Bounds for the rainbow disconnection numbers of graphs*

Xuqing Bai, Zhong Huang, Xueliang Li

Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

Email: baixuqing0@163.com, 2120150001@mail.nankai.edu.cn, lxl@nankai.edu.cn

Abstract

An edge-cut of an edge-colored connected graph is called a rainbow cut if no two edges in the edge-cut are colored the same. An edge-colored graph is rainbow disconnected if for any two distinct vertices u and v of the graph, there exists a rainbow cut separating u and v. For a connected graph G, the rainbow disconnection number of G, denoted by $\mathrm{rd}(G)$, is defined as the smallest number of colors required to make G rainbow disconnected.

In this paper, we first give some upper bounds for $\mathrm{rd}(G)$, and moreover, we completely characterize the graphs which meet the upper bounds of the Nordhaus-Gaddum type result obtained early by us. Secondly, we propose a conjecture that for any connected graph G, either $\mathrm{rd}(G) = \lambda^+(G)$ or $\mathrm{rd}(G) = \lambda^+(G) + 1$, where $\lambda^+(G)$ is the upper edge-connectivity, and prove that the conjecture holds for many classes of graphs, which supports this conjecture. Moreover, we prove that for an odd integer k, if G is a k-edge-connected k-regular graph, then $\chi'(G) = k$ if and only if $\mathrm{rd}(G) = k$. It implies that there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G)$ for odd k, and also there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G) + 1$ for odd k. For $k = 3$, the result gives rise to an interesting result, which is equivalent to the famous Four-Color Problem. Finally, we give the relationship between $\mathrm{rd}(G)$ of a graph G and the rainbow vertex-disconnection number $\mathrm{rvd}(L(G))$ of the line graph $L(G)$ of G.

Keywords: edge-coloring, edge-connectivity, rainbow disconnection coloring (number), line graph

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1 Introduction

All graphs considered in this paper are finite and undirected, and all graphs are simple unless specifically stated. Let $G = (V(G), E(G))$ be a nontrivial connected graph with vertex-set $V(G)$ and edge-set $E(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ ($N_G[v]$) denote the *degree* and the *open (closed) neighborhood* of v in G (or simply $d(v)$ and $N(v)$ ($N[v]$) respectively, when the graph G is clear from the context). We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of G, respectively. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A *u-v-path* is a path with ends u and v. For a positive integer k, we always write $[k]$ for the set $\{1, 2, \cdots, k\}$ of integers. For any notation or terminology not defined here, we follow those used in [8, 9].

Let G be a graph with an *edge-coloring* c: $E(G) \to [k]$, where adjacent edges may be colored the same. When adjacent edges of G receive different colors by c, the edge-coloring c is called *proper*. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper edge-coloring of G. A famous theorem due to Vizing [23] asserts that for any simple graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. If $\chi'(G) = \Delta(G)$, then G is said to be in *Class* 1; if $\chi'(G) = \Delta(G) + 1$, then G is said to be in *Class* 2.

We know that there are two ways to study the connectivity of graphs, one is to use paths, and the other is to use cuts. The rainbow connection using paths has been studied extensively; see for examples, papers [11, 16, 18] and book [17] and the references therein. So, it is natural to consider rainbow edge-cuts for the colored connectivity in edge-colored graphs. In [10], Chartrand et al. first discussed the rainbow edge-cuts by introducing the concept of rainbow disconnection of graphs. In [4] we call all of them global colorings of graphs since they relate global structural property: connectivity of graphs.

Recall that an *edge-cut* of a connected graph G is a set F of edges such that $G - F$ is disconnected. For two distinct vertices u and v of G, let $\lambda_G(u, v)$ (or simply $\lambda(u, v)$ when the graph G is clear from the context) denote the minimum number of edges in an edge-cut F such that u and v lie in different components of $G - F$. The minimum cardinality of an edge-cut of G is the *edge-connectivity* of G, denoted by $\lambda(G)$ (i.e., $\lambda(G)$ is the minimum value of $\lambda_G(u, v)$ taken over all pairs of distinct vertices u, v); whereas the maximum value of $\lambda_G(u, v)$ taken over all pairs of distinct vertices u, v is the *upper edge-connectivity* of G, denoted by $\lambda^+(G)$. This graph parameter $\lambda^+(G)$ was introduced and extensively studied in [6, 7]. The following proposition presents another interpretation for $\lambda(u, v)$.

Proposition 1.1 [12, 13] *For every two vertices u and v in a graph G, $\lambda(u, v)$ is equal to the maximum number of pairwise edge-disjoint u-v-paths in G.*

The following concept of rainbow disconnection of graphs was introduced by Chartrand et al. in [11]. An edge-cut R of an edge-colored connected graph G is called a *rainbow cut* if no two edges in R are colored the same. Let u and v be two distinct vertices of G. A rainbow edge-cut R of G is called a *u-v-rainbow cut* if u and v belong to different components of $G - R$. An edge-colored graph G is called *rainbow disconnected* if for every two distinct vertices u and v of G, there exists a u-v-rainbow cut in G. In this case, the edge-coloring is called a *rainbow disconnection coloring* (or rd-*coloring* for short) of G. For a connected graph G, the *rainbow disconnection number* of G, denoted by $\mathrm{rd}(G)$, is defined as the smallest number of colors required to make G rainbow disconnected. An *optimal* rd-*coloring* of G is an rd-coloring with $\mathrm{rd}(G)$ colors.

Similarly, in [3, 19] we introduce the concept of rainbow vertex-disconnection of graphs. For a connected and vertex-colored graph G, let x and y be two vertices of G. If x and y are nonadjacent, then an *x-y-vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $G - S$. If x and y are adjacent, then an *x-y-vertex-cut* is a subset S of $V(G)$ such that x and y belong to different components of $(G - xy) - S$. A vertex subset S of G is *rainbow* if no two vertices of S have the same color. An *x-y-rainbow vertex-cut* is an x-y-vertex-cut S such that if x and y are nonadjacent, then S is rainbow; if x and y are adjacent, then $S + x$ or $S + y$ is rainbow. Here one can see that if x and y are adjacent, this really causes some inconvenience for the definition of rainbow x-y-vertex-cuts. This is done just for in accordance with the common sense connectivity of graphs.

A vertex-colored graph G is called *rainbow vertex-disconnected* if for any two distinct vertices x and y of G, there exists an x-y-rainbow vertex-cut. In this case, the vertex-coloring c is called a *rainbow vertex-disconnection coloring* (or rvd-*coloring* for short) of G. For a connected graph G, the *rainbow vertex-disconnection number* of G, denoted by $\mathrm{rvd}(G)$, is the minimum number of colors required to make G rainbow vertex-disconnected. An *optimal* rvd-*coloring* of G is an rvd-coloring with $\mathrm{rvd}(G)$ colors.

This paper is organized as follows. In Section 2, we obtain some upper bounds for $\mathrm{rd}(G)$, and moreover, we completely characterize the graphs which meet the upper bound of the Nordhaus-Gaddum type result obtained early by us. In Section 3, we propose a conjecture that for any connected graph G, $\lambda^+(G) \leq \mathrm{rd}(G) \leq \lambda^+(G) + 1$, and prove that the conjecture holds for many classes of graphs, which supports this conjecture. Furthermore, for all odd $k \geq 1$, we give a sufficient and necessary condition for a k-edge-connected k-regular graph G with $\mathrm{rd}(G) = k$, from which we get that there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G)$ for odd k, and also there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G)+1$ for odd k. Moreover, we get a relationship between face-colorings and rainbow disconnection colorings of 3-connected cubic plane graphs. Finally, we give the relationship between $\mathrm{rd}(G)$ of G and $\mathrm{rvd}(L(G))$ of the line graph $L(G)$ of G.

2 Some upper bounds for rd(G)

In this section, we obtain some upper bounds for the rainbow disconnection number of a graph G. Let G be a graph and X a proper subset of $V(G)$. To *shrink* X is to delete all the edges between vertices of X and then identify the vertices of X into a single vertex. We denote the resulting graph by G/X. For each vertex x of G, let E_x be a set of all edges incident with x in G. For an edge-colored graph G, a vertex v of G is *proper* if the colors of edges incident with v are distinct in G. Now we give some upper bounds for $\mathrm{rd}(G)$ in terms of the upper edge-connectivity. First, we give some useful lemmas and introduce a shrinking operation.

Lemma 2.1 [10] *If G is a nontrivial connected graph, then*

$$\lambda(G) \leq \lambda^+(G) \leq \mathrm{rd}(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Lemma 2.2 [10] (i) *If G is the Petersen graph, then* $\mathrm{rd}(G)$ *is* 4.

(ii) *If $W_n = C_{n-1} \vee K_1$ is the wheel of order $n \geq 4$, then* $\mathrm{rd}(W_n) = 3$.

Lemma 2.3 [2] *For a graph G, the following results hold.*

(i) *For any vertex u of G, let $H = G - u$. Then* $\mathrm{rd}(G) \leq \Delta(H) + 1$.

(ii) *If there exists a vertex u of G such that $H = G - u$ is in Class 1 and $d_H(x) \leq \Delta(H) - 1$ for each $x \in N_G(u)$, then* $\mathrm{rd}(G) \leq \Delta(H)$.

Remark 1. From the proof of Lemma 2.3 (i), we know that there exists an rd-coloring of G using colors in $[\Delta(H)+1]$ such that each vertex is proper except the vertex u.

Lemma 2.4 [21] *Let G be a loopless multigraph with maximum degree $\Delta(G)$. Then $\chi'(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$.*

Lemma 2.5 *Let G be a loopless multigraph and H be a graph by shrinking a vertex subset of G to a single vertex h. If $C_H(u,v)$ is a u-v-edge-cut in H, where $u,v \in V(H)\setminus h$, then it is also a u-v-edge-cut in G.*

Proof. Let $H = G/Y$, where $Y \subseteq V(G)$. Assume that $C_H(u,v)$ is not a u-v-edge-cut in G, namely, there exists a u-v-path P avoiding $C_H(u,v)$ in G. Then P/Y is still a u-v-path in H avoiding $C_H(u,v)$, which is a contradiction. $\square$

We define a shrinking operation on a graph G as follows.

For a given graph G, let $\lambda^+(G) = k$ and $S = \{x | d(x) \geq k+1\}$. For fixed k and S, suppose $|S| \geq 2$. Let u,v be two vertices of S. Then we can find a minimum u-v-edge-cut $C(u,v)$ such that $|C(u,v)| \leq \lambda^+(G)$ and $G \setminus C(u,v) = C_1 \cup C_2$. Then we define the two operations o and O as follows:

$$o(\{G\}) = \begin{cases} \{G/V(C_1), G/V(C_2)\}, & \text{if } |G \cap S| \geq 2, \\ \{G\}, & \text{otherwise.} \end{cases}$$

$$O(\{G_1, G_2, \cdots, G_p\}) = \cup_{i=1}^{p} o(\{G_i\}).$$

We keep the multiple edges in each operation. Since the graph is split into two pieces when we do the operation, the operation cannot last endlessly. Hence, there exists an integer r such that $O^r(\{G\}) = O^{r+1}(\{G\})$. Finally, we get a finite set of connected graphs, where each graph has exactly one vertex with degree at least $\lambda^+(G)+1$. We call this procedure of making a graph G into such pieces the *shrinking operation* on G.

Then we derive the following theorem by the shrinking operation and Lemmas 2.4 and 2.5.

Theorem 2.6 *Let G be a loopless multigraph with upper edge-connectivity $\lambda^+(G)$. Then* $\mathrm{rd}(G) \leq \lfloor \frac{3}{2}\lambda^+(G) \rfloor$.

Proof. If $\Delta(G) = \lambda^+(G)$, then the result holds by Lemma 2.1. Otherwise, suppose that we get a family of graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ by the shrinking operation on G (if only one vertex has degree at least $\lambda^+(G)+1$, then let $\mathcal{H} = \{G\}$). Let h_i be the unique vertex of H_i with $d_{H_i}(h_i) \geq \lambda^+(G)+1$ and $H_i' = H_i - h_i$ for each $i \in [t]$. Then $\Delta(H_i') \leq \lambda^+(G)$ for each $i \in [t]$. It follows from Lemma 2.4 that $\chi'(H_i') \leq \frac{3}{2}\Delta(H_i') \leq \lfloor \frac{3}{2}\lambda^+(G) \rfloor$ for each $i \in [t]$. Then there exists a proper edge-coloring f_i' for H_i' using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$ for each $i \in [t]$. For each graph H_i ($i \in [t]$), we now define a coloring f_i of H_i as follows. Let $f_i(e) = f_i'(e)$ for each $e \in H_i'$, where $i \in [t]$. For each $u \in N_{H_i}(h_i)$ ($i \in [t]$), since $d_{H'_i}(u) \leq \Delta(H'_i) < \lfloor \frac{3}{2}\lambda^+(G) \rfloor$, there is an $a_u \in [\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$ such that the color a_u is not assigned to any edge incident with u in H_i. Define $f_i(h_iu) = a_u$ for each $i \in [t]$. We show that f_i is an rd-coloring of H_i for each $i \in [t]$. Let w and z be two distinct vertices of H_i for some $i \in [t]$. Then at least one of the vertices w and z belongs to H'_i ($i \in [t]$), say $w \in V(H'_i)$. Since the set E_w separates w and z and is rainbow under f_i in H_i ($i \in [t]$), then f_i is an rd-coloring of H_i using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$, and each vertex of H_i is proper except vertex h_i under f_i. Namely, $\mathrm{rd}(H_i) \leq \lfloor \frac{3}{2}\lambda^+(G) \rfloor$ for each $i \in [t]$.

Now we claim that we can obtain an rd-coloring of G using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$ by adjusting colorings of shrunk graphs. Suppose that F_1 and F_2 are two graphs obtained by one o operation for F in terms of two vertices x_1, x_2 of F, where $d(x_1), d(x_2) \geq \lambda^+(G) + 1$ in F. Without loss of generality, let $F_i = F/V(F_i)$ and $x_i \in F_i$ ($i \in [2]$). Suppose that for each $i \in [2]$, F_i has an rd-coloring f_i using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$. Let y_i be the vertex by shrinking vertex-set $V(F_i)$ in F ($i \in [2]$). Note that $d(y_i) \leq \lambda^+(G)$ in F_i for each $i \in [2]$. So, $y_i \neq x_i$ and the vertex y_i is proper in F_i. We can exchange the colors of f_2 for F_2 such that $c(e)|_{F_2} = c(e)|_{F_1}$ for each $e \in C(x_1, x_2)$ in F and the new coloring f'_2 is still an rd-coloring of F_2 using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$. Then we obtain a coloring f of F using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$ by letting $f(e) = f_1(e)$ if $e \in F_1$ and $f(e) = f'_2(e)$ if $e \in F_2 \setminus y_2$. We now verify that the coloring f is an rd-coloring of F. For any two vertices p, q of F, if $p \in V(F_1)$ and $q \in V(F_2)$, then $C_F(x_1, x_2)$ is a p-q-rainbow cut in F; if p, q belong to one of $V(F_1)$, $V(F_2)$, without loss of generality, say $p, q \in V(F_1)$, then there exists a p-q-rainbow cut $C_{F_1}(p, q)$ in F_1 that is also a p-q-rainbow cut in F by Lemma 2.5. Repeating the above inverse shrinking procedure, we finally get an rd-coloring of G using colors from $[\lfloor \frac{3}{2}\lambda^+(G) \rfloor]$. Hence, $\mathrm{rd}(G) \leq \lfloor \frac{3}{2}\lambda^+(G) \rfloor$. Moreover, for the Petersen graph P, we have $\mathrm{rd}(P) = 4 = \lfloor \frac{3}{2}\lambda^+(P) \rfloor$ since $\lambda^+(P) = 3$. Thus, the upper bound is sharp in some sense. $\square$

Next we obtain another bound for $\mathrm{rd}(G)$.

Theorem 2.7 *Let G be a graph of order n with maximum degree $\Delta(G)$ and upper edge-connectivity $\lambda^+(G)$. Then $\mathrm{rd}(G) \leq \min\{n + \lambda^+(G) - \Delta(G) - 1, \Delta(G) + 1\}$. Furthermore, the bound is sharp in some sense.*

Proof. Let v be a vertex with degree $\Delta(G)$ and $S = V(G) \setminus N[v]$. Then there exist at most $\lambda^+(G)$ edges from x to $N[v]$ for each vertex x of $S \cup N(v)$ in G by the definition of upper edge-connectivity. Let $G' = G - v$. Observe that $\Delta(G') \leq \min\{n + \lambda^+(G) - \Delta(G) - 2, \Delta(G)\}$. So, $\mathrm{rd}(G) \leq \Delta(G') + 1 = \min\{n + \lambda^+(G) -$

$\Delta(G)-1, \Delta(G)+1\}$ by Lemma 2.3. Furthermore, it follows from Lemma 2.2 that $\mathrm{rd}(G) = n + \lambda^+(G) - \Delta(G) - 1$ if G is $K_{1,n-1}$ or W_n and $\mathrm{rd}(G) = \Delta(G) + 1$ if G is the Petersen graph. The upper bound is sharp in some sense. $\square$

In the rest of this section, we always assume that all graphs have at least four vertices, and that both G and $\overline{G}$ are connected. For any vertex $u \in V(G)$, let $\bar{u}$ denote the vertex in $\overline{G}$ corresponding to the vertex u. We then characterize the graphs which meet the upper bounds of the Nordhaus-Gaddum type result obtained early by us. The following several lemmas will be used.

Lemma 2.8 [1] *Let G be a connected graph and let G_Δ denote the subgraph of G induced by the vertices of maximum degree $\Delta(G)$. If every connected component of G_Δ is a unicyclic graph or a tree, and G_Δ is not a disjoint union of cycles, then G is in Class* 1.

Lemma 2.9 [2] *Let G be a connected graph of order n. If* $\mathrm{rd}(G) \geq n-2$, *then G has at least two vertices of degree at least $n-2$.*

Lemma 2.10 [10] *If H is a connected subgraph of a graph G, then* $\mathrm{rd}(H) \leq \mathrm{rd}(G)$.

Lemma 2.11 [10] *Let G be a connected graph, and let B be a block of G such that* $\mathrm{rd}(B)$ *is maximum among all the blocks of G. Then* $\mathrm{rd}(G) = \mathrm{rd}(B)$.

Lemma 2.12 [10] *Let G be a connected graph of order $n \geq 2$. Then* $\mathrm{rd}(G) = n-1$ *if and only if G has at least two vertices of degree $n-1$.*

Lemma 2.13 *Let G be a graph of order $n \geq 3$. Then* $\mathrm{rd}(G) = n-2$ *if and only if one of the following conditions holds.*

(i) *G has exactly one vertex of degree $n-1$ and another vertex of degree $n-2$.*

(ii) *G is a graph with $\Delta(G) = n-2$ and there are two nonadjacent vertices of degree $n-2$ in G.*

(iii) *G is a graph not in (ii) with $\Delta(G) = n-2$ and at least two maximum degree vertices. In addition, for some pair of vertices u, v of degree $n-2$, there is a vertex z of G such that $z \notin N(u) \cup N(v)$ or there are two distinct vertices x, y such that $x \in N(u) \setminus N[v]$ and $y \in N(v) \setminus N[u]$ and x, y belong to the same component of $G[V \setminus \{u, v\}]$.*

Proof. For any graph that satisfies one of the conditions (i), (ii) and (iii), we first get that $\mathrm{rd}(G) \leq n-2$ by Lemma 2.12. Furthermore, we find that $\lambda^+(G) \geq n-2$, so $\mathrm{rd}(G) \geq n-2$ by Lemma 2.1.

We now verify the converse. Assume, to the contrary, that there exists a graph G with $\mathrm{rd}(G) = n-2$ but it does not satisfy any of the conditions (i), (ii) and (iii). Note that G has at least two vertices of degree at least $n-2$ by Lemma 2.9 and G does not have two vertices of degree $n-1$. Therefore G satisfies the following two conditions:

(1) $\Delta(G) = n-2$ and there exists an edge for any two vertices of degree $n-2$ in G.

(2) G has two distinct vertices x, y such that $x \in N(u) \setminus N[v]$ and $y \in N(v) \setminus N[u]$ for any pair vertices u, v of degree $n-2$, and x, y belong to different components of $G[V \setminus \{u, v\}]$.

We will show that the rainbow disconnection number of a graph G satisfying conditions (1) and (2) is at most $n-3$. We first present a claim as follows.

Claim 1. If G be a graph satisfying conditions (1) and (2), then $d_G(a) \leq n-3$ for each $a \in V(G) \setminus \{u, v\}$.

Proof of Claim 1: If $G[V \setminus \{u, v\}]$ has at least three components or two components where each part has at least 2 vertices, then $d_G(a) \leq n-3$ for each $a \in V(G) \setminus \{u, v\}$. If $G[V \setminus \{u, v\}]$ has two components, one of which has exactly one vertex, then the vertex of the single vertex component is x or y. Without loss of generality, let x be the vertex in this single vertex component. Assume, to the contrary, that there exists a vertex w of $V(G) \setminus \{u, v\}$ with $d_G(w) = n-2$. Then v, w are two vertices of degree $n-2$ that do not meet condition (2) since $x \notin N(v) \cup N(w)$. This is a contradicton.

If x or y is a pendent vertex in G, without loss of generality, say x, then let $G' = G - x$. Then $d_{G' \setminus v}(a) \leq n-4$ for each $a \in V(G') \setminus v$. Thus, we have $\mathrm{rd}(G') \leq n-3$ by Lemma 2.3 (i). Furthermore, $\mathrm{rd}(G) = \mathrm{rd}(G') \leq n-3$ by Lemma 2.11. Otherwise, suppose that G_1^* is the component in $G[V \setminus \{u, v\}]$ that contains vertex x and G_2^* is the remaining components in $G[V \setminus \{u, v\}]$, and $|G_1^*|, |G_2^*| \geq 2$. Let $G_1 = G[V(G_1^*) \cup \{u, v\}]$ and $G_2 = G[V(G_2^*) \cup \{u, v\}]$. Observe that $\Delta(G_1), \Delta(G_2) \leq n-3$ and $\Delta(G_1 \setminus u), \Delta(G_2 \setminus v) \leq n-4$. Therefore, there is an rd-coloring c_1 (c_2) of G_1 (G_2) using colors from $[n-3]$ such that each vertex of G_1 (G_2) is proper except for the vertex u (v) by Remark 1. Then we can exchange the colors of c_2 for G_2 such that the colors of edges incident with u in $G_2 \setminus v$ are different from the colors of edges incident with v in G_1, and color the edge uv in G_2 with the same color as uv under c_1. The new coloring c_2' is still an rd-coloring of G_2 using colors from $[n-3]$ such that each vertex of G_2 is proper except for the vertex v. Then we get a coloring c of G by letting $c(e) = c_1(e)$ if $e \in G_1$ and $c(e) = c_2'(e)$ if $e \in G_2$ and $|c| = n-3$. We can verify that coloring c is an rd-coloring of G. Let p, q be two vertices of G. If there exists a vertex of $\{p, q\}$ that does not belong to $\{u, v\}$, without loss of generality, say p, then the set E_p is a p-q-rainbow cut in G. If $\{p, q\} = \{u, v\}$, then the set $E_v^1 \cup E_u^2$ is a p-q-rainbow cut in G, where E_v^1 is the set of edges incident with vertex v in G_1 and E_u^2 is the set of edges incident with vertex u in G_2. Hence, $\mathrm{rd}(G) \leq n-3$. This is a contradiction with our assumption. $\square$

In [2], we obtained the Nordhaus-Gaddum type result for $\mathrm{rd}(G)$, and examples were given to show that the upper and lower bounds are sharp. However, we are not satisfied with these examples because they are special graphs. We restate it as follows.

Lemma 2.14 [2] *If G is a connected graph such that $\overline{G}$ is also connected, then $n-2 \leq \mathrm{rd}(G) + \mathrm{rd}(\overline{G}) \leq 2n-5$ and $n-3 \leq \mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) \leq (n-2)(n-3)$. Furthermore, these bounds are sharp.*

Next we will completely characterize the graphs which meet the upper bounds in the above Nordhaus-Gaddum type result combining Lemma 2.13.

Theorem 2.15 *Let G be a graph of order $n \geq 4$. Then $\mathrm{rd}(G) + \mathrm{rd}(\overline{G}) = 2n-5$ (or $\mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) = (n-2)(n-3)$) if and only if one of G and $\overline{G}$ satisfies the following three conditions:*

(i) *condition (ii) or (iii) in Lemma 2.13 holds;*

(ii) *it has exactly two vertices of degree $n-2$, say u, v;*

(iii) *it has at least two vertices of degree 2 except $\{x, y\}$ or $\{z\}$, where $x \in N(u) \setminus N[v]$, $y \in N(v) \setminus N[u]$ and $z \notin N(u) \cup N(v)$.*

Proof. Without loss of generality, suppose that G satisfies the above three conditions. Obviously, $\mathrm{rd}(G) = n-2$ by Lemma 2.13. Since G has at least two vertices of degree 2 except $\{x, y\}$ or $\{z\}$, the graph $\overline{G} \setminus \{\bar{u}, \bar{v}\}$ is of order $n-2$ and has at least two vertices of degree $n-3$. So, $\mathrm{rd}(\overline{G}) = \mathrm{rd}(\overline{G} \setminus \{\bar{u}, \bar{v}\}) = n-3$ by Lemmas 2.11 and 2.12.

Conversely, assume that there exists a graph G with $\mathrm{rd}(G) + \mathrm{rd}(\overline{G}) = 2n-5$. Since $\overline{G}$ is connected, we have $\mathrm{rd}(G) \leq n-2$ by Lemma 2.12. Therefore, it remains to consider the case $\mathrm{rd}(G) = n-2$, $\mathrm{rd}(\overline{G}) = n-3$ by symmetry. Similarly, if $\mathrm{rd}(G) \cdot \mathrm{rd}(\overline{G}) = (n-2)(n-3)$, we only need to consider the case $\mathrm{rd}(G) = n-2$, $\mathrm{rd}(\overline{G}) = n-3$ by symmetry. Obviously, G satisfies (ii) or (iii) of Lemma 2.13. If G has more than 2 vertices with degree $n-2$, then $\overline{G}$ has at least 3 vertices with degree 1. Then $\mathrm{rd}(\overline{G}) \leq n-4$ by Lemmas 2.10 and 2.11, this is a contradiction. Thus, condition (ii) holds. To prove (iii), assume, to the contrary, that G has at most one vertex of degree 2 except $\{x, y\}$ or $\{z\}$. Then $\overline{G} \setminus \{\bar{u}, \bar{v}\}$ has at most one vertex of degree at least $n-3$. Since $d_{\overline{G}}(\bar{u}) = d_{\overline{G}}(\bar{v}) = 1$, we have $\mathrm{rd}(\overline{G}) = \mathrm{rd}(\overline{G} \setminus \{\bar{u}, \bar{v}\}) \leq n-4$ by Lemmas 2.11 and 2.12. This is a contradiction with our condition. $\square$

3 Graphs with rd$(G) \leq \lambda^+(G) + 1$

First, we recall some known results.

Lemma 3.1 [2] *If G is a connected k-regular graph, then $k \leq \mathrm{rd}(G) \leq k + 1$.*

Lemma 3.2 [2] *If $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph of order n, where $k \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_k$, then*

$$\mathrm{rd}(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} n - n_2, & \text{if } n_1 = 1, \\ n - n_1, & \text{if } n_1 \geq 2. \end{cases}$$

Lemma 3.3 [10] *The rainbow disconnection number of the grid graph $G_{m,n}$ is as follows.*

(i) *For all $n \geq 2$, $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$.*

(ii) *For all $n \geq 3$, $\mathrm{rd}(G_{2,n}) = 3$.*

(iii) *For all $n \geq 4$, $\mathrm{rd}(G_{3,n}) = 3$.*

(iv) *For all $n \geq m \geq 4$, $\mathrm{rd}(G_{m,n}) = 4$.*

Observe that $\mathrm{rd}(G) \leq \lambda^+(G) + 1$ for all connected regular graphs, complete multipartite graphs and grid graphs. Therefore, we propose the following conjecture.

Conjecture 3.4 *Let G be a connected graph with upper edge-connectivity $\lambda^+(G)$. Then $\lambda^+(G) \leq \mathrm{rd}(G) \leq \lambda^+(G) + 1$.*

Obviously, the lower bound is always true by Lemma 2.1. Furthermore, we give some classes of graphs that support the upper bound of the conjecture. The following are some useful lemmas which will be used in the sequel.

Lemma 3.5 [10] *Let G be a nontrivial connected graph. Then $\mathrm{rd}(G) = 1$ if and only if G is a tree.*

Lemma 3.6 [10] *Let G be a nontrivial connected graph. Then* $\mathrm{rd}(G) = 2$ *if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Lemma 3.7 [20] *Let G be a graph of order n ($n \geq k+2 \geq 3$). If $|E(G)| > \frac{k+1}{2}(n-1) - \frac{1}{2}\sigma_k(G)$, where $\sigma_k(G) = \sum_{\substack{x \in V(G) \\ d(x) \leq k}} (k - d(x))$, then $\lambda^+(G) \geq k+1$.*

We recall some notions of graphs from [14]. A simple graph G is *overfull* if $|E(G)| > \lfloor \frac{|V(G)|}{2} \rfloor \Delta(G)$. A graph G is *subgraph-overfull* if it has an overfull subgraph H with $\Delta(H) = \Delta(G)$. Obviously, every overfull graph is subgraph-overfull. For subgraph-overfull graphs and graphs whose maximum degree does not exceed 3, we have the following observations, which support Conjecture 3.4.

Observation 3.8 *Let G be a subgraph-overfull graph with upper edge-connectivity $\lambda^+(G)$. Then* $\mathrm{rd}(G) \leq \lambda^+(G) + 1$.

Proof. Let H be an overfull subgraph of G with $\Delta(H) = \Delta(G)$. Then $|E(H)| > \lfloor \frac{|V(H)|}{2} \rfloor \Delta(H) \geq \frac{|V(H)|-1}{2}\Delta(H)$. Combining Lemma 3.7, we have $\Delta(G) \geq \lambda^+(G) \geq \lambda^+(H) \geq \Delta(H) = \Delta(G)$. So, $\lambda^+(G) = \Delta(G)$. It follows from Lemma 2.1 that $\mathrm{rd}(G) \leq \lambda^+(G) + 1$. □

Observation 3.9 *Let G be a graph with $\Delta(G) \leq 3$. Then* $\mathrm{rd}(G) \leq \lambda^+(G) + 1$.

Proof. Obviously, $\lambda^+(G) \leq \Delta(G) \leq 3$. If $\lambda^+(G) = 1$, we get that G is a tree. It follows from Lemma 3.5 that $\mathrm{rd}(G) = 1$. If $\lambda^+(G) = 2$, G must contain a cycle and any cycle of G does not have a chord, namely, G is a graph that each block of G is a cycle or K_2 and at least one block of G is a cycle. Then $\mathrm{rd}(G) = 2$ by Lemma 3.6. If $\lambda^+(G) = 3$, we have $\mathrm{rd}(G) \leq \Delta(G) + 1 = 4$ by Lemma 2.1. □

For a graph with a large maximum degree, we get the following result.

Theorem 3.10 *Let G be a graph with order n and $\Delta(G) \geq n-3$. Then* $\mathrm{rd}(G) \leq \lambda^+(G) + 1$.

Proof. Let $d(u) = \Delta(G)$ and $G' = G - u$. Suppose $\lambda^+(G) = k$. If $\Delta(G) \geq n-2$, we have $\Delta(G') \leq k$; otherwise, let v be a vertex with $d_{G'}(v) \geq k+1$. Then we have $\lambda^+(u, v) \geq k+1$, which is a contradiction. Thus, $\mathrm{rd}(G) \leq \Delta(G') + 1 \leq k+1$ by Lemma 2.3.

If $\Delta(G) = n-3$, let $d(u) = n-3$ and let p, q be two vertices which are not adjacent to u, i.e., $V(G) = N[u] \cup \{p, q\}$. Note that $d_G(x) \leq k+2$ for each $x \in N(u)$ and $d_G(p), d_G(q) \leq k+1$ since $\lambda^+(G) = k$. Thus, $\Delta(G') \leq k+1$. We distinguish the following cases to discuss.

Case 1. $\Delta(G') \leq k$.

It follows from Lemma 2.3 that $\mathrm{rd}(G) \leq \Delta(G') + 1 \leq k+1$.

Case 2. $\Delta(G') = k+1$.

Let $D = \{x | x \in V(G') \text{ and } d_{G'}(x) = k+1\}$. If $D \subseteq \{p, q\}$, then $G'[D]$ is K_1 (otherwise, $\lambda(p, q) = k+1$, a contradiction). Thus, it follows from Lemma 2.8 that G' is in Class 1. Moreover, $d_{G'}(x) \leq \Delta(G') - 1$ for each $x \in N_G(u)$. So, $\mathrm{rd}(G) \leq \Delta(G') = k+1$ by Lemma 2.3.

Suppose $D \cap N(u) \neq \phi$. We claim $|D \cap N(u)| = 1$. Assume that there are at least two vertices in $D \cap N(u)$, say x_1, x_2. Note that $d_{N(u)}(x_1) = d_{N(u)}(x_2) = k-1$. So, $\{p, q\} \subseteq N(x_i)$ for each $i \in [2]$. Then we find $\lambda_G(x_1, x_2) \geq k+1$, which is a contradiction. Let $D \cap N(u) = \{a\}$. Then $\{p, q\} \subseteq N(a)$ since $d_{N(u)}(a) \leq k-1$. Let $R = N(u) \setminus N[a]$, $T = N(p) \cup N(q)$. Note that $R \cap T = \emptyset$ and there is no edge between R and $T \cup \{p, q\}$. Assume that $R \cap T \neq \emptyset$ or there exists a vertex of R adjacent to a vertex of $T \cup \{p, q\}$. Then we have $\lambda^+(u, a) \geq k+1$, which is a contradiction. Thus, $T \subseteq N[a]$. Let $S = N[a] \setminus T$. If there exists a vertex $s \in S$ such that s belongs to a component with a vertex of R in $G[R \cup S]$, then let $s \in S_1$ and $S_2 = S \setminus S_1$. Observe that the edge-set $E(u, S_2 \cup T) \cup E(S_1, a)$ is a u-a-edge-cut by the definitions of R, S_1 and S_2. Let $G_1 = G[R \cup S_1 \cup \{u, a\}] - ua$ and $G_2 = G[T \cup S_2 \cup \{u, p, q\}]$. Write $G_1' = G_1 - u$ and $G_2' = G_2 - a$. Observe that $\Delta(G_1'), \Delta(G_2') \leq k$. By Lemma 2.3 and Remark 1, there exists an rd-coloring c_i of G_i ($i \in [2]$) using colors from $[k+1]$, moreover, x is proper for each $x \in V(G_1) \setminus \{u\}$ ($x \in V(G_2) \setminus \{a\}$) in coloring c_1 of G_1 (c_2 of G_2). Since $|E(u, S_2 \cup T) \cup E(S_1, a)| = k$, we can exchange colors of c_2 such that $E(u, S_2 \cup T) \cup E(S_1, a)$ have distinct colors. Then we get a coloring c of G by identifying the graphs G_1 and G_2 using colors from $[k+1]$.

Furthermore, we can verify that the coloring c is an rd-coloring of G. For any two vertices w, z of G, if there exists a vertex not in $\{u, a\}$, say w, then the set E_w is a w-z-rainbow cut in G; if $\{w, z\} = \{u, a\}$, then $E(u, S_2 \cup T) \cup E(S_1, a)$ is a u-a-rainbow cut in G. Hence, $\mathrm{rd}(G) \leq k+1$. $\square$

By Observation 3.9 and Theorem 3.10, we know that Conjecture 3.4 holds for small graphs.

Corollary 3.11 *Let G be a graph of order $n \leq 7$. Then $rd(G) \leq \lambda^+(G) + 1$.*

For a k-regular graph G, we know that $\lambda^+(G) = k$ by Lemma 3.7, and then the conjecture holds by Lemma 3.1. In particular, we want to further know which k-regular graphs satisfy $\mathrm{rd}(G) = k$. In [2], we presented some results for this question. For all odd $k \geq 1$, we now deduce the following result for k-edge-connected k-regular graphs.

Theorem 3.12 *Let k be an odd integer, and G be a k-edge-connected k-regular graph. Then $\chi'(G) = k$ if and only if $\mathrm{rd}(G) = k$.*

Proof. Suppose, first, that $\chi'(G) = k$. By Lemma 2.1, we have $k = \lambda(G) \leq \mathrm{rd}(G) \leq \chi'(G) = k$. Thus, $\mathrm{rd}(G) = k$.

Conversely, suppose $\mathrm{rd}(G) = k$ and let c be an optimal rd-coloring of G. If G has a k-rainbow cut T such that $G \setminus T$ has two non-trivial components, say G_1, G_2, then we do an operation f, i.e., the graph G shrinks $V(G_1)$, $V(G_2)$ to vertices x_1, x_2, respectively. The resulting edge-colored graphs are denoted by $G/V(G_1)$, $G/V(G_2)$, respectively. Furthermore, the obtained edge-colored graphs $G/V(G_1)$ and $G/V(G_2)$ are both k-edge-connected k-regular. Assume, to the contrary, that there exists a u-v-edge-cut V in $G/V(G_1)$, where $u, v \in G/V(G_1)$ and $|V| < k$. If $x_1 \notin \{u, v\}$, then V is also a u-v-edge-cut in G by Lemma 2.5, which contradicts the condition; if one of $\{u, v\}$ is x_1, without loss of generality, say $u = x_1$, then V is a w-v-edge-cut in G, where w is any vertex of $V(G_1)$. Similarly, this is a contradiction.

Claim 1. The coloring c of G restricted to $G/V(G_1)$ is an rd-coloring of $G/V(G_1)$.

Proof of Claim 1: Note that $V(G/V(G_1)) = V(G_2) \cup \{x_1\}$. Let u, v be two vertices of $G/V(G_1)$. Suppose $u, v \in V(G_2)$. Let W be a u-v-rainbow cut in G and let W_H be the set of edges of $W \cap H$ for any subgraph H of G. Since G_1, G_2 are both $\lceil \frac{k}{2} \rceil$-connected, we have $|W_{G_2}| \geq \lceil \frac{k}{2} \rceil$. Now we show $W \subseteq G_2 \cup T$. If the remaining edges of W are all in G_1, then there still is a u-v-path in $G \setminus W$ since G_1 is $\lceil \frac{k}{2} \rceil$-connected and $|W_{G_1}| \leq \lfloor \frac{k}{2} \rfloor < \lceil \frac{k}{2} \rceil$ for k odd, which is a contradiction. If G_1 and T both have edges of W, without loss of generality, suppose that $|W_{G_1}| = s$, $|W_T| = t$ and $|W_{G_2}| = r$, where $0 < t, s < \lfloor \frac{k}{2} \rfloor$, $s + t \leq \lfloor \frac{k}{2} \rfloor$ and $r + s + t = k$. When we remove the set W_T from G, at most t u-v-paths that go through T are destroyed. However, there are $s + t$ u-v-paths going through T in G, and so at least one u-v-path goes through $T \setminus W$ in G since $s \geq 1$. Moreover, $G_1 \setminus W$ is connected since $|W_{G_1}| < \lceil \frac{k}{2} \rceil$ and G_1 is $\lceil \frac{k}{2} \rceil$-connected. So, there is at least one u-v-path in $G \setminus W$, which is a contradiction. Therefore, $W \subseteq G_2 \cup T$. Then W is a u-v-rainbow cut of $G/V(G_1)$, otherwise, if $G/V(G_1)$ has a u-v-path avoiding the set W, then there exists a u-v-path in $G \setminus W$ since G_1 is connected, which is a contradiction. If $x_1 \in \{u, v\}$, then the set E_{x_1} is a

u-v-rainbow cut of $G/V(G_1)$. The proof is complete. ∎

Repeating the operation f until the obtained edge-colored graphs do not satisfy the condition of operation f, the resulting edge-colored k-edge-connected k-regular graphs are denoted by $\mathscr{F} = \{F_i | i \in [\ell]\}$.

Claim 2. The coloring of the graph F_i in $\mathscr{F}$ is a proper coloring of F_i for each $i \in [\ell]$.

Proof of Claim 2: Assume that there exists a graph F_i for some $i \in [\ell]$ whose coloring is not proper. If F_i ($i \in [\ell]$) has two vertices, say p, q, which are not proper, then there exists a p-q-rainbow cut Z in F_i that are not E_p or E_q. Thus, we get that Z is a rainbow cut in F_i such that $F_i \setminus Z$ has two non-trivial components, which is a contradiction to our operation. Hence, F_i has at most one vertex, say b_i, which is not proper for each $i \in [\ell]$. Given an $i \in [\ell]$, let k_t ($t \in [k]$) be the number of edges incident with vertex b_i and with color t in F_i, and moreover, let F_{i,A_j} be a subgraph of F_i induced by the set of edges with colors in A_j, where A_j is the color set $[k] \setminus \{j\}$ for some $j \in [k]$. Then for the graph F_i ($i \in [\ell]$), we get $(k-1)(|F_i| - 1) + \sum_{t \in A_j} k_t \equiv 0 \pmod 2$ since the sum of degrees of vertices in F_{i,A_j} is even for each $j \in [k]$. Therefore, we have $\sum_{t \in A_1} k_t \equiv \sum_{t \in A_2} k_t \equiv \cdots \equiv \sum_{t \in A_k} k_t \equiv 0 \pmod 2$ since $(k-1)(|F_i| - 1)$ is even, which gives $k_1 \equiv k_2 \equiv \cdots \equiv k_k \pmod 2$. Since $\sum_{i=1}^{k} k_i = k$ is odd, we obtain that $k_1 \equiv k_2 \equiv \cdots \equiv k_k \equiv 1 \pmod 2$, and then $k_1 = k_2 = \cdots = k_k = 1$. So, the vertex b_i is also proper in F_i for each $i \in [\ell]$. ∎

For each vertex x of G, the colors of edges incident with vertex x are not changed in each operation f. Thus, the optimal rd-coloring c of G is a proper coloring of G, i.e., $\chi'(G) \leq k$. Combining Lemma 2.1, we get $\chi'(G) = k$. □

Remark 2. For odd k, any optimal rd-coloring of a k-edge-connected k-regular graph G is a proper coloring of G. However, for even k the argument does not follow; see the following example. Let G be a k-edge-connected k-regular graph with $V(G) = \{u_i, v_i | i \in [2r]\} \cup \{p_i, q_i | i \in [2r-2]\}$ and $E(G) = \{p_i u_j, q_i v_j, u_j v_j | i \in [2r-2], j \in [2r]\} \cup \{u_i u_{i+r}, v_i v_{i+r} | i \in [r]\}$. Observe that $k = 2r$. We give a coloring c of G as follows.

- ⋆ $c(p_1 u_i) = c(p_1 u_{i+r}) = i,\ i \in [r]$;
- ⋆ $c(u_i u_{i+r}) = r + i,\ i \in [r]$;
- ⋆ $c(u_i v_i) \equiv i + 1 \pmod{2r}, i \in [2r]$;
- ⋆ $c(u_i p_j) \equiv i + j \pmod{2r},\ 2 \leq j \leq r - 1$;
- ⋆ $c(u_i p_j) \equiv i + j + 1 \pmod{2r},\ r \leq j \leq 2r - 2$;
- ⋆ the coloring of the remaining edges can be obtained symmetrically.

Note that $|c| = 2r$ and each vertex of G is proper except vertices p_1, q_1 under the coloring c. Therefore, it is easy to verify that c is an optimal rd-coloring of G, but it is not a proper coloring of G. ∎

Remark 3. From Vizing's theorem we know that for a k-regular graph G, $\chi'(G) = k$ or $k+1$. Since for $k = 3$ determining $\chi'(G) = 3$ or 4 is NP-complete [15], there are infinitely many k-edge-connected k-regular graphs G for which $\chi'(G) = k$ for odd k, and also there are infinitely many k-edge-connected k-regular graphs G for which $\chi'(G) = k+1$ for odd k. So, from our Theorem 3.12 one can get that there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = k$ for odd k, and also there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = k+1$ for odd k. Since for k-edge-connected k-regular graphs G one has $\lambda^+(G) = k$, we then get that there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G)$ for odd k, and also there are infinitely many k-edge-connected k-regular graphs G for which $\mathrm{rd}(G) = \lambda^+(G) + 1$ for odd k. ∎

Obviously, for $k = 3$, we get the following corollary from Theorem 3.12.

Corollary 3.13 [2] *Let G be a 3-edge-connected cubic graph. Then $\chi'(G) = 3$ if and only if $\mathrm{rd}(G) = 3$.*

For planar graphs, Corollary 3.13 gives rise to an interesting result, which is equivalent to the famous Four-Color Problem. We know the Four-Color Problem is equivalent to the following statement (For a history of the Four-Color Problem, see Biggs et al. [5] or Wilson [24].)

Problem 3.14 [5, 8, 24] *Every plane graph without cut edges is 4-face-colorable.*

The Four-Color Problem is equivalent to the assertion that every 3-connected cubic plane graph is 4-face-colorable. Tait [22] found a surprising relationship between face-colorings and edge-colorings of 3-connected cubic plane graphs.

Theorem 3.15 [22] *A 3-connected cubic plane graph G is 4-face-colorable if and only it is 3-edge-colorable, i.e., $\chi'(G) = 3$.*

Moreover, we prove that a cube graph is 3-connected is equivalent to it is 3-edge-connected.

Theorem 3.16 *A cube graph G is 3-connected if and only if G is 3-edge-connected.*

Proof. First, let G be a 3-connected cube graph. Obviously, the G is 3-edge-connected. Conversely, suppose that G is a 3-edge-connected cube graph. We now prove that G is 3-connected. Assume, to the contrary, that G has a 2-vertex-cut $\{u, v\}$. Since G is a 3-regular graph, we can find a 2-edge-cut incident with vertex $\{u, v\}$. This is a contradiction with our assumption. $\square$

Therefore, we get the following result.

Lemma 3.17 *Let G be a 3-connected cubic graph. Then $\chi'(G) = 3$ if and only if $\mathrm{rd}(G) = 3$.*

Combining Theorem 3.15 and Lemma 3.17, we get a relationship between face-colorings and rainbow disconnection colorings of 3-connected cubic plane graphs.

Theorem 3.18 *A 3-connected cubic plane graph G is 4-face-colorable if and only if $\mathrm{rd}(G) = 3$.*

4 Relationship of rd(G) and rvd($L(G)$)

The *line graph* $L(G)$ of a graph G has the edges of G as its vertices, and two distinct edges of G are adjacent in $L(G)$ if and only if they share a common vertex in G. Now, we study the relationship between $\mathrm{rd}(G)$ and $\mathrm{rvd}(L(G))$.

Lemma 4.1 [3] *For an integer $n \geq 2$,*

$$\mathrm{rvd}(K_n) = \begin{cases} n-1, & \text{if } n = 2, 3, \\ n, & \text{if } n \geq 4. \end{cases}$$

Theorem 4.2 *Let G be a graph and $L(G)$ be the line graph of G. Then $\mathrm{rd}(G) \leq \mathrm{rvd}(L(G))$.*

Proof. Let c_0 be an optimal rvd-coloring of the line graph $L(G)$. Then we get an edge-coloring c of G corresponding to the vertex-coloring c_0 of $L(G)$. We can verify that c is an rd-coloring of G. For any two vertices u, v of G, if uv is not a pendent edge, we can find two edges e_1, e_2 incident with vertices u, v, respectively, and the edge e_1 (or e_2) does not have two ends as u, v. Suppose that $e_1 = ux$ and $e_2 = vy$, where $x, y \in V(G) \setminus \{u, v\}$ and x, y could be the same vertex. We know that e_1, e_2 correspond to two vertices of $L(G)$, denoted by a and b. We claim that the edge-set S of G which corresponds to an a-b-rainbow vertex-cut S' in $L(G)$ is a u-v-rainbow cut in G. Assume that there still exists a u-v-path P in G which avoids the edge-set

S of G. Then the u-v-path P in G corresponds to an a-b-path P' which avoids the vertex set S' in $L(G)$. This is a contradiction. If uv is a pendent edge of G, then uv is a u-v-rainbow cut in G. $\square$

We know that the chromatic index of G is equal to the chromatic number of $L(G)$. Similarly, we want to know whether $\mathrm{rd}(G) = \mathrm{rvd}(L(G))$ for any graph G. However, the equality is not always true. For the moment we obtain the following necessary condition for the equality.

Theorem 4.3 *Let G be a graph with $\delta(G) \geq 4$ and $L(G)$ be the line graph of G. If $\mathrm{rd}(G) = \mathrm{rvd}(L(G))$, then $\mathrm{rd}(G) = \chi'(G)$.*

Proof. Assume, to the contrary, that $\mathrm{rd}(G) < \chi'(G)$. Then $\mathrm{rvd}(L(G)) < \chi(L(G))$ since $\chi'(G) = \chi(L(G))$. Let $\mathrm{rvd}(L(G)) = t$ and c be an any vertex-coloring of $L(G)$ using colors from $[t]$. Then there exist two adjacent vertices u, v which have the same color in $L(G)$. Observe that u, v must contain in some K_ℓ ($\ell \geq 4$) of $L(G)$ since $\delta(G) \geq 4$, and the K_ℓ has at most $\ell - 1$ colors in $L(G)$. Thus, any t-vertex-coloring c of $L(G)$ is not an rvd-coloring of $L(G)$ by Lemma 4.1. This contradicts $\mathrm{rvd}(L(G)) = t$. $\square$

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