Edge-disjoint rainbow triangles in edge-colored graphs[1]

Luyi Li, Xueliang Li

Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

Email: liluyi@mail.nankai.edu.cn, lxl@nankai.edu.cn

Abstract

Let G be an edge-colored graph. A triangle of G is called *rainbow* if any two edges of the triangle have distinct colors. We use $m(G)$ and $c(G)$ to denote the number of edges of G and the number of colors appearing on the edges of G, respectively. Li et al. in 2014 proved that every edge-colored graph of order n with $m(G) + c(G) \geq n(n + 1)/2$ contains a rainbow triangle and this result is best possible. In 2019, Fujita et al. characterized all graphs G satisfying $m(G) + c(G) \geq n(n + 1)/2 - 1$ but containing no rainbow triangle. In this paper, we conjecture that every edge-colored graph of order n with $m(G) + c(G) \geq n(n + 1)/2 + 3(k - 1)$ contains k edge-disjoint rainbow triangles. We show that the conjecture holds for $k = 2$ and 3 and these results are best possible. Furthermore, we characterize all graphs G satisfying $m(G) + c(G) \geq n(n + 1)/2 + 2$ but not containing two edge-disjoint rainbow triangles. At the end, we propose a conjecture on the number of vertex-disjoint rainbow triangles in an edge-colored graph.

Keywords: edge-colored graph; rainbow triangle; edge-disjoint; vertex-disjoint.
AMS subject classification 2020: 05C15, 05C38, 05C07.

1 Introduction

We only consider simple graphs in this paper. For terminology and notation not defined here, we refer the reader to [2]. An *edge-colored graph* is a triple $G = (V(G), E(G), C)$, where $V(G)$ and $E(G)$ are the vertex-set and edge-set of G, respectively, and C is a mapping from $E(G)$ to a set $\mathbb{N}$ of colors, called an edge-coloring of G. In an edge-colored graph G, we use $C(e)$ to denote the color of an edge e and $C(G)$ to denote the set of colors of all the edges of G. For convenience, set $m(G) = |E(G)|$ and $c(G) = |C(G)|$. A

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subgraph of an edge-colored graph G is called *rainbow* if any two edges of the subgraph receive distinct colors. Let $CN_G(u)$ denote the set of colors on the edges incident with a vertex u in G, and $d^c_G(u) = |CN_G(u)|$. We use $CN_G(u)$ and $d^c_G(u)$ to denote the *color-neighborhood* and *color-degree* of a vertex u in G, respectively. When there is no confusion, we write $CN(u)$ and $d^c(u)$ instead of $CN_G(u)$ and $d^c_G(u)$, respectively. Let $\delta^c(G)$ denote the minimum value of $d^c(u)$ over all vertices u in G, called the *minimum color-degree* of the edge-colored graph G.

For two vertex-disjoint graphs G and H, we use $G \vee H$ to denote the new graph obtained by adding edges joining every vertex of G to all vertices of H. For a subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G - S$ to denote the induced subgraph $G[V(G) \setminus S]$. For any two distinct vertex subsets S and T in G, we use $E_G(S,T)$ (for short $E(S,T)$) to denote the edge subset of G such that the two ends of each edge of $E(S,T)$ are in S and T, respectively. Set $C(S,T) = \{C(e) : e \in E(S,T)\}$. If $S = \{v\}$, then we simply write $E(v,T)$ and $C(v,T)$ for $E(\{v\},T)$ and $C(\{v\},T)$, respectively.

In 1907, Mantel proved a classical result about the existence of triangle in graphs. This result says that every graph G of order n with $m(G) > \lfloor \frac{n^2}{4} \rfloor$ contains a triangle. Li et al. in 2014, showed a result on the existence of rainbow triangles in edge-colored graphs in [10]. This result can be viewed as the rainbow version of Mantel's theorem.

Theorem 1.1. *[10] Let G be an edge-colored graph on $n \geq 3$ vertices with $m(G) + c(G) \geq n(n+1)/2$. Then G contains a rainbow triangle.*

Furthermore, Fujita et al. in [8] generalized Theorem 1.1 by characterizing all graphs G satisfying $m(G) + c(G) \geq n(n+1)/2 - 1$ but containing no rainbow triangle. Next, we give a special graph class $\mathcal{G}_0$ and state their result.

Let $\mathcal{G}_0$ be the set of all edge-colored complete graphs that satisfy the following two properties:

- $K_1 \in \mathcal{G}_0$;
- For every $G \in \mathcal{G}_0$ of order $n \geq 2$ with $c(G) = n - 1$, there exists a bipartition $V(G) = V_1 \cup V_2$ such that $E(V_1, V_2)$ is monochromatic and $G[V_i] \in \mathcal{G}_0$ for $i = 1, 2$. Note that, for each $G \in \mathcal{G}_0$, G satisfies the condition $m(G) + c(G) = n(n+1)/2 - 1$ but contains no rainbow triangle.

Theorem 1.2. *[8] Let G be an edge-colored graph on $n \geq 3$ vertices with $m(G) + c(G) \geq n(n+1)/2 - 1$. If G contains no rainbow triangle, then G belongs to $\mathcal{G}_0$.*

Another significant result was proved by Erdős in [7] after Mantel' theorem in 1955: every graph G of order n with $m(G) \geq \lfloor \frac{n^2}{4} \rfloor + k$ contains at least $k \lfloor \frac{n}{2} \rfloor$ triangles for $k < \min\{4, n/2\}$. Recently, Ehard et al. [6] considered the number of rainbow triangles in edge-colored graphs and obtained the following result.

Theorem 1.3. *[6] Let k be a positive integer and G be an edge-colored graph on $n \geq 3k$ vertices with $m(G) + c(G) \geq n(n+1)/2 + k - 1$. Then G contains k rainbow triangles.*

They also found all edge-colored graphs on $n \geq 3k$ vertices with $m(G) + c(G) \geq n(n+1)/2 + k - 1$ that contain exactly k rainbow triangles. The following graph class $\mathcal{G}_1$ implies that the condition on Theorem 1.3 is best possible for $n \geq 3k$.

Let $\mathcal{G}_1$ be the set of all edge-colored complete graphs on $n \geq 3k$ vertices that are constructed recursively as follows:

• G_0 is the edge-colored complete graph with vertex-set $\{v_1, v_2, ..., v_{n-3k}\}$, where $C(v_i v_j) = i$ for all $v_i v_j \in E(G_0)$ if $i < j$;

• For $1 \leq i \leq k$, let K_3 be a rainbow triangle vertex-disjoint from G_{i-1}, and let $G_i = G_{i-1} \vee K_3$. Each of the colors of edges in K_3 does not belong to $C(G_{i-1})$, $E_{G_i}(G_{i-1}, K_3)$ is monochromatic and the color of $E_{G_i}(G_{i-1}, K_3)$ is neither used in G_{i-1} nor in K_3.

Notice that for each $G_k \in \mathcal{G}_1$, $m(G_k) + c(G_k) \geq \binom{n}{2} + k - 1$ and G_k contains exactly k rainbow triangles.

Theorem 1.4. *[6] Let G be an edge-colored graph on $n \geq 3k$ vertices. If $m(G) + c(G) \geq n(n+1)/2 + k - 1$ and G contains exactly k rainbow triangles, then $G \in \mathcal{G}_1$.*

In the past period of time, much work on the existence of rainbow triangles in edge-colored graphs has been done extensively. More results about this problem can be found in [1, 3, 4, 5]. Furthermore, one can find quite a few publications on the number of vertex-disjoint rainbow triangles in edge-colored graphs under the constraints of $\delta^c(G)$, $c(G)$ or color-neighborhood union; see [9, 11, 12, 13, 14, 15] for examples. Motivated by the above results, we consider the number of edge-disjoint rainbow triangles in an edge-colored graph G under the constraints of $m(G)$ and $c(G)$. We first pose the following conjecture.

Conjecture 1.5. *Let k be a positive integer and G be an edge-colored graph of $n \geq 4k$ vertices with $m(G)+c(G) \geq n(n+1)/2+3k-3$. Then G contains k edge-disjoint rainbow triangles.*

We confirm this conjecture for the cases $k \in \{2, 3\}$ with $n \geq k(k+4)$, and then we give a graph class $\mathcal{G}_2$ to show that the bound of conjecture 1.5 is sharp for $n \geq 4k$ if the conjecture holds.

Theorem 1.6. *Let G be an edge-colored graph on $n \geq 12$ vertices with $m(G) + c(G) \geq n(n+1)/2 + 3$. Then G contains two edge-disjoint rainbow triangles.*

Theorem 1.7. *Let G be an edge-colored graph on $n \geq 21$ vertices with $m(G) + c(G) \geq n(n+1)/2 + 6$. Then G contains three edge-disjoint rainbow triangles.*

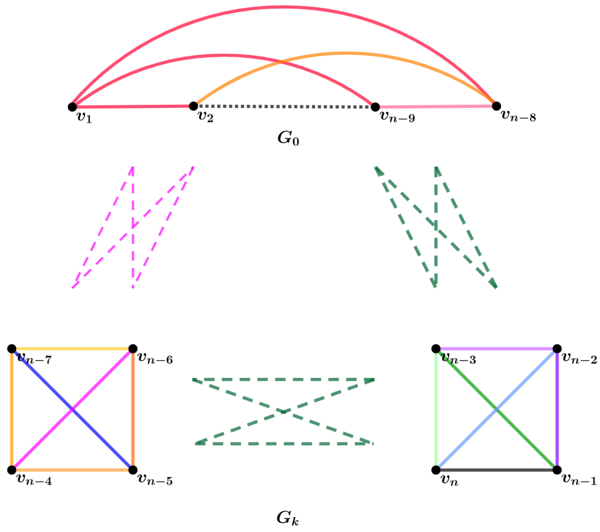

Figure 1: $G_k \in \mathcal{G}_2$ and G_k contains two edge-disjoint rainbow triangles when $k = 3$.

Let $\mathcal{G}_2$ be the set of all edge-colored complete graphs on $n \geq 4k$ vertices that are constructed recursively as follows:

- G_1 is the edge-colored complete graph with vertex-set $\{v_1, v_2, ..., v_{n-4k+4}\}$, where $C(v_iv_j) = i$ for all $v_iv_j \in E(G_1)$ if $i < j$;

- For $2 \leq i \leq k$, let K_4 be a rainbow complete graph vertex-disjoint from G_{i-1}, and let $G_i = G_{i-1} \vee K_4$. Each of the colors of edges in K_4 does not belong to $C(G_{i-1})$, $E_{G_i}(G_{i-1}, K_4)$ is monochromatic and the color of $E_{G_i}(G_{i-1}, K_4)$ is neither used in G_{i-1} nor in K_4.

Notice that for each $G_k \in \mathcal{G}_2$, $m(G_k) + c(G_k) = \binom{n}{2} + n - 4k + 4 - 1 + 7(k-1) = n(n+1)/2 + 3k - 4$. However, G_k contains exactly $k - 1$ edge-disjoint rainbow triangles. See Figure 1.

At the end of this section, we completely characterize an edge-colored graph G of order n with $m(G) + c(G) \geq n(n+1)/2 + 2$ but without two edge-disjoint rainbow triangles.

Theorem 1.8. *Let G be an edge-colored graph on $n \geq 12$ vertices with $m(G) + c(G) \geq n(n+1)/2 + 2$. Then G contains two edge-disjoint rainbow triangles or $G \in \mathcal{G}_3$.*

$\mathcal{G}_3$ is defined as the set of all edge-colored complete graphs on $n \geq 5$ vertices with the following structures:

- Let G_0 be the edge-colored complete graph in $\mathcal{G}_0$ (defined above) with vertex-set $\{v_1, v_2, ..., v_{n-3}\}$ and let K_4 be a rainbow complete graph with vertex-set $\{u_1, u_2, u_3, u_4\}$ such that $C(K_4) \cap C(G_0) = \emptyset$;

- G is an edge-colored complete graph obtained from G_0 by substituting K_4 for some vertex v_i of G_0 such that $C(u_jv_k) = C(v_iv_k)$ for all $1 \leq j \leq 4$ and every vertex $v_k \neq v_i$.

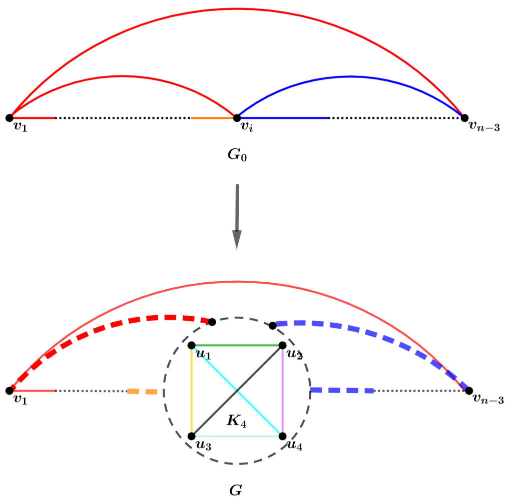

Figure 2: $G \in \mathcal{G}_3$ and G does not contains two edge-disjoint rainbow triangles.

Notice that for each $G \in \mathcal{G}_3$, $m(G) + c(G) = n(n+1)/2 + 2$. However, G contains no two edge-disjoint rainbow triangles. See Figure 2.

Because the proof of Theorem 1.7 is a little bit complicated, we give it an outline here. The proof of Theorem 1.7 goes by contradiction, which involves three main steps. In the first step, we know that G contains at least eight rainbow triangles by Theorems 1.3 and 1.4. Using Lemma 2.2 and the recoloring operation, we can deduce that G contains no rainbow clique of order five. Then it follows from Lemma 2.3 that there are two vertex-disjoint rainbow triangles, say $T_i = x_i y_i z_i x_i$ for $i = 1, 2$, in G. Further, by a short analysis for other six rainbow triangles, we can deduce that there are two copies of rainbow K_4, say $H_1 = G[\{x_1, y_1, z_1, u\}]$ and $H_2 = G[\{x_2, y_2, z_2, v\}]$. In the second step, we distinguish three cases according to the number of common vertices of H_1 and H_2. At first we obtain a new edge-colored graph G^* by the recoloring operation. By Lemma 2.2 again, one can see that this operation does not create new rainbow triangles and breaks all the rainbow triangles belonging to H_i for $i = 1, 2$. Then by Theorem 1.1 we can find a rainbow triangle T^* not belonging to H_i for $i = 1, 2$ in G^*. Finally, it is easy to find three edge-disjoint rainbow triangles that include T^* in G.

This paper is organized as follows: In Section 2, we first build up basic terminology and significant conditions. In Section 3, we give the proofs of Theorems 1.6, 1.7 and 1.8. At the last section, Section 4, we pose a conjecture and construct a special graph class to show that the bound of this conjecture is sharp if it holds.

2 Terminology and lemmas

At the beginning of this section, we give two sufficient conditions to guarantee the existence of three edge-disjoint rainbow triangles in an edge-colored graph. Furthermore, note that Theorem 1.6 will be used in the proof of Lemma 2.1. We will give the proof of Theorem 1.6 in Section 3 independent of Lemma 2.1.

Lemma 2.1. *Let G be an edge-colored graph on $n \geq 12$ vertices with $m(G) + c(G) \geq n(n+1)/2+6$. Suppose that G contains five distinct rainbow triangles T_i, $1 \leq i \leq 5$, that satisfy one of the following two conditions:*

(1) T_1, T_2, T_3, T_4 and T_5 contain a common edge in G;

(2) T_1, T_2, T_3 and T_4 contain a common edge and they are edge-disjoint from T_5.

Then G contains three edge-disjoint rainbow triangles.

Proof. First, assume that T_i satisfies the condition (1) for $i = 1, 2, 3, 4, 5$. Set $T_i = xyz_ix$ for $1 \leq i \leq 5$ and $G_0 = G - \{xy\}$. Note that $m(G_0) + c(G_0) \geq n(n + 1)/2 + 6 - 2 = n(n+1)/2+4$. From Theorem 1.6, there exist two edge-disjoint rainbow triangles T and T_0 in G_0. It is easily seen that there is at least one rainbow triangle T_i such that T_i is edge-disjoint from T and T_0 in G, where $1 \leq i \leq 5$.

Next, we suppose that T_i satisfies the condition (2) for $i = 1, 2, 3, 4, 5$ and G does not contain three edge-disjoint rainbow triangles. Assume that $T_i = xyz_ix$ for $1 \leq i \leq 4$, $T_5 = uvwu$ and $G_0 = G-\{xy\}$. Since $m(G_0)+c(G_0) \geq n(n+1)/2+6-2 = n(n+1)/2+4$, from Theorem 1.6 there exist two edge-disjoint rainbow triangles T' and T'' in G_0. To avoid that G contains three edge-disjoint rainbow triangles, $E(T') \cup E(T'')$ uses exactly one edge of T_i for some $1 \leq i \leq 4$. Without loss of generality, set $T' = xz_1z_2x$ and $T'' = xz_3z_4x$. If T_5 is edge-disjoint from T' and T'', then T_5, T' and T'' are three edge-disjoint rainbow triangles in G, a contradiction. Recall that T_5 is edge-disjoint from T_i for $1 \leq i \leq 4$. We may assume $T_5 = wz_1z_2w$. Then T'', T_2 and T_5 are three edge-disjoint rainbow triangles, a contradiction. The proof is thus complete. □

Next we introduce an important lemma of this paper, which will be used frequently in the following proofs. In this lemma, we mainly consider the number of times that each of the three colors of a rainbow triangle appears in G.

Lemma 2.2. *Let G be an edge-colored graph on $n \geq 21$ vertices with $m(G) + c(G) \geq n(n + 1)/2 + 6$. If G does not contain three edge-disjoint rainbow triangles, then each of the three colors in every rainbow triangle appears only once in G.*

Overview: Because the proof of this lemma is quite complicated, we first give it an outline. The proof goes by contradiction. At first we fix a rainbow triangle $T = xyzx$ in G and set $G_0 = G - E(T)$. Since $C(T) \cap C(G_0) \neq \emptyset$, we distinguish three cases according to the value of $|C(T) \cap C(G_0)|$ being 1,2,3, respectively. The cases of $|C(T) \cap C(G_0)|$ being 1,2 are easy to be settle done. Only he case of $|C(T) \cap C(G_0)|$ being 3 is hard to be fixed, for which we use three main steps. In the first step, we find two distinct vertices w_1 and w_2 such that w_1yzw_1 and w_2xzw_2 are two rainbow triangles different from T in G. In the second step, we show that neither $G_0[\{x, y, z, w_1, w_2\}]$ nor $G_1[\{x, y, z, w_1, w_2\}]$ contains an rainbow triangle, where $G_1 = G - \{xy, zw_1, zw_2\}$. This means that every rainbow triangle contains at least one edge of $E(T) \cup \{zw_1, zw_2\}$ in G. In the third step, by a short analysis we can find five rainbow triangles such that they satisfy the condition (2) of Lemma 2.1, which implies the existence of three edge-disjoint rainbow triangles in G.

Next we proceed to giving the details of our proof.

Proof of Lemma 2.2: Suppose not, choose an arbitrary rainbow triangle $T = xyzx$ in G with $C(T) \cap C(G_0) \neq \emptyset$, where $G_0 = G - E(T)$. Note that G_0 does not contain two edge-disjoint rainbow triangles. Next, we consider the following three cases, depending on the value of $|C(T) \cap C(G_0)|$.

Case 1. $|C(T) \cap C(G_0)| = 3$.

Note that $m(G_0)+c(G_0) \geq n(n+1)/2+6-3 = n(n+1)/2+3$. By Theorem 1.6, G_0 contains two edge-disjoint rainbow triangles. Hence, the two edge-disjoint rainbow triangles together with T compose three edge-disjoint rainbow triangles in G, a contradiction.

Case 2. $|C(T) \cap C(G_0)| = 2$.

Without loss of generality, set $C(T) \cap C(G_0) = \{C(xy), C(yz)\}$ and $G_1 = G - \{xy, xz\}$. Since $m(G_1) + c(G_1) \geq n(n+1)/2 + 6 - 3 = n(n+1)/2 + 3$, by Theorem 1.6 G_1 contains two edge-disjoint rainbow triangles. If neither of them contains yz, then these two edge-disjoint rainbow triangles together with T compose three edge-disjoint rainbow triangles in G, a contradiction. Hence, we assume that one of the two rainbow triangles contains yz, say w_1yzw_1. By the same discussions for $G - \{xy, yz\}$ and $G - \{xz, yz\}$, we can obtain two rainbow triangles w_2xzw_2 and w_3xyw_3 in G. If w_1, w_2 and w_3 are pairwise different, then w_1yzw_1, w_2xzw_2 and w_3xyw_3 are three edge-disjoint rainbow triangles in G, a contradiction.

Assume $w_1 = w_3 \neq w_2$. Note that $m(G_0)+c(G_0) \geq n(n+1)/2+6-4 = n(n+1)/2+2$. The fact that G_0 is not complete, together with Theorems 1.3 and 1.4, implies that G_0 contains at least four distinct rainbow triangles. It is clear that each rainbow triangle of G_0 uses at least one edge in $\{w_1y, w_2x, w_2z\}$. Otherwise, we can easily find a rainbow triangle that is edge-disjoint from w_1yzw_1(or w_3xyw_3) and w_2xzw_2 in G, a contradiction.

Let $H_0 = G_0[\{x, y, z, w_1, w_2\}]$. It can be seen that H_0 contains at most three triangles: xw_1w_2x, zw_1w_2z and yw_1w_2y; see Figure 3. Hence, there is at least one rainbow triangle T_1 of G_0 that does not belong to H_0. Obviously, T_1 uses exactly one edge e_1 of $\{w_1y, w_2x, w_2z\}$. In fact, regardless of $e_1 = w_2x, w_2z$ or w_1y, our discussion is similar. Hence, we only consider the case that $e_1 = w_2x$. Without loss of generality, suppose $T_1 = u_1xw_2u_1$.

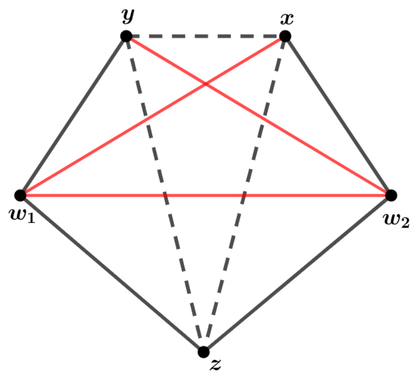

Figure 3: H_0 is an induced subgraph of G_0, where the black solid edges must belong to H_0, the black dotted edges do not belong to H_0 and the red solid edges possibly exist in H_0.

Recall that T and T_1 are two edge-disjoint rainbow triangles. If zw_1w_2z or yw_1w_2y is rainbow, then we can find three edge-disjoint rainbow triangles in G, a contradiction. Hence, H_0 contains at most one rainbow triangle, which implies that there are at least two other rainbow triangles T_2 and T_3 in G_0 that do not belong to H_0. Similarly, we assume that T_i contains $e_i \in \{w_1y, w_2x, w_2z\}$ for $i = 2, 3$. For $i = 2$ or 3, if $e_i = w_1y$, then T, T_1 and T_i are three edge-disjoint rainbow triangles in G, a contradiction. If $e_1 = e_2 = e_3 = w_2x$, then T_1, T_2, T_3, w_2xzw_2 contains a common edge w_2x and they are edge-disjoint from xyw_1x. It follows from the condition (2) of Lemma 2.1 that G contains three edge-disjoint rainbow triangles, a contradiction. If $e_2 = e_3 = zw_2$ or $\{e_2, e_3\} = \{w_2x, w_2z\}$, it is not difficult to find that there are three edge-disjoint rainbow triangles in G, a contradiction. Similarly, we can prove the cases that $w_1 \neq w_2 = w_3$ and $w_1 = w_2 \neq w_3$.

If $w_1 = w_2 = w_3$, set $G_1 = G - \{xw_1, yz\}$. Since $m(G_1) + c(G_1) \geq n(n+1)/2 + 6 - 3 = n(n+1)/2 + 3$, by Theorem 1.6 there are two edge-disjoint rainbow triangles T_1 and T_2 in G_1. To avoid that G contains three edge-disjoint rainbow triangles, one of the edges xy and xz is contained in T_1 or T_2. Without loss of generality, assume $T_1 = wxyw$. It is clear that $w \neq w_3$. Then we find two distinct vertices $w_1 = w_2$ and $w(\neq w_1)$ such that w_1yzw_1, w_1xzw_1 and $wxyw$ are three distinct rainbow triangles in G. Consequently, by a similar discussion with the case of $w_1 = w_3 \neq w_2$, we can find a contradiction.

Case 3. $|C(T) \cap C(G_0)| = 1$.

Assume $C(T) \cap C(G_0) = \{C(xy)\}$. At first, we prove the following claims.

Claim 1. There are two distinct vertices w_1 and w_2 such that w_1yzw_1 and w_2xzw_2 are two rainbow triangles different from T in G.

Proof. If not, by a similar argument for $G - \{xy, xz\}$ and $G - \{xy, yz\}$ in Case 2, we can find two rainbow triangles u_1yzu_1 and u_2xzu_2 distinct from T in G. Clearly, $u_1 = u_2$. Set $G_1 = G - \{u_1z, xy\}$. Since $m(G_1) + c(G_1) \geq n(n+1)/2 + 3$, by Theorem 1.6 G_1 contains two edge-disjoint rainbow triangles T_1 and T_2. If $E(T_1) \cap E(T_2)$ contains no xz or yz, then T_1, T_2 and T are edge-disjoint rainbow triangles in G, a contradiction. Without loss of generality, we suppose that T_1 contains xz in G_1, say $T_1 = u_3xzu_3$. Clearly, u_1yzu_1 and u_3xzu_3 are rainbow triangles in G and $u_3 \neq u_1$, a contradiction. The claim thus follows. □

Choose two vertices w_1 and w_2 that satisfy Claim 1 in G. Let $G_1 = G - \{xy, zw_1, zw_2\}$. Since $m(G_1) + c(G_1) \geq n(n+1)/2 + 6 - 5 = n(n+1)/2 + 1$ and G_1 is not complete, by Theorems 1.3 and 1.4 we can find three rainbow distinct triangles in G_1. For convenience, set $H_1 = G_1[\{x, y, z, w_1, w_2\}]$; see Figure 4. Note that H_1 contains at most two triangles yw_1w_2y and xw_1w_2x. Hence, there is at least one rainbow triangle that does not belong to H_1. In fact, every rainbow triangle not belonging to H_1 must contain an edge in $\{w_2x, xz, zy, yw_1\}$. Otherwise, this rainbow triangle, and xzw_2x and yzw_1y are three edge-disjoint rainbow triangles, a contradiction.

Claim 2. H_1 contains no rainbow triangle.

Proof. Suppose to the contrary that H_1 contains at least one rainbow triangle. choose a rainbow triangle T_1 that does not belong to H_1, and set $V(T_1) - V(H_1) = \{u_1\}$. Then T_1 contains an edge e_1 of $\{w_2x, xz, zy, yw_1\}$. By symmetry, we only need to consider the cases $e_1 \in \{w_2x, xz\}$. If H_1 contains two rainbow triangles, then both yw_1w_2y and xw_1w_2x are rainbow. If $e_1 = w_2x$, then T, T_1 and yw_1w_2y are three edge-disjoint rainbow triangles, a contradiction. If $e_1 = xz$, then T_1, yzw_1y and xw_1w_2x are three edge-disjoint rainbow triangles, a contradiction.

If H_1 contains only one rainbow triangle, without loss of generality, assume that yw_1w_2y is a rainbow triangle. Thus, there is another rainbow triangle T_2 that does not belong to H_1. Similarly, T_2 contains an edge e_2 of $\{w_2x, xz, zy, yw_1\}$ and suppose that $V(T_2) - V(H_1) = \{u_2\}$. If $e_i = xw_2(yz)$, then yw_1w_2y, T_i and $T(xzw_2x)$ are three edge-disjoint rainbow triangles for $i = 1, 2$, a contradiction. If $e_1 = e_2 = xz$, then T, T_1, T_2, xzw_2x and yw_1w_2y satisfy the condition (2) of Lemma 2.1. Then G contains three edge-disjoint rainbow triangles, a contradiction. Similarly, we can find a contradiction when $e_1 = e_2 = yw_1$. Hence, we have $\{e_1, e_2\} = \{xz, yw_1\}$. By a similar analysis for $G - \{yw_1, xz\}$, we can always deduce a contradiction, and the claim thus follows. □

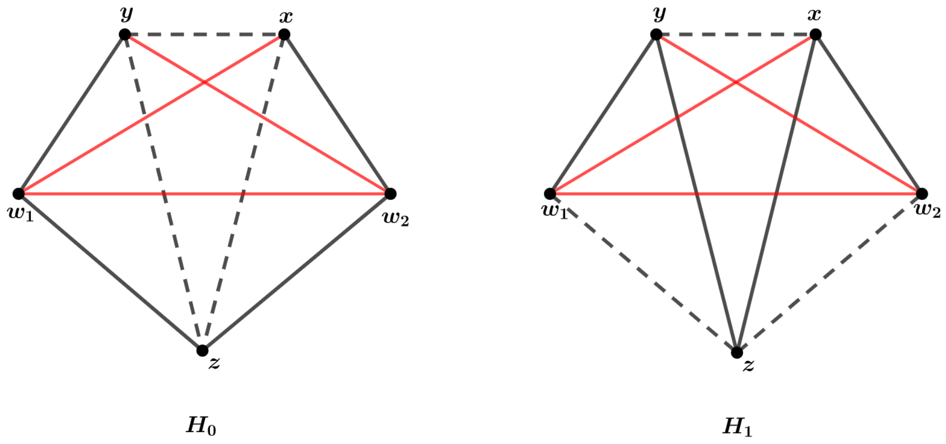

Figure 4: H_i is an induced subgraph of G_i, where the black solid edges must belong to H_i, the black dotted edges do not belong to H_i and the red solid edges possibly exist in H_i for $i = 1, 2$.

Assume that T_1, T_2 and T_3 are three rainbow triangles that does not belong to H_1 in G_1. Let $e_i \in E(T_i) \cap \{w_2x, xz, zy, yw_1\}$ and $V(T_i) - V(H_1) = \{u_i\}$ for $1 \leq i \leq 3$.

Recall that $G_0 = G - \{xy, yz, xz\}$ and $m(G_0) + c(G_0) \geq n(n+1)/2 + 1$. Since G_0 is not complete, by Theorems 1.3 and 1.4 we can find three distinct rainbow triangles in G_0. Set $H_0 = G_0[\{x, y, z, w_1, w_2\}]$; see Figure 4. Note that H_0 contains at most three triangles yw_1w_2y, xw_1w_2x and zw_1w_2z. If yw_1w_2y or xw_1w_2x is a triangle in H_0, then it also belongs to H_1. From Claim 1, we know that neither yw_1w_2y nor xw_1w_2x is a rainbow triangle. Hence, H_0 contains at most one rainbow triangle. In fact, every rainbow triangle not belonging to H_0 must contain one edge in $\{xw_2, zw_2, zw_1, yw_1\}$. Otherwise, this rainbow triangle, and xzw_2x and yzw_1y are three edge-disjoint rainbow triangles, a contradiction.

Claim 3. H_0 contains no rainbow triangle.

Proof. Suppose that H_0 contains exactly one rainbow triangle. This implies that zw_1w_2z is the unique rainbow triangle in H_0 and there are at least two rainbow triangles T_1^* and T_2^* that do not belong to H_0. Similarly, we know that T_i^* contains exactly one edge e_i^* from $\{xw_2, zw_2, zw_1, yw_1\}$, and set $V(T_i^*) - V(H_0) = \{v_i\}$ for $i = 1, 2$. If $e_i^* = xw_2$(or yw_1), then T, T_i^* and zw_1w_2z are three edge-disjoint rainbow triangles, a contradiction. If $e_1^* = e_2^* = zw_2(zw_1)$, we can find five rainbow triangles in G by considering $G - \{zw_2, yz\}$, and the details are omitted. Similarly, we can prove the case that $e_1^* = e_2^* = zw_1$. If $\{e_1^*, e_2^*\} = \{zw_1, zw_2\}$ and $v_1 \neq v_2$, then T, T_1^* and T_2^* are three edge-disjoint rainbow triangles in G, a contradiction. Hence, we have that $\{e_1^*, e_2^*\} = \{zw_1, zw_2\}$ and $v_1 = v_2$.

Without loss of generality, set $T_1^* = zv_1w_2z$ and $T_2^* = zw_1v_1z$. Now we go back to the graph H_1 and T_i for $i = 1, 2, 3$. If $e_i = xw_2$, then T, T_i and T_2^* are three edge-disjoint rainbow triangles in G for $i = 1, 2, 3$, a contradiction. By symmetry, we have $e_i \neq yw_1$ for $i = 1, 2, 3$, which means $\{e_1, e_2, e_3\} \subseteq \{xz, yz\}$. We assume, without loss of generality, that $e_1 = e_2 = xz$. Then we can observe that $u_1 \neq u_2$. That implies that there is a vertex, say u_1, such that $u_1 \neq v_1$. Thus, T_1, T_1^* and yzw_1y are three edge-disjoint rainbow triangles in G, a contradiction. □

After the three claims, we come back to our proof. Assume that T_1^*, T_2^* and T_3^* are three rainbow triangles that do not belong to H_0 in G_0. Let $e_i^* \in E(T_i^*) \cap \{xw_2, zw_2, zw_1, yw_1\}$ and $V(T_i^*) - V(H_0) = \{v_i\}$ for $1 \leq i \leq 3$.

We assert that $e_1^* = e_2^* = e_3^*$. If not, then we can always find two edge-disjoint rainbow triangles T_i^* and T_j^* in $\{T_1^*, T_2^*, T_3^*\}$. Thus, T, T_i^* and T_j^* are three edge-disjoint rainbow triangles in G, a contradiction.

Without loss of generality, set $e_1^* = e_2^* = e_3^* = xw_2$. Then T_1^*, T_2^*, T_3^*, xzw_2x and yzw_1y always satisfy the condition (2) of Lemma 2.1. Then, we can find three edge-disjoint rainbow triangles in G, which yields a contradiction.

Combining with the above three cases, the proof of Lemma 2.2 is now complete. □

At the end of this section, we will give a lemma on the existence of a rainbow K_5 or two vertex-disjoint rainbow triangles under the conditions that G does not contain three edge-disjoint rainbow triangles and $m(G) + c(G) \geq n(n+1)/2 + 6$, which will be used in the proof of Theorem 1.7.

Lemma 2.3. *Let G be an edge-colored graph on $n \geq 21$ vertices with $m(G) + c(G) \geq n(n+1)/2+6$. If G does not contain three edge-disjoint rainbow triangles, then G contains a rainbow K_5 or two vertex-disjoint rainbow triangles.*

Proof. Suppose that G contains no rainbow K_5 and two vertex-disjoint rainbow triangles. From Theorem 1.3, we know that G contains at least seven rainbow triangles. If G contains exactly seven rainbow triangles, then from Theorem 1.4 we have $G \in \mathcal{G}_1$. That implies that G contains three edge-disjoint rainbow triangles, a contradiction. Hence, we assume that G contains at least eight rainbow triangles $\{T_i, 1 \leq i \leq 8\}$.

The condition that G does not contain three edge-disjoint rainbow triangles implies that any five rainbow triangles in $\{T_i, 1 \leq i \leq 8\}$ do not satisfy the condition (1) of Lemma 2.1. Hence, there must exist two edge-disjoint rainbow triangles, say T_1 and T_2. It is clear that $V(T_1) \cap V(T_2) \neq \emptyset$. Next, assume that $T_1 = xyzx$, $T_2 = xuvx$ and $H = G[\{x, y, z, u, v\}]$. We first show the following claim.

Claim. H is a complete graph.

Proof. By symmetry, we only need to prove $uy \in E(G)$. If not, then H contains at most seven triangles. Hence, there is at least one rainbow triangle T_3 such that $V(T_3) \nsubseteq V(H)$. The condition that G does not contain three edge-disjoint rainbow triangles implies that T_3 contains exactly one edge e_1 from $\{xy, yz, xz, xu, uv, xv\}$. Let $V(T_3) - V(H) = \{w_1\}$.

If $e_1 = uv(yz)$, then T_1 (T_2) and T_3 are vertex-disjoint, a contradiction. Then $e_1 \in \{xy, xz, xu, xv\}$. By symmetry, we only consider the case of $e_1 = xv$.

To avoid that T_1, T_3 and $uzvu$ are three edge-disjoint rainbow triangles, $uvzu$ must not be rainbow. Hence, there is a rainbow triangle T_4 such that $T_4 \neq T_3$ and $V(T_4) \nsubseteq V(H)$. By the same discussion for T_4, we know that T_4 contains exactly one edge e_2 from H with $e_2 \in \{xy, xz, xu, xv\}$. Let $V(T_4) - V(H) = \{w_2\}$. If $e_2 = xu$ and $w_1 \neq w_2$, then T_1, T_3 and T_4 are three edge-disjoint rainbow triangles, a contradiction. If $e_2 = xu$ and $w_1 = w_2$, from Lemma 2.2 we have that $G[\{u, v, x, w_1\}]$ is rainbow. Hence, uvw_1u is a rainbow triangle vertex-disjoint from T_1, a contradiction. If $e_2 = xz$ and $w_1 = w_2$, then $zv \notin E(G)$. Otherwise, by Lemma 2.2 we have that $G[\{z, v, x, w_1\}]$ is rainbow, which implies that T_1, T_2 and zvw_1z are three edge-disjoint rainbow triangles, a contradiction. Similarly, we can deduce a contradiction when $e_2 = xy$ and $w_1 = w_2$. Since $T_3 \neq T_4$, we have $w_1 \neq w_2$ when $e_2 = xv$. Hence, we have $e_2 \in \{xv, xz, xy\}$ and $w_1 \neq w_2$.

If $e_2 = xz$ and $w_1 \neq w_2$, to avoid that T_3, T_4 and $uvzu$ are three edge-disjoint rainbow triangles in G, then $uvzu$ must not be rainbow. Hence, there is a rainbow triangle T_5 such that $T_5 \notin \{T_3, T_4\}$ and $V(T_5) \nsubseteq V(H)$. Let $V(T_5) - V(H) = \{w_3\}$. By a similar argument, we know that T_5 contains exactly one edge e_3 from H, $w_3 \notin \{w_1, w_2\}$ and $e_3 \in \{xv, xz, xy\}$. If $e_3 = xy$, then T_3, T_4 and T_5 are three edge-disjoint rainbow triangles, a contradiction. Hence, $e_3 \in \{xv, xz\}$ and $w_3 \notin \{w_1, w_2\}$. By symmetry, suppose $e_3 = xz$. We assert that $uzxu$ is not rainbow. If not, then T_1, T_4, T_5, $uzxu$ and T_3 satisfy the condition (2) of Lemma 2.1. This means that G contains three edge-disjoint rainbow triangles, a contradiction. Hence, there is a rainbow triangle T_6 such that $T_6 \notin \{T_3, T_4, T_5\}$ and $V(T_6) \nsubseteq V(H)$. Let $V(T_6) - V(H) = \{w_4\}$. It is clear that T_6 contains exactly one edge e_4 from H, $w_4 \notin \{w_1, w_2, w_3\}$ and $e_4 \in \{vx, zx\}$. If $e_4 = xz$, then we can deduce a contradiction from the rainbow triangles T_1, T_4, T_5, T_6 and T_3 and Lemma 2.1. If $e_4 = xv$, we can also deduce a contradiction by repeating the above analyses.

If $e_2 \in \{xv, xy\}$ and $w_1 \neq w_2$, by a similar discussion we can always find five rainbow triangles such that they satisfy the condition (2) of Lemma 2.1. Consequently, G contains three edge-disjoint rainbow triangles, a contradiction. The claim thus follows. $\square$

After the claim we come back to the proof. Since H is complete, and $yx \in T_1$ and $ux \in T_2$, from Lemma 2.2 both $C(yx)$ and $C(ux)$ appear only once in G. Then, $xyux$ is also a rainbow triangle. By repeat use of Lemma 2.2, we can show that H is a rainbow complete graph, a contradiction. The proof of the lemma is thus complete. $\square$

3 Proofs of our main results

After the above preparations, we are ready to give the proofs of our main results.

Proof of Theorem 1.6: Suppose to the contrary that G does not contain two edge-disjoint rainbow triangles. From Theorem 1.3, G contains at least four distinct rainbow triangles. Since G does not contain two edge-disjoint rainbow triangles, any two of these rainbow triangles have one common edge. Hence, we need to consider the following two cases:

Case 1. There are three rainbow triangles with a common edge in G.

Assume that the common edge is xy, and $T_i = xyv_ix$, $i = 1, 2, 3$, are three distinct rainbow triangles in G. Set $G_0 = G - \{xy\}$. It can be seen that $m(G_0) + c(G_0) \geq n(n+1)/2+3-2 = n(n+1)/2+1$. Thus, by Theorem 1.1 G_0 contains a rainbow triangle T_0. Clearly, T_0 must be edge-disjoint from one of $\{T_1, T_2, T_3\}$ in G.

Case 2. Any three rainbow triangles have no common edges in G.

Without loss of generality, suppose that $T_1 = xyzx$, $T_2 = xzwx$ and $T_3 = xywx$ are three distinct rainbow triangles in G. Assume that $G_0 = G - \{xy, wz\}$. Thus, $m(G_0) + c(G_0) \geq n(n + 1)/2 + 3 - 4 = n(n + 1)/2 - 1$. Note that G_0 is not a complete graph, by Theorem 1.2 G_0 contains a rainbow triangle T_0. It is clear that T_0 must be edge-disjoint from at least one of $\{T_1, T_2, T_3\}$ in G.

Combining the above two cases, the proof is thus complete. □

To prove Theorem 1.8, we first recall an important operation on graphs. Let G be a graph and X be a proper subset of the vertex-set of G. To *shrink* X is to delete all edges between the vertices of X and then identify the vertices of X into a single vertex. The resulting graph is denoted by G/X.

Proof of Theorem 1.8: Suppose that the statement is false. From Theorem 1.1, we know that G contains a rainbow triangle. For each rainbow triangle of G, we have the following claim.

Claim 1. Each of the three colors in every rainbow triangle appears only once in G.

Proof. Choose an arbitrary rainbow triangle $T = xyzx$ in G and set $G_0 = G - \{xy, yz, zx\}$. The hypothesis implies that G_0 contains no rainbow triangles. Suppose to the contrary, that there is a color $C(xy)$ in T that appears at least twice in G. This means that $C(xy) \in C(T) \cap C(G_0)$. Let $G_1 = G - \{xy, yz\}$. Notice that $m(G_1) + c(G_1) \geq m(G) - 2 + c(G) - 1 \geq n(n + 1)/2 - 1$. Since G_1 is not a complete graph, by Theorem 1.2 G_1

contains a rainbow triangle T_1. Obviously, T_1 contains xz; since otherwise, G contains two edge-disjoint rainbow triangles T and T_1, a contradiction. Assume that $T_1 = xzwx$ and $G_2 = G - \{xz\}$. Noticing that $m(G_2) + c(G_2) \geq m(G) - 1 + c(G) - 1 \geq n(n+1)/2$, by Theorem 1.1 G_2 contains a rainbow triangle T_2.

We assert that $V(T_2) \subseteq \{x, y, z, w\}$. If not, T_2 is a rainbow triangle edge-disjoint from T or T_1 in G, a contradiction. Thus, T_2 has two choices in G_2, i.e. $T_2 = xywx$ or $T_2 = ywzy$. Without loss of generality, set $T_2 = xywx$ and $G_3 = G - \{xy, wz\}$. Recall that $C(xy) \in C(T) \cap C(G_0)$. Then, $m(G_3)+c(G_3) \geq m(G)-2+c(G)-1 \geq n(n+1)/2-1$. Since G_3 is not a complete graph, by Theorem 1.2 again G_3 contains a rainbow triangle T_3. It can be easily seen that T_3 must be edge-disjoint from one of the rainbow triangles from $\{T, T_1, T_2\}$, a contradiction. The claim thus follows. □

We need to show more claims before proceeding our proof.

Claim 2. G contains a rainbow clique of order four.

Proof. From Theorem 1.3, we may assume that T_1, T_2 and T_3 are three distinct rainbow triangles in G. If T_1, T_2 and T_3 contain a common edge e in G, let $G_0 = G - \{e\}$. Note that $m(G_0) + c(G_0) \geq n(n+1)/2 + 2 - 2 = n(n+1)/2$. Hence, by Theorem 1.1 G_0 contains a rainbow triangle T_4, and it is not difficult to see that T_4 must be edge-disjoint from one of the triangles from $\{T_1, T_2, T_3\}$, a contradiction.

Hence, we may assume that $T_1 = xyzx$, $T_2 = xzwx$ and $T_3 = xywx$. It is clear that $G[\{x, y, z, w\}]$ is complete. Note that each edge of $G[\{x, y, z, w\}]$ is contained in a rainbow triangle. From Claim 1, $G[\{x, y, z, w\}]$ is a rainbow complete graph, and the claim thus follows. □

Next, let $K_4 = G[\{x, y, z, w\}]$ be a rainbow complete graph, and let G^* be the edge-colored graph obtained from G by recoloring xz, yw and yz with $C(xw)$.

Claim 3. G^* does not contain rainbow triangles and $G^* \in \mathcal{G}_0$.

Proof. From Claim 1, one can see that each color of K_4 appears only one time in G and this operation of recoloring does not create new rainbow triangles and breaks all rainbow triangles belonging to K_4. Note that each rainbow triangle of G^* is also rainbow in G. Suppose that G^* contains a rainbow triangle. Then G contains a rainbow triangle not belonging to K_4, which implies that we can find two edge-disjoint rainbow triangles in G, a contradiction. Hence, H contains no rainbow triangles. Note that $m(G^*) + c(G^*) \geq m(G) + c(G) - 3 = n(n+1) - 1$. From Theorem 1.2, we have $G^* \in \mathcal{G}_0$, and the claim thus follows. □

It follows from $G^* \in \mathcal{G}_0$ that G^* is complete. Note that the operation of recoloring does not change any edge of G. Then G is also complete.

Claim 4. $C(v, K_4)$ is monochromatic for each vertex $v \in G - K_4$.

Proof. Choose an arbitrary $v \in G - K_4$. Recalling that G is complete, without loss of generality, suppose $C(vx) \neq C(vy)$. By Claim 1, we know that $vxyv$ is a rainbow triangle, which implies that $vxyv$ and $xzwx$ are two edge-disjoint rainbow triangles in G, a contradiction. □

After the claims, we come back to our proof. From Claim 2, take a rainbow K_4 from G. Let H be the edge-colored graph obtained from G by shrinking $V(K_4)$ to a vertex u such that $C(vu) = C(v, K_4)$ for each vertex v in $G - K_4$. It is not difficult to see that if G^* contains no rainbow triangles, then H contains no rainbow triangles. From Claims 1 and 4, we have $C(H) = C(G) \setminus C(K_4)$. Note that H is complete and $|V(H)| = n - 3$. Then, $m(H) + c(H) = m(G) + c(G) - 6 - 6 - 3(n - 4) \geq (n - 2)(n - 3)/2 - 1$. From Theorem 1.2, we know that $H \in \mathcal{G}_0$, which implies that $G \in \mathcal{G}_3$, a contradiction. The proof is now complete. □

Proof of Theorem 1.7: Suppose to the contrary, that G does not contain three edge-disjoint rainbow triangles. From Theorem 1.3, we know that G contains at least seven rainbow triangles. If G contains exactly seven rainbow triangles, then from Theorem 1.4 we have $G \in \mathcal{G}_1$, which implies that G contains three edge-disjoint rainbow triangles, a contradiction. Hence, we suppose that G contains at least eight rainbow triangles $\{T_i, 1 \leq i \leq 8\}$.

Claim. G contains no rainbow clique of order five.

Proof. Suppose to the contrary that $G[V_0]$ is a rainbow K_5 with $V_0 = \{x, y, z, u, v\}$. Let G^* be an edge-colored graph obtained from G by recoloring xy, xz, xu, xv and zv with $C(yu)$. From Lemma 2.2, one can see that this operation does not create new rainbow triangles and breaks all rainbow triangles belonging to $G[V_0]$. Since $m(G^*) + c(G^*) = m(G) + c(G) - 5 \geq n(n+1)/2 + 1$, by Theorem 1.1 G^* contains a rainbow triangle T^* not belonging to $G^*[V_0]$. It is not difficult to see that each rainbow triangle of G^* corresponds to a rainbow triangle of G. Thus, T^* is also a rainbow triangle not belonging to $G[V_0]$ in G. Then, we can easily find three edge-disjoint rainbow triangles that include T^* in G, a contradiction. □

From the above Claim and Lemma 2.3, there are two vertex-disjoint rainbow triangles, say $T_i = x_i y_i z_i x_i$ for $i = 1, 2$. To avoid that G contains three rainbow edge-disjoint triangles, each T_i contains at least one edge of $E(T_1) \cup E(T_2)$ for $3 \leq i \leq 8$. It follows from the fact that T_1 and T_2 have no common vertices that each T_i can not simultaneously contain an edge of T_1 and an edge of T_2 for $3 \leq i \leq 8$. If there are four rainbow triangles T_{i_1}, T_{i_2}, T_{i_3} and T_{i_4} in $\{T_i, 3 \leq i \leq 8\}$ such that T_{i_j} contains an edge of T_1 for

$1 \leq j \leq 4$, then it is not difficult to see that there must exist two edge-disjoint rainbow triangles T' and T'' in $\{T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}\}$. Since T' and T'' are edge-disjoint from T_2, we get three edge-disjoint rainbow triangles in G, a contradiction. Hence, for the remaining six rainbow triangles $\{T_i, 3 \leq i \leq 8\}$, we suppose that T_i contains one edge of T_1 for $i = 3, 4, 5$ and T_j contains one edge of T_2 for $j = 6, 7, 8$. For convenience, we suppose that $V(T_i) - V(T_1) = \{u_i\}$ and $V(T_j) - V(T_2) = \{v_j\}$ for $i = 3, 4, 5$ and $j = 6, 7, 8$. If $u_i \neq u_j$ for $3 \leq i \neq j \leq 5$, then T_i, T_j and T_2 are three edge-disjoint rainbow triangles in G, a contradiction. Then, $u_3 = u_4 = u_5$. Similarly, we can get $v_6 = v_7 = v_8$. Set $u = u_3$ and $v = v_6$. Thus, one can see that both $G[\{x_1, y_1, z_1, u\}]$ and $G[\{x_2, y_2, z_2, v\}]$ are rainbow copy of K_4.

Let $H_1 = G[\{x_1, y_1, z_1, u\}]$ and $H_2 = G[\{x_2, y_2, z_2, v\}]$. From Lemma 2.2, each color of H_1 and H_2 appears only one time in G. Note that H_1 and H_2 have at most two common vertices. If H_1 and H_2 have two common vertices, without loss of generality, set $u = x_2$ and $x_1 = v$. Let $G_1 = G - \{x_1u, y_1z_1, y_2z_2\}$. Note that this operation breaks triangles T_i for $1 \leq i \leq 8$. Since $m(G_1) + c(G_1) \geq n(n+1)/2$, by Theorem 1.1 G_1 contains a rainbow triangle T' not belonging to H_1 or H_2. One can easily get three edge-disjoint rainbow triangles in G, a contradiction.

If H_1 and H_2 have only one common vertex, set $u = v$. We assert that $E_G(T_1, T_2) = \emptyset$. In fact, by symmetry we only consider the case $x_1x_2 \notin E(G)$. If not, from Lemma 2.2 we know that $C(ux_i)$ appears only once in G for $i = 1, 2$. Hence, ux_1x_2u is a rainbow triangle edge-disjoint from T_1 and T_2 in G, a contradiction.

Let G^* be the edge-colored graph obtained from G by recoloring x_1y_1, y_1z_1 and z_1u with $C(ux_1)$ and recoloring x_2y_2, y_2z_2 and z_2u with $C(ux_2)$. From Lemma 2.2, one can see that this operation does not create new rainbow triangles and breaks all the rainbow triangles belonging to H_i for $i = 1, 2$. Since $m(G^*) + c(G^*) = m(G) + c(G) - 6 \geq n(n+1)/2$, by Theorem 1.1 G^* contains a rainbow triangle T^* not belonging to H_i for $i = 1, 2$. It is easy to find three edge-disjoint rainbow triangles that include T^* in G, a contradiction.

The proof for the case that H_1 and H_2 have no common vertices is similar to the above discussions, and the details are omitted. The proof is now complete. □

4 Concluding remarks

At the end of the paper, we pose a conjecture about the number of vertex-disjoint rainbow triangles in an edge-colored graph G under the constraints of $m(G)$ and $c(G)$. We also construct a graph class $\mathcal{G}_4$ to show that the bound in the conjecture is sharp for $n \geq 5k$ if it holds.

Conjecture 4.1. *Let G be an edge-colored graph on $n \geq 5k$ vertices with $m(G) + c(G) \geq n(n+1)/2 + 6k - 6$. Then G contains k vertex-disjoint rainbow triangles.*

Let $\mathcal{G}_4$ be the set of all edge-colored complete graphs on $n \geq 5k$ vertices that are constructed recursively as follows:

• G_0 is the edge-colored complete graph with vertex-set $\{v_1, v_2, ..., v_{n-5k+5}\}$, where $C(v_iv_j) = i$ for all $v_iv_j \in E(G_0)$ if $i < j$;

• For $1 \leq i \leq k-1$, let K_5 be a rainbow complete graph vertex-disjoint from G_{i-1}, and let $G_i = G_{i-1} \vee K_5$. Each of the colors of edges in K_5 does not belong to $C(G_{i-1})$, $E_{G_i}(G_{i-1}, K_5)$ is monochromatic and the color of $E_{G_i}(G_{i-1}, K_5)$ is neither used in G_{i-1} nor in K_5.

Notice that for each $G_k \in \mathcal{G}_4$, $m(G_k) + c(G_k) \geq \binom{n}{2} + n - 5k + 5 - 1 + 11(k-1) = n(n+1)/2 + 6k - 7$ and G_k contains exactly $k-1$ vertex-disjoint rainbow triangles.

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