Graphs with Minimum Vertex-Degree Function-Index for Convex Functions[1]

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Abstract

An (n,m)-graph is a graph with n vertices and m edges. The vertex-degree function-index $H_f(G)$ of a graph G is defined as $H_f(G) = \sum_{v\in V(G)} f(d(v))$, where f is a real function. Recently, Tomescu considered the upper bound of $H_f(G)$ and got the connected (n,m)-graph G with $m \geq n$ which maximizes $H_f(G)$ if $f(x)$ is strictly convex with two special properties. He also characterized all (n,m)-graphs G with $1 \leq m \leq n$ satisfying that $H_f(G) \leq f(m) + mf(1) + (n-m-1)f(0)$ if $f(x)$ is strictly convex and differentiable and its derivative is strictly convex. In this paper, we will consider the lower bound of $H_f(G)$ and show that every (n,m)-graph with $1 \leq m \leq n(n-1)/2$ satisfies that $H_f(G) \geq rf(k+1) + (n-r)f(k)$ if $f(x)$ is strictly convex, where $k = \lfloor 2m/n \rfloor$ and $r = 2m - nk$. Moreover, the equality holds if and only if $G \in \mathcal{G}(n,m)$, where $\mathcal{G}(n,m)$ is the family of all (n,m)-graphs G satisfying that the vertex-degree $d(v) \in \{\lfloor \frac{2m}{n} \rfloor, \lceil \frac{2m}{n} \rceil\}$ for all $v \in V(G)$. Under the same condition on f we also obtain a result for the minimum of $H_f(G)$ among all connected (n,m)-graphs. It is easy to see that if $f(x)$ is strictly concave, we can get the maximum case for $H_f(G)$.

1 Introduction

We only consider simple and finite graphs in this paper. For terminology and notation not defined here, we refer the reader to [2, 20]. We use $V(G)$ and $E(G)$ to denote

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the vertex-set and edge-set of a graph G, respectively. An (n, m)-*graph* is a graph $G = (V(G), E(G))$, where $m = |E(G)|$ and $n = |V(G)|$. Let $G(n, m)$ represent the collection of all (n, m)-graphs. For any two vertices u and v, if u is adjacent to v, we denote it by $u \sim v$. A graph G is called k-regular if the degree $d(v) = k$ for every $v \in V(G)$. We denote a complete graph with n vertices by K_n. Moreover, we use C_n and P_n to denote a cycle and a path on n vertices, respectively.

For two disjoint graphs G and H, the union $G \cup H$ of G and H is a new graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For two disjoint graphs G and H, we use $G \vee H$ to denote a new graph obtained by adding edges joining every vertex of G to every vertex of H. For a subset F of $E(G)$, we use $G - F$ to denote the subgraph of G obtained by deleting all edges of F from G, whereas for a subset S of $V(G)$, we use $G - S$ to denote the subgraph of G induced by $V \backslash S$ in G. If M is a matching of G, we use $|M|$ to denote the number of edges in M.

Denote the degree of a vertex v in G also by d_v, and denote the sequence of degrees of a graph G with n vertices by $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. In this paper, we will study a kind of general chemical index, called the vertex-degree function-index $H_f(G)$ of a graph G with function $f(x)$, which was first introduced by Linial and Rozenman in [14], and is defined as follows:

$$H_f(G) = \sum_{v \in V(G)} f(d_v).$$

Another topological function-index TI was introduced by Gutman in [5]. For a symmetric real function $f(x, y)$ and a graph G, the topological index is defined as

$$TI(G) = \sum_{uv \in E(G)} f(d_u, d_v).$$

This was also called the *bond-incident-degree index* $BID(G)$ by Vukičević and Durdević in [21]. Notice that by taking the symmetric real function equals to $f(x)/x + f(y)/y$ for some function $f(x)$, one could deduce that $H_f(G)$ is a special case of $TI(G)$. For more knowledge on TI we refer to [4, 5, 10, 16, 21], and we denoted $TI(G)$ by $IT_f(G)$ in [10].

In the past years, many researchers have done a lot of work on chemical indices, including Zagreb indices; see [3, 6, 8, 9, 11–13, 17] and the references therein. Recently, Tomescu [18, 19] studied $H_f(G)$ for convex function f. He gave some upper bounds for the function-index $H_f(G)$ and the function f is required to satisfy some other properties except for the convexity. Their results are stated as follows.

Theorem 1.1. *[Lemma 2.2 [18]] If $G \in G(n,m)$ maximizes (minimizes) $H_f(G)$ where $f(x)$ is strictly convex (concave), then G has at most one nontrivial connected component C and C has a vertex of degree $|V(C)| - 1$.*

Theorem 1.2. *[Theorem 2.3 [19]] Let $n \geq 2$ and $G \in G(n,m)$ such that $1 \leq m \leq n-1$. If $f(x)$ is a strictly convex function having property that $f(x)$ is differentiable and its derivative is strictly convex, then it holds that*

$$H_f(G) \leq f(m) + mf(1) + (n-m-1)f(0),$$

with equality if and only if $G = S_{m+1} \cup (n-m-1)K_1$.

Theorem 1.3. *[Theorem 2.4 [19]] If $n \geq 3$, $n \leq m \leq 2n-3$, $f(x)$ is a strictly convex function having property that $f(x)$ is differentiable and its derivative is strictly convex, and $G \in G(n,m)$ is connected, then it holds that*

$$H_f(G) \leq f(n-1) + f(m-n+2) + (m-n+1)f(2) + (2n-m-3)f(1),$$

with equality if and only if $G = K_1 \vee (K_{1,m-n+1} \cup (2n-m-3)K_1)$.

As one can see, Tomescu's results are all about the upper bound of $H_f(G)$. Ali et al. in [1] gave the following lower bound for connected (n,m)-graphs under some constraints on n and m.

Theorem 1.4. *[Theorem 1 [1]] If $n \geq 4$, $3n/2 \geq m \geq n+1$ and $f(x)$ is a convex function, then among all connected (n,m)-graphs, graphs in $\mathcal{G}(n,m)$ attain the minimum value of $H_f(G)$, where the graph family $\mathcal{G}(n,m)$ is defined in the following Definition 1.5.*

In this paper, we will further study the minimum (maximum) values of $H_f(G)$ among all (n,m)-graphs with the property that f is strictly convex (concave). Moreover, we will give a same result among all connected (n,m)-graphs. Note that our result Theorem 1.7 will cover the result Theorem 1.4. Before proceeding, we give the definition of our extremal graphs as follows.

Definition 1.5. *Given $n \geq 2$ and $1 \leq m \leq n(n-1)/2$, define $\mathcal{G}(n,m)$ to be the family of all (n,m)-graphs G satisfying that $d(v) \in \{\lfloor \frac{2m}{n} \rfloor, \lceil \frac{2m}{n} \rceil\}$ for all $v \in V(G)$.*

For an (n,m)-graph G, let $k = \lfloor 2m/n \rfloor$ and $r = 2m - kn \in \{0, 1, \ldots, n-1\}$, then G belongs to $\mathcal{G}(n,m)$ if and only if G has r vetices of degree k and $n-r$ vertices of degree $k+1$. Note that for some given m and n, the graph family $\mathcal{G}(n,m)$ contains both connected and disconnected graphs. We give an example in Figure 1.

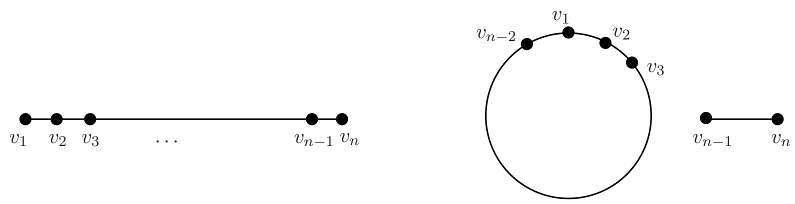

Figure 1. Graphs P_n and $C_{n-2} \cup K_2$ in $\mathcal{G}(n,m)$ for $m = n-1$ and $n \geq 5$.

Our main results are stated as follows.

Theorem 1.6. *Let $n \geq 2$ and G be an (n,m)-graph with $1 \leq m \leq n(n-1)/2$, and let $k = \lfloor 2m/n \rfloor$ and $r = 2m - kn$. If f is a strictly convex function, then it holds that*

$$H_f(G) \geq rf(k+1) + (n-r)f(k),$$

and the equality holds if and only if $G \in \mathcal{G}(n,m)$.

We will construct some graphs to show that for $n \leq m \leq n(n-1)/2$, there are connected graphs $G \in \mathcal{G}(n,m)$, and for $m = n-1$, we have the path $P_n \in \mathcal{G}(n, n-1)$. Therefore, if we consider only connected (n,m)-graphs, we also have the following result.

Theorem 1.7. *Let $n \geq 2$ and G be a connected (n,m)-graph with $n-1 \leq m \leq n(n-1)/2$, and let $k = \lfloor 2m/n \rfloor$ and $r = 2m - kn$. If f is a strictly convex function, then it holds that*

$$H_f(G) \geq rf(k+1) + (n-r)f(k),$$

and the equality holds if and only if G is connected and $G \in \mathcal{G}(n, m)$.

Our results can cover some previous known results. For example, for the general zeroth-order Randić index ${}^0R_\alpha(G)$, the function $f(x) = x^\alpha$ is strictly convex for $\alpha > 1$. Then we can obtain a lower bound of Randić index ${}^0R_\alpha(G)$ by Theorem 1.6, and moreover, ${}^0R_\alpha(G)$ attains the minimum if and only if $G \in \mathcal{G}(n, m)$.

2 Preliminaries

At first we recall an important inequality, the *Jensen inequality.* which states that

$$\sum_{i=1}^{n} f(x_i) \geq nf(\frac{\sum_{i=1}^{n} x_i}{n})$$

for any $x_1, x_2, \ldots, x_n \in [a, b]$ if f is a convex function on an interval $[a, b]$. Using this inequality, we can get the following lemma.

Lemma 2.1. *Let $n \geq 1$, $m \geq 0$ be integers and f be a strictly convex function. Suppose that $s_1, s_2, \ldots, s_n$ is a sequence of non-negative integers such that $\sum_{i=1}^{n} s_i = 2m$. Let $k = \lfloor 2m/n \rfloor$ and $r = 2m - nk$. Then we have*

$$\sum_{i=1}^{n} f(s_i) \geq rf(k+1) + (n-r)f(k).$$

Proof. If $r = 0$, then by the convexity of f and the *Jensen inequality*, we have

$$\sum_{i=1}^{n} f(s_i) \geq nf(\frac{\sum_{i=1}^{n} s_i}{n}) = nf(\frac{2m}{n}) = nf(k).$$

It remains to show that the result is true for any $r \in \{1, 2, \ldots, n-1\}$. Suppose that $\{s_i\}_{i=1}^{n}$ is a sequence of integers such that $\sum_{i=1}^{n} f(s_i)$ is minimal. We claim that $s_i \in \{k, k+1\}$ for all $1 \leq i \leq n$. If the claim does not hold, without loss of generality, suppose that $s_1 \geq s_2 \geq \cdots \geq s_n$. Since $1 \leq r \leq n-1$, we have $s_1 \geq k+1$ and $s_n \leq k$. Then, there would be some $s_i \notin \{k, k+1\}$ such that either $s_1 \geq k+2$ or $s_n \leq k-1$. Thus, $s_1 - s_n - 1 \geq 1$. Let $s_1' = s_1 - 1$, $s_i' = s_i$ for $2 \leq i \leq n-1$ and $s_n' = s_n + 1$. Since $s_1 - s_n - 1 \geq 1$, $s_1' \neq s_n$ and $s_n' \neq s_1$, it shows that $\{s_i'\}_{i=1}^{n}$ is a different sequence from

$\{s_i\}_{i=1}^n$. Since f is a strictly convex function, then $f(x+1)-f(x)$ is strictly monotone increasing. So, we would obtain that

$$\sum_{i=1}^{n} f(s_i') - \sum_{i=1}^{n} f(s_i) = [f(s_n+1)-f(s_n)] - [f(s_1)-f(s_1-1)] < 0,$$

which contradicts the minimality of $\sum_{i=1}^{n} f(s_i)$.

The proof is thus complete. □

We prove Theorem 1.7 by constructing a connected (n,m)-graph G such that $d(v) \in \{\lfloor \frac{2m}{n} \rfloor, \lceil \frac{2m}{n} \rceil\}$ for all $v \in V(G)$. In order to make our construction more consistent and reasonable, we need the following two lemmas.

Lemma 2.2. *Let $\lfloor 2m/n \rfloor = k$ and $r = 2m - nk$, where r is even and $r \neq 0$. Then there is a k-regular graph with n vertices and $m - r/2$ edges, and its complement has a matching with $r/2$ edges.*

Proof. Since r is even, it shows that nk is also even. Note that $r \neq 0$. Then $k < n-1$. We consider the following three cases.

Case 1. k is even and n is odd.

Consider a graph G_1 with vertex-set $\{v_1, v_2, \dots v_n\}$ and $v_i \sim v_j$ if and only if $|i-j|$ is congruent modulo n with a number belonging to the set $\{-k/2, -k/2+1, \dots, -1, 1, \dots, k/2\}$. Then G_1 is a k-regular graph with $m - r/2$ edges. By the construction of G_1, there is a matching M_1 in the complement of G_1 with edge-set $\{v_i v_{i+\frac{n-1}{2}} : 1 \leq i \leq (n-1)/2\}$ satisfying $|M_1| = (n-1)/2$. Note that $k/2 < (n-1)/2$. Then these edges do not appear in G_1. That is, M_1 is a matching with $(n-1)/2$ edges in the complement of G_1. Since $r \leq n-1$, G_1 is a required graph.

Case 2. Both k and n are even.

Consider the graph G_1 we constructed above. Then there is a matching M_2 with edge-set $\{v_i v_{i+\frac{n}{2}} : 1 \leq i \leq n/2\}$ in the complement of G_1. Note that $|M_2| = n/2$ and $r \leq n-1$. Then G_1 is also a required graph.

Case 3. k is odd and n is even.

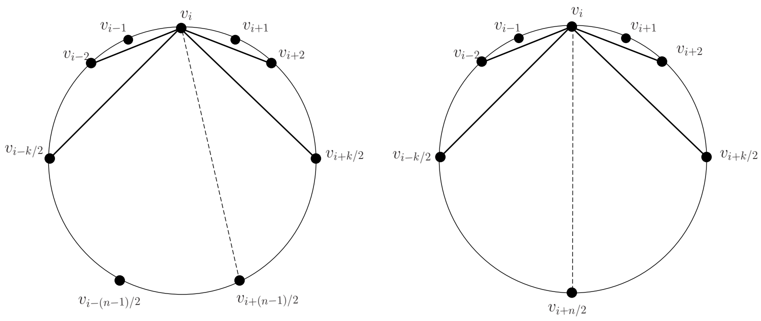

Figure 2. G_1 for k is even.

Consider a graph G_3 with vertex-set $\{v_1, v_2, \dots v_n\}$ and $v_i \sim v_j$ if and only if $|i-j|$ is congruent modulo n with a number belonging to the set $\{-(k-1)/2, -(k-1)/2+1, \dots, -1, 1, \dots, (k-1)/2\}$ or $j = i + n/2$, where $1 \le i \le n/2$. By the construction of G_3, we know that G_3 is a k-regular graph and $G_3 \in G(n, m - r/2)$, and there is a matching M_3 with edge-set $\{v_i v_{i+\frac{n}{2}-1} : 1 \le i \le n/2 - 1\}$ satisfying $|M_3| = n/2 - 1$. Note that $k < n-1$. So we get $(k-1)/2 < n/2 - 1$, which means that M_3 is a matching in the complement of G_3. Since both r and n are even and $r \le n-1$, we have $r \le n-2$. Therefore, G_3 is a required graph.

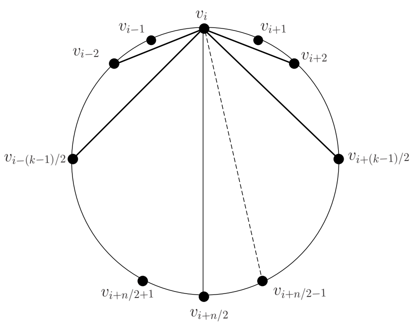

Figure 3. G_3 for k is odd and n is even.

The proof is thus complete. □

Lemma 2.3. *Let $2m = kn + 1$. Then there is a k-regular graph with $n-1$ vertices and $m - (k+1)/2$ edges, having a matching with $(n-1)/2$ edges.*

Proof. Since $2m = nk + 1$, both n and k are odd. From $k < n-1$, we deduce that $(k+1)/2 \le (n-1)/2$. Consider a k-regular graph G_4 with $n-1$ vertices as follows: $V(G_4) = \{v_1, v_2, \dots v_{n-1}\}$ and $v_i \sim v_j$ if and only if $|i-j|$ is congruent modulo $n-1$ with a number belonging to the set $\{-(k-1)/2, -(k-1)/2+1, \dots, -1, 1, \dots, (k-1)/2\}$ or

$j = i+(n-1)/2$, where $1 \le i \le (n-1)/2$. Since $2m = kn+1$, we have $2(m-(k+1)/2) = k(n-1)$. That is, G_4 is a k-regular graph and $G_4 \in G(n-1, m-(k+1)/2)$. Note that $k-1 < n-1$. Then there is a matching M_4 with edge-set $\{v_i v_{i+\frac{n-1}{2}} : 1 \le i \le (n-1)/2\}$ in G_4, such that $|M_4| = (n-1)/2$. Hence, G_4 is a required graph. □

3 Proofs of Main Results

Now we are ready to give the proofs of our main results Theorems 1.6 and 1.7.

Proof of Theorem 1.6: Since $2m = kn + r$ and $k = \lfloor 2m/n \rfloor$, noticing that $H_f(G) = \sum_{i=1}^{n} f(d_{v_i})$ and $\sum_{i=1}^{n} d_{v_i} = 2m$, by Lemma 2.1 we have

$$H_f(G) \ge rf(k+1) + (n-r)f(k).$$

Moreover, $H_f(G) = rf(k+1) + (n-r)f(k)$ if and only if the (n,m)-graph G has r vertices of degree $k+1$ and $n-r$ vertices of degree k. That is, the equality holds if and only if $G \in \mathcal{G}(n,m)$.

Now, we only need to show $\mathcal{G}(n,m) \ne \emptyset$. That is, there always exist a graph G with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k+1$ and $d_j = k$ for $1 \le i \le r$ and $r+1 \le j \le n$. In fact, it is easy to see that the degree sequence is graphical simply by verifying the conditions in [7].

Algorithm 1 Find an (n,m)-graph G with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k+1$ and $d_j = k$ for $1 \le i \le r$ and $r+1 \le j \le n$.

Input: $E^{(0)} = \emptyset$, $\boldsymbol{d}^{(0)'} = \boldsymbol{d}$ and $V^{(0)'} = (v_1^{(0)'}, v_2^{(0)'}, \ldots, v_n^{(0)'})$.

Output: An (n,m)-graph $G = (V^{(l)}, E^{(l-1)})$ with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k+1$ and $d_j = k$ for $1 \le i \le r$ and $r+1 \le j \le n$.

1: **Set** $l = 1$.
2: Find a permutation σ, such that $\sigma \boldsymbol{d}^{(l-1)'} = (d_1^{(l)}, d_2^{(l)}, \ldots, d_n^{(l)})$ is non-increasing for $\boldsymbol{d}^{(l-1)'} = (d_1^{(l-1)'}, d_2^{(l-1)'}, \ldots, d_n^{(l-1)'})$. Denote $\sigma V^{(l-1)'} = (v_1^{(l)}, v_2^{(l)}, \ldots, v_n^{(l)}) = V^{(l)}$.
3: **if** $d_1^{(l)} \ne 0$ **then**
4: **Set** $E^{(l)} = E^{(l-1)} \cup \{v_1^{(l)} v_j^{(l)} | j = 2, 3, \ldots, d_1^{(l)} + 1\}$ and $\boldsymbol{d}^{(l)'} = (0, d_2^{(l)} - 1, \ldots, d_{d_1^{(l)}+1}^{(l)} - 1, d_{d_1^{(l)}+2}^{(l)}, \ldots, d_n^{(l)})$.
5: **else** go to 7.
6: **Set** $l = l+1$ and go to 2.
7: **return** $G = (V^{(l)}, E^{(l-1)})$.

By choosing different permutations σ in Algorithm 1, we can obtain some (n,m)-graphs $G \in \mathcal{G}(n,m)$ which minimize the value of $H_f(G)$. However, from [15] we can get the following algorithm, which can generate all graphs of $\mathcal{G}(n,m)$.

Algorithm 2 Find all (n,m)-graphs with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k+1$ and $d_j = k$ for $1 \le i \le r$ and $r+1 \le j \le n$.

Input: n, m and $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k+1$ and $d_j = k$ for $1 \le i \le r$ and $r+1 \le j \le n$.

Output: $\mathcal{G}(n,m)$ for any given n and m.

1: Construct a complete n-partite graph $H = (P_1, P_2, \ldots, P_n)$, such that each P_i for $1 \le i \le r$ has $k+1$ vertices and each P_j for $r+1 \le j \le n$ has k vertices.
2: Find all perfect matchings in H, denoted by $\{M_1, M_2, \ldots, M_l\}$.
3: **Set** $\mathcal{G}(n,m) = \emptyset$ and $s = 1$.
4: **while** $s \le l$ **do**
5:     Construct a new graph G_s with vertex-set $\{p_1, p_2, \ldots, p_n\}$ and $p_i \sim p_j$ if and only if there is an edge between P_i and P_j in M_s.
6:     **if** G_s does not have multiple edges and $G_s \ncong G$ for any $G \in \mathcal{G}(n,m)$ **then**
7:         **Set** $\mathcal{G}(n,m) = \mathcal{G}(n,m) \bigcup \{G_s\}$.
8:     **else** $\mathcal{G}(n,m) = \mathcal{G}(n,m)$.
9:     **Set** $s = s+1$ and go to 4.
10: **return** $\mathcal{G}(n,m)$.

□

Note that to check that $G_s \ncong G$ for any $G \in \mathcal{G}(n,m)$ is a very hard nut to crack. Although this algorithm can be used to generate all graphs of $\mathcal{G}(n,m)$, it cannot guarantee the existence of any graph in $\mathcal{G}(n,m)$.

Proof of Theorem 1.7: By the proof of Theorem 1.6, we only need to show that there is a connected (n,m)-graph belonging to $\mathcal{G}(n,m)$ for any given n and m such that $n-1 \le m \le n(n-1)/2$.

If $m = n-1$ we have the path $P_n \in \mathcal{G}(n, n-1)$, which is connected, as required.

If $n \le m \le n(n-1)/2$, then $k = \lfloor \frac{2m}{n} \rfloor \ge 2$. Noticing that $2m = kn + r$, we distinguish the following three cases to discuss.

Case 1. $r = 0$, *i.e.*, $2m = nk$.

In this case, we need to find a connected k-regular (n,m)-graph. From the condition

[2] for a sequence to be graphical, we know that a k-regular graph with n vertices exists if and only if $n \geq k+1$ and nk is even. Noticing that $m \leq n(n-1)/2$, there must be a k-regular (n, m)-graph which satisfies $2m = nk$. Moreover, it is easy to know that there also exists a connected k-regular (n, m)-graph G which satisfies $2m = nk$. That is, $G \in \mathcal{G}(n, m)$ and G is connected.

Case 2. r is even and $r \neq 0$.

From $2m = nk + r$, we obtain $2(m - r/2) = kn$. By Lemma 2.2, there is a k-regular graph H^* with n vertices and $m - r/2$ edges, and its complement has a matching M^* with $r/2$ edges. Adding all $r/2$ edges that appear in M^* to the graph H^*, we then get a new graph, called G. One can see that $G \in G(n, m)$ and $H_f(G) = rf(k+1) + (n-r)f(k)$. That is, $G \in \mathcal{G}(n, m)$. From our construction, there is an n-cycle $v_1 v_2 \dots v_n v_1$ in G, and so G is also connected.

Case 3. r is odd.

Note that $k < n-1$. First, we show that it is true for $r = 1$. By Lemma 2.3, there is a k-regular graph $H^{**} \in G(n-1, m-(k+1)/2)$, which contains a matching M^{**} with $(k+1)/2$ edges. Deleting all $(k+1)/2$ edges in M^{**} from H^{**} and adding a new vertex such that this vertex is adjacent to all $k+1$ vertices of M^{**}, we get a graph $G \in G(n, m)$, which satisfies $H_f(G) = f(k+1) + (n-1)f(k)$. By our construction, the graph G is also connected.

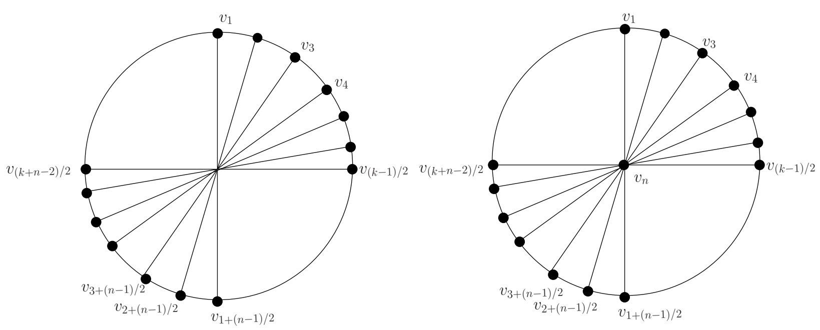

Figure 4. H^{**} and G for $r = 1$.

It remains to show that the result is true for $r \geq 3$ and r is odd. The equality can be written as $2(m - (r-1)/2) = nk + 1$. By Lemma 2.3, there is a k-regular graph $D_1 \in G(n-1, m-(k+r)/2)$, which contains a matching N_1 with $(k+1)/2$ edges.

Deleting all $(k+1)/2$ edges in N_1 from D_1 and adding a new vertex such that this vertex is adjacent to all $k+1$ vertices of N_1, we get a graph $D_2 \in G(n, m-(r-1)/2)$ and $H_f(D_2) = f(k+1)+(n-1)f(k)$. If $r-1 \leq k+1$, we can add any $(r-1)/2$ edges in N_1 to D_2. Thus, we find a graph $G \in G(n,m)$ satisfying $H_f(G) = rf(k+1)+(n-r)f(k)$. If $r-1 > k+1$, we denote $s = r-k-2$. Notice that $2(m-(r-1)/2) = nk+1$. Since r is odd, then both n and k are odd. That is, both $n-1$ and $k+r$ are even. From the construction we give above, in fact, by the proof of Case 3 in Lemma 2.2, there is a k-regular graph $D_3 \in G(n-1, m-(k+r)/2)$, whose complement has a matching N_2 with $(n-3)/2$ edges. Note that $s = r-k-2 \leq n-3-2 = n-5 < n-3$. So we can add any $s/2$ edges in matching N_2 to D_3. In this way, we obtain a graph D_4 with $n-1$ vertices and $m-(k+1)$ edges. Moreover, it has s vertices of degree $k+1$ and $n-1-s$ vertices of degree k. Add a new vertex to D_4 such that the new vertex is adjacent to any $k+1$ of the remaining $n-1-s$ vertices. It does works since $n-1-s = n-1-(r-k-2) \geq n-1-(n-2-k-2) = k+3$. Hence, we get a graph $G \in G(n,m)$ satisfying $H_f(G) = rf(k+1)+(n-r)f(k)$. It is easy to see from our construction that G is also connected. That is, there is a connected graph $G \in \mathcal{G}(n,m)$ when r is odd.

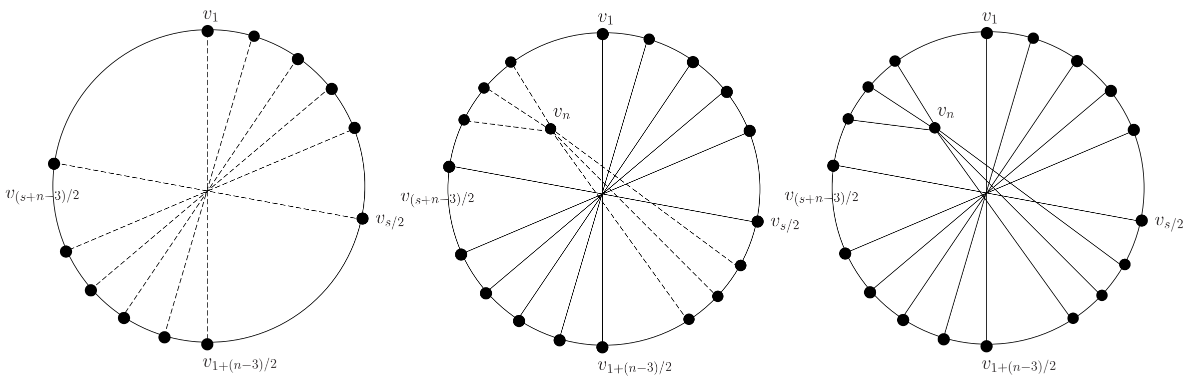

Figure 5. Graphs for $r \geq 3$ and $r-1 > k+1$.

The above proof can guarantee the existence of connected graphs in $\mathcal{G}(n,m)$. The following Algorithm 3 (similar to Algorithm 2) can be used to find all connected graphs in $\mathcal{G}(n,m)$.

Algorithm 3 Find all connected (n, m)-graphs with degree sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k + 1$ and $d_j = k$ for $1 \leq i \leq r$ and $r + 1 \leq j \leq n$.

Input: n, m and $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ where $d_i = k + 1$ and $d_j = k$ for $1 \leq i \leq r$ and $r + 1 \leq j \leq n$.

Output: All connected graphs in $\mathcal{G}(n, m)$ for any given n and m, denoted by $\mathcal{G}^*(n, m)$.

1: Construct a complete n-partite graph $H = (P_1, P_2, \ldots, P_n)$, such that each P_i for $1 \leq i \leq r$ has $k + 1$ vertices and each P_j for $r + 1 \leq j \leq n$ has k vertices.
2: Find all perfect matchings in H, denoted by $\{M_1, M_2, \ldots, M_l\}$.
3: **Set** $\mathcal{G}^*(n, m) = \emptyset$ and $s = 1$.
4: **while** $s \leq l$ **do**
5: Construct a new graph G_s with vertex-set $\{p_1, p_2, \ldots, p_n\}$ and $p_i \sim p_j$ if and only if there is an edge between P_i and P_j in M_s.
6: **if** G_s is **connected** with no multiple edges and $G_s \ncong G$ for any $G \in \mathcal{G}^*(n, m)$ **then**
7: **Set** $\mathcal{G}^*(n, m) = \mathcal{G}^*(n, m) \bigcup \{G_s\}$.
8: **else** $\mathcal{G}^*(n, m) = \mathcal{G}^*(n, m)$.
9: **Set** $s = s + 1$ and go to 4.
10: **return** $\mathcal{G}^*(n, m)$.

□

Note that although this algorithm can be used to generate all connected graphs of $\mathcal{G}(n, m)$, it cannot guarantee the existence of any connected graph in $\mathcal{G}(n, m)$.

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