ is proper. Then $\{v_1\}$ is a v_2-v_3 proper vertex-cut. Similarly, $\{v_2\}$ is a v_1-v_3 proper vertex-cut and $\{v_3\}$ is a v_1-v_2 proper vertex-cut. Thus c' is a pvd-coloring of G and $pvd(G) \leq 2$. □

Theorem 2.8 *Let G be a connected graph. Then $pvd(G) = 1$ if and only if G is triangle-free.*

Proof. Suppose $pvd(G) = 1$. If G contains a triangle, then $pvd(G) \geq 2$ by Lemmas 2.1 and 2.7, a contradiction. On the contrary, suppose that G is triangle-free. Then $N(v)$ is an independent set for each $v \in V(G)$. Define a vertex coloring c of G such that $c(v) = 1$ for every $v \in V(G)$. For any two vertices x, y of G, if x, y are nonadjacent, then $N(x)$ is an x-y proper vertex-cut. If x, y are adjacent, consider the set $F = N(x)\setminus\{y\}$. Obviously, F is an x-y vertex-cut. Since $N(x)$ is independent, $F \cup \{y\}$ is proper. Then F is an x-y proper vertex-cut. Thus c is a pvd-coloring, and so $pvd(G) = 1$. □

It is well known that for any graph G of order n, $\chi(G) = n$ if and only if G is a complete graph. We have the same result for graphs G with $pvd(G) = n$.

Theorem 2.9 *Let G be a connected graph of order $n \geq 4$. Then $pvd(G) = n$ if and only if G is a complete graph.*

Proof. If $pvd(G) = n$, then G is a complete graph by Inequality (1) and Theorem 2.8. If G is complete, then $pvd(G) = \chi(G) = n$ by Lemma 2.6. □

Let G be a connected graph of order n. It is known that $\chi(G) \geq w(G)$ the clique number of G. Next we study the relation between $pvd(G)$ and $w(G)$, and use it to characterize the graphs with pvd-numbers $n - 1$ and $n - 2$.

Lemma 2.10 *Let G be a connected graph. Then the following conditions hold.*

(*i*) *If $w(G) < 4$, then $pvd(G) \geq w(G) - 1$.*

(*ii*) *If $w(G) \geq 4$, then $pvd(G) \geq w(G)$.*

Proof. Suppose $w(G) < 4$. If $w(G) = 2$, then G is triangle-free and $pvd(G) = 1$ by Theorem 2.8. If $w(G) = 3$, then G contains at least one triangle, and $pvd(G) \geq 2$ by Lemmas 2.1 and 2.7. Hence, $pvd(G) \geq w(G) - 1$. Suppose $w(G) \geq 4$, then $K_{w(G)}$ is a subgraph of G. Then $pvd(G) \geq pvd(K_{w(G)}) = w(G)$ by Theorem 2.9. □

Two known and useful results are stated as follows.

Theorem 2.11 *[15] Let G be a graph of order n. Then $\chi(G) = n - 1$ if and only if $w(G) = n - 1$.*

Theorem 2.12 *[16] Let G be a graph of order $n \geq 5$. Then $\chi(G) = n - 2$ if and only if $w(G) = n - 2$ or G is isomorphic to $C_5 \vee K_{n-5}$.*

Theorem 2.13 *Let G be a connected graph of order $n \geq 5$. Then $pvd(G) = n-1$ if and only if $w(G) = n-1$.*

Proof. Note that G is noncomplete. If $pvd(G) = n-1$, then $\chi(G) \geq n-1$ by Theorem 2.6. Since $\chi(G) = n$ if and only if G is complete, we have $\chi(G) = n-1$, and then $w(G) = n-1$ by Theorem 2.11. If $w(G) = n-1 \geq 4$, then $\chi(G) = n-1$ by Theorem 2.11. Then $pvd(G) = n-1$ by Theorem 2.6 and Lemma 2.10. $\square$

Theorem 2.14 *Let G be a connected graph of order $n \geq 7$. Then $pvd(G) = n-2$ if and only if $w(G) = n-2$ or G is isomorphic to $C_5 \vee K_{n-5}$.*

Proof. Note that G is noncomplete. If $pvd(G) = n-2$, then $\chi(G) \geq n-2$ and $w(G) \leq n-2$. Then $\chi(G) = n-2$ by Theorem 2.11. Hence $w(G) = n-2$ or G is isomorphic to $C_5 \vee K_{n-5}$ by Theorem 2.12. On the contrary, if $w(G) = n-2 \geq 5$, then $pvd(G) \geq n-2$ by Lemma 2.10. By Theorems 2.9 and 2.13, we have $pvd(G) = n-2$. If G is isomorphic to $C_5 \vee K_{n-5}$, then any two adjacent vertices of G has at least two common neighbors. Then $pvd(G) = \chi(G) = n-2$ by Lemma 2.6. $\square$

3 Sufficient conditions for $pvd(G) = \chi(G)$

In this section, we give some sufficient conditions for a graph G with $pvd(G) = \chi(G)$ and prove that almost all graphs G have $pvd(G) = \chi(G)$ and $pvd(\overline{G}) = \chi(\overline{G})$. In addition, we present an equivalent assertion of the Four Color Theorem.

Theorem 3.1 *Let G be a connected graph. If $\delta(G) \geq \frac{n+2}{2}$, then $pvd(G) = \chi(G)$.*

Proof. If $\delta(G) \geq \frac{n+2}{2}$, then for any two adjacent vertices of G, there are at least $\frac{n+2}{2} \times 2 - n = 2$ common neighbors if n is even, and at least $\frac{n+3}{2} \times 2 - n = 3$ common neighbors if n is odd. Then by Lemma 2.6 we have $pvd(G) = \chi(G)$. $\square$

A graph G is *color critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such graphs were first investigated by Dirac ([14]). Here, for simplicity, we abbreviate the term "color critical" to "critical". A k-critical graph is one that is k-chromatic and critical. Two useful results are stated as follows.

Theorem 3.2 *[14] A k-chromatic graph contains a k-critical subgraph.*

Theorem 3.3 *[14] Let G be a connected $(k+1)$-critical graph. Then $\delta(G) \geq k$.*

Theorem 3.4 *Let G be a connected graph of order n. If $\chi(G) \geq \frac{n+4}{2}$, then $pvd(G) = \chi(G)$.*

Proof. Let $\chi(G) = k$. Then G has a k-critical subgraph by Theorem 3.2. Denote the subgraph by H. Then $\delta(H) \geq k-1$ by Theorem 3.3. Since $\chi(G) = k \geq \frac{n+4}{2}$, we have $\delta(H) \geq k-1 \geq \frac{n+2}{2} \geq \frac{|V(H)|+2}{2}$. Then $pvd(H) = \chi(H)$ by Theorem 3.1. Since $pvd(H) \leq pvd(G) \leq \chi(G)$ by Lemma 2.1 and Theorem 2.3, and $\chi(G) = \chi(H)$, we have $pvd(G) = \chi(G)$. □

We can immediately get an equivalent statement of Theorem 3.4 by Theorem 2.3.

Theorem 3.5 *Let G be a connected graph of order n, and k be an integer with $k \geq \frac{n+4}{2}$. Then $pvd(G) = k$ if and only if $\chi(G) = k$.*

Note that since the converse proposition of Theorem 3.4 is not necessarily true, there may be some omissive conditions in the use of Theorem 3.5 to characterize the graphs with some specific pvd-numbers. For example, for the case $k = n-1$, $pvd(G) = n-1$ if and only if $\chi(G) = n-1$ when $n-1 \geq \frac{n+4}{2}$, that is, $n \geq 6$. However, when $n = 5$, the proposition that $pvd(G) = 4$ if and only if $\chi(G) = 4$ still holds by Theorem 2.13.

Let G be a nontrivial connected graph. For each edge $e = xy \in E(G)$, we add two new vertices u, v such that u, v are common neighbors of x, y and u, v are nonadjacent. Denote the set of new added vertices to an edge xy by S_{xy}. The resulted graph is called the *associate graph* of G and denote it by G^*. Note that the set of all new added vertices to G is independent. In addition, a graph G is called a *wing graph* if there exists a graph H such that $G = H^*$, namely that any two adjacent vertices of degrees at least 3 of G have two common neighbors of degrees 2. Then we have the following result.

Lemma 3.6 *If G is a connected graph, then $pvd(G^*) = \chi(G)$.*

Proof. Since G is nontrivial and connected, we have $\chi(G) \geq 2$. Let $V(G^*) = V(G) \cup V'$, where V' is set of all new added vertices. Since any two adjacent vertices of G has at least two common neighbors in G^*, they receive different colors in any optimal pvd-coloring of G^* by Lemma 2.5. Then $\chi(G) \leq pvd(G^*)$. Next we show $pvd(G^*) \leq \chi(G)$. Let c be a proper coloring G. Define a coloring c^* of G^* such that $c^*(v) = c(v)$ for $v \in V(G)$ and $c^*(v) = 1$ for $v \in V'$. Noth that $|\Gamma(c^*)| = |\Gamma(c)|$. Let x, y be any two vertices of G^*. Suppose $\{x, y\} \subseteq V(G)$. If x, y are nonadjacent, then $N_G(x)$ is an x-y proper vertex-cut. If x, y are adjacent, consider the set $F = \{N_G(x) \setminus \{y\}\} \cup S_{xy}$. It is obvious that F is an x-y vertex-cut. Since each vertex in S_e is only adjacent to x and y, then F is proper. Since $c^*(x) \neq c^*(y)$, we have at least one vertex of $\{x, y\}$, say x, whose color is different from 1. Then $F \cup \{x\}$ is proper and F is an x-y proper vertex-cut. Suppose that one vertex of $\{x,y\}$, say x, belongs to V'. If x,y are nonadjacent, then $N_{G^*}(x)$ is an x-y proper vertex-cut. If x,y are adjacent, let z be another neighbor of x. We have $c^*(z) \neq c^*(y)$ by the definition of c^*. Then $\{z\}$ is an x-y proper vertex-cut. Hence c^* is a pvd-coloring of G^* and then $pvd(G^*) \leq \chi(G)$. $\square$

Corollary 3.7 *Let G be a connected graph. If $\chi(G) \geq 3$, then $pvd(G^*) = \chi(G^*)$.*

Proof. Let c be a proper coloring of G. Define a coloring c^* of G^* as follows. Let $c^*(v) = c(v)$ for $v \in V(G)$. For each edge xy of G, since $|\Gamma(c)| \geq 3$, we can find a color of $\Gamma(c) \setminus \{c(x), c(y)\}$. Assign the color to each vertex of S_{xy}. Then $\Gamma(c^*) = \Gamma(c)$. It is easy to check that c^* is a proper coloring of G^*, and then $\chi(G^*) \leq \chi(G)$. Since G is a subgraph of G^*, we have $\chi(G) \leq \chi(G^*)$. Then $\chi(G) = \chi(G^*)$. Hence $\chi(G^*) = pvd(G^*)$ by Theorem 3.6 . $\square$

The authors of [4, 22] introduced and studied the rainbow vertex-disconnection of graphs. For a connected graph G, it is obvious that $pvd(G) \leq rvd(G)$. However, there is no directed relation between $rvd(G)$ and $\chi(G)$. We continue to study the graphs G with $pvd(G) = \chi(G)$ with the help of $rvd(G)$. The following results are from [4].

Theorem 3.8 *[4] Let G be a connected graph of order n. Then $rvd(G) = n$ if and only if any two vertices has at least two common neighbors.*

Theorem 3.9 *[4] Almost all graphs G of order n have $rvd(G) = rvd(\overline{G}) = n$.*

By Lemma 2.6, Theorems 3.8 and 3.9, we can get the following two results immediately.

Corollary 3.10 *Let G be a connected graph of order n. If $rvd(G) = n$, then $pvd(G) = \chi(G)$.*

Theorem 3.11 *Almost all graphs G have $pvd(G) = \chi(G)$ and $pvd(\overline{G}) = \chi(\overline{G})$.*

Referring to [6], we know that for a graph G, the pvd-coloring of G is a global coloring and the classical proper vertex coloring is a local coloring. It is somewhat interesting that $pvd(G) = \chi(G)$ holds in so many classes of graphs G. We next give an equivalent statement of the Four Color Theorem. It is known that there are several equivalent statements of the Four Color Theorem in graph theory, one of which is as follows.

Theorem 3.12 *[8] The Four Color Theorem*
Every loopless planar graph is 4-colorable.

If G is a loopless planar graph, then G^* is also a loopless planar graph by the definition of an associate graph. By Lemma 3.6, we have the following equivalent statement.

Theorem 3.13 *Every loopless planar wing graph is 4-proper vertex-disconnection colorable.*

4 The relation between $pd(G)$ and $pvd(L(G))$

Given a connected graph G and its line graph $L(G)$, the authors of [4] showed that $rd(G) \leq rvd(L(G))$. We next give the same relation between $pd(G)$ and $pvd(L(G))$.

Theorem 4.1 *Let G be a connected graph and $L(G)$ be the line graph of G. Then $pd(G) \leq pvd(L(G))$.*

Proof. Let c' be a pvd-coloring of the line graph $L(G)$ of G. Since there is a one-to-one correspondence between the edge-coloring of G and the vertex-coloring of $L(G)$, we can get an edge-coloring c of G according to the coloring c'. Next we will show that c is a proper disconnection coloring of G. For any two vertices u, v, if one vertex is of degree 1, then the pendent edge is an x-y proper cut. If $d(u) = d(v) = 2$ and u, v are in a triangle T, let $V(T) = \{u, v, z\}$. Then z is the common neighbor of u and v. Let $e_1 = uv$, $e_2 = uz$ and $e_3 = vz$. Let v_{e_i} $(i \in [3])$ be the vertex in $L(G)$ corresponding to the edge e_i in G. Then the vertex subset $\{v_{e_1}, v_{e_2}, v_{e_3}\}$ induces a triangle in $L(G)$. Let S be a v_{e_2}-v_{e_3} proper vertex-cut in $L(G)$. Then $v_{e_1} \in S$, and $S + v_{e_2}$ or $S + v_{e_3}$ is a proper vertex subset. Assume that $S + v_{e_2}$ is proper. Then $\{e_1, e_2\}$ is a u-v proper cut in G. For the other cases of u and v, we can always find two nonadjacent edges e_1 and e_2 in $E(G)$ such that e_1 is incident to u and e_2 is incident to v. Then v_{e_1}, v_{e_2} are nonadjacent in $L(G)$. Let S be a v_{e_1}-v_{e_2} proper vertex-cut in $L(G)$. Then $\{v_{e_1}, v_{e_2}\} \nsubseteq S$. We can get that the edge subset F, to which S corresponds in G, is a u-v proper cut. Otherwise, assume that there is a u-v path P in $G - F$. Since e_1, e_2 belong to $G - F$, the path $P + e_1 + e_2$ in $G - F$ corresponds to a v_{e_1}-v_{e_2} path in $L(G) - S$, a contradiction. $\square$

For a graph G, it is known that $\chi'(G) = \chi(L(G))$. By above theorem we have $pd(G) \leq pvd(L(G))$. However, the equality is not always true. Next we will give a sufficient condition for $pd(G) < pvd(L(G))$, and an example graph G such that the difference of $pd(G)$ and $pvd(L(G))$ can be very large. Three useful results are stated as follows.

Lemma 4.2 *[3] Let G be a connected graph, and let B be a block of G such that $pd(B)$ is maximum among all blocks of G. Then $pd(G) = pd(B)$.*

Theorem 4.3 *[3] If G is a connected graph, then $pd(G) \leq \chi'(G) - 1$.*

Theorem 4.4 *[3] Let $K_{m,n}$ be a complete bipartite graph with $2 \leq m \leq n$. Then $pd(K_{m,n}) = \lceil \frac{n}{2} \rceil$.*

Corollary 4.5 *Let G be a connected graph and $L(G)$ be the line graph of G. If $\delta(G) \geq 4$, then $pd(G) < pvd(L(G))$.*

Proof. By contradiction, suppose $pd(G) = pvd(L(G))$. Since $\chi'(G) = \chi(L(G))$, we have $pvd(L(G)) = pd(G) \leq \chi'(G) - 1 = \chi(L(G)) - 1$ by Theorem 4.3. Then $pvd(L(G)) < \chi(L(G))$. Hence for any optimal pvd-coloring c of $L(G)$, there are two adjacent vertices x, y in $L(G)$ such that $c(x) = c(y)$. Since $\delta(G) \geq 4$, the vertices x, y belong to some clique K_t ($t \geq 4$) in $L(G)$. By Theorem 2.9, the vertices x and y should have different colors under the coloring c, a contradiction, and so $pd(G) < pvd(L(G))$. □

Theorem 4.6 *For any integers a and b with $4 \leq 2a \leq b$, there is a connected graph G such that $pd(G) = a$ and $pvd(L(G)) = b$.*

Proof. Let $K_{2,2a}$ be a complete bipartite graph with bipartition (V_1, V_2), where $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, \cdots, v_{2a}\}$. Let $V_3 = \{v_{2a+1}, \cdots, v_b\}$ be a set of vertices. Note that V_3 is empty if $b = 2a$. Joint v_i to u_1 for each $2a + 1 \leq i \leq b$. The resulted graph is denoted by G. Then $pd(G) = a$ by Theorems 4.2 and 4.4. Now consider the line graph $L(G)$. Denote the vertex, to which the edge u_1v_i ($i \in [b]$) of G corresponds in $L(G)$, by p_i, and the vertex, to which the edge u_2v_i ($i \in [2a]$) of G corresponds in $L(G)$, by z_i. Then $E(L(G)) = \{p_ip_j, z_lz_s, p_lz_l, : 1 \leq i, j \leq b, 1 \leq l, s \leq 2a\}$. Since $w(L(G)) = b \geq 4$, we have $pvd(L(G)) \geq b$. Define a coloring c of $L(G)$ such that $c(p_i) = i$ for $i \in [b]$ and $c(z_j) = j$ for $j \in [2a]$. Observe that $N(v)$ is proper for each vertex $v \in L(G)$. It is easy to check that c is a pvd-coloring of $L(G)$, and then $pvd(L(G)) \leq b$. □

Since $a \leq \frac{b}{2}$, we have $b - a \geq \frac{b}{2}$. So for this graph G, we have $pvd(L(G)) - pd(G) = b - a \geq \frac{b}{2}$, which can be arbitrarily large.

5 Hardness results

In this section, we first show some NP-hardness results for the proper vertex-disconnection of graphs. Then we show that it is solvable in polynomial time to determine the pvd-number for a graph with maximum degree less than four .

5.1 NP-hardness results

The hardness results for rainbow (vertex-)disconnection and proper disconnection have been studied, see [2, 13]. The proof of the following results uses a similar technique.

At first we introduce some notation and terminology. Let X be a finite set of Boolean variables. A *truth assignment* for X is a function $T : X \mapsto \{true, false\}$. We extend T to the set $L := X \cup \{\overline{x} : x \in X\}$ by setting $T(\overline{x}) := true$ if $T(x) := false$ and vice versa ($\overline{x}$ can be regarded as the negation of x). The elements of L are called the *literals* over X. A *clause* over X is a disjunction of literals and is *satisfied* by a truth assignment if and only if at least one of its members is true. A *Boolean formula in conjunctive normal form*, abbreviated as *CNF formula*, over X is a conjunction of clauses and is *satisfiable* if and only if there is a truth assignment simultaneously satisfying all of its clauses. A CNF formula is a *3CNF formula* if each of its clauses consists of exactly three literals.

Lemma 5.1 *Given a vertex-colored graph G and two vertices s,t of G, deciding whether there is an s-t proper vertex-cut in G is NP-complete.*

Proof. Since deciding whether a given vertex subset of G is an s-t proper vertex-cut can be done in polynomial-time, this problem is in NP. We now show that the problem is NP-complete by giving a polynomial reduction from the 3-SAT problem to this problem. Given a 3CNF formula $\phi = \wedge_{i=1}^{m} c_i$ over n variables $x_1, x_2, \cdots, x_n$, we construct a graph G_ϕ with two special vertices s,t and a vertex-coloring f such that there is an s-t proper vertex-cut in G_ϕ if and only if ϕ is satisfiable. Let $L_{c_i}(x)$ denote the location of literal x in clause c_i for $i \in [m]$.

We define G_ϕ as follows:

$$V(G_\phi) = \{c_i, u_{i,k}, v_{i,k}, z_{i,k} : i \in [m], k \in [3]\} \cup \{x_j, \overline{x}_j : j \in [n]\} \cup \{s, t, b\}.$$

$$\begin{aligned} E(G_\phi) = \{&x_j v_{i,k}, \overline{x}_j u_{i,k} : \text{If } x_j \in c_i \text{ and } L_{c_i}(x_j) = k, i \in [m], j \in [n], k \in [3]\} \\ \cup \{&x_j u_{i,k}, \overline{x}_j v_{i,k} : \text{If } \overline{x}_j \in c_i \text{ and } L_{c_i}(\overline{x}_j) = k, i \in [m], j \in [n], k \in [3]\} \\ \cup \{&x_j \overline{x}_j, s x_j, s \overline{x}_j, s z_{i,1} : i \in [m], j \in [n]\} \\ \cup \{&c_i u_{i,k}, c_i v_{i,k}, c_i z_{i,3}, c_i b : i \in [m], k \in [3]\} \\ \cup \{&v_{i,k} z_{i,k}, z_{i,1} z_{i,2}, z_{i,2} z_{i,3} : i \in [m], k \in [3]\} \\ \cup \{&t c_i, sb, bt : i \in [m]\}. \end{aligned}$$

Now we define a vertex-coloring f of G_ϕ as follows: For $i \in [m], j \in [n]$ and $k \in [3]$, let $f(x_j) = f(\overline{x}_j) = r_j$, $f(u_{i,k}) = f(v_{i,k}) = f(z_{i,k}) = r_{i,k}$, $f(s) = f(t) = r$, and $f(c_i) = f(b) = r'$. All those colors are distinct, see Figure 1.

We claim that there is an s-t proper vertex-cut in G_ϕ if and only if ϕ is satisfiable.

Suppose that there is an s-t proper vertex-cut S in G_ϕ. Since the vertex b is the common neighbor of s and t in G_ϕ, we have $b \in S$. Then $c_i \notin S$ for each $i \in [m]$ by the fact that $c_i b \in E(G_\phi)$ and $f(c_i) = f(b)$. Thus S also separates s and c_i. There exists at least one j ($j \in [n]$) such that $x_j \in S$ or $\overline{x}_j \in S$. Otherwise, we have that $\{v_{i,k} : k \in [3]\} \subseteq S$. Since $f(v_{i,k}) = f(z_{i,k})$ and $v_{i,k} z_{i,k} \in E(G_\phi)$, we have $\{z_{i,k} : k \in [3]\} \nsubseteq S$. Then there is an s-t path in $G \setminus S$, a contradiction. Since $f(x_j) = f(\overline{x}_j)$ and $x_j \overline{x}_j \in E(G_\phi)$, the vertices x_j and $\overline{x}_j$ can not simultaneously belong to S. If $x_j \in S$, set $x_j = 1$. If $\overline{x}_j \in S$, set $x_j = 0$.

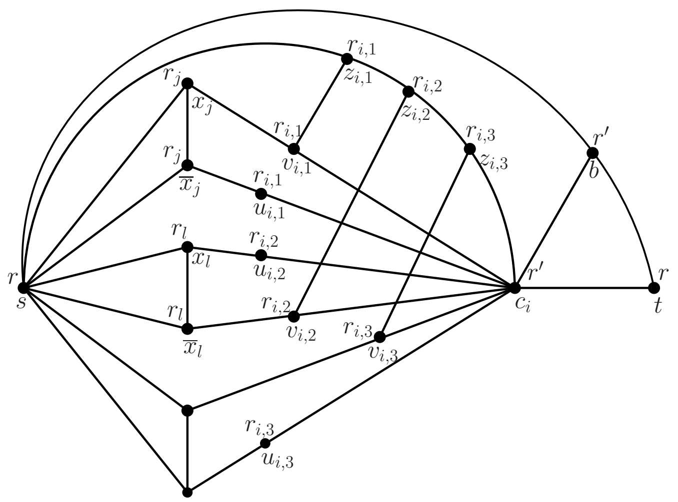

Figure 1: The literals $x_j, \overline{x}_l$ of c_i and $x_j, \overline{x}_l$ are the first and second literals, respectively.

Furthermore, if the literal associated with x_j in clause c_i is false, then $v_{i,l} \in S$ where l is the location of literal in clause c_i. Then from above discussion, we obtain that the three literals of c_i cannot be false simultaneously. Therefore, ϕ is satisfiable.

Suppose that ϕ is satisfiable. We now try to find an s-t proper vertex-cut S in G_ϕ under the coloring f. Clearly, $b \in S$ and $c_i \notin S$ for each $i \in [m]$. For any variable x_j ($j \in [n]$), if $x_j = 0$, let $\overline{x}_j \in S$. In this case, if $x_j \in c_i$, then x_j is adjacent to $v_{i,k}$ in G_ϕ and let $v_{i,k} \in S$. If $\overline{x}_j \in c_i$, then x_j is adjacent to $u_{i,k}$ in G_ϕ and let $\{u_{i,k}, z_{i,k}\} \subseteq S$. For any variable x_j ($j \in [n]$), if $x_j = 1$, let $x_j \in S$. In this case, if $x_j \in c_i$, then $\overline{x}_j$ is adjacent to $u_{i,k}$ in G_ϕ and let $\{u_{i,k}, z_{i,k}\} \subseteq S$. If $\overline{x}_j \in c_i$, then $\overline{x}_j$ is adjacent to $v_{i,k}$ in G_ϕ and let $v_{i,k} \in S$. Therefore, for any literal x of c_i with $L_{c_i}(x) = k$, if x is false, then $v_{i,k} \in S$; if x is true, then $\{u_{i,k}, z_{i,k}\} \subseteq S$. Note that there is always one vertex of $\{v_{i,k}, z_{i,k}\}$ belonging to S. Since each c_i is satisfied, the set S is an s-t vertex-cut. Observe that S is proper. Then S is an s-t proper vertex-cut. The proof is complete. □

Theorem 5.2 *Given a vertex-colored graph G, deciding whether G is proper vertex-disconnected is NP-complete.*

Proof. Let G be a given vertex-colored graph. Note that a graph is proper vertex-disconnected if and only if for any two vertices there is a proper vertex-cut separating them. For any two vertices u, v, deciding whether a given vertex subset of G is a u-v proper vertex-cut can be done in polynomial time. Hence this problem is in NP.

For the vertex-colored graph G_ϕ defined above, we can get that G_ϕ is proper vertex-disconnected if and only if G_ϕ has an s-t proper vertex-cut. Since the necessity is obvious, we show the sufficiency below. For any two vertices x, y of G_ϕ, if one vertex, say y, belongs to $\{x_j, \overline{x}_j, u_{i,k}, v_{i,k}, z_{i,k}, b : j \in [n], i \in [m], k \in [3]\}$, observe that $N(y)$ is proper.

Then if $x \notin N(y)$, then the set $N(y)$ is an x-y proper vertex-cut; if $x \in N(y)$, then the set $N(y) \setminus \{x\}$ is an x-y proper vertex-cut. If $x = c_i$ and $y = c_j$ for $i, j \in [m]$, consider the clause c_i. Let x be the first literal of clause c_i, where $x = x_l$ or $\overline{x}_l$ for some $l \in [n]$. Then the set $F_i = \{u_{i,1}, u_{i,2}, u_{i,3}, z_{i,1}, v_{i,2}, v_{i,3}, x, t, b\}$ is a c_i-c_j $(i \neq j)$ proper vertex-cut. Furthermore, F_i is also an s-c_i proper vertex-cut, and $F_i \setminus \{t\}$ is a t-c_i proper vertex-cut. Thus, any pair of vertices have a proper vertex-cut in G_ϕ. The proof is complete by Lemma 5.1. □

In the classical vertex-coloring, for a fixed positive integer k, the *graph k-colorability problem* is the problem to determine whether G is k-colorable. We now define the *graph k-proper vertex-disconnection colorability problem* as the problem to determine whether G is k-proper vertex-disconnection colorable for a fixed positive integer k. Referring to [17, 18], there are the following results.

Theorem 5.3 *[18] For a fixed integer $k \geq 3$, the graph k-colorability problem is NP-complete.*

Theorem 5.4 *[17] Graph 3-colorability is NP-complete, even though the graph G is planar with $\Delta = 4$.*

Theorem 5.5 *For a fixed positive integer $k \geq 3$, the graph k-proper vertex-disconnection colorable problem is NP-hard, even if $k = 3$ and the graph G is a planar graph with $\Delta(G) = 12$.*

Proof. Firstly, this problem is not in NP. For a fixed positive integer $k \geq 3$, let a graph G be a yes-instance of the problem. The solution of the instance is a pvd-coloring c of G with $|\Gamma(c)| \leq k$. However, deciding whether the coloring c is a pvd-coloring of G, namely whether G is proper vertex-disconnected, cannot be solved in polynomial time by Theorem 5.2. Secondly, note that a graph G is k-colorable if and only if $\chi(G) \leq k$. By Lemma 3.6, we have $pvd(G^*) = \chi(G)$. Then a graph G is k-colorable if and only if G^* is k-proper vertex-disconnection colorable. At this time, if G is a planar graph with $\Delta(G) = 4$, the corresponding associate graph G^* is a planar graph with $\Delta(G^*) = 12$. Then the proof is complete by Theorems 5.3 and 5.4. □

5.2 Polynomial time solvable results

At first, we give the pvd-number for a 3-regular noncomplete graph.

Lemma 5.6 *If G is a 3-regular noncomplete graph, then $pvd(G) \leq 2$.*

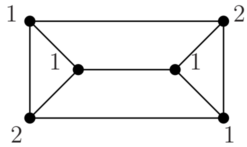

Figure 2: The graph G_0 and the vertex-coloring.

Proof. Firstly, if G is the graph G_0 (see Figure 2), then we color G_0 with two colors as shown in Figure 2. It is easy to check that G is proper vertex-disconnected. Next, we consider the graph $G \neq G_0$. If G has no triangle, then $pvd(G) = 1$ by Theorem 2.8. Suppose that G has at least one triangle. We say that a triangle is *isolated* in G if there is no triangle in G having common vertex with it. If there are two triangles sharing a same edge, then denote the structure by *θ-type structure.* Because G is 3-regular and noncomplete, then for any triangle, it either is isolated or belongs to a θ-type structure. In addition, observe that any two different θ-type structures are vertex-disjoint. Let $T = \{T_1, T_2, \cdots, T_s\}$ be the set of all isolated triangles and θ-type structures of G, where T_i is an isolated triangle or a θ-type structure and $V(T_i) \cap V(T_j) = \emptyset$ $(i \neq j)$. Next we give a vertex-coloring $c : V(G) \mapsto [2]$ as follows.

Coloring Algorithm

Input: A connected 3-regular noncomplete graph G with $T = \{T_1, T_2, \cdots, T_s\}$.
Output: A coloring c of G and vertex subsets U_1, U_2, V_2.
1: Set $i = 1$ and the vertex subsets $U_1 = U_2 = V_2 = \emptyset$.
2: **For** $i = 1$ **to** s do:
3: **If** T_i is a triangle, then
4: **if** we can find a vertex, say v_i, of T_i nonadjacent to each vertex of V_2, then assign v_i with color 2. Set $V_2 := V_2 \cup \{v_i\}$ and $i := i + 1$.
5: **else** choose two vertices, say v_i, z_i, of T_i and assign color 2 to them. Assign color 1 to the last vertex, say u_i, of T_i. Set $U_1 := U_1 \cup \{u_i\}$, $U_2 := U_2 \cup \{v_i, z_i\}$, and $i := i + 1$.
6: **else** T_i is a θ-type structure, then find a vertex, say v_i of degree 3 in T_i and assign it with color 2. Set $V_2 := V_2 \cup \{v_i\}$ and $i := i + 1$.
7: Assign color 1 to the remaining vertices.

If coloring c is a pvd-coloring of G, then we have $pvd(G) \leq 2$. Next we begin to check it.

By the Coloring Algorithm, we know that no triangle in G is monochromatic. A triangle is called *general* if it is isolated, and has two vertices with color 1 and one vertex with color 2. A triangle is called *particular* if it is isolated, and has two vertices with

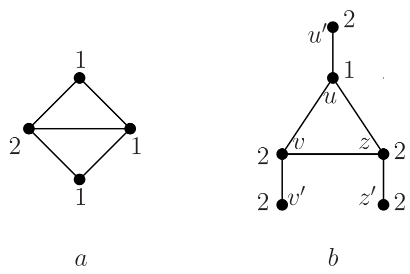

Figure 3: (a) A θ-type structure, and (b) a particular triangle under the coloring c.

color 2 and one vertex with color 1. Let $V(G) = U_1 \cup U_2 \cup V_1 \cup V_2$, where U_1, U_2 and V_2 are obtained by above algorithm. Note that U_i ($i \in [2]$) consists of all vertices with color i in particular triangles, and V_2 consists of all vertices with color 2 in general triangles and θ-type structures. Furthermore, the subsets U_1 and V_2 are separately independent. Let T_v ($v \in V(G)$) denote the isolated triangle to which the vertex v belongs. For any isolated triangle T, suppose that $V(T) = \{u, v, z\}$. Denote the third neighbor of u, v and z by u',v' and z'. We call $\{u', v', z'\}$ the *neighbor set* of T. Then we have the following claims.

Claim 1: the neighbor set of a particular triangle is independent.

Suppose that T is a particular triangle with $V(T) = \{u, v, z\}$ and neighbor set $\{u', v', z'\}$. Then $c(u') = c(v') = c(z') = 2$ by Step 6, as shown in Figure 3(b). We first show that each vertex of $\{u', v', z'\}$ is in a general triangle. Since $c(u') = c(v') = c(z') = 2$, each vertex of $\{u', v', z'\}$ is in a triangle. If one vertex, say u', belongs to a θ-type structure T, then $d_T(u') = 3$ by Step 7. Then $d_G(u') \geq 4$, a contradiction. Hence each vertex of $\{u', v', z'\}$ is in an isolated triangle. When we color T_u, these triangles $T_{u'}, T_{v'}$ and $T_{z'}$ have already been colored. Because when we begin to color $T_{u'}$, none of the vertices in $N(u')$ are colored. Then we can assign color 2 to u' by Step 5, and triangle $T_{u'}$ is general. Similarly, we can get that the triangles $T_{v'}$ and $T_{z'}$ are general. Since T_u is an isolated triangle, the vertices u', v' and z' are different. Suppose there are two vertices in $\{u', v', z'\}$ are adjacent, say that u', v' are adjacent. Since $d(u') = d(v') = 3$, we have $T_{u'} = T_{v'}$. Since $c(u') = c(v') = 2$, the triangle $T_{u'}$ is particular, a contradiction. Thus $\{u', v', z'\}$ is an independent set.

Claim 2: For each vertex $v \in V_1 \cup U_2$, the set $N(v)$ is proper.

Suppose that $v \in V_1$. Then $c(v) = 1$. If v is not in a triangle, then $N(v)$ is independent and proper. If v belongs to exactly one triangle, then the triangle T_v is general by the definition of V_1. Let $V(T_v) = \{v, v_1, v_2\}$ and the third neighbor of v be v_3. Then $\{c(v_1), c(v_2)\} = \{1, 2\}$ and v_3 is nonadjacent to v_1 and v_2. Then $N(v) = \{v_1, v_2, v_3\}$ is proper. If v belongs to two triangles, then v belongs to a θ-type structure T. Assume that $N(v) = \{v_1, v_2, v_3\}$ and $d_T(v_2)$=3. Then T is colored as shown in Figure 3(a), namely that $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$. Hence $N(v)$ is proper. Suppose $v \in U_2$. Then

$c(v) = 2$ and v is in a particular triangle. Then $N(v)$ consists of two nonadjacent vertices with color 2 and one vertex with color 1 (see Figure 3(b)). Thus $N(v)$ is proper.

Now we show that c is a pvd-coloring of G. For any two vertices x, y, we discuss them case by case. If one vertex, say x, is in $V_1 \cup U_2$, then $N(x)$ is proper by Claim 2. Then $N(x)$ is an x-y proper vertex-cut if x, y are nonadjacent, and $N(x) \setminus \{y\}$ is an x-y proper vertex-cut if x, y are adjacent. If one vertex, say x, is in U_1, then x is in a particular triangle. Let $V(T_x) = \{x, v, z\}$ and the neighbor set of T_x be $\{x', v', z'\}$ such that p, p' are adjacent for each vertex $p \in \{x, v, z\}$. If $y = x'$, then $\{v', z'\}$ is an x-y proper vertex-cut by Claim 1. If $y = v'$, then $\{x', v, z'\}$ is an x-y proper vertex-cut. If $y = z'$, then $\{x', v', z\}$ is an x-y proper vertex-cut. If $y \notin \{x', v', z'\}$, then $\{x', v', z'\}$ is an x-y proper vertex-cut.

If $x, y \in V_2$, then x, y are nonadjacent. If one vertex, say x, is in a θ-type structure T, then v is a vertex of degree 3 in T by the fact that $c(v) = 2$. Denote these two vertices of degree 2 in T by u, z. Then $\{u, z\}$ is an x-y proper vertex-cut. Now assume that x and y belong to two different general triangles. Let $V(T_x) = \{x, u, v\}$ and the neighbor set of T_x be $\{x', u', v'\}$ such that p, p' are adjacent for each vertex $p \in \{x, u, v\}$. Then x', u' and v' are three different vertices. Suppose $y = u'$. If x', v' are nonadjacent, then $\{u, x', v'\}$ is an x-y proper vertex-cut. If x', v' are adjacent, consider the adjacent relation between x' and u'. If x', u' are adjacent, then v', u' are nonadjacent by the fact that $G \neq G_0$. Denote the third neighbor of v' by z, then $\{u, x', z\}$ is an x-y proper vertex-cut. If x', u' are nonadjacent, denote the third neighbor of x' by z. If z, v' are nonadjacent, then $\{u, v', z\}$ is an x-y proper vertex-cut. If z', v' are adjacent, then $\{u, z\}$ is an x-y proper vertex-cut. For the case $y = v'$, we can discuss it similarly as above. Next assume $y \notin \{u', v'\}$. If x', v' are nonadjacent, then $\{u, x', v'\}$ is an x-y proper vertex-cut. If x', v' are adjacent, consider the adjacent relation of x', u'. If x', u' are nonadjacent, then $\{x', u', v\}$ is an x-y proper vertex-cut. If x', u' are adjacent, then u', v' are nonadjacent by the fact that $G \neq G_0$. Then $\{u', v'\}$ is an x-y proper vertex-cut. Thus c is a pvd-coloring of G. $\square$

Let H be a graph as shown in Figure 4, in which v is called the *key vertex* of H. Now we give the pvd-number of a general noncomplete graph with maximum degree less than four .

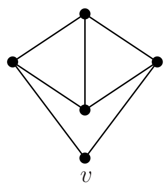

Figure 4: The graph H

Lemma 5.7 *If G is a connected noncomplete graph with $\Delta(G) \leq 3$, then $pvd(G) \leq 2$.*

Proof. If G is triangle-free, then $pvd(G) = 1$ by Theorem 2.8. Suppose that G has at least one triangle. Let G' be a maximal connected subgraph of G with $\delta(G') \geq 2$. Then $pvd(G) = pvd(G')$ by Lemma 2.2. Let $\{u_1, \cdots, u_t\}$ be the set of vertices of degree 2 in G' and $H_1, \cdots, H_t$ be t copies of H such that the key vertex of H_i is v_i ($i \in [t]$). We construct a new graph G'' by connecting v_i and u_i for each $i \in [t]$. Then G'' is a 3-regular graph. By Lemma 5.6, $pvd(G'') \leq 2$. Since G' is a subgraph of G'', we have $pvd(G) = pvd(G') \leq 2$. □

Theorem 5.8 *Let G be a connected graph with $\Delta(G) \leq 3$. Then $pvd(G) = 4$ if and only if G is isomorphic to K_4.*

Proof. The sufficiency is obvious by Theorem 2.9. Suppose $pvd(G) = 4$. Then G is complete. Otherwise, we have $pvd(G) \leq 2$ by Lemma 5.7, a contradiction. Since $\Delta(G) \leq 3$, G is isomorphic to K_4. □

Theorem 5.9 *Determining the pvd-number of a graph with maximum degree less than four is solvable in polynomial time.*

Proof. Let G be a connected graph with $\Delta(G) \leq 3$. Then $1 \leq pvd(G) \leq \chi(G) \leq \Delta(G) + 1 \leq 4$ by Theorem 2.3. Firstly, decide whether G is isomorphic to K_4. If G is isomorphic to K_4, then $pvd(G) = 4$; If not, decide whether G has triangles. If G is triangle-free, then $pvd(G) = 1$ by Theorem 2.8. If G has at least one triangles, then $pvd(G) = 2$ by Lemma 2.7 and Theorem 5.7. Note that there is no graph G with $pvd(G) = 3$. □

Furthermore, we can also obtain a polynomial time algorithm to find a pvd-coloring for a connected graph with maximum degree less than four . Suppose that G is a connected graph with $\Delta(G) \leq 3$. If G is triangle-free, then the coloring c, in which $c(v) = 1$ for each $v \in V(G)$, is a pvd-coloring of G. If G is complete, then the coloring such that each vertex has a different color is a pvd-coloring of G. Now suppose that G is noncomplete and has at least one triangle. Let G' and G'' be the two graphs as shown in the proof of Lemma 5.7. Then we can obtain a pvd-coloring c'' of G'' by the Coloring Algorithm in the proof of Lemma 5.6. Because G' is a subgraph of G'', let c' be the coloring that is obtained by restricting c'' to G'. Then c' is a pvd-coloring of G'. Let $V(G) = V(G') \cup V'$, and let c be a coloring of G such that $c(v) = c'(v)$ if $v \in V(G')$ and $c(v) = 1$ if $v \in V'$. It is easy to check that c is a pvd-coloring of G. Thus we obtain a pvd-coloring of G in polynomial time.

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