Note on rainbow triangles in edge-colored graphs

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Abstract

Let G be a graph with an edge-coloring c, and let $\delta^c(G)$ denote the minimum color-degree of G. A subgraph of G is called rainbow if any two edges of the subgraph have distinct colors. In this paper, we consider color-degree conditions for the existence of rainbow triangles in edge-colored graphs. At first, we give a new proof for characterizing all extremal graphs G with $\delta^c(G) \geq \frac{n}{2}$ that do not contain rainbow triangles, a known result due to Li, Ning, Xu and Zhang. Then, we characterize all complete graphs G without rainbow triangles under the condition $\delta^c(G) = log_2 n$, extending a result due to Li, Fujita and Zhang. Hu, Li and Yang showed that G contains two vertex-disjoint rainbow triangles if $\delta^c(G) \geq \frac{n+2}{2}$ when $n \geq 20$. We slightly refine their result by showing that the result also holds for $n \geq 6$, filling the gap of n from 6 to 20. Finally, we prove that if $\delta^c(G) \geq \frac{n+k}{2}$ then every vertex of an edge-colored complete graph G is contained in at least k rainbow triangles, generalizing a result due to Fujita and Magnant. At the end, we mention some open problems.

Keywords: edge-coloring; edge-colored complete graph; rainbow triangle; color-degree condition

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1 Introduction

An edge-coloring of a graph G is a mapping $c : E(G) \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. A graph G is called an *edge-colored graph* if G is assigned an edge-coloring.

A subset F of edges of G is called *rainbow* if no pair of edges in F receive the same color, and a subgraph of G is called *rainbow* if the edge-set of the subgraph is rainbow.

The notions of "color-degree" and "minimum color-degree" appeared in [3]. Let G be an edge-colored graph. For a vertex $v \in V(G)$, the *color-degree* of v in G is the number of distinct colors assigned to the edges incident to v, denoted by $d^c_G(v)$. We use $\delta^c(G) := \min\{d^c_G(v) : v \in V(G)\}$ to denote the *minimum color-degree* of G.

In this paper, we will consider the existence of rainbow triangles in edge-colored graphs under color-degree conditions. This topic has received much attention recent years. For related references and the recent development, we refer the reader to surveys [8, 11] and the references [4, 6, 15].

A starting point of extremal graph theory might be the Mantel's theorem, which states that every graph G on n vertices contains a triangle if $e(G) > \frac{n^2}{4}$. Although it can be proved directly, an obvious but well-known corollary of Mantel's theorem is that the conclusion still holds under the weaker condition that "the minimum degree $\delta(G) > \frac{n+1}{2}$". In 2012, Li and Wang [14] considered the corresponding color-degree version, i.e., a color-degree condition for the existence of rainbow triangles in an edge-colored graph. They proved that every edge-colored graph G on n vertices contains a rainbow triangle if $\delta^c(G) \geq \frac{(\sqrt{7}+1)n}{6}$. Furthermore, they conjectured that the color-degree condition can be weakened to $\delta^c(G) \geq \frac{n+1}{2}$, and if true, it is best possible. This conjecture was confirmed by Li [13] in 2013. Aiming to attack Li and Wang's conjecture, independently, Li, Ning, Xu and Zhang [12] proved two stronger results in 2014 by completely different methods. We list these results as follows.

Theorem 1 ([13]). *Let G be an edge-colored graph of order n. If $\delta^c(G) \geq \frac{n+1}{2}$, then G contains a rainbow triangle.*

Theorem 2 ([12]). *Let G be an edge-colored graph of order n. If $\sum_{v\in V(G)} d^c_G(v) \geq \frac{n(n+1)}{2}$, then G contains a rainbow triangle.*

Theorem 3 ([12]). *Let G be an edge-colored graph of order n. If $\delta^c(G) \geq \frac{n}{2}$ and G contains no rainbow triangles, then n is even and G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, unless $G = K_4 - e$ or K_4 when $n = 4$.*

We will give Theorem 3 a new proof in next section. We also study the existence of rainbow triangles in an edge-colored complete graph. In this direction, Fujita, Li and Zhang [6] obtained the following result.

Theorem 4 ([6]). *Let G be an edge-colored complete graph of order n. If $\delta^c(G) > log_2 n$, then G contains a rainbow triangle, and the bound for $\delta^c(G)$ is tight.*

We will characterize the extremal graphs in Theorem 4, with the aid of the so-called Gallai partition. A partition of a graph G is a family of subsets $V_1, V_2, \cdots, V_q$ of $V(G)$ satisfying that $\bigcup_{1\leq i\leq q} V_i = V(G)$ and $V_i \bigcap V_j = \emptyset$ for $1 \leq i < j \leq q$.

Definition 5 ([9]). (Gallai partition) Let G be an edge-colored complete graph. A partition $V_1, V_2, \cdots, V_q$ of G is called a Gallai partition if $q \geq 2$, $|\bigcup_{1\leq i<j\leq q} C(V_i, V_j)| \leq 2$ and $|C(V_i, V_j)| = 1$ for $1 \leq i < j \leq q$.

Our characterization result for extremal graphs is as follows.

Theorem 6. *Let G be an edge-colored complete graph of order n. If $\delta^c(G) \geq log_2 n$ and G contains no rainbow triangles, then the following statements hold.*

(1) $d^c_G(v) = log_2 n$ for all $v \in V(G)$;

(2) G has a Gallai partition $(V_1, V_2, \cdots, V_q)$ where $q = 2$ or 4;

(3) $C(V_i, V_j) \bigcap (C(V_i) \bigcup C(V_j)) = \emptyset$ for all $1 \leq i < j \leq q$.

For the existence of rainbow triangles going through each vertex of an edge-colored complete graph, Fujita and Magnant [7] showed the following result.

Theorem 7 ([7]). *Let G be an edge-colored complete graph of order n. If $\delta^c(G) \geq \frac{n+1}{2}$, then every vertex of G is contained in a rainbow triangle.*

The lower bound on $\delta^c(G)$ in Theorem 7 is sharp. To see this, we present the following example.

Example 8. *Consider a complete graph $G = K_{2n}$. Let v be a vertex of G such that $d^c_G(v) = n$. Set $|N_1(v)| = 1$ and $|N_i(v)| = 2$ for $2 \leq i \leq n$. Color the edges between $N_1(v)$ and $N_i(v)$ by i for $2 \leq i \leq n$. For any vertex $u \in N_i(v)$, color two edges between u and $N_j(v)$ by i and j, respectively, for $2 \leq i \neq j \leq n$. Color the edge in $G[N_i(v)]$ by a new color different from the colors of edges incident with v. Then we get an edge-colored complete graph G with $\delta^c(G) = n$; (see Figure 1).*

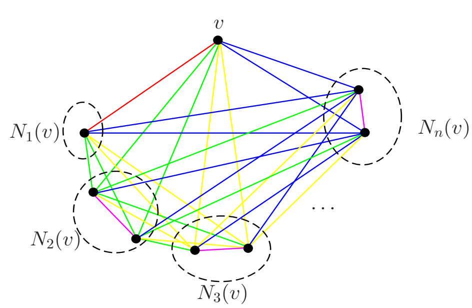

Figure 1: The structure of $G = K^c_n$ in Example 8

We will prove the following generalization of Theorem 7.

Theorem 9. *Let G be an edge-colored complete graph of order n. If $\delta^c(G) \geq \frac{n+k}{2}$, then every vertex of G is contained in at least k rainbow triangles.*

In 2020, Hu, Li and Yang [10] considered the color-degree condition for the existence of two vertex-disjoint rainbow triangles in an edge-colored graph, and proposed a general conjecture.

Conjecture 10 ([10])**.** *Let G be an edge-colored graph of order $n \geq 3k$, where $k \geq 1$ is an integer. If $\delta^c(G) \geq \frac{n+k}{2}$, then G contains k vertex-disjoint rainbow triangles.*

They gave examples to show that if the conjecture is true then it is best possible. They also confirmed the conjecture for the case of $k = 2$ when $n \geq 20$. Actually, Conjecture 10 is a rainbow version of an old result of Dirac [5] on degree conditions for vertex-disjoint triangles. We will give a complete proof for Hu et al.'s result by filling in the gap of n from 6 to 20.

Theorem 11. *Let G be an edge-colored graph of order $n \geq 6$. If $\delta^c(G) \geq \frac{n+2}{2}$, then G contains two vertex-disjoint rainbow triangles.*

Before proceeding, we introduce some additional terminology and notation. The color of an edge e in an edge-colored graph G and the set of colors assigned to $E(G)$ are denoted by $c(e)$ and $C(G)$, respectively. The set of colors appearing on the edges between two vertex subsets V_1 and V_2 in G is denoted by $C(V_1, V_2)$. When $V_1 = \{v\}$, we simply use $C(v, V_2)$ instead of $C(\{v\}, V_2)$. The set of colors appearing on the edges of a subgraph H of G is denoted by $C(H)$, and if $H = G[V_1]$ then we simply write $C(V_1)$ for $C(G[V_1])$. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. For $1 \leq i \leq d^c_G(v)$, let $N_i(v)$ denote the set of vertices connecting v with edges of the same color i, that is, $N_i(v) = \{u \in N_G(v) : c(uv) = i\}$. Let $d^{mon}(v)$ be the maximum number of edges incident with v with the same color, that is, $d^{mon}(v) = \max\{|N_i(v)| : 1 \leq i \leq d^c_G(v)\}$. The *monochromatic-degree* of G, denoted by $\Delta^{mon}(G)$, is defined as $\max\{d^{mon}(v) : v \in V(G)\}$. For other notation and terminology not defined here, we refer to [2].

2 Our proofs

In this section, we will give proofs of Theorems 3, 6, 9 and 11, respectively. We first rewrite Theorem 3 in a more compact form (Theorem 12) as follows, whose proof uses a result due to Andrásfai et al.

Theorem 12. *Let G be an edge-colored graph on $n \geq 5$ vertices. If $\delta^c(G) \geq \frac{n}{2}$, then G contains a rainbow triangle, unless G is a properly colored $K_{\frac{n}{2},\frac{n}{2}}$ where n is even.*

Theorem 13 (Andrásfai, Erdős, Sós [1])**.** *Let G be a triangle-free graph on n vertices. If $\delta(G) > \frac{2n}{5}$, then G is bipartite.*

Proof of Theorem 12. Our proof is somewhat inspirited by the proof of Theorem 6 in [13]. Suppose G contains no rainbow triangles. Let G be such a graph with $\delta^c(G) \geq \frac{n}{2}$ and the number $e(G)$ of edges of G is as small as possible. Clearly, G contains no monochromatic paths of length three. Otherwise, let $P := v_0v_1v_2v_3$ be such a path. Then the new graph $G' := G - v_1v_2$ satisfies that $\delta^c(G') = \delta^c(G) \geq \frac{n}{2}$ and $e(G') = e(G) - 1$, contradicting the choice of G.

Now assume $\triangle^{mon} := \triangle^{mon}(G) \geq 2$. Let $v \in V(G)$ such that $d^{mon}(v) = \triangle^{mon} = k$, and let $U := \{u_1, u_2, \ldots, u_k\} \subseteq N_G(v)$ such that all vu_i, $i \in [1,k]$, have the same color. Suppose that $X = \{x_1, x_2, \ldots, x_t\} \subseteq N(v)\backslash U$ is a maximum subset such that for any $i \in [1,t]$, $c(x_iv) = c_i$. Set $c(u_jv) = c_0$ for $j \in [1,k]$. Since G contains no rainbow triangles, for any $i \neq j$ with $1 \leq i, j \leq k$ we have $c(x_ix_j) \in \{c_i, c_j\}$. Furthermore, since G contains neither rainbow triangles nor monochromatic paths of length 3, we get the conclusion $(*)$ that for any $i \in [1,k]$, if $u_ix_j \in E(G)$ then $c(u_ix_j) = c_j$ for $j \in [1,t]$.

Now we construct a digraph D_1 with $V(D_1) = X$ and $A(D_1) = \{x_ix_j : 1 \leq i, j \leq t, c(x_ix_j) = c_j\}$. For any x_j, notice that $N^+_{D_1}(x_j)$ corresponds to a rainbow set of neighbors of x_j in X, in which each color is different from $c(vx_j)$. Hence, we obtain

$$\begin{aligned}
n = |V(G)| &\geq |U| + |\{v\}| + |X| + (d^c(x_j) - d^+_{D_1}(x_j) - |\{c_j\}|)\\
&= \triangle^{mon} + 1 + d^c(x_j) - 1 + d^c(x_j) - d^+_{D_1}(x_j) - 1\\
&= \triangle^{mon} + n - 1 - d^+_{D_1}(x_j),
\end{aligned}$$

which implies that $d^+_{D_1}(x_j) \geq \triangle^{mon} - 1$ for any $j \in [1,t]$. By the definition of D_1, $d^-_{D_1}(x_j) \leq \triangle^{mon} - 1$. Hence, $d^+_{D_1}(x_j) = d^-_{D_1}(x_j) = \triangle^{mon} - 1$ for any $j \in [1,t]$.

If $E(U,X) \neq \emptyset$, then $d^{mon}(x_j) \geq d^-_D(x_j) + 2 = \Delta^{mon} + 1$ by the above conclusion $(*)$, a contradiction. This proves that $E(U,X) = \emptyset$. In the following, we divide the proof into two cases. First, assume $|U| \geq 3$. If $G[U]$ is properly colored, then $G[U]$ is triangle-free. Hence, there exist two vertices, say $u_1, u_2 \in U$, such that $u_1u_2 \notin E(G)$. It follows from the fact $N(u_1) \cap \{u_1, u_2, X\} = \emptyset$ that $n - 2 - (d^c(v) - 1) \geq d^c(u_1)$, which implies that $d^c(u_1) \leq \frac{n-1}{2}$, a contradiction. If $G[U]$ is not properly colored, then there exists a vertex $u \in U$ such that $d^{mon}_{G[U]}(u) \geq 2$, say uu_i, uu_j are colored with the same color. Then, $d^c(u) - 1 \leq |G - (X \cup \{u, u_i, u_j\})|$, which implies that $d^c(u) \leq \frac{n}{2} - 1$, a contradiction. Next we consider the case $|U| = 2$ and assume $U = \{u_1, u_2\}$. Clearly, $u_1u_2 \in E(G)$; since otherwise, $d^c(u_i) \leq \frac{n}{2} - 1$, for $i = 1, 2$. Let $Y = V(G) - (U \cup X \cup \{v\})$. To guarantee that $d^c(u_i) \geq \frac{n}{2}$ for $i = 1, 2$, the set of edges incident with u_i and Y are rainbow, and $\frac{n}{2} - 2 \leq |Y| \leq n - \delta^c(G) - 2$. This implies that $|Y| = \frac{n}{2} - 2$ and $|X| = \frac{n}{2}$. Suppose $c(u_1y_i) = c'_i$ for $y_i \in Y$. Since G contains no rainbow triangles, we have $c(y_iy_j) \in \{c'_i, c'_j\}$. Now we define a digraph D_2 as follows: $V(D_2) = Y$ and $\overrightarrow{y_iy_j}$ exists if and only if $c(y_iy_j) = c'_j$. Hence, there exists a vertex $y \in Y$ such that $d^+_{D_2}(y) \leq \frac{|Y|-1}{2}$, which implies that $d^c_{Y \cup U}(y) \leq \frac{|Y|-1}{2} + 1$. Moreover, for all $x \in X$, since $d^+_{D_1}(x) = \Delta^{mon} - 1 = 1$, D_1 has a 1-factor and $d^c_{X \cup \{v\}}(x) = 2$. Hence, $d^c_Y(x) = |Y|$ and $C(x, X) \cap C(x, Y) = \emptyset$ for all $x \in X$. Let ω be the number of components of D_1. Then, $d^c_X(y) \leq \omega \leq |X|/3$. Thus, $d^c(y) \leq \frac{\frac{n}{2}-3}{2} + \frac{\frac{n}{2}-1}{3} + 2 < \frac{n}{2}$, a contradiction.

If $\Delta^{mon} = 1$, then G is properly colored. Since G contains no rainbow triangles, G is triangle-free. Since $\delta(G) = \delta^c(G) \geq \frac{n}{2} > \frac{2n}{5}$, by Theorem 13 we infer that G is bipartite. By the condition $\delta^c(G) \geq \frac{n}{2}$, G is a properly colored $K_{\frac{n}{2},\frac{n}{2}}$. The proof is complete. $\square$

The following result will be used in the proof of Theorem 6.

Lemma 14 ([9])**.** *Let G an edge-colored complete graph. If G contains no rainbow triangles, then G has a Gallai partition.*

Proof of Theorem 6. Let G be a graph satisfying the assumptions of Theorem 6. Note that if log_2n is not an integer, then $\delta^c(G) \geq \lceil log_2n \rceil > log_2n$. By Theorem 4, G contains a rainbow triangle, a contradiction. Thus log_2n is an integer and $\delta^c(G) = log_2n$. Since G contains no rainbow triangles, G has a Gallai partition, say $V = (V_1, V_2, \cdots, V_q)$. Note that if $q = 3$ then we can take V_1 so that only one color is used on the edges between V_1 and $V(G) \setminus V_1$. Then (V_1, V_2, V_3) can be seen as $(V_1, V(G) \setminus V_1)$. Thus we may assume that $q = 2$ or $q \geq 4$.

We proceed the proof by induction on $\delta^c(G)$. The induction base is that $\delta^c(G) = 2$ and $n = 4$. If $q = 2$, then $\delta^c(G[V_i]) \geq 1 = log_2\frac{4}{2}$, $i = 1, 2$. Thus, $|V_1| = |V_2| = 2$; otherwise, $\delta^c(G[V_i]) > log_2|V_i|$, which means that there is a rainbow triangle in $G[V_i]$, $i = 1, 2$, a contradiction. Then, $\delta^c(G[V_i]) = 1$, $i = 1, 2$. Since $\delta^c(G) = 2$, we have $C(v, V_1) \cap C(v, V_2) = \emptyset$ for all $v \in V(G)$. Thus, $d^c_G(v) = 3$ for all $v \in V(G)$ and $C(V_1, V_2) \cap (C(V_1) \cup C(V_2)) = \emptyset$. If $q = 4$, then let $|V_i| = 1$ for $1 \leq i \leq 4$, and the result follows evidently.

Now let $\delta^c(G) = t \geq 3$ and assume that Theorem 6 is true for $\delta^c(G) \leq t - 1$.

If $q = 2$, we can infer that there is a Gallai partition (V_1, V_2) of G such that $|V_1| = |V_2| = \frac{n}{2}$ and $\delta^c(G[V_i]) = \delta^c(G) - 1 = log_2\frac{n}{2}$; since otherwise, there exists an $i \in [1, 2]$ such that $|V_i| < \frac{n}{2}$, say $i = 1$. Since $\delta^c(G[V_1]) \geq \delta^c(G) - 1 \geq log_2\frac{n}{2}$, by Theorem 4 there exists a rainbow triangle, a contradiction. Since $G[V_i]$ contains no rainbow triangles and $\delta^c(G[V_i]) = t - 1$, by the induction hypothesis, $d^c_{G[V_i]}(v) = t - 1$ for $i = 1, 2$. Since $\delta^c(G) = t$ and $|C(V_1, V_2)| = 1$, we have $C(v, V_1) \cap C(v, V_2) = \emptyset$ for all $v \in V(G)$. Hence, $d^c_G(v) = t$ for all $v \in V(G)$ and $C(V_1, V_2) \cap (C(V_1) \cup C(V_2)) = \emptyset$.

If $q \geq 4$, then choose V_k such that $|V_k| = \min\{|V_i| : 1 \leq i \leq q\}|$. Thus, $|V_k| \leq \frac{n}{4}$. Then, $\delta^c(V_k) = \delta^c(G) - 2 = log_2\frac{n}{4}$, and hence, $|V_k| = \frac{n}{4}$. Therefore, $q = 4$ and $|V_i| = \frac{n}{4}$, $1 \leq i \leq 4$. Thus, $\delta^c(G[V_i]) = log_2\frac{n}{4} = t - 2$, $1 \leq i \leq 4$. By the induction hypothesis, $d^c_{G[V_i]}(v) = t - 2$ for all $v \in V_i$, $1 \leq i \leq 4$. Since $\delta^c(G) = t$ and $|\bigcup_{1 \leq i < j \leq 4} C(V_j, V_i)| \leq 2$, we have $|C(v, V(G) \setminus V_s)| = 2$, where $V_s \in \{V_1, V_2, \ldots, V_q\}$ such that $v \in V_s$. Then, $C(v, V_s) \cap C(v, V_i) = \emptyset$, for $i \neq s$. Thus, $d^c_G(v) = t$ for all $v \in V(G)$ and $C(V_s, V_i) \cap (C(V_s) \cup C(V_i)) = \emptyset$ for $i \neq s$. Moreover, $C(V_i, V_j) \cap (C(V_i) \cup C(V_j)) = \emptyset$ for $1 \leq i \neq j \leq 4$. This completes the proof. $\square$

Proof of Theorem 9. Let G be a graph satisfying the assumptions of Theorem 9 and $v \in V(G)$, and let $t = d^c_G(v)$. Suppose $|N_1(v)| = \cdots = |N_s(v)| = 1$ and $2 \leq |N_{s+1}(v)| \leq \cdots \leq$

$|N_t(v)|$. Clearly,

$$s \geq k+1 \text{ and } t-s \leq \frac{n-1-s}{2}. \tag{1}$$

Let $S_1 = \bigcup_{1\leq i\leq s} N_i(v)$ and $S_2 = \bigcup_{s+1\leq i\leq t} N_i(v)$. Suppose that $R(v)$ is a maximum such subset of $E(G)$ that for any edge $xy \in R(v)$, $vxyv$ is rainbow. Then the number of rainbow triangles containing v is equal to $|R(v)|$.

Now we define an oriented graph D on S_1 in such a way that for any edge xy, (1) if $c(xy) = c(vx)$, then the orientation of the edge is from y to x; (2) if $xy \in R(v)$, then we give the orientation arbitrarily.

Clearly, all out-arcs from a vertex $u \in S_1$ are assigned colors different from $c(uv)$. For $u \in S_1$, let

$$R_u(v) = \{uw \in R(v) \mid \overleftarrow{uw} \in D\},$$

and

$$R'_u(v) = \{uw \in R(v) \mid w \in S_2\}.$$

Then according to the definition of D, we have

$$d^c_{S_1\cup\{v\}}(u) \leq d^+_D(u) + 1 + |R_u(v)|.$$

Since for any edge $xy \in E(G) \setminus R(v)$, $vxyv$ is not rainbow, we have $c(xy) \in \{c(vx), c(vy)\}$. Therefore, $|C(u, S_2) \setminus \{c(uv)\}| \leq t - s + |R'_u(v)|$. Hence, we have

$$d^c(u) \leq d^+_D(u) + 1 + |R_u(v)| + |R'_u(v)| + t - s. \tag{2}$$

Now we proceed the proof of Theorem 9 by induction on k. The case $k = 1$ follows directly from Theorem 7. Let $k \geq 2$ and assume Theorem 9 holds for $k-1$. Suppose to the contrary, that $|R(v)| < k$. Since $\delta^c(G) \geq \frac{n+k}{2} \geq \frac{n+k-1}{2}$, we have $|R(v)| = k-1$ by hypothesis. Because $|R_u(v)| + |R'_u(v)| \leq k-1$, if $d^+_D(u) \leq \frac{s-k}{2}$, we have $d^c(u) \leq \frac{n+k-1}{2}$ by Inequalities (1) and (2), a contradiction. Therefore, $d^+_D(u) \geq \frac{s-k+1}{2}$ for $u \in S_1$. Let w be a vertex with minimum out-degree in D. Then, $\frac{s-k+1}{2} \leq d^+_D(w) \leq \frac{s-1}{2}$. Assume that $d^+_D(w) = \frac{s-k+a}{2}$, $1 \leq a \leq k-1$. Then,

$$|R_w(v)| + |R'_w(v)| \geq k - \frac{a+1}{2}; \tag{3}$$

otherwise, $d^c(w) < \frac{n+k}{2}$ by Inequality (2). For $u \in S_1 \bigcup(\psi(R_w(v))\setminus\{w\})$, since $\overrightarrow{uw}$ is an out-arc from u, we have $R_u(v) \bigcap R_w(v) = \emptyset$. Therefore, for $u \in S_1 \setminus \{w\}$, $(R_w(v) \bigcup R'_w(v)) \bigcap (R_u(v) \bigcup R'_u(v)) = \emptyset$. Then by Inequality (3), $|R_u(v) \bigcup R'_u(v)| \leq k-1-(k-\frac{a+1}{2}) \leq \frac{a-1}{2}$. By Inequality (2), to guarantee that $d^c(u) \geq \delta^c(G) \geq \frac{n+k}{2}$, we have $d^+_D(u) \geq \frac{s+k-a}{2}$ for all $u \in S_1 \setminus \{w\}$. Hence, $\sum_{u\in S_1} d^+_D(u) \geq (s-1)\frac{s+k-a}{2} + \frac{s-k+a}{2}$. Since $s \geq k+1 \geq 3$ and $1 \leq a \leq k-1$, we have

$\sum_{u\in S_1} d_D^+(u) > \frac{s(s-1)}{2}$, a contradiction. The proof is now complete. $\square$

Before giving the proof of Theorem 11, we need more preparations. For a vertex $v \in V$, we say that $xy \in E(G)$ is a *good edge* for v if $vxyv$ is a rainbow triangle. Let $R(v)$ denote the set of all good edges for v, and $r(v) = |R(v)|$. The following lemma from [10] gives a lower bound on the number of rainbow triangles going through the vertex with maximum monochromatic degree in an edge-colored graph.

Lemma 15 ([10])**.** *Let G be an edge-colored graph of order n and $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$. Then $r(v) \geq \frac{1}{2}(2\delta^c(G) - n)(\delta^c(G) + \Delta^{mon}(G) - 1)$.*

The following notation is from [7]. Let G be an edge-colored graph and v be a vertex of G. A subset A of $N_G(v)$ is said to have the *dependence property* with respect to a vertex $v \notin A$, denoted by DP_v, if $c(aa') \in \{c(va), c(va')\}$ for all $aa' \in E(G[A])$. Using the same method as in the proof of Fact 1 of [7] one can directly get the following lemma.

Lemma 16. *If a subset A of vertices in an edge-colored graph G has the DP_v, then there exists a vertex $x_0 \in A$ such that the number of colors different from $c(vx_0)$ on the edges incident with x_0 in $G[A]$ is at most $\frac{|A|-1}{2}$, and moreover, if it attains the maximum, then $G[A]$ is a complete subgraph, and $d^{mon}(x_0) \geq \frac{|A|+1}{2}$.*

Lemma 17. *Let G be an edge-colored graph of order n. If $\delta^c(G) \geq \frac{n+2}{2}$ and there are two vertices y, z such that $G' = G - \{y, z\}$ has no rainbow triangles, then G has two vertex-disjoint rainbow triangles containing y and z, respectively.*

Proof. Since $\delta^c(G) \geq \frac{n+2}{2}$, we have $\delta^c(G') \geq \delta^c(G) - 2 \geq \frac{|G'|}{2}$. By Theorem 3, n is even and G' is a properly colored $K_{\frac{n-2}{2},\frac{n-2}{2}}$ when $n \geq 8$, say $G' = G'[A, B]$, or an edge-colored K_4 or $K_4 - e$ with $d^c_{G'}(v) = 2$ for all $v \in V(G')$ when $n = 6$. Since $\delta^c(G) \geq \frac{n+2}{2}$, the edges from every vertex in G' to z and y are assigned two different fresh colors, and the edges from y and z, respectively, to $V(G')$ are assigned at least $\frac{n}{2}$ different colors. If $n = 6$, then we can easily find two vertex-disjoint rainbow triangles containing z and y, respectively. Thus, we assume $n \geq 8$. In fact, since $|C(y, V(G'))| \geq \frac{n}{2} \geq 4$ and $|C(z, V(G'))| \geq \frac{n}{2} \geq 4$, and $|A| = |B| = \frac{n}{2} - 1 \geq 3$, we can easily find two vertex-disjoint rainbow triangles containing z and y, respectively. The proof is thus complete. $\square$

Now we are ready for the proof of our last result Theorem 11.

Proof of Theorem 11. Suppose to the contrary, that G is a counterexample with the smallest number of edges. Since $\delta^c(G) \geq \frac{n+2}{2}$, G contains a rainbow triangle. Let $T(G)$ be the set of all rainbow triangles in G. Now we proceed by proving the following claims.

Claim 1. For any rainbow triangle $uvwu$ in $T(G)$, each of the vertices u, v and w is contained in a rainbow triangle which is edge-disjoint from $uvwu$.

Proof. Since G contains no vertex-disjoint rainbow triangles, each triangle in $T(G)$ meets at least one of u, v, w. Let T_u (*resp.*, T_v, T_w) denote the subset of $T(G) \setminus \{uvwu\}$, in which every triangle contains u (*resp.*, v, w). Thus, $T(G) = T_u \cup T_v \cup T_w \cup \{uvwu\}$. If $T_u = \emptyset$, then there is no rainbow triangle in $G' = G - \{v, w\}$. From Lemma 17, we can find two vertex-disjoint rainbow triangles in G, a contradiction. By symmetry, we have that $T_v \neq \emptyset$ and $T_w \neq \emptyset$.

Suppose to the contrary, w.l.o.g, that there is no rainbow triangle containing u which is edge-disjoint from $uvwu$. This implies that $T_u \subseteq T_v \cup T_w$. Therefore, there is no rainbow triangle in $G' = G - \{v, w\}$. From Lemma 17, we can find two vertex-disjoint rainbow triangles in G, a contradiction. By symmetry, each of u, v, w is contained in a rainbow triangle which is edge-disjoint from $uvwu$. The proof is thus complete. $\square$

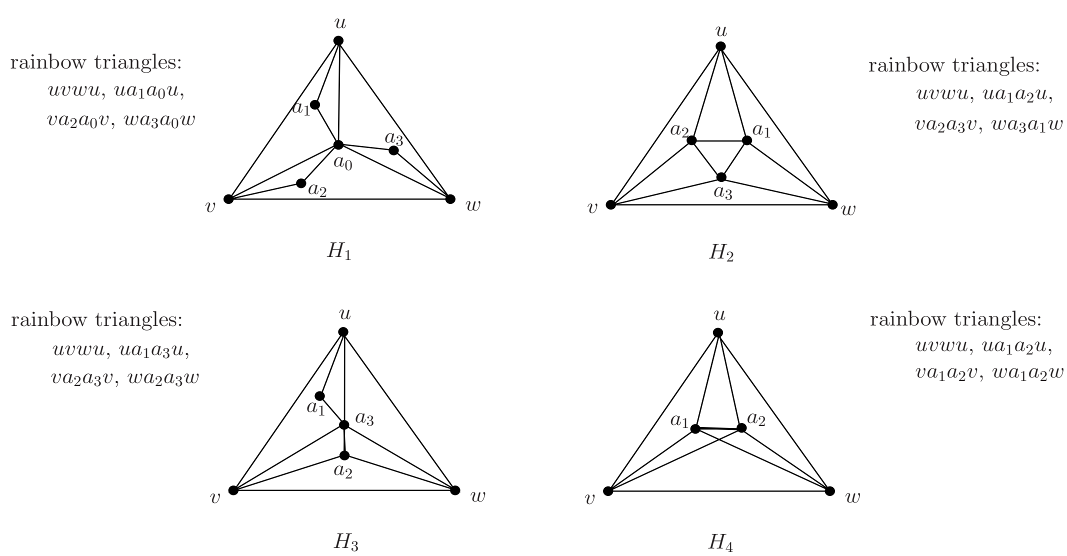

Figure 2: Four edge-colored graphs

For convenience, we construct four edge-colored graphs H_1, H_2, H_3 and H_4 as shown in Figure 2. Since G contains no vertex-disjoint rainbow triangles, by Claim 1 we know that any rainbow triangle $uvwu$ in $T(G)$ must be contained in a copy of H_1, H_2, H_3 or H_4 in G as an outer triangle shown in Figure 2. Let $\mathcal{H}_{uvwu}$ be the set of all subgraphs of G isomorphic to H_i (or simply, H_i-subgraphs) for $1 \leq i \leq 4$, such that $uvwu$ is the outer triangle of these subgraphs.

Claim 2. Let $uvwu$ be a rainbow triangle. Then for any $H \in \mathcal{H}_{uvwu}$, we have the following statements.

(1) If H is an H_1-subgraph of G, then $R(u) \cup R(v) \cup R(w) \subseteq G[V(H)]$.

(2) If H is an H_2 or H_3-subgraph, and $R(u) \cup R(v) \cup R(w) \subsetneq G[V(H)]$, then there is an H_1-subgraph of G in $\mathcal{H}_{uvwu}$.

(3) If H is an H_4-subgraph and $R(u) \cup R(v) \cup R(w) \subsetneq G[V(H)]$, then there is an H_2 or H_3-subgraph of G in $\mathcal{H}_{uvwu}$.

Proof. Suppose that there exists a vertex $x \in V(G) \setminus V(H)$ such that x is contained in a rainbow triangle. Since G contains no vertex-disjoint rainbow triangles, the rainbow triangle containing x must also contain a vertex of each rainbow triangle in $G[H]$.

(1) We may assume $xua_0x \in T(G)$. Then by Claim 1, there exists a rainbow triangle containing x which is edge-disjoint from xua_0x. Thus by symmetry, at least one of $xwvx$, xwa_2x, xwa_1x is in $T(G)$, which would guarantee that G contains two vertex-disjoint rainbow triangles, a contradiction.

(2) If H is an H_2-subgraph of G, assume $xua_3x \in T(G)$. Clearly, $H - ua_1 - ua_2 - a_1a_2 + ua_3 + xa_3 + ux$ is an H_1-subgraph of G. If H is an H_3-subgraph of G, then at least one of $xua_3x, xua_2x, xva_3x, xwa_3x$ is in $T(G)$. If $xua_3x \in T(G)$, by Claim 1 there exists a rainbow triangle containing x which is edge-disjoint from xua_3x. Thus by symmetry, at least one of $xwvx$, xwa_2x, xwa_1x is in $T(G)$, which would guarantee that G contains two vertex-disjoint rainbow triangles, a contradiction. If $xua_2x \in T(G)$, by Claim 1 there exists a rainbow triangle which contains x but is edge-disjoint from xua_2x. Since G contains no vertex-disjoint rainbow triangles, xwa_3x is in $T(G)$. So, $H - wa_2 + xa_3 + xw$ is an H_1-subgraph of G.

(3) By symmetry, we may assume $xua_2x \in T(G)$. Then $H - ua_1 + ux + xa_2$ is an H_3-subgraph of G.

The proof is now complete. $\square$

Let u be a vertex of G with $d^{mon}(u) = \Delta^{mon}(G)$. By Lemma 15, we have $r(u) \geq 4$. Then u must be contained in a rainbow triangle $uvwu$.

Claim 3. If $R(u) \cup R(v) \cup R(w) \subseteq G[V(H)]$, then $6 \leq n \leq 10$ and $r(u) \leq 7$.

Proof. We distinguish the following three cases.

Case 1. There exists an $H \in \mathcal{H}_{uvwu}$ such that H is an H_1-subgraph of G.

By Lemma 17, we know that $G - \{u, a_0\}$ has at least one rainbow triangle. By symmetry, we assume that one of va_1a_3v, va_1wv is in $T(G)$. Clearly, each rainbow triangle with vertex u must intersect with all rainbow triangles in $T(G)$. Thus, $r(u) \leq 6$ by enumeration. Since $va_1 \in E(G)$ but $va_1 \notin R(u)$, we have $\Delta^{mon}(G) \geq 2$. By Lemma 15, $r(u) \geq \frac{n+4}{2}$, and thus $7 \leq n \leq 8$.

Case 2. There exists an $H \in \mathcal{H}_{uvwu}$ such that H is an H_2 or H_3-subgraph of G.

If $H \cong H_2$, then by Lemma 17, $G - \{u, a_3\}$ contains at least one rainbow triangle. If H is an H_3-subgraph shown as Figure 3 (a), then by Lemma 17 we know that $G - \{u, a_3\}$ contains at least one rainbow triangle. If H is an H_3-subgraph shown as Figure 3 (b), then by Lemma 17 we know that $G - \{u, a_3\}$ contains at least one rainbow triangle. Thus, $r(u) \leq 7$ by enumeration.

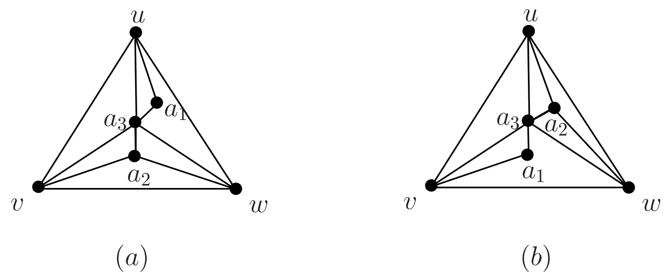

Figure 3: H_2-subgraphs of G

Since $G[V(H)]$ contains a non-rainbow triangle, we have $\Delta^{mon}(G) \geq 2$. Applying Lemma 15, we have $r(u) \geq \frac{n+4}{2}$, and thus $6 \leq n \leq 10$.

Case 3. There exists an $H \in \mathcal{H}_{uvwu}$ such that H is an H_4-subgraph of G.

Since G contains no vertex-disjoint rainbow triangles, $R(u) \subseteq \{va_1, va_2, vw, a_1a_2, a_1w, a_2w\}$. Thus, $r(u) \leq 6$. Then by Lemma 15, we have $n \leq 10$. The proof is thus complete. $\square$

By Claim 2, w.l.o.g., we assume that $R(u) \cup R(v) \cup R(w) \subseteq G[V(H)]$. If there exists a vertex $x \in V(G) \setminus V(H)$, let A be a subset of $N^c(x)$ such that $|A| = d^c(x)$. We can get that x is not contained in any rainbow triangle, and A has the DP_x. Hence by Lemma 16, there exists a vertex $a \in A$ such that

$$d^c_{A\cup\{x\}}(a) \leq \frac{d^c(x)+1}{2}. \tag{4}$$

If $n = 6$ with $V(H) = 5$ or $n = 7$ with $V(H) = 6$, then there is a vertex in A with color-degree less than $\delta^c(G)$, a contradiction.

When $n = 6$ and $H = G$, by Case 2 of Claim 3, $r(u) \geq 5$. Note that each non-rainbow triangle contributes a vertex with degree 5 in G. Then G is a complete graph. For any rainbow triangle T in G, we know that there exists a vertex in $V(G-T)$ with color-degree at most 1 in $G-T$. Since $|T(G)| \geq r(u)+2 \leq 7$, there is a vertex in G with color-degree less than 4.

If $n = 7$ with $H = G$, then $e(G) \geq 18$ since $\delta^c(G) \geq 5$, which implies that $d(a) \geq 5$ for $a \in G$. Hence for any rainbow triangle T in G, by Theorems 1 and 4 we know that each vertex in $V(G-T)$ has a color-degree at most 2 in $G-T$. Hence, $d^c_G(u) \leq 4$, a contradiction.

If $n = 8$, then the equality in Inequality (4) holds; otherwise, $d^c(a) < \frac{d^c(x)+1}{2} + n - 1 - d^c(x) \leq \frac{n+2}{2}$, a contradiction. Hence, $G[A]$ is a complete graph and each vertex $a \in A$ has $d^c_{A\cup\{x\}}(a) = 3$ by Lemma 16. Let $\{y, z\} \subseteq V(G) \setminus (A \cup \{x\})$. Then $c(ay)$ and $c(az)$ are two fresh colors for vertex a in $G[A \cup \{x\}]$. Since $d^c(y)$ and $d^c(z)$ are larger than 5, there must exist two vertex-disjoint rainbow triangles containing y and z, respectively, a contradiction.

If $n = 9$, then $d^c(a) \leq \frac{d^c(x)+1}{2} + n - 1 - d^c(x) < \frac{n+2}{2}$, a contradiction.

If $n = 10$, then $\Delta^{mon}(G) \leq 2$; otherwise, $r(u) \geq 8$, a contradiction. Then, $d^c_{A\cup\{x\}}(a) \leq \lfloor \frac{|A|+1}{2} \rfloor \leq 3$. If $d^c_{A\cup\{x\}}(a) = 3$, then each vertex b in A has at least 2 distinct colors different from $c(xb)$. Hence, the average value of the monochromatic-degrees of vertices in $A \cup \{x\}$ is 3, a contradiction.

The proof of Theorem 11 is now complete. □

3 Concluding remarks

In this paper we mainly study color-degree conditions forcing rainbow triangles in edge-colored graphs. Many unsolved problems are left.

Theorem 3 characterizes all edge-colored graphs on n vertices satisfying $\delta^c(G) \geq \frac{n}{2}$ without rainbow triangles. It is natural to characterize those edge-colored graphs containing no rainbow triangles with the weaker condition that "$\delta^c(G) \geq \frac{n-1}{2}$".

We conclude our paper with two open problems recently proposed in [4]: (1) Find tight color-degree conditions forcing a larger rainbow clique. (2) Find tight color-degree conditions forcing a rainbow cycle of length at most r for $r \geq 4$.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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