ROGERS–RAMANUJAN TYPE IDENTITIES AND CHEBYSHEV POLYNOMIALS OF THE THIRD KIND

LISA H. SUN

Abstract. It is known that q-orthogonal polynomials play an important role in the field of q-series and special functions. During studying Dyson's "favorite" identity of Rogers–Ramanujan type, Andrews pointed out that the classical orthogonal polynomials also have surprising applications in the world of q. By introducing Chebyshev polynomials of the third and the fourth kinds into Bailey pairs, Andrews derived a family of Rogers–Ramanujan type identities and also results related to mock theta functions and Hecke–type series. In this paper, by constructing a new Bailey pair involving Chebyshev polynomials of the third kind, we further extend Andrews' way in the studying of Rogers–Ramanujan type identities. By inserting this Bailey pair into various weak forms of Bailey's lemma, we obtain a companion identity for Dyson's favorite identity and a number of Rogers–Ramanujan type identities. As consequences, we also obtain results related to Appell–Lerch series and the generalized Hecke–type series. Furthermore, our key Bailey pair also fits in the bilateral versions of Bailey's lemma due to Andrews and Warnaar, which leads to more identities on the generalized Hecke–type series and false theta functions.

Keywords. Rogers–Ramanujan type identities, Dyson's favorite identity, Bailey pair, Bailey's lemma, Chebyshev polynomials, Appell–Lerch series, Hecke–type series, false theta functions

1. Introduction

Freeman Dyson, in his article, A Walk Through Ramanujan's Garden [15], describes his study of Rogers–Ramanujan type identities during the dark days of World War II. Among these identities he found his favorite one as follows

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+q^j+q^{2j})}{(q;q)_{2n+1}}=\prod_{n=1}^{\infty}\frac{(1-q^{9n})}{(1-q^n)}. \tag{1.1}$$

Dyson's proof of (1.1) and the proof subsequently provided by Slater [32, p. 161] are based on what has become known as Bailey's lemma [13].

In the treatment of q-series, the q-orthogonal polynomials have been successfully applied to study various problems, especially to Rogers–Ramanujan type identities, see, for example [6, 7, 9, 14, 19]. Recently, Andrews [8] pointed out that the classical orthogonal polynomials also could enter naturally into the world of q.

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Denote the nth classical Chebyshev polynomials of the third kind by $V_n(x)$, see Section 2 for definition. By verifying the following identity [8, Theorem 3.1] involving $V_n(x)$,

$$\prod_{j=1}^{n}(1+2xq^j+q^{2j})=\sum_{j=0}^{n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}2n+1\\ n-j\end{bmatrix}, \tag{1.2}$$

Andrews found a Bailey pair

$$\left(\frac{q^{\binom{n+1}{2}}V_n(x)}{1-q},\frac{\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}}\right)$$

at $a=q$. Then by fitting the above Bailey pair into a consequence of Bailey's lemma, Andrews [8, (4.2)] derived the following generalization of Dyson's favorite identity (1.1)

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}}=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{3\binom{n+1}{2}}V_n(x), \tag{1.3}$$

which reduces to many Rogers–Ramanujan type identities.

In this paper, we further apply Chebyshev polynomials of the third kind to study a companion identity of Dyson's favorite one (1.1)

$$\sum_{n\geq 0}\frac{q^{2n^2}\prod_{i=1}^{n}(1+q^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}}=\frac{(q,q^5,q^6;q^6)_\infty(q^9;q^{18})_\infty}{(q;q)_\infty}, \tag{1.4}$$

which can be found in Ramanujan's lost notebook [11, p.103, Entry 5.3.4].

By using Chebyshev polynomials of the third kind, we show that

$$\left(q^{n^2}\Big(V_n(x)+V_{n-1}(x)\Big),\frac{\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}\right) \tag{1.5}$$

form a Bailey pair at $a=1$. Based on Bailey's lemma [13, (3.1)], we obtain a generalization of (1.4) as follows, which reduces to several Rogers–Ramanujan type identities.

Theorem 1.1. *We have*

$$\sum_{n\geq 0}\frac{q^{2n^2}\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{3n^2}(V_n(x)+V_{n-1}(x)). \tag{1.6}$$

By fitting our key Bailey pair (1.5) into different weak forms of Bailey's lemma, we are led to other identities in the similar form as (1.6). As their consequences, we obtain more Rogers–Ramanujan type identities and also a number of results involving Appell–Lerch series, the generalized Hecke–type series and false theta functions.

This paper is organized as follows. In Section 2, we recall some basic definitions and properties of Chebyshev polynomials of the third kind, Bailey pair, Bailey's lemma and the bilateral versions of Bailey's lemma. In Section 3, we devote to construct our key Bailey pair involving Chebyshev polynomials of the third kind. In Section 4, by inserting the Bailey pair into a weak form of Bailey's lemma, we derive (1.6). We also give the detailed procedures to obtain the companion identity (1.4) when x is taken to be $\frac{1}{2}$. In Section 5, we consider another weak form of Bailey's lemma, from which we obtain more Rogers–Ramanujan type identities. In Section 6, we study some of the Appell–Lerch series arising as consequences of our main results. In Section 7, based on Bailey's lemma, we restrict our attention to the results where Chebyshev polynomials have been inserted into the generalized Hecke–type series. In Section 8, we apply the bilateral versions of Bailey pair and Bailey's lemma to derive identities on the generalized Hecke–type series and false theta functions. At last, in Section 9, we reveal a connection between our work and Andrews' result (1.2), which leads to an identity on Gaussian polynomials by applying the orthogonality of Chebyshev polynomials of the third kind.

2. Chebyshev polynomials and Bailey's lemma

Throughout this paper, we adopt standard notations and terminologies for q-series [16]. We assume that $|q| < 1$. The q-shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1. \end{cases}$$

We also use the notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

There are more compact notations for the multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The Gaussian polynomials, or q-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } k < 0 \text{ or } k > n, \\ \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{otherwise.} \end{cases}$$

We also denote q-binomial coefficients when q is replaced by q^ℓ by $\begin{bmatrix} n \\ k \end{bmatrix}_{q^\ell}$.

In [8], Andrews pointed out that not only q-orthogonal polynomials but also the classical orthogonal polynomials can be naturally applied in the study of q-series. Recall that the Chebyshev polynomial of the first kind is defined by

$$T_n(x) = \cos n\theta,$$

where $x = \cos\theta$. By combining the trigonometric identities, it is direct to derive that $T_n(x)$ satisfies the fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

for $n > 1$ with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Moreover, the Chebyshev polynomial of the third kind $V_n(x)$ is given by

$$V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta}$$

which can be determined by

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x) \tag{2.1}$$

for $n > 1$ together with the initial conditions $V_0(x) = 1$ and $V_1(x) = 2x-1$. For convenience, we also set $V_n(x) = 0$ for $n < 0$. These two kinds of Chebyshev polynomials are closely related. To be more precisely, we have for $n \geq 1$

$$2T_n(x) = V_n(x) + V_{n-1}(x). \tag{2.2}$$

Thereby equivalently, we will present our results in terms of $V_n(x)$ in this paper.

In [8, Lemma 4.1], Andrews stated some special values of $V_n(x)$ which can be easily derived by using the mathematical induction based on the recurrence relation (2.1).

Lemma 2.1. *For $n \geq 0$,*

$$V_n(-1) = (-1)^n(2n+1), \tag{2.3a}$$

$$V_n\Big(-\frac{1}{2}\Big) = \begin{cases} -2, & \textit{if } n \equiv 1 \pmod 3, \\ 1, & \textit{otherwise}, \end{cases} \tag{2.3b}$$

$$V_n(0) = \begin{cases} 1, & \textit{if } n \equiv 0, 3 \pmod 4, \\ -1, & \textit{otherwise}, \end{cases} \tag{2.3c}$$

$$V_n\Big(\frac{1}{2}\Big) = \begin{cases} 1, & \textit{if } n \equiv 0, 5 \pmod 6, \\ 0, & \textit{if } n \equiv 1, 4 \pmod 6, \\ -1, & \textit{if } n \equiv 2, 3 \pmod 6, \end{cases} \tag{2.3d}$$

$$V_n(1) = 1, \tag{2.3e}$$

$$V_n\Big(\frac{3}{2}\Big) = F_{2n+1}, \tag{2.3f}$$

$$V_n\Big(-\frac{3}{2}\Big) = (-1)^n L_{2n+1}, \tag{2.3g}$$

where F_n and L_n are the Fibonacci and Lucas numbers which are defined by the recurrence relations

$$F_n = F_{n-1} + F_{n-2},$$

$$L_n = L_{n-1} + L_{n-2}$$

for $n > 1$ combined with the initial values $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively.

A popular method to prove identities of Rogers–Ramanujan type is based on Bailey's lemma, see [5,13,20,37]. During studying Rogers' work on Ramanujan's identities [10,11,27,28], Bailey [13] discovered the underlying mechanism which was named "Bailey transform". The most famous specialization of Bailey transformation now known as "Bailey pair" which is given as a pair of sequence of rational functions $(\alpha_n(a,q),\beta_n(a,q))_{n\geq 0}$ with respect to a such that

$$\beta_n(a,q)=\sum_{j=0}^{n}\frac{\alpha_j(a,q)}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{2.4}$$

Bailey [13] provided the following fundamental result for producing an infinite family of identities out of one identity, see also Andrews [5, Theorem 3.3].

Lemma 2.2 (Bailey's Lemma)**.** *If $\alpha_n(a,q),\beta_n(a,q)$ form a Bailey pair, then*

$$\begin{aligned}&\frac{1}{(aq/\rho_1,aq/\rho_2;q)_n}\sum_{j=0}^{n}\frac{(\rho_1,\rho_2;q)_j(aq/\rho_1\rho_2;q)_{n-j}}{(q;q)_{n-j}}\left(\frac{aq}{\rho_1\rho_2}\right)^j\beta_j(a,q)\\ &=\sum_{j=0}^{n}\frac{(\rho_1,\rho_2;q)_j}{(q;q)_{n-j}(aq;q)_{n+j}(aq/\rho_1,aq/\rho_2;q)_j}\left(\frac{aq}{\rho_1\rho_2}\right)^j\alpha_j(a,q). \end{aligned}\tag{2.5}$$

There are some special weak forms of Bailey's lemma which attracts more attention since they are more direct to derive Rogers–Ramanujan type identities from Bailey pairs. By collecting a list of 96 Bailey pairs, and using some weak forms of Bailey's lemma, Slater compiled her famous list of 130 identities of Rogers–Ramanujan type [31,32]. We are mainly concerned with the following weak forms of Bailey's lemma.

Lemma 2.3. *We have*

$$\sum_{n\geq 0}q^{n^2}\beta_n(1,q)=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n^2}\alpha_n(1,q), \tag{2.6a}$$

$$\sum_{n\geq 0}q^{n^2}(-q;q^2)_n\beta_n(1,q^2)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{n^2}\alpha_n(1,q^2), \tag{2.6b}$$

$$\sum_{n\geq 0}q^{n(n+1)/2}(-1;q)_n\beta_n(1,q)=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n(n+1)/2}(-1;q)_n}{(-q;q)_n}\alpha_n(1,q). \tag{2.6c}$$

The above three weak forms can be obtained from Bailey's lemma 2.2 by taking $a=1$, $n\to\infty$, and ρ_1,ρ_2 to be certain special values. More precisely, (2.6a) is obtained by setting $\rho_1,\rho_2\to\infty$, (2.6b) is derived by taking $q\to q^2,\rho_1\to\infty,\rho_2\to -q$, and (2.6c) is derived by taking $\rho_1\to\infty,\rho_2\to -1$. For more details, see, for example, [17,21,30].

By applying Bailey's lemma iteratively to an appropriate Bailey pair in the simple sum case, one can obtain multi-analog identities of Rogers–Ramanujan type straightforwardly. Let us take the following one [4] as an illustration.

Lemma 2.4. *Let $(\alpha_n(a,q),\beta_n(a,q))$ be a Bailey pair, then*

$$\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{a^{n_1+\cdots+n_k}q^{n_1^2+\cdots+n_k^2}\beta_{n_1}(a,q)}{(q)_{n_k-n_{k-1}}(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}}=\frac{1}{(aq)_\infty}\sum_{n\geq 0}q^{kn^2}a^{kn}\alpha_n(a,q). \tag{2.7}$$

Obviously, when $a=1$ and $k=1$, the above identity reduces to (2.6a).

Moreover, Andrews and Warnaar [12] introduced a symmetric bilateral version of the classic Bailey transform and gave the following two bilateral versions of Bailey's lemma to study identities on false theta functions.

Lemma 2.5 (Theorem 5, [12])**.** *If for n nonnegative integer*

$$\overline{\beta}_n=\sum_{r=-n}^{n}\frac{\overline{\alpha}_r}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r}}, \tag{2.8}$$

then

$$\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{2n}q^n\overline{\beta}_n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}q^{n(n+1)}\sum_{j=-n}^{n}\overline{\alpha}_jq^{-j^2} \tag{2.9}$$

subject to conditions on $\overline{\alpha}_n$ and $\overline{\beta}_n$ that make the series absolutely convergent.

Lemma 2.6 (Theorem 10, [12])**.** *If $\overline{\alpha}_n$ and $\overline{\beta}_n$ are given as* (2.8)*, then*

$$\sum_{n=0}^{\infty}(q;q)_{2n}q^n\overline{\beta}_n=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}\overline{\alpha}_jq^{-2j^2} \tag{2.10}$$

subject to conditions on $\overline{\alpha}_n$ and $\overline{\beta}_n$ that make the series absolutely convergent.

Note that a pair $(\overline{\alpha}_n,\overline{\beta}_n)$ in the form of (2.8) is a bilateral version of the classical Bailey pair (2.4) with respected to $a=1$ and $q\to q^2$. Actually, if we denote $\alpha_0=\overline{\alpha}_0$ and for $n>0$, $\alpha_n=\overline{\alpha}_n+\overline{\alpha}_{-n}$, then $(\alpha_n,\overline{\beta}_n)$ forms a classical Bailey pair.

3. The key Bailey pair involving $V_n(x)$

The object of this section is to construct the Bailey pair which bring Chebyshev polynomials of the third kind $V_n(x)$ into the field of q-series. By inserting this Bailey pair into the weak forms of Bailey's lemma, it will lead to various Rogers–Ramanujan type identities.

Based on the three term recurrence relation (2.1) of $V_n(x)$, we obtain the following result.

Theorem 3.1. *We have*

$$\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})=\sum_{j=0}^{n}q^{j^2}\begin{bmatrix}2n\\n-j\end{bmatrix}_{q^2}\Big(V_j(x)+V_{j-1}(x)\Big). \tag{3.1}$$

Proof. For brevity, we denote $v_n(x) = V_n(x) + V_{n-1}(x)$. Obviously, $v_0(x) = 1$, $v_1(x) = V_1(x) + V_0(x) = 2x$, and $v_2(x) = V_2(x) + V_1(x) = 4x^2 - 2$. It is easy to see that $\{v_n(x)\}_{n\geq 0}$ form a basis for the polynomials in x over $\mathbb{C}$.

Since for $n > 1$, $V_n(x)$ satisfies the three-term recurrence relation

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x),$$

it is easy to show that, for $n > 2$,

$$v_n(x) = 2xv_{n-1}(x) - v_{n-2}(x). \tag{3.2}$$

Denote the left and the right hand sides of (3.1) by $L_n(x)$ and $R_n(x)$, respectively. Notice that $L_n(x)$ is uniquely determined by the recurrence relation

$$L_n(x) = (1 + 2xq^{2n-1} + q^{4n-2})L_{n-1}(x) \tag{3.3}$$

for $n \geq 1$ combined with $L_0(x) = 1$. Clearly, $R_0(x) = 1$. Therefore, to show that $L_n(x) = R_n(x)$, it is sufficient to prove that for $n \geq 1$, $R_n(x)$ satisfies the same recurrence relation (3.3), which can be rewritten as follows

$$2xq^{2n-1}R_{n-1}(x) = R_n(x) - (1 + q^{4n-2})R_{n-1}(x). \tag{3.4}$$

By (3.2), it directly leads to that for $j > 1$

$$2xv_j(x) = v_{j+1}(x) + v_{j-1}(x).$$

Substituting the above relation into (3.4), it becomes to the following form

$$\begin{aligned} &q^{2n-1}\left(\begin{bmatrix}2n-2\\ n-1\end{bmatrix}_{q^2} 2x + q\begin{bmatrix}2n-2\\ n-2\end{bmatrix}_{q^2} 2xv_1(x) + \sum_{j=2}^{n-1} q^{j^2}\begin{bmatrix}2n-2\\ n-1-j\end{bmatrix}_{q^2} (v_{j+1}(x) + v_{j-1}(x))\right) \\ &\quad = \sum_{j\geq 0} q^{j^2}\left(\begin{bmatrix}2n\\ n-j\end{bmatrix}_{q^2} - (1+q^{4n-2})\begin{bmatrix}2n-2\\ n-1-j\end{bmatrix}_{q^2}\right) v_j(x). \end{aligned}$$

Then in the first two terms on the left hand side of the above identity, we can replace $2x$ and $2xv_1(x)$ by $v_1(x)$ and $v_2(x) + 2v_0(x)$, respectively. Since $\{v_n(x)\}_{n\geq 0}$ form a basis for the polynomials in x, to verify $L_n(x) = R_n(x)$, it is sufficient to prove the coefficients of $v_j(x)$ on both sides of the above identity coincide with each other, that is, for $j \geq 0$,

$$\begin{aligned} &q^{2n-1+(j-1)^2}\begin{bmatrix}2n-2\\ n-j\end{bmatrix}_{q^2} + q^{2n-1+(j+1)^2}\begin{bmatrix}2n-2\\ n-2-j\end{bmatrix}_{q^2} \\ &\quad = q^{j^2}\begin{bmatrix}2n\\ n-j\end{bmatrix}_{q^2} - q^{j^2}(1+q^{4n-2})\begin{bmatrix}2n-2\\ n-1-j\end{bmatrix}_{q^2}, \end{aligned}$$

which can be confirmed by direct simplification, and thereby (3.1) is valid. □

Now, if we rewrite (3.1) as follows

$$\frac{\prod_{j=1}^{n}(1 + 2xq^{2j-1} + q^{4j-2})}{(q^2;q^2)_{2n}} = \sum_{j=0}^{n} \frac{q^{j^2}(V_j(x) + V_{j-1}(x))}{(q^2;q^2)_{n+j}(q^2;q^2)_{n-j}},$$

then by definition (2.4), it immediately implies our key Bailey pair

$$\left(q^{n^2}\Big(V_n(x)+V_{n-1}(x)\Big),\frac{\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}\right) \tag{3.5}$$

relative to $a=1$ and $q\to q^2$. By fitting this Bailey pair (3.5) into different weak forms of Bailey's lemma 2.3, we will obtain a family of Rogers–Ramanujan type identities and also identities related to Appell–Lerch series, Hecke–type series and false theta functions.

4. The weak form (2.6a) of Bailey's lemma

In this section, we will show how to derive the companion identity (1.4) for Dyson's favourite identity (1.1) by using the key Bailey pair (3.5) associated with $V_n(x)$. Meanwhile, by taking x to be some other special values, we will obtain more Rogers–Ramanujan type identities. We also consider the multisum generalization of these identities.

First, by setting $q\to q^2$ in the weak form (2.6a) of Bailey's lemma and with the aid of the Bailey pair (3.5), we directly obtain Theorem 1.1. Now, we will show how to derive Ramanujan's identity (1.4) from Theorem 1.1.

Theorem 4.1. *Ramanujan's identity* (1.4) *is valid.*

Proof. Denote the left hand side of identity (1.4) by L. In Theorem 1.1, by taking $x=\frac{1}{2}$ and using the special value of $V_n(x)$ (2.3d), we have

$$\begin{aligned}
L&=\sum_{n\geq 0}\frac{q^{2n^2}\prod_{j=1}^{n}(1+q^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}\\
&=\frac{1}{(q^2;q^2)_\infty}\left(1+\sum_{n\geq 1}q^{3n^2}\left(V_n\Big(\frac{1}{2}\Big)+V_{n-1}\Big(\frac{1}{2}\Big)\right)\right)\\
&=\frac{1}{(q^2;q^2)_\infty}\Bigg(1+2\sum_{n\geq 1}q^{3(6n)^2}+\sum_{n\geq 0}q^{3(6n+1)^2}-\sum_{n\geq 0}q^{3(6n+2)^2}\\
&\qquad -2\sum_{n\geq 0}q^{3(6n+3)^2}-\sum_{n\geq 0}q^{3(6n+4)^2}+\sum_{n\geq 0}q^{3(6n+5)^2}\Bigg).
\end{aligned}$$

Taking the parity of n into consideration, we see that

$$\begin{aligned}
\sum_{n\geq 1}q^{3(6n)^2}-\sum_{n\geq 0}q^{3(6n+3)^2}&=\sum_{n\geq 1}(-1)^n q^{3(3n)^2},\\
\sum_{n\geq 0}q^{3(6n+1)^2}-\sum_{n\geq 0}q^{3(6n+4)^2}&=\sum_{n\geq 0}(-1)^n q^{3(3n+1)^2},\\
\sum_{n\geq 0}q^{3(6n+2)^2}-\sum_{n\geq 0}q^{3(6n+5)^2}&=\sum_{n\geq 0}(-1)^n q^{3(3n+2)^2}.
\end{aligned}$$

Therefore, it implies that

$$\begin{aligned}
L &= \frac{1}{(q^2;q^2)_\infty}\Big(1+2\sum_{n\geq 1}(-1)^n q^{3(3n)^2}+\sum_{n\geq 0}(-1)^n q^{3(3n+1)^2}-\sum_{n\geq 0}(-1)^n q^{3(3n+2)^2}\Big)\\
&= \frac{1}{(q^2;q^2)_\infty}\Big(\sum_{n=-\infty}^{\infty}(-1)^n q^{3(3n)^2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{3(3n+1)^2}\Big)\\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}e^{\frac{2n\pi i}{3}}, \qquad (4.1)
\end{aligned}$$

in which the last step can be affirmed by considering the above sum according to the residues of n modulo 3. Then by applying the famous Jacobi's triple product identity which is for $z,q\in\mathbf{C}$ and $z\neq 0$,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}z^n=(q,z,q/z;q)_\infty, \tag{4.2}$$

we further obtain that

$$\begin{aligned}
L &= \frac{1}{(q^2;q^2)_\infty}(q^3e^{\frac{2\pi i}{3}},q^3e^{-\frac{2\pi i}{3}},q^6;q^6)_\infty\\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\prod_{n\geq 0}(1-q^{3+6n}e^{\frac{2\pi i}{3}})(1-q^{3+6n}e^{-\frac{2\pi i}{3}})\\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\prod_{n\geq 0}\big(1-q^{3+6n}(e^{\frac{2\pi i}{3}}+e^{-\frac{2\pi i}{3}})+q^{6+12n}\big)\\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\prod_{n\geq 0}(1+q^{3+6n}+q^{6+12n})\\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\prod_{n\geq 0}\frac{(1-q^{9+18n})}{(1-q^{3+6n})}\\
&= \frac{(q^6;q^6)_\infty(q^9;q^{18})_\infty}{(q^2;q^2)_\infty(q^3;q^6)_\infty},
\end{aligned}$$

which completes the proof by multiplying both of the numerator and the denominator by $(q,q^5;q^6)_\infty$ and then simplifying. □

Noting that by setting $x=-\frac{1}{2}$ in (1.6) and following the similar procedures as above, we can also get (1.4) after replacing q by $-q$.

As more consequences of Theorem 1.1, let us consider the cases corresponding to other special values of $V_n(x)$ as given in Lemma 2.1. The special value of $V_n(x)$ at $x=-1$ (or equivalently, $x=1$) yields the following Ramanujan's identity.

Theorem 4.2 (Entry 5.3.3, [11, P. 102])**.**

$$\sum_{n\geq 0}\frac{q^{2n^2}(q;q^2)_n^2}{(q^2;q^2)_{2n}}=\frac{(q^3;q^3)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty}. \tag{4.3}$$

Proof. Taking $x=-1$ in (1.6) and using the special value of $V_n(x)$ at $x=-1$ (2.3a), we have

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{2n^2}\prod_{j=1}^{n}(1-2q^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{3n^2}\big(V_n(-1)+V_{n-1}(-1)\big)\\&=\frac{1}{(q^2;q^2)_\infty}\big(1+2\sum_{n\geq 1}(-1)^n q^{3n^2}\big)\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}.\end{aligned}$$

Then the proof is complete by using Jacobi's triple product identity (4.2) and simplifying. □

When x is taken to be zero, we obtain Entry 5.3.2 in Ramanujan's lost notebook.

Theorem 4.3 (Entry 5.3.2, [11, P. 101])**.**

$$\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}}=\frac{(q^6;q^{12})_\infty(q^6;q^6)_\infty}{(q;q)_\infty}. \tag{4.4}$$

Proof. By setting $x=0$ in (1.6) and using the special value of $V_n(x)$ (2.3c), we obtain

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{2n^2}\prod_{j=1}^{n}(1+q^{4j-2})}{(q^2;q^2)_{2n}}&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{3n^2}\big(V_n(0)+V_{n-1}(0)\big)\\&=\frac{1}{(q^2;q^2)_\infty}\Big(1+2\sum_{n\geq 1}q^{3(4n)^2}-2\sum_{n\geq 0}q^{3(4n+2)^2}\Big)\\&=\frac{1}{(q^2;q^2)_\infty}\Big(1+2\sum_{n\geq 1}(-1)^n q^{3(2n)^2}\Big)\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2}.\end{aligned}$$

By Jacobi's triple product identity (4.2), it leads to that

$$\sum_{n\geq 0}\frac{q^{2n^2}(-q^2;q^4)_n}{(q^2;q^2)_{2n}}=\frac{(q^{12},q^{12},q^{24};q^{24})_\infty}{(q^2;q^2)_\infty}=\frac{(q^{12};q^{24})_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty}, \tag{4.5}$$

which completes the proof by replacing q with $q^{\frac{1}{2}}$ in the above identity. □

For identity (4.4), it is also contained in Slater's list [32, p.155, (29)], and one can see also Andrews and Berndt [10, p. 254, Entry 11.3.1]. For the above two identities (4.3) and (4.4), Andrews also considered the theta expansions of their left hand sides, which were given by equations $(3.1)_R$ and $(3.2)_R$ in [2], respectively.

Taking $x = \frac{3}{2}$ and $-\frac{3}{2}$ in (1.6), respectively, we obtain the following two identities immediately.

Theorem 4.4. *We have*

$$\sum_{n\geq 0} \frac{q^{2n^2}\prod_{j=1}^{n}(1+3q^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0} q^{3n^2}(F_{2n+1}+F_{2n-1}), \tag{4.6}$$

$$\sum_{n\geq 0} \frac{q^{2n^2}\prod_{j=1}^{n}(1-3q^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty}\Big(1+\sum_{n\geq 1}(-1)^n q^{3n^2} L_{2n}\Big). \tag{4.7}$$

Remark that from [33, A000032], we see that for $n \geq 1$, $L_n = 2F_{n+1} - F_n$. It clearly implies that for $n \geq 1$

$$L_{2n} = F_{2n+1} + F_{2n-1}. \tag{4.8}$$

Consequently, the identities (4.6) and (4.7) are equivalent by substituting $q \to -q$.

Now, let us consider the multi-analog of Rogers–Ramanujan type identities. By inserting the key Bailey pair (3.5) into (2.7) with $a = 1$ and $q \to q^2$, we obtain the following generalization of Theorem 1.1.

Theorem 4.5. *We have*

$$\begin{aligned} &\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0} q^{(2k+1)n^2}(V_n(x)+V_{n-1}(x)) \\ &= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{2(n_1^2+\cdots+n_k^2)}\prod_{j=1}^{n_1}(1+2xq^{2j-1}+q^{4j-2})}{(q^2;q^2)_{n_k-n_{k-1}}(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_2-n_1}(q^2;q^2)_{2n_1}}. \end{aligned} \tag{4.9}$$

Obviously, when $k = 1$, we are led to Theorem 1.1 immediately. Note that the summation on the left hand side of the above identity can be obtained by substituting $q \to q^{\frac{2k+1}{3}}$ into the right hand side of identity (1.6). Therefore, based on Theorems 4.1–4.4 and by taking x to be 0 (with $q^2 \to q$), 1/2 (or equivalently $-1/2$), 1 (or equivalently -1), and 3/2 (or equivalently $-3/2$), respectively, we can obtain the following identities immediately.

Corollary 4.6. *We have*

$$\begin{aligned} &\frac{(q^{4k+2},q^{4k+2},q^{8k+4};q^{8k+4})_\infty}{(q;q)_\infty} \\ &= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_1^2+\cdots+n_k^2}(-q;q^2)_{n_1}}{(q;q)_{n_k-n_{k-1}}(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{2n_1}}, \end{aligned} \tag{4.10a}$$

$$\frac{(q^{4k+2};q^{4k+2})_\infty(q^{6k+3};q^{12k+6})_\infty}{(q^2;q^2)_\infty(q^{2k+1};q^{4k+2})_\infty}$$

(4.10b)

$$= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{2(n_1^2+\cdots+n_k^2)}(q^3;q^6)_{n_1}}{(q^2;q^2)_{n_k-n_{k-1}}(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_2-n_1}(q^2;q^2)_{2n_1}(q;q^2)_{n_1}},$$

$$\frac{(q^{2k+1},q^{2k+1},q^{4k+2};q^{4k+2})_\infty}{(q^2;q^2)_\infty}$$

(4.10c)

$$= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{2(n_1^2+\cdots+n_k^2)}(q;q^2)_{n_1}^2}{(q^2;q^2)_{n_k-n_{k-1}}(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_2-n_1}(q^2;q^2)_{2n_1}},$$

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0} q^{(2k+1)n^2}(F_{2n+1}+F_{2n-1})$$

(4.10d)

$$= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{2(n_1^2+\cdots+n_k^2)}\prod_{j=1}^{n_1}(1+3q^{2j-1}+q^{4j-2})}{(q^2;q^2)_{n_k-n_{k-1}}(q^2;q^2)_{n_{k-1}-n_{k-2}}\cdots(q^2;q^2)_{n_2-n_1}(q^2;q^2)_{2n_1}}.$$

Specially, when $k=1$, the above four identities reduce to (4.4), (1.4), (4.3) and (4.6), respectively.

5. The weak form (2.6b) of Bailey's lemma

In this section, we consider the applications of the weak form (2.6b) of Bailey's lemma. Firstly, by inserting our key Bailey pair (3.5) into the second weak form (2.6b), it leads to the following result.

Theorem 5.1. *We have*

$$\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n\prod_{i=1}^{n}(1+2xq^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0} q^{2n^2}\big(V_n(x)+V_{n-1}(x)\big). \tag{5.1}$$

Employing the similar procedures as given in Section 4 and taking x to be $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$ and $\frac{3}{2}$ (or equivalently, $-\frac{3}{2}$), respectively, the above identity reduces to the following Rogers–Ramanujan type identities.

Corollary 5.2. *We have*

$$\sum_{n\geq 0}\frac{q^{n^2}(q;q^2)_n}{(q^4;q^4)_n} = \frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty}, \tag{5.2a}$$

$$\sum_{n\geq 0}\frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \frac{(-q;q)_\infty(-q^6;q^{12})_\infty}{(-q^2;q^4)_\infty}, \tag{5.2b}$$

$$\sum_{n\geq 0}\frac{q^{n^2}(-q^2;q^4)_n}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q;q^2)_\infty(q^8;q^8)_\infty(q^8;q^{16})_\infty}{(q^2;q^2)_\infty}, \tag{5.2c}$$

$$\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n(q^3;q^6)_n}{(q^2;q^2)_{2n}(q;q^2)_n}=\frac{(q^4;q^4)_\infty(q^6;q^{12})_\infty}{(q;q)_\infty}, \tag{5.2d}$$

$$\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n^3}{(q^2;q^2)_{2n}}=\frac{(-q^2;q^4)_\infty^2}{(q;q^2)_\infty}, \tag{5.2e}$$

$$\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n\prod_{i=1}^n(1+3q^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{2n^2}(F_{2n+1}+F_{2n-1}). \tag{5.2f}$$

We remark that when $x=-1,0,\frac{1}{2}$ and $\frac{3}{2}$, the summation on the right hand sides of (5.1) can be deduced by setting $q\to q^{\frac{2}{3}}$ in (4.3), (4.5), (1.4) and (4.6), which leads to (5.2a), (5.2c), (5.2d), and (5.2f), respectively. Now, we show how to derive (5.2b) and (5.2e) from (5.1).

Proof of (5.2b) *and* (5.2e). Taking $x=-\frac{1}{2}$ in (5.1) and with the aid of (2.3b), we have

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{2n^2}\big(V_n(-\frac{1}{2})+V_{n-1}(-\frac{1}{2})\big)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(1+2\sum_{n\geq 1}q^{2(3n)^2}-\sum_{n\geq 0}q^{2(3n+1)^2}-\sum_{n\geq 0}q^{2(3n+2)^2}\Big)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(\sum_{n=-\infty}^{\infty}q^{2(3n)^2}-\sum_{n=-\infty}^{\infty}q^{2(3n+1)^2}\big)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2}e^{\frac{2n\pi i}{3}}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(-q^2e^{\frac{2\pi i}{3}},-q^2e^{-\frac{2\pi i}{3}},q^4;q^4)_\infty\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{(-q^6;q^{12})_\infty(q^4;q^4)_\infty}{(-q^2;q^4)_\infty},
\end{aligned}$$

which completes the proof of identity (5.2b) by simplifying.

Taking $x=1$ in (5.1) and employing (2.3e), we obtain that

$$\begin{aligned}
\sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n^3}{(q^2;q^2)_{2n}} &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{2n^2}\big(V_n(1)+V_{n-1}(1)\big)\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(1+2\sum_{n\geq 1}q^{2n^2}\big)
\end{aligned}$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2},$$

which implies (5.2e) by applying Jacobi's triple product identity (4.2) and simplifying. $\square$

For these identities, one can see that (5.2a) is contained in Slater's list [32, (4)] with $q \to -q$ and also can be found in [21, P.10, (2.4.2)]; the identity (5.2b) is Entry 5.3.8 in Ramanujan's lost notebook [11, P. 105]; (5.2c) can be found in [21, P.21, (2.16.4)] and [29, P. 16, (5.5)]; (5.2d) is Entry 5.3.9 in [11, P. 105]. Specially, (5.2e) seems to be new, which can be seen as a missing member of modulo 4 identities in Slater's list, see [32, P. 153] and [21, P. 11].

6. The weak form (2.6c) of Bailey's lemma and Appell–Lerch series

In this section, we will study the application of the weak form (2.6c) of Bailey's lemma in deriving identities on Appell–Lerch serires.

Recall that an Appell–Lerch series is of the following form

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} b^n}{1-aq^n}, \tag{6.1}$$

which was first studied by Appell and Lerch. After multiplying the series (6.1) by the factor $a^{\ell/2}$ and viewing it as a function in the variables a, b and q, it is also refereed as an Appell function of level ℓ.

By inserting the Bailey pair (3.5) into the weak form (2.6c) with $q \to q^2$, we obtain the following result, from which several identities involving Appell–Lerch series are derived.

Theorem 6.1. *We have*

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-1;q^2)_n \prod_{i=1}^{n}(1+2xq^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}}$$
$$= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+n}(-1;q^2)_n}{(-q^2;q^2)_n} \big(V_n(x)+V_{n-1}(x)\big). \tag{6.2}$$

By taking $x=1$ (or equivalently, $x=-1$) in the above identity, we obtain the following result.

Corollary 6.2. *We have*

$$\sum_{n\geq 0} \frac{q^{n^2+n}(-1;q^2)_n(-q;q^2)_n^2}{(q^2;q^2)_{2n}} = 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+q^{2n}}. \tag{6.3}$$

Proof. The left hand side of (6.3) can be obtained directly by setting $x=1$ in (6.2). For the summation on the right hand side of (6.2), with the aid of (2.3e), we see that

$$\sum_{n\geq 0} \frac{(-1;q^2)_n q^{2n^2+n}}{(-q^2;q^2)_n} \big(V_n(1)+V_{n-1}(1)\big)$$

$$
\begin{aligned}
&= 1 + 2\sum_{n\geq 1} \frac{(-1;q^2)_n q^{2n^2+n}}{(-q^2;q^2)_n} \\
&= 1 + 4\sum_{n\geq 1} \frac{q^{2n^2+n}}{1+q^{2n}} \\
&= 2\Big(\frac{1}{2} + \sum_{n\geq 1} \frac{q^{2n^2+n}}{1+q^{2n}} + \sum_{n=-\infty}^{-1} \frac{q^{2n^2+n}}{1+q^{2n}}\Big) \\
&= 2\sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+q^{2n}},
\end{aligned}
$$

which completes the proof. □

It is notable that the summation on the right hand side of the above identity is closely related to the following mock theta function of order 2

$$
\mu^{(2)}(q) = 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+q^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q^2;q^2)_n^2},
$$

see [24, 25]. From (6.3), we obtain another expression of $\mu^{(2)}(q)$ as follows

$$
\mu^{(2)}(q) = \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q^2)_n (-q;q^2)_n^2 q^{n^2+n}}{(q^2;q^2)_{2n}}. \tag{6.4}
$$

In (6.2), by substituting $x = -\frac{1}{2}$ (or equivalently, $x = \frac{1}{2}$), we obtain the following result.

Corollary 6.3. *We have*

$$
\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2+n}(-1;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n}(-q;q^2)_n} &= 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\Big(\sum_{n=-\infty}^{\infty} \frac{q^{18n^2+3n}}{1+q^{6n}} - \sum_{n=-\infty}^{\infty} \frac{q^{18n^2+15n+3}}{1+q^{2(3n+1)}}\Big) \\
&= 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{e^{\frac{2n\pi i}{3}} q^{2n^2+n}}{1+q^{2n}}.
\end{aligned} \tag{6.5}
$$

Proof. It is direct to obtain the left hand side of (6.5) by substituting $x = -\frac{1}{2}$ into (6.2). For the summation on the right hand side, by using (2.3b), we have that

$$
\begin{aligned}
&\sum_{n\geq 0} \frac{q^{2n^2+n}(-1;q^2)_n}{(-q^2;q^2)_n}\Big(V_n\big(-\frac{1}{2}\big) + V_{n-1}\big(-\frac{1}{2}\big)\Big) \\
&\quad = 1 + 4\sum_{n\geq 1} \frac{q^{2(3n)^2+3n}}{1+q^{6n}} - 2\sum_{n\geq 0} \frac{q^{2(3n+1)^2+3n+1}}{1+q^{2(3n+1)}} - 2\sum_{n\geq 0} \frac{q^{2(3n+2)^2+3n+2}}{1+q^{2(3n+2)}} \\
&\quad = 2\sum_{n=-\infty}^{\infty} \frac{q^{2(3n)^2+3n}}{1+q^{6n}} - 2\sum_{n\geq 0} \frac{q^{2(3n+1)^2+3n+1}}{1+q^{2(3n+1)}} - 2\sum_{n=-\infty}^{-1} \frac{q^{2(3n+1)^2+3n+1}}{1+q^{2(3n+1)}}
\end{aligned}
$$

$$= 2\sum_{n=-\infty}^{\infty}\frac{q^{2(3n)^2+3n}}{1+q^{6n}} - 2\sum_{n=-\infty}^{\infty}\frac{q^{2(3n+1)^2+3n+1}}{1+q^{2(3n+1)}},$$

which completes the proof by further considering the residue classes of n modulo 3. $\square$

Taking $x=0$ in (6.2) and then setting $q^2\to q$, we obtain the following identity involving Appell–Lerch series.

Corollary 6.4. *We have*

$$\sum_{n\geq 0}\frac{q^{\frac{n^2+n}{2}}(-1;q)_n(-q;q^2)_n}{(q;q)_{2n}} = 2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{4n^2+n}}{1+q^{2n}}. \tag{6.6}$$

Proof. When $x=0$ in (6.2), by using (2.3c), the summation on the right hand side becomes

$$\sum_{n\geq 0}\frac{q^{2n^2+n}(-1;q^2)_n}{(-q^2;q^2)_n}\big(V_n(0)+V_{n-1}(0)\big)$$

$$= 1+4\sum_{n\geq 1}\frac{q^{2(4n)^2+4n}}{1+q^{8n}} - 4\sum_{n\geq 0}\frac{q^{2(4n+2)^2+4n+2}}{1+q^{2(4n+2)}}$$

$$= 2\sum_{n=-\infty}^{\infty}\frac{q^{2(4n)^2+4n}}{1+q^{8n}} - 2\sum_{n\geq 0}\frac{q^{2(4n+2)^2+4n+2}}{1+q^{2(4n+2)}} - 2\sum_{n=-\infty}^{-1}\frac{q^{2(4n+2)^2+4n+2}}{1+q^{2(4n+2)}}$$

$$= 2\sum_{n=-\infty}^{\infty}\frac{q^{2(4n)^2+4n}}{1+q^{8n}} - 2\sum_{n=-\infty}^{\infty}\frac{q^{2(4n+2)^2+4n+2}}{1+q^{2(4n+2)}}$$

$$= 2\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2(2n)^2+2n}}{1+q^{4n}}.$$

Then the proof is complete by replacing q^2 by q. $\square$

When $x=\frac{3}{2}$ (or equivalently, $x=-\frac{3}{2}$) in (6.2), we obtain the following result.

Corollary 6.5. *We have*

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-1;q^2)_n\prod_{i=1}^{n}(1+3q^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}} = 2\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2+n}}{1+q^{2n}}\big(F_{2n+1}+F_{2n-1}\big). \tag{6.7}$$

Proof. It is direct to obtain the left hand side of (6.7) by setting $x=\frac{3}{2}$ in (6.2). For the summation on the right hand side, by using (2.3f), we have that

$$\sum_{n\geq 0}\frac{q^{2n^2+n}(-1;q^2)_n}{(-q^2;q^2)_n}\Big(V_n\big(\frac{3}{2}\big)+V_{n-1}\big(\frac{3}{2}\big)\Big)$$

$$= 1 + 2\sum_{n\geq 1} \frac{q^{2n^2+n}}{1+q^{2n}}\big(F_{2n+1}+F_{2n-1}\big)$$

$$= 2\sum_{n\geq 0} \frac{q^{2n^2+n}}{1+q^{2n}}\big(F_{2n+1}+F_{2n-1}\big),$$

which completes the proof. □

7. Generalized Hecke–type Series

In this section, we will use identity (3.1) to establish identities related to generalized Hecke–type series involving indefinite quadratic forms.

Recall that a series is of Hecke–type if it has the following form

$$\sum_{(n,j)\in D} (-1)^{H(n,j)} q^{Q(n,j)+L(n,j)},$$

where H and L are linear forms, Q is a quadratic form, and D is some subset of $\mathbb{Z}\times\mathbb{Z}$ such that $Q(n,j)\geq 0$ for any $(n,j)\in D$. Hecke–type series have received extensive attention since the study of Jacobi and Hecke, see, for example [3, 18, 22, 34, 35]. In [8], Andrews introduced the generalized Hecke–type series in which the restriction $Q(n,j)\geq 0$ is removed. In the same paper, Andrews also stated the following result

$$\sum_{n\geq 0} \frac{q^{n^2+\alpha n}}{(q;q)_n (q;q)_{n+\beta}} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} \frac{(q^{\alpha-\beta};q)_n (-1)^n q^{\beta n + \binom{n+1}{2}}}{(q;q)_n}, \tag{7.1}$$

which can be derived from Heine's second transformation [16, p. 241, (III.2)] by taking $a=b=\frac{1}{\tau}$, $z=q^{\alpha+1}\tau^2$, $c=q^{\beta+1}$, and then letting $\tau\to 0$.

With the light of some special cases of Andrews' identity (7.1), from identity (3.1) we can deduce the following results on generalized Hecke–type series.

Theorem 7.1. *We have*

$$\sum_{n\geq 0} \frac{q^{2n^2+2n}\prod_{i=1}^{n}(1+2xq^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} q^{-j^2}\big(V_j(x)+V_{j-1}(x)\big). \tag{7.2}$$

Proof. Denote the left hand side of (7.2) by L and $v_n(x)=V_n(x)+V_{n-1}(x)$ as given in the proof of Theorem 3.1. Using identity (3.1), we have

$$L = \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q^2;q^2)_{2n}} \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} v_j(x)$$

$$= \sum_{j\geq 0}\sum_{n\geq 0} \frac{q^{2(n+j)^2+2(n+j)+j^2} v_j(x)}{(q^2;q^2)_n (q^2;q^2)_{n+2j}}$$

$$= \sum_{j\geq 0} q^{3j^2+2j} v_j(x) \sum_{n\geq 0} \frac{q^{2n^2+2(2j+1)n}}{(q^2;q^2)_n (q^2;q^2)_{n+2j}}.$$

By applying (7.1) with $q \to q^2, \alpha = 2j+1, \beta = 2j$, and then dividing the summation according to the parity of n, we get

$$\begin{aligned}
L &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2+2j} v_j(x) \sum_{n\geq 0} (-1)^n q^{n^2+n+4nj} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2+2j} v_j(x) \sum_{n\geq 0} q^{4n^2+2n+8nj} (1 - q^{4n+4j+2}) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n} (1 - q^{4n+2}) \sum_{j=0}^{n} q^{-j^2} v_j(x) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n} (1 - q^{4n+2}) \sum_{j=0}^{n} q^{-j^2} (V_j(x) + V_{j-1}(x)), \qquad (7.3)
\end{aligned}$$

which completes the proof by reconsidering the parity of the variable n. □

When x is taken to be special values, we can obtain some identities on generalized Hecke–type series. Let us take $x = -1$ first. It is also equivalent to the case when $x = 1$ and then $q \to -q$.

Corollary 7.2. *We have*

$$\sum_{n\geq 0} \frac{q^{2n^2+2n} (q;q^2)_n^2}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{-j^2}.$$

Proof. Taking $x = -1$ in (7.2), we obtain

$$\sum_{n\geq 0} \frac{q^{2n^2+2n} (q;q^2)_n^2}{(q^2;q^2)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{-j^2} \big(V_j(-1) + V_{j-1}(-1)\big).$$

By using the special values of $V_n(x)$ at $x = -1$ (2.3a), we are led to

$$\begin{aligned}
\sum_{n\geq 0} \frac{q^{2n^2+2n} (q;q^2)_n^2}{(q^2;q^2)_{2n}} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \big(1 + 2 \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{-j^2}\big) \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n^2+n} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{-j^2},
\end{aligned}$$

which completes the proof. □

By setting $x = 0$ in (7.2) and then replacing q^2 by q, we obtain the following result.

Corollary 7.3. *We have*

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-q;q^2)_n}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}}\sum_{j=-\lfloor\frac{n}{4}\rfloor}^{\lfloor\frac{n}{4}\rfloor}(-1)^j q^{-2j^2}.$$

Proof. To be more direct, we start from the expression (7.3). By substituting $x=0$ in (7.3), we obtain

$$\sum_{n\geq 0}\frac{q^{2n^2+2n}(-q^2;q^4)_n}{(q^2;q^2)_{2n}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{4n^2+2n}(1-q^{4n+2})\sum_{j=0}^{n}q^{-j^2}(V_j(0)+V_{j-1}(0)).$$

By employing (2.3c), it leads to that

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{2n^2+2n}(-q^2;q^4)_n}{(q^2;q^2)_{2n}}&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{4n^2+2n}(1-q^{4n+2})\Big(1+2\sum_{j=1}^{\lfloor\frac{n}{4}\rfloor}q^{-(4j)^2}-2\sum_{j=0}^{\lfloor\frac{n-2}{4}\rfloor}q^{-(4j+2)^2}\Big)\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{4n^2+2n}(1-q^{4n+2})\Big(\sum_{j=-\lfloor\frac{n}{4}\rfloor}^{\lfloor\frac{n}{4}\rfloor}q^{-(4j)^2}-\sum_{j=-\lfloor\frac{n-2}{4}\rfloor-1}^{\lfloor\frac{n-2}{4}\rfloor}q^{-(4j+2)^2}\Big)\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{4n^2+2n}(1-q^{4n+2})\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}(-1)^j q^{-(2j)^2}\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{n(n+1)}\sum_{j=-\lfloor\frac{n}{4}\rfloor}^{\lfloor\frac{n}{4}\rfloor}(-1)^j q^{-(2j)^2},\end{aligned}$$

where the last step follows by taking the parity of n into consideration. Then the proof is complete by setting $q^2\to q$. □

By taking $x=-\frac{1}{2}$ in (7.2), we obtain the following identity. It is also equivalent to the result given by setting $x=\frac{1}{2}$ and then $q\to -q$.

Corollary 7.4. *We have*

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{2n^2+2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n}(-q;q^2)_n}&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}q^{4n^2+2n}(1-q^{4n+2})\Big(\sum_{j=-\lfloor\frac{n}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}q^{-(3j)^2}-\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n-1}{3}\rfloor}q^{-(3j+1)^2}\Big)\\&=\frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 0}(-1)^n q^{n(n+1)}\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}e^{\frac{2\pi ij}{3}}q^{-j^2}.\end{aligned}$$

Proof. As in the above corollary, we take $x=-\frac{1}{2}$ in (7.3). With the help of (2.3b), we get

$$\sum_{n\geq 0}\frac{q^{2n^2+2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n}(-q;q^2)_n}$$

$$
\begin{aligned}
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=0}^{n} q^{-j^2}\big(V_j(-\frac{1}{2})+V_{j-1}(-\frac{1}{2})\big)\\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n}(1-q^{4n+2})\Big(1+2\sum_{j=1}^{\lfloor \frac{n}{3}\rfloor} q^{-(3j)^2} - \sum_{j=0}^{\lfloor \frac{n-1}{3}\rfloor} q^{-(3j+1)^2} - \sum_{j=0}^{\lfloor \frac{n-2}{3}\rfloor} q^{-(3j+2)^2}\Big)\\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n}(1-q^{4n+2})\Big(\sum_{j=-\lfloor \frac{n}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} q^{-(3j)^2} - \sum_{j=-\lfloor \frac{n+1}{3}\rfloor}^{\lfloor \frac{n-1}{3}\rfloor} q^{-(3j+1)^2}\Big)\\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=-n}^{n} e^{\frac{2\pi i j}{3}} q^{-j^2}\\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{n(n+1)} \sum_{j=-\lfloor \frac{n}{2}\rfloor}^{\lfloor \frac{n}{2}\rfloor} e^{\frac{2\pi i j}{3}} q^{-j^2},
\end{aligned}
$$

where the last step is derived by taking the parity of n into consideration. □

Applying Andrews' rewritten form (7.1) of Heine's transformation formula, we also obtain the following result on the generalized Hecke–type series.

Theorem 7.5. *We have*

$$
\begin{aligned}
&\sum_{n\geq 1} \frac{q^{2n^2-2n}\prod_{i=1}^{n}(1+2xq^{2i-1}+q^{4i-2})}{(q^2;q^2)_{2n-1}}\\
&\qquad = \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2-2n}(1-q^{12n+6}) \sum_{j=0}^{n} q^{-j^2}\big(V_j(x)+V_{j-1}(x)\big). \qquad (7.4)
\end{aligned}
$$

Proof. Denote the left hand side of (7.4) by L and let $v_j(x) = V_j(x) + V_{j-1}(x)$ for $j \geq 0$. Using identity (3.1) and noting that $\frac{1}{(q^2;q^2)_{-1}} = 0$, we obtain

$$
\begin{aligned}
L &= \sum_{n\geq 0} \frac{q^{2n^2-2n}}{(q^2;q^2)_{2n-1}} \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} v_j(x)\\
&= \sum_{j\geq 0}\sum_{n\geq 0} \frac{q^{2(n+j)^2-2(n+j)+j^2} v_j(x)(1-q^{4n+4j})}{(q^2;q^2)_n (q^2;q^2)_{n+2j}}\\
&= \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \sum_{n\geq 0} \frac{q^{2n^2+2(2j-1)n}(1-q^{4n+4j})}{(q^2;q^2)_n (q^2;q^2)_{n+2j}}\\
&= \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \Big(\sum_{n\geq 0} \frac{q^{2n^2+2(2j-1)n}}{(q^2;q^2)_n (q^2;q^2)_{n+2j}} - q^{4j} \sum_{n\geq 0} \frac{q^{2n^2+2(2j+1)n}}{(q^2;q^2)_n (q^2;q^2)_{n+2j}}\Big)
\end{aligned}
$$

By applying (7.1) with $q \to q^2, \beta = 2j$, and substituting α by $2j-1$ and $2j+1$, respectively, the above result becomes

$$\begin{aligned} L &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \left(\sum_{n\geq 0} \frac{(q^{-2};q^2)_n (-1)^n q^{n^2+n+4nj}}{(q^2;q^2)_n} - q^{4j} \sum_{n\geq 0} (-1)^n q^{n^2+n+4nj} \right) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \left(1 + q^{4j} - q^{4j} \sum_{n\geq 0} (-1)^n q^{n^2+n+4nj} \right) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \left(1 - q^{4j} \sum_{n\geq 1} (-1)^n q^{n^2+n+4nj} \right) \end{aligned}$$

By dividing the above sum on n into two parts according to the parity of n, it implies that

$$\begin{aligned} L &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \left(1 - \sum_{n\geq 0} q^{(2n+2)^2+(2n+2)+4(2n+2)j+4j} + \sum_{n\geq 1} q^{(2n-1)^2+(2n-1)+4(2n-1)j+4j} \right) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \left(\sum_{n\geq 0} q^{4n^2-2n+8nj} - \sum_{n\geq 0} q^{4n^2+10n+8nj+12j+6} \right) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2-2j} v_j(x) \sum_{n\geq 0} q^{4n^2-2n+8nj} (1 - q^{12n+12j+6}) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2-2n} (1 - q^{12n+6}) \sum_{j=0}^{n} q^{-j^2} v_j(x), \end{aligned}$$

which completes the proof. $\square$

Following the similar procedures as given in deducing the corollaries of Theorem 7.1, and taking $x = -1$ (or equivalently, $x = 1$), $x = 0$ (with $q^2 \to q$) and $x = -\frac{1}{2}$ (or equivalently, $x = \frac{1}{2}$) in (7.4), respectively, we obtain the following results on generalized Hecke–type series.

Corollary 7.6. *We have*

$$\sum_{n\geq 1} \frac{q^{2n^2-2n}(q;q^2)_n^2}{(q^2;q^2)_{2n-1}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2-2n} (1 - q^{12n+6}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

$$\sum_{n\geq 1} \frac{q^{n^2-n}(q;q^2)_n}{(q;q)_{2n-1}} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{2n^2-n} (1 - q^{6n+3}) \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{-2j^2},$$

$$\begin{aligned} \sum_{n\geq 1} \frac{q^{2n^2-2n}(-q^3;q^6)_n}{(q^2;q^2)_{2n-1}(-q;q^2)_n} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2-2n} (1 - q^{12n+6}) \Big(\sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} q^{-(3j)^2} - \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n-1}{3} \rfloor} q^{-(3j+1)^2} \Big) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{4n^2-2n} (1 - q^{12n+6}) \sum_{j=-n}^{n} e^{\frac{2\pi i j}{3}} q^{-j^2}. \end{aligned}$$

8. The bilateral versions of Bailey's lemma

In this section, we aim to apply the bilateral versions of Bailey's lemma due to Andrews and Warnaar [12] to derive identities on false theta functions and the generalized Hecke–type series.

Recall that false theta functions were introduced by Rogers in 1917 [28] that appear like series for classical theta functions except for incorrect signs of some of the terms in the series. In his notebooks as well as in the Lost Notebook [26], Ramanujan gave many examples of identities for false theta functions.

Let's rewrite the identity (3.1) as follows

$$\frac{\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2})}{(q^2;q^2)_{2n}}=\sum_{j=0}^{n}\frac{(V_j(x)+V_{j-1}(x))q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}. \tag{8.1}$$

When x is taken to be some special values, it allows us to transform the above identity into the bilateral versions of the form (2.8). Then inserting the resulting bilateral Bailey pairs into the bilateral versions of Bailey's lemma (2.9) and (2.10), we will obtain identities on false theta functions and the generalized Hecke–type series, respectively.

Firstly, let us consider the case $x=-1$, which leads to the following false theta function identity on Page 13 in Ramanujan's Lost Notebook [26].

Theorem 8.1 (Ramanujan [26, p. 13])**.**

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)}. \tag{8.2}$$

Proof. Taking $x=-1$ in (8.1) and by using the special value of $V_n(x)$ at $x=-1$ (2.3a), we have

$$\begin{aligned}\frac{(q;q^2)_n^2}{(q^2;q^2)_{2n}}&=\sum_{j=0}^{n}\frac{\big(V_j(-1)+V_{j-1}(-1)\big)q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}\\&=\frac{1}{(q^2;q^2)_n^2}+2\sum_{j=1}^{n}\frac{(-1)^jq^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}\\&=\sum_{j=-n}^{n}\frac{(-1)^jq^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}.\end{aligned}$$

It follows that

$$\overline{\alpha}_n=(-1)^nq^{n^2},\qquad \overline{\beta}_n=\frac{(q;q^2)_n^2}{(q^2;q^2)_{2n}} \tag{8.3}$$

form a bilateral Bailey pair as given by (2.8). Then fitting it into the bilateral version of Bailey's lemma (2.9), we obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}q^{n(n+1)}\sum_{j=-n}^{n}(-1)^j$$

$$= \sum_{n=0}^{\infty} q^{n(n+1)} \Big(1 + 2\sum_{j=1}^{n} (-1)^j\Big)$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)},$$

which completes the proof. □

We remark that the bilateral Bailey pair (8.3) is a special case of the one given by Andrews and Warnaar [12, (4.5a, 4.5b)], from which they obtained a generalized form of (8.2)

$$\sum_{n=0}^{\infty} \frac{(-qz;q^2)_n(-q/z;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{z^{-n} - z^{n+1}}{1-z} q^{n(n+1)}. \tag{8.4}$$

Moreover, for the right hand side of (8.2), it can be rewritten in the form of classical false theta functions as follows

$$\Big(\sum_{n\geq 0} - \sum_{n<0}\Big) q^{2n(2n+1)}.$$

For the related results on it, see also Andrews [1, (6.2)], Andrews and Berndt [10, Entry 9.3.4], Andrews and Warnaar [12, eq. (1.1b)], and Wang [36, (1.2)].

If we insert the bilateral Bailey pair (8.3) into (2.10), it directly implies the following identity on the generalized Hecke–type series.

Theorem 8.2. *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{-j^2}. \tag{8.5}$$

Similarly, when $x = -\frac{1}{2}$, we are led to the following two identities.

Theorem 8.3. *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^3;q^6)_n q^n}{(-q;q^2)_n(-q;q)_{2n+1}} = \big(\sum_{n\geq 0} - \sum_{n<0}\big) q^{9n^2+3n}, \tag{8.6}$$

$$\sum_{n=0}^{\infty} \frac{(-q^3;q^6)_n q^n}{(-q;q^2)_n(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \Big(\sum_{j=-\lfloor \frac{n}{6} \rfloor}^{\lfloor \frac{n}{6} \rfloor} q^{-(3j)^2} - \sum_{j=-\lfloor \frac{n+2}{6} \rfloor}^{\lfloor \frac{n-2}{6} \rfloor} q^{-(3j+1)^2} \Big) \tag{8.7}$$

$$= \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \sum_{j=-\lfloor \frac{n}{2} \rfloor}^{\lfloor \frac{n}{2} \rfloor} e^{\frac{2\pi i}{3} j} q^{-j^2}. \tag{8.8}$$

Proof. Substituting $x = -\frac{1}{2}$ into (8.1) and with the help of (2.3b), we obtain that

$$\frac{\prod_{j=1}^{n}(1 - q^{2j-1} + q^{4j-2})}{(q^2;q^2)_{2n}} = \sum_{j=0}^{n} \frac{\big(V_j(-\frac{1}{2}) + V_{j-1}(-\frac{1}{2})\big) q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}} = \sum_{j=-n}^{n} \frac{q^{j^2} e^{\frac{2\pi i}{3} j}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}.$$

It yields a bilateral Bailey pair as follows

$$\overline{\alpha}_n = e^{\frac{2\pi i}{3}n} q^{n^2}, \qquad \overline{\beta}_n = \frac{(-q^3;q^6)_n}{(-q;q^2)_n(q^2;q^2)_{2n}}. \tag{8.9}$$

Then fitting it into the bilateral version of Bailey's lemma (2.9), we obtain

$$\sum_{n=0}^{\infty} \frac{(-q^3;q^6)_n q^n}{(-q;q^2)_n(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n(n+1)} \sum_{j=-n}^{n} e^{\frac{2\pi i}{3}j}$$

By noting the fact that for any integer m, $\sum_{j=3m}^{3m+2} e^{\frac{2\pi i}{3}j} = 0$ and according to the residue classes of n modulo 3, the sum on the right hand side of the above identity can be divided into three parts as follows

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{3n(3n+1)} + \sum_{n=0}^{\infty} q^{(3n+1)(3n+2)}(1 + e^{\frac{2\pi i}{3}(3n+1)} + e^{\frac{2\pi i}{3}(-3n-1)}) \\
&\qquad + \sum_{n=0}^{\infty} q^{(3n+2)(3n+3)}(1 + e^{\frac{2\pi i}{3}(3n+1)} + e^{\frac{2\pi i}{3}(-3n-1)} + e^{\frac{2\pi i}{3}(3n+2)} + e^{\frac{2\pi i}{3}(-3n-2)}) \\
&= \sum_{n=0}^{\infty} q^{3n(3n+1)} - \sum_{n=0}^{\infty} q^{(3n+2)(3n+3)} \\
&= \sum_{n=0}^{\infty} q^{9n^2+3n} - \sum_{n=-\infty}^{-1} q^{9n^2+3n},
\end{aligned}$$

which completes the proof of (8.6).

Then inserting the bilateral Bailey pair (8.9) into (2.10), we obtain (8.8) by direct simplification and (8.7) can be derived by separating the sum according to the residues of n modulo 3. □

Note that for the false theta function on the right hand side of (8.6) after q being replaced by $q^{\frac{1}{6}}$, Rogers [28] gave the following identity

$$\Big(\sum_{n\geq 0} - \sum_{n<0}\Big) q^{\frac{n(3n+1)}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(-q;q)_n}.$$

See also [10, Sec. 9.4] for other expressions of the above false theta functions.

To derive our next results, let us consider the case $x = 0$ in (8.1).

Theorem 8.4. *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{2n(2n+1)}(1 + q^{4n+2}), \tag{8.10}$$

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \sum_{j=-\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{n}{4} \rfloor} (-1)^j q^{-4j^2}. \tag{8.11}$$

Proof. Taking $x = 0$ in (8.1) and combining with the special value of $V_n(x)$ at $x = 0$ (2.3c), we obtain that

$$\frac{(-q^2;q^4)_n}{(q^2;q^2)_{2n}} = \sum_{j=0}^{n} \frac{\big(V_j(0)+V_{j-1}(0)\big)q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}$$

$$= \frac{1}{(q^2;q^2)_n^2} + 2\sum_{\substack{j=4\\ j\equiv 0 \pmod 4}}^{n} \frac{q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}} - 2\sum_{\substack{j=2\\ j\equiv 2 \pmod 4}}^{n} \frac{q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}}$$

$$= \sum_{\substack{j=-n\\ j\equiv 0 \pmod 4}}^{n} \frac{q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}} - \sum_{\substack{j=-n\\ j\equiv 2 \pmod 4}}^{n} \frac{q^{j^2}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j}},$$

which leads to a bilateral Bailey pair $(\overline{\alpha}_n, \overline{\beta}_n)$ with

$$\overline{\alpha}_n = \begin{cases} (-1)^{\frac{n}{2}} q^{n^2}, & \text{if } n \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \qquad \overline{\beta}_n = \frac{(-q^2;q^4)_n}{(q^2;q^2)_{2n}}. \tag{8.12}$$

By fitting it into the bilateral version of Bailey's lemma (2.9), we obtain

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n(n+1)} \sum_{\substack{j=-n\\ j\equiv 0 \pmod 2}}^{n} (-1)^{\frac{j}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} q^{n(n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{2n(2n+1)}(1+q^{4n+2}),$$

which completes the proof of (8.10).

Then inserting the bilateral Bailey pair (8.12) into (2.10), we have that

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \sum_{\substack{j=-\lfloor \frac{n}{2} \rfloor\\ j\equiv 0 \pmod 2}}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\frac{j}{2}} q^{-j^2}$$

$$= \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \sum_{j=-\lfloor \frac{n}{4} \rfloor}^{\lfloor \frac{n}{4} \rfloor} (-1)^j q^{-4j^2},$$

which completes the proof of (8.11). ∎

Similarly with the case of $x = -\frac{1}{2}$, we find that when $x = \frac{1}{2}$,

$$\overline{\alpha}_n = (-1)^n e^{\frac{2\pi i}{3} n} q^{n^2}, \qquad \overline{\beta}_n = \frac{(q^3;q^6)_n}{(q;q^2)_n (q^2;q^2)_{2n}} \tag{8.13}$$

form a bilateral Bailey pair. It leads to the following identities which can be seen as companion forms of identities (8.6) and (8.8), respectively.

Theorem 8.5. *We have*

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^n}{(q;q^2)_n(-q;q)_{2n+1}}=\Big(\sum_{n\geq 0}-\sum_{n<0}\Big)(-1)^nq^{9n^2+3n}(1+q^{6n+2}), \tag{8.14}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^n}{(q;q^2)_n(-q;q)_{2n}}&=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\Bigg(\sum_{j=-\lfloor\frac{n}{6}\rfloor}^{\lfloor\frac{n}{6}\rfloor}(-1)^jq^{-(3j)^2}+\sum_{j=-\lfloor\frac{n+2}{6}\rfloor}^{\lfloor\frac{n-2}{6}\rfloor}(-1)^jq^{-(3j+1)^2}\Bigg)\\&=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}(-1)^je^{\frac{2\pi i}{3}j}q^{-j^2}.\end{aligned} \tag{8.15}$$

Proof. By fitting the bilateral Bailey pair (8.13) into (2.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^n}{(q;q^2)_n(-q;q)_{2n+1}}&=\sum_{n=0}^{\infty}q^{n(n+1)}\sum_{j=-n}^{n}(-1)^je^{\frac{2\pi i}{3}j}\\&=\sum_{n=0}^{\infty}q^{n(n+1)}\Big(1+2\sum_{j=1}^{\lfloor\frac{n}{3}\rfloor}(-1)^j+\sum_{j=0}^{\lfloor\frac{n-1}{3}\rfloor}(-1)^j-\sum_{j=0}^{\lfloor\frac{n-2}{3}\rfloor}(-1)^j\Big).\end{aligned}$$

Then separating the above sum according to the residue classes of n modulo 6, it equals to

$$\begin{aligned}&\sum_{n\geq 0}q^{6n(6n+1)}+2\sum_{n\geq 0}q^{(6n+1)(6n+2)}+\sum_{n\geq 0}q^{(6n+2)(6n+3)}\\&\qquad-\sum_{n\geq 0}q^{(6n+3)(6n+4)}-2\sum_{n\geq 0}q^{(6n+4)(6n+5)}-\sum_{n\geq 0}q^{(6n+5)(6n+6)}\\&=\sum_{n\geq 0}(-1)^nq^{3n(3n+1)}+2\sum_{n\geq 0}(-1)^nq^{(3n+1)(3n+2)}+\sum_{n\geq 0}(-1)^nq^{(3n+2)(3n+3)}\\&=\Big(\sum_{n\geq 0}-\sum_{n<0}\Big)(-1)^nq^{3n(3n+1)}+\Big(\sum_{n\geq 0}-\sum_{n<0}\Big)(-1)^nq^{(3n+1)(3n+2)},\end{aligned}$$

which leads to (8.14) by simplification.

Moreover, by inserting the bilateral Bailey pair (8.13) into (2.10), (8.15) can be derived by separating the sum according to the residues of n modulo 3. □

As last, by taking $x=1$ in (8.1), it implies that

$$\overline{\alpha}_n=q^{n^2},\qquad \overline{\beta}_n=\frac{(-q;q^2)_n^2}{(q^2;q^2)_{2n}} \tag{8.16}$$

form a bilateral Bailey pair. Then fitting it into the bilateral versions of Bailey's lemma (2.9) and (2.10), we derive the following two identities.

Theorem 8.6. *We have*

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^n}{(-q^2;q^2)_n(1+q^{2n+1})}=\sum_{n=0}^{\infty}(2n+1)q^{n(n+1)}, \tag{8.17}$$

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}\sum_{j=-\lfloor\frac{n}{2}\rfloor}^{\lfloor\frac{n}{2}\rfloor}q^{-j^2}. \tag{8.18}$$

Note that identity (8.17) also can be derived by setting $z\to 1$ in (8.4), see also Andrews and Warnaar [12, (1.4)].

9. Relations with Andrews' result

As a concluding observation, by comparing our result (3.1) with Andrews' identity (1.2), and using the orthogonality of Chebyshev polynomials of the third kind, we obtain an identity on q-binomial coefficients.

It's known that the orthogonality on $V_n(x)$ is given as follows

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}V_n(x)V_m(x)\mathrm{d}x=\begin{cases}0, & \text{if } m\neq n,\\ \pi, & \text{if } m=n,\end{cases} \tag{9.1}$$

see, for example, [23, Sec. 4.2.2]

Theorem 9.1. *We have*

$$\sum_{j=-n-1}^{n}q^{2j^2+j}\begin{bmatrix}2n+1\\ n-j\end{bmatrix}_{q^2}^{2}=(1+q^{4n+1})\begin{bmatrix}4n\\ 2n\end{bmatrix}. \tag{9.2}$$

Proof. Replacing n by $2n$ in Andrews' identity (1.2), we obtain that

$$\prod_{j=1}^{2n}(1+2xq^j+q^{2j})=\sum_{j=0}^{2n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}4n+1\\ 2n-j\end{bmatrix}. \tag{9.3}$$

Apparently, the left hand side of the above identity can be rewritten as

$$\prod_{j=1}^{2n}(1+2xq^j+q^{2j})=\prod_{j=1}^{n}(1+2xq^{2j}+q^{4j})\prod_{j=1}^{n}(1+2xq^{2j-1}+q^{4j-2}).$$

By replacing the left hand side of the above identity with (9.3), and the two product terms on the right hand side by (1.2) (with $q\to q^2$) and (3.1), respectively, it turns out that

$$\sum_{j=0}^{2n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}4n+1\\ 2n-j\end{bmatrix}=\sum_{j=0}^{n}q^{j^2+j}\begin{bmatrix}2n+1\\ n-j\end{bmatrix}_{q^2}V_j(x)\cdot\sum_{j=0}^{n}q^{j^2}\begin{bmatrix}2n\\ n-j\end{bmatrix}_{q^2}\big(V_j(x)+V_{j-1}(x)\big).$$

Then multiplying both hand sides of the above identity by $\sqrt{\frac{1+x}{1-x}}$, and calculating the integrals on $x \in [-1,1]$ by employing the orthogonal property (9.1), we are led to

$$\begin{aligned}
\begin{bmatrix} 4n+1 \\ 2n \end{bmatrix} &= \sum_{j=0}^{n} q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} + \sum_{j=0}^{n-1} q^{j^2+j+(j+1)^2} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2} \begin{bmatrix} 2n \\ n-j-1 \end{bmatrix}_{q^2} \\
&= \sum_{j=0}^{n} q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2}^2 \frac{(1+q^{2j+1})}{(1+q^{2n+1})} \\
&= \frac{1}{1+q^{2n+1}} \left(\sum_{j=0}^{n} q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2}^2 + \sum_{j=0}^{n} q^{(j+1)(2j+1)} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2}^2 \right) \\
&= \frac{1}{1+q^{2n+1}} \left(\sum_{j=0}^{n} q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2}^2 + \sum_{j=1}^{n+1} q^{j(2j-1)} \begin{bmatrix} 2n+1 \\ n-j+1 \end{bmatrix}_{q^2}^2 \right) \\
&= \frac{1}{1+q^{2n+1}} \sum_{j=-n-1}^{n} q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}_{q^2}^2,
\end{aligned}$$

which completes the proof by further simplifying. $\square$

Note that identity (9.2) can be seen as a finite form of Jacobi's triple product identity (4.2) with $q \to q^4$ and $z \to -q^3$. In fact, by taking $n \to \infty$ in (9.2) and simplifying, we obtain that

$$\sum_{j=-\infty}^{\infty} q^{2j^2+j} = (-q, -q^3, q^4; q^4)_\infty.$$

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References

[1] G.E. Andrews, Ramanujan's "lost" notebook, I: partial theta functions, Adv. Math. 41 (1981), 137–172.
[2] G.E. Andrews, Ramanujan's "Lost" Notebook. II: θ-functions expansions, Adv. Math. 41 (1981), 173–185.
[3] G.E. Andrews, Hecke modular forms and the Kac-Peterson identities, Trans. Amer. Math. Soc. 283 (2) (1984), 451–458.
[4] G.E. Andrews, Multiple series Rogers–Ramanujan type identities, Pacific J. Math. 114 (2) (1984), 267–283.
[5] G.E. Andrews, q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, *CBMS Regional Conference*, Vol. 66, Amer. Math. Soc., Providence, RI, 1986.
[6] G.E. Andrews, Parity in partition identities, Ramanujan J. 23 (1-3) (2010), 45–90.

[7] G.E. Andrews, q-Orthogonal polynomials, Rogers–Ramanujan identities, and mock theta functions, Proc. Steklov Inst. Math. 276 (1) (2012), 21–32.
[8] G.E. Andrews, Dyson's "Favorite" identity and Chebyshev polynomials of the third and fourth kind, Ann. Comb. 23 (2019), 443–464.
[9] G.E. Andrews and R. Askey, Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials, In: M. Aigner, (ed.) Higher Combinatorics, pp. 3–26, Reidel, Dordrecht, Holland, 1977.
[10] G.E. Andrews and B.C. Berndt, Ramanujan's lost notebook, part I, Springer, 2005.
[11] G.E. Andrews and B.C. Berndt, Ramanujan's lost notebook, part II, Springer, 2009.
[12] G.E. Andrews and S.O. Warnaar, The Bailey transform and false theta functions, Ramanujan J. 14 (2007), 173–188.
[13] W.N. Bailey, Identities of the Rogers–Ramanujan type, Proc. London Math. Soc. 50 (2) (1949), 421–435.
[14] I. Cherednik and B. Feigin, Rogers–Ramanujan type identities and Nil-DAHA, Adv. Math. 248 (2013), 1050–1088.
[15] F.J. Dyson, A walk through Ramanujan's garden. In: G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, R.A. Rankin, (eds.), Ramanujan Revisited, Academic Press, Boston, MA (1988), pp. 7–28.
[16] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications, (2nd ed.), Cambridge University Press, Cambridge, 2004.
[17] N.S.S. Gu and J. Liu, Families of multisums as mock theta functions, Adv. Appl. Math. 79 (2016), 98–124.
[18] D. Hickerson and E. Mortenson, Hecke–type double sums, Appell–Lerch sums and mock theta functions, I, Proc. Lond. Math. Soc. 109 (3) (2014), 382–422.
[19] M.E.H. Ismail and R.M. Zhang, q-Bessel functions and Rogers–Ramanujan type identities, Proc. Amer. Math. Soc. 146 (2018), 3633–3646.
[20] C. Jennings-Shaffer, Exotic Bailey–Slater SPT–functions I: Group A., Adv. Math. 305 (2017), 479–514.
[21] J.M. Laughlin, A.V. Sills, and P. Zimmer, Rogers–Ramanujan-Slater type identities, Electronic J. Comb. 15 (2008), #DS15.
[22] J. Lovejoy, Bailey pairs and indefinite quadratic forms, J. Math. Anal. Appl. 410 (2014), 1002–1013.
[23] J. Mason and D. Handscomb, Chebyshev Polynomials, CRC Press, New York, 2003.
[24] R.J. McIntosh, Second order mock theta functions, Canad. Math. Bull. 50 (2) (2007), 284–290.
[25] R.J. McIntosh, New mock theta conjectures, Part I, Ramanujan J. 46 (2018), 593–604.
[26] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[27] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25 (1894), 318–343.
[28] L.J. Rogers, On two theorems of combinatory analysis and some allied identities, Proc. London Math. Soc. 16 (1917), 315–336.
[29] A.V. Sills, Identities of the Rogers–Ramanujan-Slater type, Int. J. Number Theory 3 (2007), 293–323.
[30] A.V. Sills, An Invitation to the Rogers–Ramanujan Identities, CRC Press, New York, 2018.
[31] L.J. Slater, A new proof of Rogers's transformations of infinite series, Proc. London Math. Soc. 53 (2) (1951), 460–475.
[32] L.J. Slater, Further identities of the Rogers–Ramanujan type, Proc. London Math. Soc. 54 (1952), 147–167.
[33] N.J.A. Sloane, On-line encyclopedia of integer sequences, http://oeis.org/.
[34] C. Wang and A.J. Yee, Truncated Hecke–Rogers type series, Adv. Math. 365 (2020), 107051.
[35] C. Wang and A.J. Yee, Truncated Hecke–Rogers type series–part II, Ramanujan J., DOI: 10.1007/s11139-021-00407-w.
[36] L. Wang, New proofs of Ramanujan's identities on false theta functions, Ramanujan J. 50 (2) (2019), 423–431.
[37] S.O. Warnaar, 50 Years of Bailey's Lemma, Algebraic Combinatorics and Applications, Springer-Verlag, 2001, pp. 333–347.

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China
E-mail address: sunhui@nankai.edu.cn