On k-uniform random hypergraphs without generalized fans

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Abstract

Let the k-uniform hypergraph Fan^k consist of k edges that pairwise intersect exactly in one vertex x, plus one more edge intersecting each of these edges in a vertex different from x. Mubayi and Pikhurko [A new generalization of Mantel's theorem to k-graphs, J. Combin. Theory B 97(2007), 669–678] determined the exact Turán number $ex(n, \mathrm{Fan}^k)$ of Fan^k for sufficiently large n, which provides a generalization of Mantel's theorem. In this paper, we give a sparse version of Mubayi and Pikhurko's result. For a fixed integer k ($k \geq 3$), let $G^k(n,p)$ be a probability space consisting of k-uniform hypergraphs with n vertices, in which each element of $\binom{[n]}{k}$ occurring independently as an edge with probability p. We show that there exists a positive constant K such that with high probability the following is true. If $p > K/n$, then every maximum Fan^k-free subhypergraph of $G^k(n,p)$ is k-partite for $k \geq 4$; and if $p > K(\log n)^\gamma/n$, where $\gamma > 0$ is an absolute constant, then every maximum Fan^3-free subhypergraph of $G^3(n,p)$ is tripartite. Our result is an exact version of a random analogue of the stability result of Fan^k-free k-graphs, which can be obtained by

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using the transference theorem given by Samotij [Stability results for random discrete structures, Random Struct. Algor. 44(2014), 269–289].

Keywords: Mantel's theorem; random hypergraphs; Turán number.

1 Introduction

Mantel's theorem [19] is known as a cornerstone result in extremal combinatorics, which shows that every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges and the unique triangle-free graph that achieves this bound is the asymptotically balanced complete bipartite graph. In other words, every maximum (with respect to the number of edges) triangle-free subgraph of the n-vertex complete graph K_n^2 is bipartite.

There are several possible generalizations of this problem to k-uniform hypergraphs (k-graphs for short). One was suggested by Katona [16] and Bollobás [2], and further studied by Frankl and Füredi [11, 12], De Caen [8], Sidorenko [27], Shearer [26], Keevash and Mubayi [18], Pikhurko [23] and Goldwasser [13], etc.. Another extension, the so-called expanded triangle, was studied by Frankl [10], Keevash and Sudakov [17] etc.. In this paper, we study another generalization of triangles introduced by Mubayi and Pikhurko [20].

Let Fan^k be the k-graph consisting of $k+1$ edges $e_1, \ldots, e_k, e$, with $e_i \cap e_j = \{x\}$ for all $i \neq j$, where $x \notin e$, and $|e_i \cap e| = 1$ for all i. In other words, k edges share a single common vertex x and the last edge intersects each of the other edges in a single vertex different from x. Call the vertex x the *center* of Fan^k. Observe that Fan^2 is simply a triangle, and in this sense Fan^k generalizes the definition of K_3^2.

Given a k-uniform ($k \geq 2$) hypergraph H, the *Turán number* of H, denoted by $ex(n, H)$, is the maximum number of edges in a k-uniform hypergraph on n vertices which does not contain any copy of H. Unlike the graph case, if $k \geq 3$, then even the asymptotic behavior of the function $ex(n, H)$ is not known apart from some very specific hypergraphs H. Still, for an arbitrary H, it makes sense to define the *Turán density* of H, denoted $\pi(H)$, by

$$\pi(H) = \lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{k}}.$$

We call a k-uniform hypergraph H is *k-partite* if its vertex set can be partitioned into k classes, such that every edge intersects every partition class in precisely one vertex. Let *Turán hypergraph* $T_k^k(n)$ be the complete n-vertex k-uniform k-partite hypergraph whose partite sets are as equally-sized as possible. Let $t_k^k(n)$ denote the number of edges of $T_k^k(n)$. In particular, Mantel's theorem states that $ex(n, \mathrm{Fan}^2) = t_2^2(n)$ for all positive n and the maximum triangle-free graph on n vertices is $T_2^2(n)$. Mubayi and Pikhurko [20] generalized this to $k > 2$, for large n.

Theorem 1.1 *[20] Let $k \geq 3$. Then, for all sufficiently large n, the maximum number of edges in an n-vertex k-graph containing no copy of* Fan^k *is* $t_k^k(n) = \prod_{i=1}^{k} \lfloor \frac{n+i-1}{k} \rfloor$. *The only k-graph for which equality holds is $T_k^k(n)$.*

Their key approach is to consider the stability of Fan^k. Two k-graphs F and G of the same order are called *m-close* if we can add or remove at most m edges from the first graph and make it isomorphic to the second. They proved that

Theorem 1.2 *[20] For any $k \geq 2$ and $\delta > 0$ there exist $\epsilon > 0$ and N such that the following holds for all $n > N$: if G is an n-vertex* Fan^k*-free k-graph with at least $t_k^k(n) - \epsilon n^k$ edges, then G is δn^k-close to $T_k^k(n)$.*

Nowadays, transferring extremal combinatorial results from the deterministic to the probabilistic setting arouses the interest of researchers. Besides all the generalizations of Mantel's theorem, there are some sparse versions of Mantel's theorem as well. Let $G(n, p)$ be the Erdős-Rényi random graph model consisting of graphs with n vertices, in which the edges are chosen independently with probability p. An event occurs *with high probability* (w.h.p.) if the probability of that event approaches 1 as n tends to infinity. DeMarco and Kahn [9] proved that if $p > K\sqrt{\log n/n}$ for some constant K, then every maximum triangle-free subgraph of $G(n, p)$ is w.h.p. bipartite, and this bound is best possible up to a constant multiple. Problems of this type were first considered by Babai, Simonovits and Spencer [3]. Then Brightwell, Panagiotou, and Steger [5] proved the existence of a constant c, depending only on ℓ, such that whenever $p \geq n^{-c}$, w.h.p. every maximum K_ℓ^2-free subgraph of $G(n, p)$ is $(\ell - 1)$-partite.

For $n \in \mathbb{Z}$ and $p \in (0, 1)$, let $G^k(n, p)$ be a probability space consisting of k-uniform hypergraphs with n vertices, in which each element of $\binom{[n]}{k}$ occurs independently as an edge with probability p. Note that in particular, $G^2(n, p) = G(n, p)$, the usual graph case.

Balogh et al. [4] studied this type of problem in random 3-uniform hypergraphs to extend Mantel's theorem, which can be treated as a sparse version of Frankl and Füredi's result given in [12]. In [15], the similar problem in random 4-uniform hypergraphs are studied.

In this paper, we study an extremal problem concerning Fan^k in random k-uniform hypergraphs, and obtain the following theorem.

Theorem 1.3 *For $k \geq 3$, there exists a positive constant K such that w.h.p. the following is true. If $p > K/n$, then every maximum* Fan^k*-free subhypergraph of $G^k(n,p)$ is k-partite for $k \geq 4$; and if $p > K(\log n)^\gamma/n$, where $\gamma > 0$ is an absolute constant, then every maximum* Fan^3*-free subhypergraph of $G^3(n,p)$ is tripartite.*

In fact, we can prove that if $p = \frac{1}{n^{3/2}\log n}$, then w.h.p. there is a maximum Fan^3-free subhypergraph of $G^3(n,p)$ that is not tripartite. To see this, let T be a 3-graph on vertex set $[5]$, with three edges $\{1,2,3\}$, $\{3,4,5\}$ and $\{1,2,4\}$. Note that T is not tripartite. If $p = \frac{1}{n^{3/2}\log n}$, then using the second moment method, we can prove w.h.p. there are $n^{1/3}$ vertex disjoint copies of T in $G^3(n,p)$. Then applying Janson's inequality, we can prove that one from them satisfies that its edges are not in any copy of Fan^3. Hence a maximum Fan^3-free subhypergraph of $G^3(n,p)$ will contain this T, so it is not tripartite. The computation is tedious, so we omit the details. We can see there is still a gap between the probabilities above and the one in Theorem 1.3. Following a similar strategy, one may also obtain the corresponding probabilities when w.h.p. there is a maximum Fan^k-free subhypergraph of $G^k(n,p)$ that is not k-partite for $k \geq 4$. To be more precise, for some appropriate p', one may expect to find some k-graph T^k which is not k-partite, such that w.h.p. in $G^k(n,p')$, there exists a maximum Fan^k-free subhypergraph containing T^k. However, finding such T^k is not easy for $k \geq 4$.

To prove Theorem 1.3, we need to transfer Theorem 1.2 to the probability setting, to derive the asymptotic stability of Fan^k. Such asymptotic stability result of Fan^k can be conclude from a celebrated transference theorem established by Samotij [24], which builds on the work of Schacht [25], and also a weaker version proved by Conlon and Gowers [7].

Let H be a k-uniform hypergraph with at least $k+1$ vertices. The *k-density* of H, denoted by $m_k(H)$, is defined as follows,

$$m_k(H) = \max\left\{\frac{e(L)-1}{v(L)-k} : L \subseteq H \text{ with } v(L) \geq k+1\right\}.$$

Applying Samotij's transference theorem (Theorem 3.4 in [24]) to Fan^k, one can obtain the following result concerning the stability of Fan^k in random k-uniform hypergraphs.

Theorem 1.4 *For every $\delta > 0$ and $k \geq 3$, there exist positive constants K and ϵ such that if $p_n \geq Kn^{-1/m_k(\mathrm{Fan}^k)}$, then w.h.p. the following holds. Every* Fan^k*-free* *subhypergraph of* $G^k(n,p_n)$ *with at least* $(\pi(\mathrm{Fan}^k)-\epsilon)\binom{n}{k}p_n$ *edges admits a partition* $(V_1,V_2,\ldots,V_k)$ *of* $[n]$ *such that all but at most* $\delta n^k p_n$ *edges have one vertex in each* V_i*.*

The derivation of Theorem 1.4 is similar to that of Theorems 1.5 and 1.8 in [24], we include the proof of Theorem 1.4 in the appendix.

As we can see, Theorem 1.4 implies that the largest Fan^k-free subhypergraph of $G^k(n,p_n)$ is almost k-partite, thus our result Theorem 1.3 is an exact version of Theorem 1.4 to this point.

The rest of the paper is organized as follows. In Section 2, we introduce some more notation and preliminaries. We present the proof of Theorem 1.3 in Section 3, and we provide the proof of Theorem 1.4 in Appendix. To simplify the formulas, we shall omit floor and ceiling signs when they are not crucial. In this paper, we will always assume that n is the variable that tends to infinity. Undefined notation and terminology can be found in [6].

2 Preliminaries

Let $G\sim G^k(n,p)$. The *size* of a hypergraph H, denoted by $|H|$, is the number of edges it contains. We simply write $x=(1\pm\epsilon)y$ when $(1-\epsilon)y\le x\le(1+\epsilon)y$. Given an n-vertex k-graph H and a partition $\Pi=(A_1,A_2,\cdots,A_k)$ of the vertex set $V(H)$ of H, we say that an edge e of H is *crossing* if $e\cap A_i$ is non-empty for every i. We use $H[\Pi]$ to denote the set of crossing edges of H. A vertex set partition $\Pi=(A_1,A_2,\cdots,A_k)$ is *asymptotically balanced* if $|A_i|=(1\pm 10^{-10})n/k$ for all i.

The *link hypergraph* $L(x)$ of a vertex x in G is the $(k-1)$-graph with vertex set $V(G)$ and edge set $\{x_1x_2\ldots x_{k-1} : xx_1x_2\ldots x_{k-1}$ is an edge of $G\}$. The *crossing link hypergraph* $L_\Pi(x)$ of a vertex x is the subhypergraph of $L(x)$ whose edge set is $\{x_1x_2\ldots x_{k-1} : xx_1x_2\ldots x_{k-1}$ is a crossing edge of $G\}$. The *degree* $d(x)$ of x is the size of $L(x)$, while the *crossing degree* $d_\Pi(x)$ of x is the size of $L_\Pi(x)$. Given two vertices x and y, their *co-neighborhood* $N(x,y)$ is the set $\{x_1x_2\ldots x_{k-2} : xyx_1x_2\ldots x_{k-2}$ is an edge of $G\}$; the *co-degree* of x and y is the number of elements in their co-neighborhood.

We denote by $q_k(G)$ the size of a largest k-partite subhypergraph of G. We say a vertex partition Π with k classes, which we will call a k-partition, is *maximum* if $|G[\Pi]|=q_k(G)$. Let F be a maximum Fan^k-free subhypergraph of G. Since Fan^k is not k-partite, we have $q_k(G)\le|F|$. Thus, to prove Theorem 1.3, we will show that w.h.p. $|F|\le q_k(G)$. Further, we will prove that w.h.p. F is k-partite.

We will use the following Chernoff-type bound in our proofs.

Lemma 2.1 *[1] Let Y be the sum of mutually independent indicator random variables, and let $\mu = \mathbb{E}[Y]$, the expectation of Y. For all $\epsilon > 0$,*

$$Pr[|Y - \mu| > \epsilon\mu] < 2e^{-c_\epsilon \mu},$$

where $c_\epsilon = \min\{-\ln(e^\epsilon(1+\epsilon)^{-(1+\epsilon)}), \epsilon^2/2\}$.

In the sequel, we use c_ϵ to denote the constant in Lemma 2.1.

Note that the number of vertices of Fan^k is $k(k-1)+1$, and the number of edges of Fan^k is $k+1$, moreover, let

$$h(t) = \begin{cases} k(k-1)+1-t(k-2) & \text{if } 0 \le t \le 2, \\ \min\{k(k-1)+1-(t-1)(k-1),\ k(k-1)+1-t(k-2)\} & \text{if } 3 \le t < k. \end{cases}$$

For a subhypergraph L of Fan^k with $k+1-t$ edges, $0 \le t < k$, the number of vertices of L is at least $h(t)$. Thus, we have

$$m_k(\mathrm{Fan}^k) \le \frac{k+1-t-1}{h(t)-k} \le \max\{\frac{1}{k-1}, \frac{k-t}{(k-t)(k-2)+1}\} < 1.$$

Let

$$p_0 = \begin{cases} (\log n)^\gamma / n & \text{if } k = 3, \\ 1/n & \text{if } k \ge 4, \end{cases}$$

where $\gamma > 0$ is an absolute constant. We have $p_0 \ge 1/n \gg n^{-1/m_k(\mathrm{Fan}^k)}$.

Some propositions for $G^k(n,p)$ will be stated below. The following two propositions are obtained by standard argument, so we skip the proofs.

Proposition 2.1 *For any $0 < \epsilon < 1$, there exists a constant K such that if $p > K/n$, then w.h.p. for any vertex v of G, its degree $d(v)$ is $(1 \pm \epsilon)\frac{1}{(k-1)!}n^{k-1}p$.*

Proposition 2.2 *For any $0 < \epsilon < 1$, there exists a constant K such that if $p > K/n$, then w.h.p. for any k-partition $\Pi = (A_1, A_2, \dots, A_k)$ with $|A_2|, \dots, |A_k| \ge \frac{n}{k^k}$, and any vertex $v \in A_1$ we have $d_\Pi(v) = (1 \pm \epsilon)p|A_2||A_3| \dots |A_k|$.*

From Theorem 1.1, it is easy to get that $\pi(\mathrm{Fan}^k) = \frac{k!}{k^k}$. With Propositions 2.1, 2.2 and Theorem 1.4, we obtain the following proposition.

Proposition 2.3 *For any $0 < \epsilon < 1$, there exists a constant K such that if $p > K/n$, then w.h.p. the following holds: if F is a maximum* Fan^k*-free subhypergraph of G and Π is a k-partition maximizing $|F[\Pi]|$, then $|F| \geq \left(\pi(\mathrm{Fan}^k) - \epsilon\right)\binom{n}{k}p = \left(\frac{k!}{k^k} - \epsilon\right)\binom{n}{k}p$, and Π is an asymptotically balanced partition.*

Proof. It suffices to prove the statement when ϵ is small. For a partition $\Pi = (A_1, A_2, \ldots, A_k)$, it is clear that $|F| \geq q_k(G) \geq |G[\Pi]|$. And Proposition 2.2 implies that w.h.p. $|G[\Pi]| = (1 \pm \epsilon)p|A_1||A_2| \ldots |A_k|$ if $|A_2|, |A_3|, \ldots, |A_k| \geq \frac{n}{k^k}$. Consider a partition $\Gamma = (A_1', A_2', \ldots, A_k')$ satisfying $|A_1'| = |A_2'| = \ldots = |A_k'| = \frac{n}{k}$, then we have $|F| \geq |G[\Gamma]| \geq \left(\frac{k!}{k^k} - \epsilon\right)\binom{n}{k}p$. Also, by applying Theorem 1.4 with $\delta = \frac{\epsilon}{2(k!)}$, we obtain that if Π maximizes $|F[\Pi]|$, then we have

$$|G[\Pi]| \geq |F[\Pi]| \geq \left(\frac{k!}{k^k} - \epsilon\right)\binom{n}{k}p - \delta n^k p \geq \left(\frac{k!}{k^k} - 2\epsilon\right)\binom{n}{k}p. \tag{1}$$

Now we prove that Π is an asymptotically balanced partition. Suppose on the contrary, Π is not asymptotically balanced.
(i) If Π is not an asymptotically balanced partition and $|A_1|, |A_2|, \ldots, |A_k| \geq \frac{n}{k^k}$, then we claim that w.h.p. $|G[\Pi]| \leq (1 + \epsilon)p|A_1||A_2| \ldots |A_k| < \left(\frac{k!}{k^k} - 2\epsilon\right)\binom{n}{k}p$. Indeed, let $|A_i| = a_i n + o(n)$ for $i = 1, \ldots, k$, we have $\sum_{i=1}^k a_i = 1$ since $\sum_{i=1}^k |A_i| = n$. Applying the arithmetic-geometric mean inequality, we have $\Pi_{i=1}^k a_i \leq \left(\frac{\sum_{i=1}^k a_i}{k}\right)^k$, the equality holds if and only if $a_1 = \ldots = a_k$. Note that Π is not asymptotically balanced implies that there exists some j such that $a_j > (1+10^{-10})/k$ or $a_j < (1-10^{-10})/k$. Therefore there exist $i \neq i'$ such that $a_i \neq a_{i'}$. Consequently, for sufficiently small ϵ,

$$\Pi_{i=1}^k a_i < \left(\frac{\sum_{i=1}^k a_i}{k}\right)^k - 2\epsilon = \frac{1}{k^k} - 2\epsilon. \tag{2}$$

Realize that $|A_1| \ldots |A_k| = \Pi_{i=1}^k a_i n^k + o(n^k)$, combining with (2), we have $|A_1| \ldots |A_k| < \left(\frac{1}{k^k} - 2\epsilon\right) n^k + o(n^k)$. Hence,

$$|G[\Pi]| \leq (1+\epsilon)p|A_1||A_2| \ldots |A_k| < (1 + \epsilon)\left(\frac{1}{k^k} - 2\epsilon\right) pn^k + o(pn^k) < \left(\frac{k!}{k^k} - 2\epsilon\right)\binom{n}{k}p,$$

which contradicts (1).
(ii) If Π is not asymptotically balanced and one of $|A_1|, |A_2|, \ldots, |A_k|$ is less than $\frac{n}{k^k}$, then Proposition 2.1 implies that

$$|G[\Pi]| < (1 + \epsilon)\frac{n}{k^k}\frac{1}{(k-1)!}pn^{k-1}. \tag{3}$$

For small enough ϵ,

$$\left(1+\frac{6k^k}{k!}\right)\epsilon < 1. \tag{4}$$

Since $(k-1)! \geq 2$ for $k \geq 3$, we have

$$\frac{1+\epsilon}{k^k(k-1)!} \leq \frac{1+\epsilon}{2k^k}. \tag{5}$$

Note that

$$\left(1+\frac{6k^k}{k!}\right)\epsilon - 1 = \epsilon + 1 - \left(2 - \frac{6k^k}{k!}\epsilon\right) = 2k^k\left(\frac{1+\epsilon}{2k^k} - \left(\frac{1}{k^k} - \frac{3}{k!}\epsilon\right)\right),$$

and so we have

$$\frac{1+\epsilon}{2k^k} < \left(\frac{1}{k^k} - \frac{3}{k!}\epsilon\right) \tag{6}$$

by (4). Hence, combining (3), (5) and (6), we obtain that $|G[\Pi]| < \left(\frac{k!}{k^k} - 3\epsilon\right)\frac{n^k}{k!} < \left(\frac{k!}{k^k} - 2\epsilon\right)\binom{n}{k}p$, which contradicts (1). Therefore, if Π maximizes $|F[\Pi]|$, then Π is asymptotically balanced. $\square$

3 Proof of Theorem 1.3

Let F be a Fan^k-free subhypergraph of G. We want to show that $|F| \leq q_k(G)$. We now state a key lemma. The lemma proves $|F| \leq q_k(G)$ with some additional conditions on F.

Lemma 3.1 *Let F be a Fan^k-free subhypergraph of G and $\Pi = (A_1, A_2, \ldots, A_k)$ be an asymptotically balanced partition maximizing $|F[\Pi]|$. For $1 \leq i \leq k$, let $B_i = \{e \in F : |e \cap A_i| \geq 2\}$. There exist positive constants K and δ such that if $p > K/n$ for $k \geq 4$; and $p > K(\log n)^\gamma/n$ for $k = 3$, where $\gamma > 0$ is an absolute constant, and the following conditions hold:*

(i) $|\bigcup_{i=1}^{k} B_i| \leq \delta p n^k$,

(ii) $B_1 \neq \emptyset$,

then w.h.p. $|F[\Pi]| + k|B_1| < |G[\Pi]|$.

Remark. Note that by relabeling if necessary, Condition (ii) is satisfied provided that there exists some j such that $B_j \neq \emptyset$. We point out that if Condition (ii) does not hold, i.e., $|B_1| = 0$, then clearly $|F[\Pi]| + k|B_1| \leq |G[\Pi]|$. Lemma 3.1 states that $|F[\Pi]| + k|B_1| \leq |G[\Pi]|$, while Condition (ii) implies the strict inequality.

The proof of Lemma 3.1 is presented in the next subsection. Now we use Lemma 3.1 to prove Theorem 1.3.

Let $\tilde{F}$ be a maximum Fan^k-free subhypergraph of G, so

$$|\tilde{F}| \geq q_k(G). \tag{7}$$

To prove Theorem 1.3, we will show that w.h.p. $|\tilde{F}| \leq q_k(G)$ and $\tilde{F}$ is k-partite. Let $\Pi = (A_1, A_2, \ldots, A_k)$ be a k-partition maximizing $|\tilde{F}[\Pi]|$. From Proposition 2.3, we get that Π is asymptotically balanced for sufficiently large K. For $1 \leq i \leq k$, let $\tilde{B}_i = \{e \in \tilde{F}, |e \cap A_i| \geq 2\}$. Without loss of generality, we may assume $|\tilde{B}_1| \geq |\tilde{B}_2|, \ldots, |\tilde{B}_k|$. Then we have

$$\sum_{i=1}^{k} |\tilde{B}_i| \leq k|\tilde{B}_1|.$$

Consequently,

$$|\tilde{F}| \leq |\tilde{F}[\Pi]| + \sum_{i=1}^{k} |\tilde{B}_i| \leq |\tilde{F}[\Pi]| + k|\tilde{B}_1|. \tag{8}$$

Combine with (7), we have

$$q_k(G) \leq |\tilde{F}| \leq |\tilde{F}[\Pi]| + k|\tilde{B}_1| \leq |G[\Pi]| \leq q_k(G),$$

where the penultimate inequality follows from Lemma 3.1 and its remark, hence, $|\tilde{F}| = q_k(G)$. So the equalities hold throughout the inequalities. If $\tilde{B}_1 \neq \emptyset$, then by Lemma 3.1, $|\tilde{F}[\Pi]| + k|\tilde{B}_1| < |G[\Pi]|$, a contradiction. So $\tilde{B}_1$ is an empty set. Since we assume that $|\tilde{B}_1| \geq |\tilde{B}_2|, \ldots, |\tilde{B}_k|$, we have that $|\tilde{B}_1| = |\tilde{B}_2| = \ldots = |\tilde{B}_k| = 0$, which implies that $\tilde{F}$ is k-partite.

The proof is thus complete. ∎

3.1 Proof of Lemma 3.1

Let

$$c = \begin{cases} \frac{1}{16^2(k-1)k^{k-1}} & \text{if } k \geq 4, \\ \min\{\frac{1}{2\times 48^2}, (\ln 4 - 1)\gamma\} & \text{if } k = 3. \end{cases}$$

And let $c' = \frac{1}{16k^2(k-1)}c$, $K = \frac{32(k+1)^4}{c}$, $\delta = \frac{(c')^k}{4k}$, and $c_1 = \frac{2k^2\delta}{c}$.

Call the edges in $G[\Pi] \setminus F$ *missing*. To prove Lemma 3.1, it suffices to prove that the number of missing edges is larger than $k|B_1|$. So we assume for contradiction that the number of missing edges w.h.p. is at most $k|B_1|$. Since Conditions (i) and (ii) of Lemma 3.1 tell us that $0 < |B_1| \leq |\bigcup_{i=1}^{k} B_i| \leq \delta p n^k$, our hypothesis implies that

$$\text{the number of missing edges is at most } k\delta p n^k. \tag{9}$$

We call a pair of vertices $\{u, v\}$ a *bad pair* if u and v belong to the same part A_i and are covered by an edge of F. For distinct vertices u and v, call the pair $\{u, v\}$ *sparse* if the co-degree of u and v in F is at most $\frac{c}{8k(k-1)} p n^{k-2}$, otherwise, call $\{u, v\}$ *dense*. For the sparse pairs, we derive the following claim.

Claim 3.1 *If we have a set $C = \{x, y_1, \ldots, y_k\} \subseteq V(G)$, which contains at least one edge $d = \{y_1, \ldots, y_k\}$ in F, then at least one pair of vertices $\{u, v\}$ from $\binom{C}{2} \setminus \binom{d}{2}$ is sparse, i.e., at least one of $\{x, y_i\}$ for $1 \leq i \leq k$ is sparse. Moreover, if there is an edge $e \in F$, such that $x \in e$ and $e \cap d = \{y_1\}$, then at least one of $\{x, y_i\}$ for $2 \leq i \leq k$ is sparse.*

Proof. For the first part, we assume on the contrary that there is no sparse pair in $\binom{C}{2} \setminus \binom{d}{2}$. Consider the pair $\{x, y_1\}$; since it is not sparse, there are more than $\frac{c}{8k(k-1)} p n^{k-2}$ edges containing $\{x, y_1\}$. Note that among those edges, there are at most $\sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i}$ edges containing vertices in $\{y_2, \ldots, y_k\}$. We claim that

$$\frac{c}{8k(k-1)} p n^{k-2} - \sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i} > 1. \tag{10}$$

In fact, for $k \geq 4$ and sufficiently large n,

$$\sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i} = (k-1) \binom{n-(k+1)}{k-3} + O(n^{k-4}) < \frac{2(k-1)}{(k-3)!} n^{k-3}.$$

Therefore,

$$\frac{c}{8k(k-1)} p n^{k-2} - \sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i} > \frac{4(k+1)^3}{k} n^{k-3} - \frac{2(k-1)}{(k-3)!} n^{k-3} > 1.$$

For $k = 3$, we have $\sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i} = 2$. So

$$\frac{c}{8k(k-1)} p n^{k-2} - \sum_{i=1}^{k-2} \binom{k-1}{i} \binom{n-(k+1)}{k-2-i} > \frac{c}{8k(k-1)} K (\log n)^{\gamma} - 2 > 1.$$

From (10), we can select an edge e_1 containing $\{x, y_1\}$, and disjoint with $\{y_2, \ldots, y_k\}$. Repeat that process, suppose we have found edges $e_1, e_2, \ldots, e_t$ ($1 \leq t < k$) such that

$\{x, y_i\} \subseteq e_i$ for $1 \leq i \leq t$, $(e_i \cap e_j) \setminus \{x\} = \emptyset$ for any $i \neq j$, and $\{y_{t+1}, y_{t+2}, \ldots, y_k\} \cap \bigcup_{i=1}^{t} e_i = \emptyset$. Let M denote the number of vertices in $\left(\bigcup_{i=1}^{t} e_i\right) \cup d$, then $M < k(k-1)+1$. We claim that

$$\frac{c}{8k(k-1)} pn^{k-2} - \sum_{i=1}^{k-2} \binom{M}{i} \binom{n-M}{k-2-i} > 1. \tag{11}$$

Indeed, since $M < k(k-1)+1$, for $k \geq 4$ and sufficiently large n,

$$\sum_{i=1}^{k-2} \binom{M}{i} \binom{n-M}{k-2-i} = M \binom{n-M}{k-3} + O(n^{k-4}) < \frac{2M}{(k-3)!} n^{k-3}.$$

Hence,

$$\frac{c}{8k(k-1)} pn^{k-2} - \sum_{i=1}^{k-2} \binom{M}{i} \binom{n-M}{k-2-i} > \frac{4(k+1)^3}{k} n^{k-3} - \frac{2M}{(k-3)!} n^{k-3} > 1.$$

For $k = 3$, we have $\sum\limits_{i=1}^{k-2} \binom{M}{i}\binom{n-M}{k-2-i} = M$. So

$$\frac{c}{8k(k-1)} pn^{k-2} - \sum_{i=1}^{k-2} \binom{M}{i} \binom{n-M}{k-2-i} > \frac{c}{8k(k-1)} K(\log n)^{\gamma} - M > 1.$$

Note that $\{x, y_{t+1}\}$ is not sparse, by (11), we can select an edge e_{t+1} such that $\{x, y_i\} \subseteq e_i$ for $1 \leq i \leq t+1$, $(e_i \cap e_j) \setminus \{x\} = \emptyset$ for any $i \neq j$, and $\{y_{t+2}, y_{t+3}, \ldots, y_k\} \cap \bigcup_{i=1}^{t+1} e_i = \emptyset$. That implies we can greedily build a copy of Fan^k in F with the center x.

For the second part, since $e \cap d = \{y_1\}$, let $e_1 = e$ in the procedure above, we can still greedily build a copy of Fan^k in F. $\square$

Let W consist of vertices which are incident to at least $cn^{k-1}p$ missing edges. We have w.h.p.

$$|W| \leq c_1 n, \tag{12}$$

for otherwise, we encounter at least $\frac{c_1 n \cdot cn^{k-1}p}{k} > k\delta pn^k$, a contradiction to (9).

Claim 3.2 *If $\{v_0, v_1\}$ is a bad pair, then w.h.p. $\{v_0, v_1\}$ intersects W.*

Proof. Since $\{v_0, v_1\}$ is a bad pair, we have $\{v_0, v_1\} \subseteq A_i$ for some $i \in [k]$. Assume without loss of generality that $v_0, v_1 \in A_1$ and are covered by edge e. Consider any edge of G of the form $\{v_1, v_2, \ldots, v_k\}$ where for every $2 \leq i \leq k$, $v_i \in A_i \setminus e$. By Claim 3.1 with $C = \{v_0, v_1, \ldots, v_k\}$ and $d = \{v_1, \ldots, v_k\}$, either at least one pair $\{v_0, v_j\}$ with $j \neq 1$ is sparse or the k-tuple $\{v_1, v_2, \ldots, v_k\}$ is missing. From Lemma

2.1, the number of choices of such k-tuple $\{v_1, v_2, \ldots, v_k\}$ is w.h.p. at least $\frac{1}{2}\left(\frac{n}{k}\right)^{k-1}p$. If the latter case occurs at least half of the time, then w.h.p. v_1 belongs to at least $\frac{1}{4}\left(\frac{n}{k}\right)^{k-1}p > cn^{k-1}p$ missing edges. That implies $v_1 \in W$. So let us suppose that the former case occurs at least half of the time. Let $u \neq v_1$ be a vertex belonging to a k-tuple we chose above, such that $\{v_0, u\}$ is a sparse pair. By Lemma 2.1, with probability at most $2e^{-c_\epsilon\left(\frac{n}{k}\right)^{k-2}p}$, $\{v_0, u\}$ appears in at least $2\left(\frac{n}{k}\right)^{k-2}p$ k-tuples. Since the number of choices of u is less than n, with the union bound we derive that, with probability at least $1 - n \cdot 2e^{-c_\epsilon\left(\frac{n}{k}\right)^{k-2}p}$, every sparse pair $\{v_0, v_j\}$ appears at most $2\left(\frac{n}{k}\right)^{k-2}p$ times. By the definition of p, $1 - n \cdot 2e^{-c_\epsilon\left(\frac{n}{k}\right)^{k-2}p} = 1 - o(1)$ for $k \geq 3$, so we obtain that every sparse pair $\{v_0, v_j\}$ w.h.p. appears at most $2\left(\frac{n}{k}\right)^{k-2}p$ times. Hence v_0 is in at least $\frac{\frac{1}{4}\left(\frac{n}{k}\right)^{k-1}p}{2\left(\frac{n}{k}\right)^{k-2}p} = \frac{n}{8k}$ sparse pairs.

In fact, those sparse pairs yield many missing edges. For any sparse pair $\{v_0, v_j\}$, the number of crossing edges containing v_0 and v_j in F is at most $\frac{c}{8k(k-1)}pn^{k-2}$. From Lemma 2.1 and the union bound, the number of crossing edges containing v_0 and v_j in $G[\Pi]$ is w.h.p. at least $\frac{1}{2}\left(\frac{n}{k}\right)^{k-2}p$. Hence, the number of missing edges containing v_0 and v_j is w.h.p. at least $\frac{1}{2}\left(\frac{n}{k}\right)^{k-2}p - \frac{c}{8k(k-1)}pn^{k-2} \geq \frac{1}{4}\left(\frac{n}{k}\right)^{k-2}p$. On the other hand, any such missing edge contains at most $k-1$ such sparse pairs $\{v_0, v_i\}$. It follows that v_0 belongs to at least

$$\frac{\frac{n}{8k} \cdot \frac{1}{4}\left(\frac{n}{k}\right)^{k-2}p}{k-1} \geq cn^{k-1}p$$

missing edges, that implies v_0 belongs to W. $\square$

From the definition of W, we get that the number of missing edges is at least $\frac{1}{k}|W|cn^{k-1}p$. Combining with our hypothesis that w.h.p. there are at most $k|B_1|$ missing edges, we obtain that $|B_1| \geq \frac{1}{k^2}|W|cn^{k-1}p$. For any edge e in B_1 such that $x \in A_1 \cap e$, there are at most $k-1$ ways to choose a bad pair $\{x, y\} \subset e$. Then by Claim 3.2, there exists a vertex x such that

$$x \in W \cap A_1 \text{ belonging to at least } \frac{|B_1|}{(k-1)|W|} \geq \frac{1}{k^2(k-1)}cn^{k-1}p \text{ edges in } B_1. \tag{13}$$

Note that each of such edges contains some vertex in $A_1 \setminus \{x\}$.

Let Y_1 consist of those $y \in A_1$, such that $\{x, y\}$ is a bad pair. Let $Z_1 \subseteq Y_1$ be the set of vertices z for which $\{x, z\}$ is dense. Since $|Y_1 \setminus Z_1| \leq |A_1|$, the number of edges in B_1 containing x and some vertex of $Y_1 \setminus Z_1$ is w.h.p. at most

$$\frac{n}{k} \cdot \frac{c}{8k(k-1)}pn^{k-2} < \frac{1}{2k^2(k-1)}cn^{k-1}p.$$

Thus, the number of edges in B_1 containing x and some vertex of Z_1 is w.h.p. at least $\frac{1}{2k^2(k-1)}cn^{k-1}p$. For any $z \in Z_1$, by Lemma 2.1 and the union bound, the number of edges in B_1 containing x and z is w.h.p. at most $2\binom{n-2}{k-2}p < 2\binom{n}{k-2}p < 2n^{k-2}p$. Thus, w.h.p.

$$|Z_1| \geq \frac{\frac{1}{2k^2(k-1)}cn^{k-1}p}{2n^{k-2}p} = \frac{1}{4k^2(k-1)}cn > c'n.$$

Let Z_i consist of those $z \in A_i$ for which $\{x, z\}$ is dense, $2 \leq i \leq k$. If $|Z_i| \geq c'n$ for every $2 \leq i \leq k$, then by Lemma 2.1 there are w.h.p. at least $\frac{1}{2}(c'n)^k p$ k-tuples $\{z_1, z_2, \ldots, z_k\}$ in G with $z_j \in Z_j$ for $1 \leq j \leq k$. And every such k-tuple $\{z_1, z_2, \ldots, z_k\}$ is a missing edge by Claim 3.1. So we obtain at least $\frac{1}{2}(c'n)^k p > k\delta n^k p$ missing edges, a contradiction.

Therefore we assume without loss of generality that $|Z_2| < c'n$. Thus, w.h.p. there are less than $\frac{n}{k}\frac{c}{8k(k-1)}pn^{k-2} + 2c'n \cdot n^{k-2}p = 4c'n^{k-1}p$ edges in F containing x and intersecting A_2.

Let us consider partition Π' obtained from Π by moving x from A_1 to A_2. If $e \in F[\Pi] \setminus F[\Pi']$, we call it a crossing edge we *lose*; if $e \in F[\Pi'] \setminus F[\Pi]$, we call it a crossing edge we *gain*. By the above arguments, we lose less than $4c'n^{k-1}p$ crossing edges. Note that Π was chosen to maximize $F[\Pi]$, so

$$\text{we must gain fewer than } 4c'n^{k-1}p \text{ crossing edges.} \tag{14}$$

However, we will show that it is not the case, thus obtain a contradiction.

If there are at least $2c_1n^{k-1}p$ edges f containing x such that $f \setminus \{x\}$ intersects some part A_i, $i \in [k]$, in at least two vertices, then this creates at least $2c_1n^2p$ bad pairs (since each bad pair appears in at most $\binom{n}{k-3} < n^{k-3}$ such edges f). By Claim 3.2, those bad pairs in turn force W to have size at least $2c_1np$ (since every vertex in W belongs to less than n bad pairs). Therefore we obtain a contradiction to (12). So let us assume otherwise. That is, there are less than $2c_1n^{k-1}p$ edges f containing x such that $f \setminus \{x\}$ intersects some part A_i, $i \in [k]$, in at least two vertices. Recall that we have proved w.h.p. there are less than $4c'n^{k-1}p$ edges in F containing x and intersecting A_2, combining with (13), we have that w.h.p. there are at least

$$\frac{1}{k^2(k-1)}cn^{k-1}p - 4c'n^{k-1}p - 2c_1n^{k-1}p > 4c'n^{k-1}p$$

edges h in F containing x, disjoint with A_2, and satisfying $|h \cap (A_i \setminus \{x\})| = 1$ and $|h \cap A_j| \leq 1$ for $3 \leq j \leq k$, which contradicts (14). Thus we complete the proof. ∎

Acknowledgments. The authors are grateful to the anonymous referees and editor for very constructive and insightful suggestions and comments, which helped to improve the presentation of the paper.

Ran Gu was partially supported by National Natural Science Foundation of China (No. 11701143) and the Fundamental Research Funds for the Central Universities. Hui Lei was partially supported by the National Natural Science Foundation of China (No. 12001296) and the Fundamental Research Funds for the Central Universities, Nankai University (Nos. 63201163 and 63211093). Yongtang Shi was partially supported by the National Natural Science Foundation of China (No. 11922112), Natural Science Foundation of Tianjin (Nos. 20JCZDJC00840, 20JCJQJC00090).

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Appendix A Proof of Theorem 1.4

We need some more notation and definitions before the derivation of Theorem 1.4. In the sequel, we employ the notation and definitions used by Samotij in [24].

Given a hypergraph H, for a set $U \subseteq V(H)$, we write $H[U]$ to denote the subhypergraph of H induced by U, i.e., the hypergraph on the vertex set U whose edges are all the edges of H that are fully contained in U.

We use the notational convention that the sequences are denoted by boldface letters, e.g., $\mathbf{p}$ stands for (p_n), that is, the sequence $p : \mathbb{N} \to [0,1]$ indexed by the set of natural numbers. The only exception is that, due to typesetting limitations, the sequence $(\mathcal{B}_n)$ will be denoted by $\mathfrak{B}$.

Definition A.1 *[24] Let $\mathbf{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let α be a positive real number, and let $\mathfrak{B} = (\mathcal{B}_n)_{n\in\mathbb{N}}$, where $\mathcal{B}_n$ is a family of subsets of $V(H_n)$. We say that $\mathbf{H}$ is $(\alpha, \mathfrak{B})$-stable if for every positive δ, there exist positive ϵ and N such that for every n with $n \geq N$ and every $U \subseteq V(H_n)$ with $|U| \geq (\alpha-\epsilon)|V(H_n)|$, we have either $|H_n[U]| \geq \epsilon|H_n|$ or $|U \setminus B| \leq \delta|V(H_n)|$ for some $B \in \mathcal{B}_n$.*

For a hypergraph H, a vertex $v \in V(H)$, and a set $U \subseteq V(H)$, let $\deg_i(v,U)$ denote the number of edges of H containing v and at least i vertices in $U \setminus \{v\}$. More precisely, let

$$\deg_i(v,U) = |\{e \in H : v \in e \text{ and } |e \cap (U \setminus \{v\})| \geq i\}|.$$

For $q \in [0,1]$, let $\mu_i(H,q)$ denote the expected value of the sum of squares of such degrees over all $v \in V(H)$ with U replaced by the q-random subset V_q of $V(H)$, namely,

$$\mu_i(H,q) = \mathbb{E}\left[\sum_{v\in V(H)} \deg_i^2(v,V_q)\right].$$

Definition A.2 *[24] Let* $\mathbf{H}$ *be a sequence of* k*-uniform hypergraphs, let* $\mathbf{p}$ *be a sequence of probabilities, and let* K *be a positive constant. We say that* $\mathbf{H}$ *is* $(K, \mathbf{p})$*-bounded if for every* $i \in \{0, \dots, k-1\}$*, there exists an* N *such that for every* n *with* $n \geq N$ *and every* $q \in [0, 1]$ *with* $q \geq p_n$*, we have*

$$\mu_i(H_n, q) \leq K q^{2i} \frac{|H_n|^2}{|V(H_n)|}.$$

The following transference theorem is the key tool to prove Theorem 1.4.

Theorem A.1 *[Theorem 3.4 in [24]] Let* $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$ *be a sequence of* k*-uniform hypergraphs be a sequence of* k*-uniform hypergraphs, let* α *be a positive real number, and let* $\mathfrak{B} = (\mathcal{B}_n)_{n \in \mathbb{N}}$*, where* $\mathcal{B}_n$ *is a family of subsets of* $V(H_n)$*, and suppose that* $\mathbf{H}$ *is* $(\alpha, \mathfrak{B})$*-stable. Furthermore, let* K *be a positive real and let* $\mathbf{p}$ *be a sequence of probabilities such that* $p_n^k |H_n| \to \infty$ *as* $n \to \infty$*,* $\mathbf{H}$ *is* $(K, \mathbf{p})$*-bounded, and* $|\mathcal{B}_n| = \exp(o(p_n |V(H_n)|))$*. Then for every positive* δ*, there exist positive* ξ, b, C*, and* N *such that for every* n *with* $n \geq N$ *and every* q *satisfying* $Cp_n \leq q \leq 1$*, the following holds with probability at least* $1 - \exp(-bq|V(H_n)|)$*: Every subset* $W \subseteq V(H_n)_q$ *with* $|W| \geq (\alpha - \xi) q |V(H_n)|$ *that satisfies* $|W \setminus B| \geq \delta q |V(H_n)|$ *for every* $B \in \mathcal{B}_n$ *satisfies* $|H[W]| \geq \xi q^k |H_n| > 0$*.*

By Theorem 1.2, we can easily obtain the following stability result on Fan^k.

Theorem A.2 *For every positive constant* δ*, there exists a positive constant* ϵ *such that the following holds: For every* k*-uniform hypergraph with at least* $\left(\pi(\mathrm{Fan}^k) - \epsilon\right) \binom{n}{k}$ *edges that does not contain* Fan^k*, there exists a partition of* $[n]$ *into sets* V_1*,* V_2*,* $\dots V_k$ *such that all but at most* δn^k *edges have one point in each* V_i*.*

We also need the following result obtained by Gowers [14], Nagle, Rödl, and Schacht [21], and Tao [28].

Theorem A.3 *[Theorem 2.5 in [24]] For an arbitrary* k*-uniform hypergraph* H *and any positive constant* δ*, there exists a positive constant* ϵ *such that every* k*-uniform hypergraph on* n *vertices with at most* $\epsilon n^{v(H)}$ *copies of* H *may be made* H*-free by removing from it at most* δn^k *edges.*

Proof Theorem 1.4. For an application of Theorem A.1, we consider the sequence of $e(\mathrm{Fan}^k)$-uniform hypergraphs $\mathbf{H} = (H_n = (V_n, E_n))_{n \in \mathbb{N}}$ where $V_n = E(K_n^k)$ (i.e. the set of edges of K_n^k), and the edges of E_n correspond to copies of Fan^k in K_n^k. Moreover, we set $p_n = n^{-1/m_k(\mathrm{Fan}^k)}$ and $\alpha = \pi(\mathrm{Fan}^k)$. Let $\mathcal{B}_n$ be the family of edge sets of all complete k-partite k-uniform hypergraphs on the vertex set $[n]$. Observe

that if the assumptions of Theorem A.1 are satisfied, then we can immediately derive Theorem 1.4 by applying Theorem A.1. Thus, in order to complete the proof, we verify the following assumptions of Theorem A.1.
(a) $\mathbf{H}$ is $(\alpha, \mathfrak{B})$-stable.
(b) $p_n^k|H_n| \to \infty$ as $n \to \infty$.
(c) $\mathbf{H}$ is $(K, \mathbf{p})$-bounded.
(d) $|\mathcal{B}_n| = \exp(o(p_n|V(H_n)|))$.

For any constant δ^*, let $\epsilon_{A.2}(\delta^*)$ equal ϵ obtained by applying Theorem A.2 with $\delta = \delta^*$, and let $\epsilon_{A.3}(\delta^*)$ equal ϵ obtained by applying Theorem A.3 with $\delta = \delta^*$.

(a) Fix a positive δ, let $\delta'' = \frac{\delta}{3k!}$, $\epsilon' = \epsilon_{A.2}(\delta'')$, $\delta' = \min\{\delta'', \epsilon'/2\}$ and $\epsilon = \min\{\epsilon'/2, \epsilon_{A.3}(\delta')\}$. Let G be a subhypergraph of K_n^k with at least $(\alpha - \epsilon)\binom{n}{k}$ edges that cannot be made k-partite by removing from it $\delta\binom{n}{k}$ edges. We claim that it contains at least $\epsilon n^{v(\text{Fan}^k)}$ copies of Fan^k. Indeed, if it did not, then by Theorem A.3, removing at most $\delta' n^k$ edges from G would make it into a Fan^k-free k-uniform hypergraph G. Since such G would still have at least $ex(n, \text{Fan}^k) - (\epsilon + \delta')n^k$ edges, by Theorem A.2, it could be made k-partite by removing from it some further $\delta'' n^k$ edges. Hence, G could be made k-partite by removing at most $2\delta'' n^k$ edges, which is fewer than $\delta\binom{n}{k}$ edges, contradicting our assumption. Therefore, (a) is verified.

(b) Since $\frac{e(\text{Fan}^k)-1}{v(\text{Fan}^k)-k} = \frac{k}{(k-1)^2}$, we have

$$m_k(\text{Fan}^k) \geq \frac{k}{(k-1)^2} > \frac{1}{k-1}. \tag{15}$$

Note that $p_n^k|H_n| \geq p_n^k\binom{n}{v(\text{Fan}^k)} = \Omega\left(\frac{n^{k(k-1)+1}}{n^{1/m_k(\text{Fan}^k)}}\right)$, combining with (15), we obtain that $p_n^k|H_n| \to \infty$ as $n \to \infty$.

(c) Schacht in [25] proved that for every k-uniform hypergraph F with at least one vertex contained in at least two edges, $\mathbf{H}$ is $(K, \mathbf{p}^*)$-bounded for some $K \geq 1$, where $p_n^* = n^{-1/m_k(F)}$. Here we include the proof (when F is Fan^k) for completeness.

Observe that H_n is a regular hypergraph with $\binom{n}{k}$ vertices and every vertex is contained in $\Theta(n^{v(\text{Fan}^k)-k})$ edges and $|E_n| = \Theta(n^{v(\text{Fan}^k)})$. We will show that for $q \geq n^{-1/m_k(\text{Fan}^k)}$ and $i = 1, 2, \ldots, e(\text{Fan}^k) - 1$ we have

$$\mu_i(H_n, q) = \mathbb{E}\left[\sum_{v \in V_n} \deg_i^2(v, V_{n,q})\right] = \sum_{v \in V_n} \mathbb{E}\left[\deg_i^2(v, V_{n,q})\right] = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right).$$

Recall the definition of $\mathbf{H}$, every $v \in V_n$ corresponds to an edge $e(v)$ in K_n^k. Therefore, the number $\mathbb{E}\left[\deg_i^2(v, V_{n,q})\right]$ is the expected number of pairs (F_1, F_2) of copies F_1 and F_2 of Fan^k in K_n^k satisfying $e(v) \in E(F_1) \cap E(F_2)$ and both copies F_1 and F_2 have at least i edges in $E(G^k(n,q) \setminus \{e(v)\}$. Summing over all such pairs F_1 and F_2 we obtain

$$
\begin{aligned}
\mathbb{E}\left[\deg_i^2(v, V_{n,q})\right] &\leq \sum_{F_1,F_2:e(v)\in E(F_1)\cap E(F_2)} \sum_{j=0}^{|E(F_1)\cap E(F_2)|-1} q^{2i-j} \\
&= O\left(\sum_{F_1,F_2:e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)}\right) \qquad (16)
\end{aligned}
$$

since $q \leq 1$. Furthermore,

$$
\sum_{F_1,F_2:e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)} = O\left(\sum_{J:e(v)\in E(J)} n^{2v(\mathrm{Fan}^k)-2v(J)} q^{2i-(e(J)-1)}\right), \qquad (17)
$$

where the sum on the right-hand side is indexed all hypergraphs $J \subseteq K_n^k$ which contain $e(v)$ and which are isomorphic to a subhypergraph of Fan^k. It follows from the definition of $m_k(\mathrm{Fan}^k)$ and $q \geq n^{-1/m_k(\mathrm{Fan}^k)}$ that $n^{v(J)}q^{e(J)} = \Omega(qn^k)$. Combining this with (16) and (17) we obtain

$$
\begin{aligned}
\mathbb{E}\left[\deg_i^2(v, V_{n,q})\right] &= O\left(\sum_{J:e(v)\in E(J)} n^{2v(\mathrm{Fan}^k)-2v(J)} q^{2i-(e(J)-1)}\right) \\
&= O\left(\sum_{J:e(v)\in E(J)} n^{2v(\mathrm{Fan}^k)-v(J)-k} q^{2i}\right).
\end{aligned}
$$

Moreover, since $v(J) \geq k$ we have

$$
\mathbb{E}\left[\deg_i^2(v, V_{n,q})\right] = O\left(\sum_{J:e(v)\in E(J)} n^{2v(\mathrm{Fan}^k)-2k} q^{2i}\right),
$$

consequently,

$$
\mu_i(H_n, q) = \sum_{v\in V_n} O\left(n^{2v(\mathrm{Fan}^k)-2k} q^{2i}\right) = O\left(n^{2v(\mathrm{Fan}^k)-2k} q^{2i}\right) = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right)
$$

Thus completes the verification of (c).

(d) Note that

$$
|\mathcal{B}_n| \leq k^n = \exp(n \log k), \qquad (18)
$$

and

$$
p_n|V(H_n)| = p_n\binom{n}{k}. \qquad (19)
$$

Moreover, we have

$$\frac{n \log k}{p_n \binom{n}{k}} = O\left(\frac{1}{n^{k-1-1/m_k(\mathrm{Fan}^k)}}\right),$$

which approaches 0 as $n \to \infty$ by (15). Combining (18) and (19), we have that $|\mathcal{B}_n| = \exp(o(p_n|V(H_n)|))$. And we conclude the proof of Theorem 1.4.