A sharp lower bound for the spectral radius in K_4-saturated graphs

Jaehoon Kim[*], Alexandr V. Kostochka[†], Suil O[‡], Yongtang Shi[§] and Zhiwen Wang[¶]

Abstract

For given graphs G and H, the graph G is H-saturated if G does not contain H as a subgraph but for any $e \in E(\overline{G})$, $G + e$ contains H. In this note, we prove that if G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$,

$$\sum_{w \in N(v)} d_G(w) \geq (r-2)d(v) + (r-1)(n-r+1),$$

then $\rho(G) \geq \rho(S_{n,r})$, where $S_{n,r}$ is the graph obtained from a copy of K_{r-1} with vertex set S by adding $n-r+1$ vertices, each of which has neighborhood S. This provides a sharp lower bound for the spectral radius in an n-vertex K_{r+1}-saturated graph for $r = 2, 3$, verifying a special case of a conjecture by Kim, Kim, Kostochka and O.

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1 Introduction

A key theme in extremal graph theory is to study the relations between the number of edges of graphs and the substructures they have. This study goes back to 1941 when Turán [16]

[*]Mathematical Sciences Department, KAIST, jaehoon.kim@kaist.ac.kr. Research supported by the POSCO Science Fellowship of POSCO TJ Park Foundation.

[†]Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia, kostochk@math.uiuc.edu. Research of this author is supported in part by NSF RTG Grant DMS-1937241.

[‡]Department of Applied Mathematics and Statistics, The State University of New York, Korea, Incheon, 21985, suil.o@sunykorea.ac.kr. Research supported by NRF-2020R1F1A1A01048226, NRF-2021K2A9A2A06044515, and NRF-2021K2A9A2A1110161711.

[§]Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China, shi@nankai.edu.cn. Research supported by NSFC Nos. 12161141006 and 12111540249.

[¶]School of Mathematics Sciences and LPMC, Nankai University, Tianjin, 300071, China, walkerwzw@163.com. Research supported by CPSF No. 2021M691671.

proved that the n-vertex complete r-partite graph is the unique graph having maximum number of edges among all n-vertex graphs not having K_{r+1} as a subgraph. Since then, the studies on the *extremal number*, denoted $ex(n, H)$, which is defined to be the maximum number of edges in an n-vertex graph not containing H as a subgraph, was extensively done.

As the extremal number is defined in terms of the maximality, it naturally implies that any addition of edge to an extremal example creates a copy of K_{r+1}. This motivates to give a name to such a concept of maximality with respect to edge addition, introducing the notion of *saturation*. For given graphs G and H, the graph G is *H-saturated* if H is not a subgraph of G but for any $e \in E(\overline{G})$, H is a subgraph of $G + e$. With this definition, the extremal number $ex(n, H)$ can be treated as the maximum number of edges in an n-vertex H-saturated graph.

The *saturation number of H*, written $sat(n, H)$, is defined to be the minimum number of edges in an n-vertex H-saturated graph and was also extensively studied. The first result on saturation numbers was proved in 1964 [5]. Erdős, Hajnal and Moon [5] determined the saturation number of K_{r+1} and characterized the extremal graphs. We let $S_{n,r}$ be the n-vertex graph obtained from a copy of K_{r-1} with the vertex set S by adding $n - r + 1$ vertices, each of which has neighborhood S.

Theorem A [5]. *If $2 \le r < n$, then $sat(n, K_{r+1}) = (r-1)(n-r+1) + \binom{r-1}{2}$. The only n-vertex K_{r+1}-saturated graph with $sat(n, K_{r+1})$ edges is the graph $S_{n,r}$.*

For history and exciting developments on the theory of saturation number, we refer the reader to an excellent survey [6] by Faudree, Faudree, and Schmitt.

Note that for a graph with given number of vertices, the average degree $\overline{d}(G) = \frac{2|E(G)|}{|V(G)|}$ carries the same information with $|E(G)|$, so the Turán's theorem can be restated in terms of the average degree. As the relations between the average degree $\overline{d}(G)$ and subgraph structures of G have been explored, it is natural to ask what will happen if we replace $\overline{d}(G)$ with another parameter?

For a graph G, let $A(G)$ denote its adjacency matrix and let $\rho(G)$ denote the spectral radius of maximum of $A(G)$, that is, $\rho(G) = \max\{|\lambda_i| : 1 \le i \le n\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. Since $A(G)$ is real-valued and symmetric, all λ_is are real numbers, and we may assume $\lambda_1 \ge \cdots \ge \lambda_n$. By the Perron-Frobenius Theorem (see [8, 9]), we have $\rho(G) = \lambda_1$.

It is well-known that the maximum degree of G plus one bounds from above the chromatic number of the graph. Wilf [18] improved this fact by replacing the maximum degree by its spectral radius, showing that the chromatic number is at most its spectral radius plus one. In this result, the spectral radius plays a similar role with the maximum degree, hinting that the spectral radius of a graph might be a right parameter for replacing the average degree in Turán's theorem.

Indeed, Nikiforov [13] proved that the spectral radius behaves like the average degree in terms of Turán's theorem: if G is an n-vertex K_{r+1}-free graph, then $\rho(G) \le \rho(T_{n,r})$. Since each K_{r+1}-saturated graph is K_{r+1}-free, his theorem implies the following.

Theorem B [12]. *If G is a K_{r+1}-saturated graph with n vertices, then*

$$\rho(G) \leq \rho(T_{n,r}).$$

Similarly, one can naturally ask whether the spectral radius verison of the Erdős-Hajnal-Moon theorem is true, in other words, "what is the minimum possible spectral radius $\rho(G)$ of an n-vertex K_{r+1}-saturated graph?" Indeed, Kim, Kim, Kostochka, and O [11] conjectured as follows.

Conjecture 1.1. *[11] If G is an n-vertex K_{r+1}-saturated graph, then $\rho(G) \geq \rho(S_{n,r})$.*

Furthermore, they supported this conjecture by giving an asymptotically tight lower bound of $\rho(S_{n,r}) + \frac{r-2}{2} + \Theta(\frac{r^{1.5}}{\sqrt{n}})$. In particular, their bound is tight for $r = 2$, verifying the conjecture for $r = 2$.

Theorem C [11]. *If G is an n-vertex K_3-saturated graph, then $\rho(G) \geq \rho(S_{n,2})$; equality holds only when G is $S_{n,2}$ or a Moore graph.*

In this note, we prove that if G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$, $\sum_{w \in N(v)} d(w) \geq (r-2)d(v) + (r-1)(n-r+1)$, then $\rho(G) \geq \rho(S_{n,r})$. By using this, we give a simpler proof of Theorem C and also prove Conjecture 1.1 for $r = 3$.

For undefined terms of graph theory, see West [17]. For basic properties of spectral graph theory, see Brouwer and Haemers [2] or Godsil and Royle [8].

2 Results and proofs

We first prove Theorem 2.2. Note that the spectral radius of $S_{n,r}$ is as follows.

Proposition 2.1. [7, 11, 15] *For integers $2 \leq r < n$,*

$$\rho(S_{n,r}) = \frac{r - 2 + \sqrt{(r-2)^2 + 4(r-1)(n-r+1)}}{2}.$$

Theorem 2.2. *If G is an n-vertex K_{r+1}-saturated graph such that for each vertex $v \in V(G)$,*

$$\sum_{w \in N(v)} d(w) \geq (r-2)d(v) + (r-1)(n-r+1),$$

then $\rho(G) \geq \rho(S_{n,r})$.

Proof. Let A be the adjacency matrix of G and let $\mathbf{x} = (x_1, \dots, x_n)^T$ be the Perron vector corresponding to the spectral radius of G, say ρ. Note that $\mathbf{x}$ has all positive entries by the

Perron-Frobenius theorem. Without loss of generality, we may assume that $\sum_{i=1}^{n} x_i = 1$. Suppose that $p(x) = x^2 - (r-2)x - (r-1)(n-r+1)$. Then we have

$$p(A)\mathbf{x} = \left[A^2 - (r-2)A - (r-1)(n-r+1)I\right]\mathbf{x} = p(\rho)\mathbf{x}.$$

Thus we have

$$\begin{aligned} p(\rho) = p(\rho)\left(\sum_{v\in V(G)} x_v\right) &= \sum_{v\in V(G)} p(\rho)x_v = \sum_{v\in V(G)}\sum_{u\in V(G)} p(A)_{vu}x_u = \sum_{v\in V(G)} x_v \sum_{u\in V(G)} p(A)_{uv} \\ &\geq \min_{v\in V(G)} \sum_{u\in V(G)} p(A)_{uv} = \min_{v\in V(G)} \sum_{u\in V(G)} (A^2 - (r-2)A - (r-1)(n-r+1)I)_{uv} \\ &= \min_{u\in V(G)} \left[\left(\sum_{w\in N(u)} d(w)\right) - (r-2)d(u) - (r-1)(n-r+1)\right] \geq 0, \end{aligned}$$

which yields that $\rho(G) \geq \rho(S_{n,r})$. $\square$

With Theorem 2.2, we now give a simpler proof of Theorem C.

Proof of Theorem C. We may assume $n \geq 3$, as it is otherwise trivial. By Theorem 2.2, it suffices to show that for each vertex $v \in V(G)$,

$$\sum_{w\in N(v)} d(w) \geq n-1.$$

As G is K_3-saturated and $n \geq 3$, the graph G has diameter two. Moreover, G is K_3-free, so a breadth first search yields $\sum_{w\in N(v)} d(w) \geq d(v) + (n-1-d(v)) = n-1$. Equality in the bound holds only when for every vertex $v \in V(G)$, and every vertex $x \in V(G)\backslash N[v]$, we have $|N(v)\cap N(x)| = 1$. This yields that G does not have a cycle of length at most 4. If $V(G)\backslash N[v] = \varnothing$ for some vertex v, then G is $S_{n,2}$. Otherwise, the girth of G is exactly 5, which implies that it is a Moore graph by Hoffman and Singleton [10] (see also [9, 4]). $\square$

For a vertex $v \in V(G)$, let $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(u)\cup\{v\}$. Next, we prove a sharp lower bound for the spectral radius in an n-vertex K_4-saturated graph.

Theorem 2.3. *If G is an n-vertex K_4-saturated graph, then $\rho(G) \geq \rho(S_{n,3})$; equality holds only when G is $S_{n,3}$.*

Proof. By Theorem 2.2, it suffices to show that for each vertex $v \in V(G)$,

$$\sum_{w\in N(v)} d(w) \geq d(v) + 2(n-2).$$

We consider the following two types of vertices v separately.

Case 1. The graph induced by the closed neighborhood $N[v]$ is K_4-saturated. Since $G[N[v]]$ is K_4-saturated, $G[N(v)]$ is K_3-saturated. As an addition of vw creates a new K_4 in G for all $w \notin N[v]$, such a w has at least two neighbors in $N(v)$, so we have

$$\sum_{w\in N(v)} d(w) \geq d(v) + 2|E(G[N(v)])| + 2(n - d(v) - 1) \tag{1}$$

By Theorem A, we have

$$|E(G[N(v)])| \geq d(v) - 1. \tag{2}$$

Thus, we have

$$\sum_{w\in N(v)} d(w) \geq d(v) + 2(d(v) - 1) + 2(n - d(v) - 1) = d(v) + 2(n - 2).$$

Case 2. The graph induced by the closed neighborhood $N[v]$ is not K_4-saturated.

If there is a vertex $w \neq v$ with $d(w) = n - 1$, then w is also adjacent to v. Then K_4-freeness of G implies that $G[N(v)]$ is a star with the center w and $G[N[v]] = S_{d(v)+1,2}$, a contradiction that $G[N[v]]$ is not K_4-saturated. Hence, we may assume $\Delta(G) \leq n - 2$.

Moreover, we may assume $\delta(G) \geq 4$ as well. Indeed, if there exists a vertex w of degree two, all the vertices in $V(G) - G[N[w]]$ must be adjacent to the two vertices in $N(w)$ and $G[N(w)] = K_2$ since G is K_4-saturated. This yields $\Delta(G) = n - 1$, a contradiction. If there exists a vertex w with $d(w) = 3$, then $G[N(w)]$ is not K_3, which implies that there are two vertices $u, u' \in N(w)$ such that u and u' are not adjacent. Also $G[N(w)]$ is not trivial since the addition of an edge from w to one of its non-neighbors creates a K_4. Thus the remaining vertex u'' in $N(w)$ is adjacent to at least one of u and u'. Every vertex in $V(G) - N[w]$ must be adjacent to the vertex u'' since the addition of an edge from w to one of its non-neighbors creates a K_4. If u'' is adjacent to both u and u', then we have $\Delta(G) = n - 1$, contradiction. Now, we may assume that u'' is adjacent to u'. Then similarly, u' is adjacent to all the vertices in $V(G) - N[w]$, which implies that $V(G) - N[w]$ is independent. Then we cannot create a copy of K_4 by adding an edge uu' to G. Hence we indeed may assume $\delta(G) \geq 4$.

Let $G_1, \ldots, G_t$ be the components of $G[N(v)]$ having at least one edge, and let

$$N_1 = \bigcup_{1\leq i\leq t} V(G_i), \quad N_2 = N(v) - N_1, \quad \text{and} \quad N_3 = V(G) - N[v].$$

Note that $N(v) = N_1 \cup N_2$ and $N_3 \neq \varnothing$. As adding an edge vw for some $w \in N_3$ yields a copy of K_4 in G, we have $E(G[N(v)]) \neq \varnothing$, so $t \geq 1$. Also, since $\delta(G) \geq 4$ and $|E(G[N_2])| = 0$, we have $|N(x) \cap N_3| \geq 3$ for all vertices $x \in N_2$. Thus we have

$$\begin{aligned}\sum_{w\in N(v)} d(w) &= d(v) + 2|E(G[N_1])| + |[N_1, N_3]| + |[N_2, N_3]| \\ &\geq d(v) + 2(d(v) - |N_2| - t) + |[N_1, N_3]| + 3|N_2| \\ &\geq 3d(v) + |N_2| - 2t + |[N_1, N_3]|. \end{aligned} \tag{3}$$

Now we need to estimate $|[N_1, N_3]|$. For each $u \in N_3$, choose $i(u) \in [t]$ such that there is an edge $w_1w_2 \in E(G_{i(u)})$ such that $w_1, w_2 \in N(u)$. As adding uv to G creates K_4, such an $i(u)$ must exist. Let E_1 be the set of such edges uw_1, uw_2, then we have $E_1 \subseteq [N_1, N_3]$ and $|[N_1, N_3]| \geq |E_1| = 2|N_3| = 2(n - d(v) - 1)$.

If $t = 1$, then (3) yields

$$\begin{aligned}\sum_{w \in N(v)} d(w) &\geq 3d(v) + |N_2| - 2t + 2(n - d(v) - 1) \\ &\geq d(v) + 2(n-2) + |N_2| \geq d(v) + 2(n-2), \end{aligned} \tag{4}$$

as desired.

If $t \geq 2$, then take an edge $x_iy_i \in E(G_i)$ for each $i \in [t]$, and let $x_{t+1} = x_1$. Adding an edge y_ix_{i+1} to G yields a copy of K_4 in G, we know that there are two adjacent vertices u and u' in N_3 such that $\{y_i, x_{i+1}\} \subseteq N(u) \cap N(u')$. No matter what $i(u), i(u')$ are, this copy of $K_4 - e$ contains at least two edges not in E_1. As these edges outside E_1 we obtain are distinct for all $i \in [t]$, we obtain $2t$ edges in $[N_1, N_3] \setminus E_1$, hence $|[N_1, N_3]| \geq 2(n - d(v) - 1) + 2t$. This together with (3) yields

$$\sum_{w \in N(v)} d(w) \geq 3d(v) + |N_2| + 2(n - d(v) - 1) = d(v) + 2(n-1) + |N_2| > d(v) + 2(n-2), \tag{5}$$

as desired.

Assume that we have $\sum_{w \in N(v)} d(w) = d(v) + 2(n-2)$ for all $v \in V(G)$. Then we have an equality in each step of the computation.

If there exists a vertex v for which $G[N[v]]$ is K_4-saturated, then an equality in (2) gives $|E(G[N(v)])| = d(v) - 1$, which implies that $G[N(v)]$ is a $S_{d(v),2}$ by Theorem A, since $G[N(v)]$ is K_3-saturated. This implies that $G[N[v]] = S_{d(v)+1,3}$, and every vertex in $V(G) - N[v]$ is adjacent to exactly two vertices in $N(v)$ to have an equality in (1). Let x be the center of the star, $S_{d(v),2}$. Note that any edge of $G[N(v)]$ is incident to x. Thus we have $d(x) = n - 1$ since if $x' \in V(G) - N[v]$ is not adjacent to x, then adding vx' to G does not yield a copy of K_4, a contradiction. As $G[N[x]] = G$ is K_4-saturated, we again have $G = S_{n,3}$ by letting x play the role of v.

Now, assume that for every vertex v, $G[N[v]]$ is not K_4-saturated while $\sum_{w \in N(v)} d(w) = d(v) + 2(n-2)$ for all $v \in V(G)$. Then we have $\Delta(G) \leq n - 2$. Otherwise, $G[N[v]]$ is K_4-saturated for some vertex v with $\Delta(G) = d(v)$, which applies to Case 1.

As the second equality in (5) is strict, we must have $t = 1$ and the inequalities in (4) are equalities for every vertex $v \in V(G)$. Moreover, for any $v \in V(G)$, we have $N_2 = \varnothing$ and $G[N(v)]$ must be a tree. If $G[N(v)]$ has two vertices u, u' of distance at least three within the tree $G[N(v)]$, adding an edge uu' to G yields a copy of K_4 with vertices u, u', w, w' for some $w, w' \in N_3$. However, the edges wx, wx' in E_1 incident with w belongs to a triangle wxx', while wuu' does not form a triangle. Hence at least one of wu and wu' are not in E_1, implying that $|[N_1, N_3]| \geq |E_1| + 1 > 2(n - d(v) - 1)$. Then we have a strict inequality in

(4), a contradiction. Hence $G[N(v)]$ must be a tree of diameter two, a star. However, this shows that $G[N[v]]$ is K_4-saturated, which contradicts the assumption.

Thus, we can conclude that $G = S_{n,3}$ if we have $\sum_{w \in N(v)} d(w) = d(v) + 2(n-2)$ for all $v \in V(G)$. $\square$

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