Poincaré Polynomials of Odd Diagram Classes

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Abstract. An odd diagram class is a set of permutations with the same odd diagram. Brenti, Carnevale and Tenner showed that each odd diagram class is an interval in the Bruhat order. They conjectured that such intervals are rank-symmetric. In this paper, we present an algorithm to partition an odd diagram class in a uniform manner. As an application, we obtain that the Poincaré polynomial of an odd diagram class factors into polynomials of the form $1 + t + \cdots + t^m$. This in particular resolves the conjecture of Brenti, Carnevale and Tenner.

Keywords: Bruhat order, odd diagram class, Poincaré polynomial, palindromic polynomial, Kazhdan-Lusztig polynomial

AMS Classifications: 05A15, 06A07, 06A11

1 Introduction

Let S_n denote the symmetric group of permutations of $[n] := \{1, 2, \ldots, n\}$. For $w \in S_n$, we adopt the one-line notation, that is, we write $w = w(1)w(2)\cdots w(n)$. An *odd inversion* of w is an inversion with an additional parity condition, that is, a pair $(w(i), w(j))$ such that $1 \leq i < j \leq n$, $w(i) > w(j)$, and $i \not\equiv j \pmod 2$. The *odd length* of w is the number of odd inversions of w. This statistic was introduced by Klopsch and Voll [14] in their study of functions counting non-degenerate flags in formed spaces, see also Brenti and Carnevale [6].

The *odd diagram* of w is a diagram representation of its odd inversions, which can be viewed as an odd analogue of the classical Rothe diagram of w [5]. Specifically, the odd diagram $D_o(w)$ of w is a subset of boxes in an $n \times n$ square grid defined by

$$D_o(w) = \{(i, j)\colon w(i) > j,\, i < w^{-1}(j),\, i \not\equiv w^{-1}(j) \pmod 2\},$$

where w^{-1} is the inverse of w. Here, we use the matrix coordinates, and use (i, j) to denote the box in row i and column j. A set $D \subseteq [n] \times [n]$ is called an odd diagram if there exists $w \in S_n$ such that $D_o(w) = D$. For an odd diagram D, let $\mathrm{Perm}_n(D)$ denote the odd diagram class of D, namely,

$$\mathrm{Perm}_n(D) = \{w \in S_n\colon D_o(w) = D\}.$$

Brenti, Carnevale and Tenner [7] proved that odd diagram classes partition the symmetric group in an extremely pleasant way.

Theorem 1.1 (Brenti–Carnevale–Tenner [7, Theorem B]). *Each odd diagram class is an interval in the Bruhat order.*

They conjectured that $\mathrm{Perm}_n(D)$ satisfies a stronger symmetry property.

Conjecture 1.2 (Brenti–Carnevale–Tenner [7, Conjecture 6.12]). *Each odd diagram class is rank-symmetric in the Bruhat order.*

For a Bruhat interval $[u, v]$ in S_n, the associated *Poincaré polynomial* $P_{u,v}(t)$ is the rank generating function:

$$P_{u,v}(t) = t^{-\ell(u)} \sum_{u \le w \le v} t^{\ell(w)},$$

where $\ell(w)$ is the Coxeter length of w. In the case when $[u, v]$ is a lower interval $[e, w]$, $P_w(t^2) := P_{e,w}(t^2)$ specifies to the Poincaré polynomial of the cohomology ring of the Schubert variety X_w indexed by w.

In this paper, we prove that the Poincaré polynomial of $\mathrm{Perm}_n(D)$ admits the following factorization.

Theorem 1.3. *The Poincaré polynomial of an odd diagram class can be expressed as a product of factors of the form* $1 + t + \cdots + t^m$.

A polynomial $f(t) = a_0 + a_1 t + \cdots + a_d t^d$ of degree d is called *palindromic* if

$$t^d f(t^{-1}) = f(t).$$

Clearly, a Bruhat interval is rank-symmetric if and only if the associated Poincaré polynomial is palindromic. Theorem 1.3 obviously implies that the Poincaré polynomial of $\mathrm{Perm}_n(D)$ is palindromic, thus confirming Conjecture 1.2.

Remark. It is well known that the following conditions for $w \in S_n$ are equivalent:

(1) the Schubert variety X_w is smooth;

(2) the Poincaré polynomial $P_w(t)$ is palindromic;

(3) the Kazhdan-Lusztig polynomial associated to $[e, w]$ equals 1;

(4) w avoids the patterns 4231 and 3412, that is, there do not exits indices $i_1 < i_2 < i_3 < i_4$ such that the subsequence $w(i_1)w(i_2)w(i_3)w(i_4)$ has the same relative order as 4231 or 3412;

see for example Carrell [8] and Lakshmibai and Sandhya [15].

When X_w is smooth, $P_w(t)$ can be expressed as a product of the factors $1+t+\cdots+t^m$, see Akyildiz and Carrell [1] or Carrell [8]. A combinatorial treatment was given by Gasharov [11]. It is this fact that motivates us to consider if the Poincaré polynomial of an odd diagram class has an analogous factorization, as stated in Theorem 1.3.

This paper is structured as follows. In Section 2, we collect some notation, terminology and results used in this paper. In Section 3, we present an algorithm to give a partition of an odd diagram class. We prove that the partition is uniform. Using the results established in Section 3, we finish the proof of Theorem 1.3 in Section 4. In Section 5, we discuss problems concerning odd diagram classes, including the self-duality property and the Kazhdan-Lusztig polynomials of odd diagram classes.

2 Preliminaries

In this section, we give an overview of the Bruhat order for the symmetric group. We also describe the legal move operation introduced by Brenti, Carnevale and Tenner [7], which plays a fundamental role in the study of odd diagram classes.

The symmetric group S_n is the Coxeter group of type A_{n-1}. The reflection set is the collection $\{(i\ j)\colon 1 \le i < j \le n\}$ of transpositions, and the set $\{(i\ i+1)\colon 1 \le i \le n-1\}$ of simple transpositions constitutes a generating set. For a permutation $w \in S_n$, $w\,(i\ j)$ is the permutation obtained by swapping $w(i)$ and $w(j)$.

The Coxeter length $\ell(w)$ of $w \in S_n$ equals the number of inversion pairs of w:

$$\ell(w) = \#\{1 \le i < j \le n\colon w(i) > w(j)\}, \tag{2.1}$$

see for example Björner and Brenti [2, Proposition 1.5.2]. Notice that $\ell(w) < \ell(w\,(i\ j))$ if and only if $w(i) < w(j)$, and in this case we denote $w < w\,(i\ j)$. The transitive closure of all relations of the form $w < w\,(i\ j)$ forms the Bruhat order $\le$ on S_n.

Let us recall a combinatorial rule for deciding when two permutations are comparable, see Macdonald [16, (1.19)]. For two subsets S, T of $[n]$ with the same cardinality, write $S \le T$ if we list the elements of S and T in increasing order, say $S = \{s_1 < s_2 < \cdots < s_m\}$ and $T = \{t_1 < t_2 < \cdots < t_m\}$, then $s_i \le t_i$ for each $1 \le i \le m$. As pointed out by a reviewer, this order is called the Gale order [10].

Proposition 2.1. *Let $x, y \in S_n$. Then $x \le y$ in the Bruhat order if and only if for all $1 \le i \le n$,*

$$\{x(1), x(2), \ldots, x(i)\} \le \{y(1), y(2), \ldots, y(i)\}.$$

For $x, y \in S_n$, we use $x \lhd y$ to mean that x is covered by y, that is, there does not exist $w \in S_n$ such that $x < w < y$. The following simple criterion can be used to identify a covering relation [2, Lemma 2.1.4].

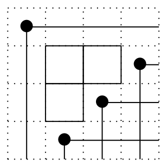
Proposition 2.2. *Let $x, y \in S_n$. Then $x \lhd y$ if and only if there exist $1 \le i < j \le n$ such that $y = x\,(i\ j)$, $x(i) < x(j)$, and for each $i < k < j$, either $x(k) < x(i)$ or $x(k) > x(j)$.*

Combining (2.1) and Proposition 2.2, it follows that $x \lhd y$ if and only if $y = x\,(i\ j)$ and $\ell(y) = \ell(x) + 1$.

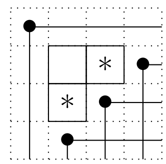
The *Rothe diagram* $D(w)$ of $w \in S_n$ is the subset

$$D(w) = \{(i, j)\colon w(i) > j,\ i < w^{-1}(j)\}$$

of an $n \times n$ grid. Alternatively, $D(w)$ can be obtained as follows. For $1 \le i \le n$, put a dot in the box $(i, w(i))$, and then delete all boxes lying on the hook with corner at the box $(i, w(i))$. Then $D(w)$ is exactly the set of the remaining boxes. Figure 2.1(a) illustrates the Rothe diagram of $w = 1432$. The odd diagram $D_o(w)$ of w is the subset



(a)



(b)

Figure 2.1: (a) $D(1432)$, (b) $D_o(1432)$

of $D(w)$ subject to the parity condition:

$$D_o(w) = \{(i,j) \in D(w) \colon i \not\equiv w^{-1}(j) \pmod 2\}.$$

Figure 2.1(b) depicts the odd diagram of $w = 1432$, where, as used in [7], the boxes in $D_o(w)$ are marked with stars.

In the remaining of this section, we give a description of the legal move operation on odd diagram classes. For a permutation $w \in S_n$, we say a transposition $(i\ j)$ is *legal* for w if w and $w\,(i\ j)$ have the same odd diagram. The following criterion for a legal transposition will be used frequently in this paper.

Theorem 2.3 (Brenti–Carnevale–Tenner [7, Theorem 4.3])**.** *Let $w \in S_n$, and $(i\ j)$ be a transposition. Set $m = \min\{w(i), w(j)\}$ and $M = \max\{w(i), w(j)\}$. Then $(i\ j)$ is legal for w if and only if the following conditions are satisfied:*

(1) i and j have the same parity;

(2) $w(p) < m$ for all $p \in \{i+1, i+3, \ldots, j-1\}$;

(3) $w(q) \notin [m, M]$ for all $q \in \{j+1, j+3, \ldots\}$.

Assume that $x \neq y \in S_n$ have the same odd diagram. It was observed in [7] that one can apply a legal move to x to obtain a permutation which is "closer" to y. Define

$$d(x,y) = \min\{i \colon x^{-1}(i) \neq y^{-1}(i)\}$$

to be the smallest value lying at different positions in x and y.

Theorem 2.4 (Brenti–Carnevale–Tenner [7, Theorem 4.9])**.** *Suppose that $x \neq y \in S_n$ have the same odd diagram. Set $i = x^{-1}(d(x,y))$ and $j = y^{-1}(d(x,y))$. Then the transposition $(i\ j)$ is legal for x.*

Based on Theorem 2.4, it can be shown that in an odd diagram class, each value appears in positions with the same parity.

Theorem 2.5 (Brenti–Carnevale–Tenner [7, Lemma 6.2])**.** *Assume that $x, y \in S_n$ have the same odd diagram. Then*

$$x^{-1}(i) \equiv y^{-1}(i) \pmod 2, \quad \textit{for } 1 \le i \le n.$$

3 A uniform partition of an odd diagram class

Throughout this section, let

$$\mathrm{Perm}_n(D) = \{w \in S_n \colon D_o(w) = D\}$$

be an odd diagram class in S_n. Our goal is to present a uniform partition of $\mathrm{Perm}_n(D)$. In other words, we shall partition $\mathrm{Perm}_n(D)$ into blocks with the same cardinality. Some properties about this uniform partition will be established, which will be used in the proof of Theorem 1.3 in Section 4.

By Theorem 1.1, $\mathrm{Perm}_n(D)$ is a Bruhat interval, say $[u, v]$. This means that if $w \in S_n$ has odd diagram D, then $u \leq w \leq v$. If $u = v$, then $\#\mathrm{Perm}_n(D) = 1$ and there is nothing to do. In the following of this section, we shall assume that $u \neq v$. Fix the following notation

$$k = d(u, v) = \min\{i \colon u^{-1}(i) \neq v^{-1}(i)\}$$

and

$$a = u^{-1}(k), \qquad b = v^{-1}(k).$$

3.1 A partition of $[u, v]$

Let us begin with the following lemma.

Lemma 3.1. *We have $a < b$ and $u(a) < u(b)$.*

Proof. Keep in mind that $u^{-1}(i) = v^{-1}(i)$ for $1 \leq i < k$. On the other hand, $u < v$ is equivalent to $u^{-1} < v^{-1}$ [2, Corollary 2.2.5]. Applying Proposition 2.1 to u^{-1} and v^{-1}, we deduce that $a < b$. Suppose now that $u(a) > u(b)$. By Theorem 2.4, the transposition $(a\ b)$ is legal for u. Thus $u\,(a\ b)$ has the same odd diagram as u. The assumption that $u(a) > u(b)$ leads to $u\,(a\ b) < u$, contradicting the minimality of u. ∎

Let

$$\{a = a_1 < a_2 < \cdots < a_m = b\} = \{a \leq i \leq b \colon u(a) \leq u(i) \leq u(b)\} \tag{3.1}$$

be the set of positions between a and b with values lying in $[u(a), u(b)]$. These positions will play a central role in the construction of the partition of $[u, v]$.

Lemma 3.2. *The subsequence $u(a_1)u(a_2)\cdots u(a_m)$ of u is increasing.*

Proof. Suppose otherwise there exists i such that $u(a_i) > u(a_{i+1})$. We claim that the transposition $(a_i\ a_{i+1})$ is legal for u. By Theorem 2.4, the transposition $(a\ b)$ is legal for u. Invoking Theorem 2.3, we see that

(1) a and b have the same parity;

(2) $u(p) < u(a)$ for all $p \in \{a+1, a+3, \ldots, b-1\}$;

(3) $u(q) \notin [u(a), u(b)]$ for all $q \in \{b+1, b+3, \ldots\}$.

By (2), the positions $a_1, \ldots, a_m$ must have the same parity as a, and so

$$a_i \equiv a_{i+1} \pmod 2.$$

It also follows from (2) that for $p \in \{a_i + 1, \ldots, a_{i+1} - 1\}$,

$$u(p) < u(a) < u(a_{i+1}).$$

To verify that $(a_i\ a_{i+1})$ is a legal transposition, it remains to check that for $q \in \{a_{i+1} + 1, a_{i+1} + 3, \ldots\}$,

$$u(q) \notin [u(a_{i+1}), u(a_i)]. \tag{3.2}$$

This can be seen as follows. Let $q \in \{a_{i+1} + 1, a_{i+1} + 3, \ldots\}$. If $q < b$, then we see from (2) that $u(q) < u(a)$, while if $q > b$, then it follows from (3) that $u(q) < u(a)$ or $u(q) > u(b)$. Since $u(a) \leq u(a_i), u(a_{i+1}) \leq u(b)$, we obtain relation (3.2). This concludes that the transposition $(a_i\ a_{i+1})$ is legal for u.

By the legality of $(a_i\ a_{i+1})$, the permutation $u\,(a_i\ a_{i+1})$ has the same odd diagram as u. However, by the assumption $u(a_i) > u(a_{i+1})$, we are led to $u\,(a_i\ a_{i+1}) < u$, which is contrary to the minimality of u. This completes the proof. ∎

The following lemma shows that for any $w \in [u, v]$, the value k appears in one of the positions $a_1, a_2, \ldots, a_m$.

Lemma 3.3. *For $w \in [u, v]$, we have*

$$w^{-1}(k) \in \{a_1, a_2, \ldots, a_m\}. \tag{3.3}$$

Proof. Write $c = w^{-1}(k)$. From Theorem 2.5, it follows that

$$c \equiv a \equiv b \pmod 2. \tag{3.4}$$

Since $u \leq w \leq v$ is equivalent to $u^{-1} \leq w^{-1} \leq v^{-1}$, it follows from Proposition 2.1 that

$$w^{-1}(i) = u^{-1}(i) = v^{-1}(i), \quad \text{for } 1 \leq i < k, \tag{3.5}$$

and

$$a \leq c \leq b. \tag{3.6}$$

By (3.4) and (3.6), we have

$$c \in \{a, a + 2, \ldots, b\}. \tag{3.7}$$

In view of the proof of Lemma 3.2, we have $\{a_1, a_2, \ldots, a_m\} \subseteq \{a, a+2, \ldots, b\}$. Our goal is to show that $c \in \{a_1, a_2, \ldots, a_m\}$. Suppose to the contrary that

$$c \in \{a, a + 2, \ldots, b\} \setminus \{a_1, a_2, \ldots, a_m\}.$$

By the definition of the set $\{a_1, a_2, \ldots, a_m\}$ as given in (3.1), we have either $u(c) < u(a) = k$ or $u(c) > u(b)$. If $u(c) < u(a) = k$, it follows from (3.5) that $u(c) = w(c) = k$, leading to a contradiction. We next consider the case $u(c) > u(b)$. The discussion is divided into two situations, according to the values $u(q)$ for $q \in \{b + 1, b + 3, \ldots, \}$, Recalling that $(a\ b)$ is a legal transposition for u, we see from Theorem 2.3 that for each $q \in \{b+1, b+3, \ldots,\}$, either $u(q) < u(a)$ or $u(q) > u(b)$.

Case 1. For each $q \in \{b+1, b+3, \ldots\}$, either $u(q) < u(a)$ or $u(q) > u(c)$. In this case, let us check that $(c\ b)$ is a legal transposition of u. By (3.7), we have $c \equiv b \pmod 2$. Using again the fact that $(a\ b)$ is a legal transposition of u, for $p \in \{c+1, \ldots, b-1\}$, we have $u(p) < u(a)$, and thus $u(p) < u(b)$. Finally, by the assumption that $u(q) < u(a)$ or $u(q) > u(c)$ for $q \in \{b+1, b+3, \ldots\}$, we see that $u(q) \notin [u(b), u(c)]$. So the transposition $(c\ b)$ is legal for u, and thus $u\,(c\ b)$ has the same odd diagram as u. However, since $u(c) > u(b)$, $u\,(c\ b)$ is smaller than u in the Bruhat order, leading to a contradiction.

Case 2. There exists $q_0 \in \{b+1, b+3, \ldots\}$ such that $u(b) < u(q_0) < u(c)$. Since $c \not\equiv q_0 \pmod 2$ and $u(c) > u(q_0)$, the box $(c, u(q_0))$ belongs to $D_o(u)$. On the other hand, noticing that

$$w(c) = k = u(a) < u(b) < u(q_0),$$

the box $(c, u(q_0))$ cannot belong to $D_o(w)$, contradicting the fact that $D_o(u) = D_o(w)$. This completes the proof. ∎

By Lemma 3.3, the interval $[u, v]$ can be partitioned according to the positions of k. Precisely, for $1 \le i \le m$, set

$$[u, v]^{(i)} = \{w \in [u, v] \colon w^{-1}(k) = a_i\}.$$

To see that each $[u, v]^{(i)}$ is indeed nonempty, we construct a specific permutation $u_i \in S_n$ belonging to $[u, v]^{(i)}$. As will be seen in Lemma 3.9, u_i is in fact the minimum element of $[u, v]^{(i)}$ in the Bruhat order.

Set $u_1 = u$. The constructions of u_i for $i = 2, \ldots, m$ rely on the increasing subsequence $u(a_1)u(a_2)\cdots u(a_m)$. For $1 \le i \le m-1$, set

$$u_{i+1} = u_i\,(a_i\ a_{i+1}). \tag{3.8}$$

For example, consider the following odd diagram class in S_9:

$$[65\mathbf{4}1\mathbf{7}2\mathbf{8}3\mathbf{9}, 958172634].$$

It is easily seen that $d(u, v) = 4$, $u^{-1}(4) = 3$ and $v^{-1}(4) = 9$, and so $m = 4$ and $(a_1, a_2, a_3, a_4) = (3, 5, 7, 9)$. The subsequence $u(3)u(5)u(7)u(9)$ is marked in boldface. Hence we have

$$u_1 = u = 65\mathbf{4}1\mathbf{7}2\mathbf{8}3\mathbf{9}, \qquad u_2 = u_1\,(3\ 5) = 65\mathbf{7}1\mathbf{4}2\mathbf{8}3\mathbf{9},$$

$$u_3 = u_2\,(5\ 7) = 65\mathbf{7}1\mathbf{8}2\mathbf{4}3\mathbf{9}, \qquad u_4 = u_3\,(7\ 9) = 65\mathbf{7}1\mathbf{8}2\mathbf{9}3\mathbf{4}.$$

Proposition 3.4. *For $1 \le i \le m$, the permutation u_i belongs to $[u, v]^{(i)}$.*

Proof. It is clear from the construction that $u_i^{-1}(k) = a_i$. We still need to show that each u_i has the same odd diagram as $u_1 = u$. To do this, we assert that for $i = 1, \ldots, m-1$, $(a_i\ a_{i+1})$ is a legal transposition of u_i.

Keep in mind that $(a\ b)$ is a legal transposition of u_1. Using similar arguments as in the proof of Lemma 3.2, we can readily deduce that $a_1 \equiv a_2 \pmod 2$, $u_1(p) < u(a_1)$ for

$p \in \{a_1+1, \ldots, a_2-1\}$, and $u_1(q) \notin [u(a_1), u(a_2)]$ for $q \in \{a_2+1, a_2+3, \ldots\}$. Hence $(a_1\ a_2)$ is a legal transposition of u_1.

Analogously, we can verify the assertion for $i = 2, \ldots, m-1$. This implies that each u_i has the same odd diagram as u, and so the proof is complete. ∎

3.2 The partition is uniform

Let us proceed to prove that the partition

$$[u, v] = \biguplus_{i=1}^{m} [u, v]^{(i)}$$

is uniform.

Proposition 3.5. *For $i = 1, \ldots, m$, the blocks $[u, v]^{(i)}$ have the same cardinality.*

To give a proof of Proposition 3.5, we construct a bijection

$$\phi_i \colon [u, v]^{(i)} \longrightarrow [u, v]^{(i+1)}$$

for $i = 1, 2, \ldots, m-1$. Let $w \in [u, v]^{(i)}$, and set

$$\phi_i(w) = w\,(a_i\ a_{i+1}).$$

Proposition 3.5 follows from the following assertion.

Proposition 3.6. *The map ϕ_i is a bijection.*

Proof. For $w \in [u, v]^{(i)}$, we first show that $\phi_i(w)$ belongs to $[u, v]^{(i+1)}$. It suffices to verify that $(a_i\ a_{i+1})$ is a legal transposition of w. This can be seen as follows. By Proposition 3.4, the permutation u_{i+1} defined in (3.8) belongs to $[u, v]^{(i+1)}$, which, together with (3.5), leads to

$$d(w, u_{i+1}) = k.$$

Applying Theorem 2.4 to the pair w and u_{i+1}, we see that $(a_i\ a_{i+1})$ is legal for w, and so $\phi_i(w) \in [u, v]^{(i+1)}$.

The reverse construction of ϕ_i is clear. Given $w' \in [u, v]^{(i+1)}$, set $\phi_i^{-1}(w') = w'\,(a_i\ a_{i+1})$. Using similar arguments as above, we can verify that $\phi_i^{-1}(w') \in [u, v]^{(i)}$. Since $\phi_i^{-1} \circ \phi_i = \phi_i \circ \phi_i^{-1}$ is the identity map, ϕ_i is a bijection. This completes the proof. ∎

Example 3.7. *Figure 3.2 depicts the following odd diagram class in S_7:*

$$[u = 5431627, v = 7461523].$$

We see that $d(u, v) = 3$, $u^{-1}(3) = 3$ and $v^{-1}(3) = 7$. Since

$$\{3 \leq i \leq 7 \colon u(3) \leq u(i) \leq u(7)\} = \{3, 5, 7\},$$

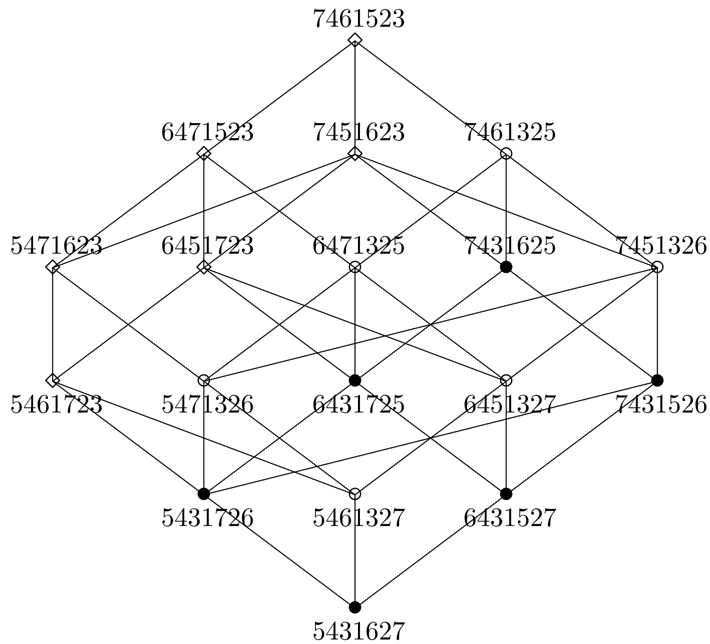


Figure 3.2: A uniform partition of an odd diagram class

$[u, v]$ *is uniformly partitioned into the following three blocks:*

$$[u,v]^{(1)} = \{w \in [u,v] \colon w^{-1}(3) = 3\},$$
$$[u,v]^{(2)} = \{w \in [u,v] \colon w^{-1}(3) = 5\},$$
$$[u,v]^{(3)} = \{w \in [u,v] \colon w^{-1}(3) = 7\},$$

which are respectively marked with solid circles, empty circles and diamond symbols.

The bijection ϕ_i enjoys the following nice property.

Proposition 3.8. *For $1 \leq i < m$, each permutation $w \in [u, v]^{(i)}$ is covered by its image $\phi_i(w)$.*

Proof. For simplicity, write $w' = \phi_i(w)$. Keep in mind that $w^{-1}(k) = a_i$ and $w'^{(-1)}(k) = a_{i+1}$. In view of (3.5), we see that $d(w, w') = k$, and so we have $w(a_{i+1}) > k$. By Proposition 2.2, we need to show that $w(t) \notin [w(a_i), w(a_{i+1})]$ for $a_i < t < a_{i+1}$.

We use proof by contradiction. Suppose otherwise that there exists $a_i < t_0 < a_{i+1}$ such that $w(a_i) < w(t_0) < w(a_{i+1})$. As explained in the proof of Proposition 3.6, the transposition $(a_i\ a_{i+1})$ is legal for w. So the conditions in Theorem 2.3 are all satisfied by w and $(a_i\ a_{i+1})$, from which we can easily check that w and $(a_i\ t_0)$ also satisfy the conditions in Theorem 2.3. Thus $(a_i\ t_0)$ is legal for w, implying that $\overline{w} = w\,(a_i\ t_0)$ belongs to $[u, v]$. However, this would lead to

$$\overline{w}^{-1}(k) = t_0 \notin \{a_1, a_2, \ldots, a_m\},$$

contrary to Lemma 3.3. This completes the proof. ∎

3.3 The blocks $[u,v]^{(i)}$ are Bruhat intervals

In this subsection, we show that each block in the partition is a Bruhat interval. This will allow us to carry out induction to complete the proof of Theorem 1.3 in Section 4. Let us first locate the minimum element of $[u,v]^{(i)}$.

Lemma 3.9. *For $1 \le i \le m$, the permutation u_i as defined in (3.8) is the minimum element of $[u,v]^{(i)}$.*

Proof. By Proposition 3.4, u_i belongs to $[u,v]^{(i)}$. It is clearly true that $u_1 = u$ is the minimum element of $[u,v]^{(1)}$. We proceed to verify the claim for u_2.

Let $w_2 \in [u,v]^{(2)}$, and set

$$w_1 = \phi_1^{-1}(w_2) \in [u,v]^{(1)}.$$

Then one can find a saturated chain from u_1 to w_1 in $[u,v]$:

$$u_1 = x_1 \lhd x_2 \lhd \cdots \lhd x_d = w_1.$$

We assert that $x_j \in [u,v]^{(1)}$ for $1 \le j \le d$. This can be seen as follows. By (3.5), we see that for $1 \le j \le d$,

$$x_j^{-1}(s) = u_1^{-1}(s), \qquad \text{for } 1 \le s < k.$$

Applying Proposition 2.1 to $x_1^{-1}, \ldots, x_d^{-1}$ together with the fact that $u_1^{-1}(k) = w_1^{-1}(k)$, we obtain that for $1 \le j \le d$, $x_j^{-1}(k) = u_1^{-1}(k) = a_1$. This verifies the assertion that $x_j \in [u,v]^{(1)}$.

For $1 \le j \le d$, define

$$y_j = \phi_1(x_j) \in [u,v]^{(2)}.$$

Note that $y_1 = u_2$ and $y_d = w_2$. We claim that $y_1, \ldots, y_d$ form a saturated chain:

$$u_2 = y_1 \lhd y_2 \lhd \cdots \lhd y_d = w_2.$$

Let us first check that $y_1 \lhd y_2$. Since $x_1 \lhd x_2$, we can find a transposition t such that $x_2 = x_1\, t$. By Proposition 3.8, we have $x_1 \lhd y_1$ and $x_2 \lhd y_2$. Write t' for the transposition $(a_1\ a_2)$. Then $y_1 = x_1\, t'$ and $y_2 = x_2\, t'$. So we have

$$y_2 = x_2\, t' = x_1\, t\, t' = x_1\, t'\, (t'\, t\, t') = y_1\, (t'\, t\, t'). \tag{3.9}$$

See Figure 3.3 for an illustration. Note that $t'\, t\, t'$ is a transposition. Moreover, we have the following length relation

$$\ell(y_2) = \ell(x_2) + 1 = (\ell(x_1) + 1) + 1 = \ell(y_1) + 1,$$

which, along with (3.9), implies that $y_1 \lhd y_2$.

Using the same arguments as above, we can verify that $y_j \lhd y_{j+1}$ for $j = 2, \ldots, d-1$, and so $y_1, \ldots, y_d$ constitute a saturated chain. This yields that $u_2 \le w_2$, and hence u_2 is the minimum of element of $[u,v]^{(2)}$.

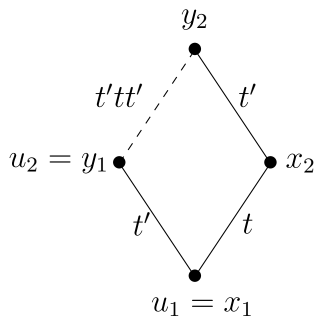


Figure 3.3: An illustration of (3.9)

To check that u_3 is the minimum of element of $[u, v]^{(3)}$, we choose $w_3 \in [u, v]^{(3)}$, and show that $u_3 \leq w_3$ by using completely analogous analysis to the proof for $u_2 \leq w_2$. Continuing this procedure, we eventually conclude that u_i is the minimum element of $[u, v]^{(i)}$ for $1 \leq i \leq m$. This completes the proof. ∎

We next determine the maximum element of $[u, v]^{(i)}$. For $i = m$, set $v_m = v$. For $1 \leq i < m$, set

$$v_i = \phi_i^{-1}(v_{i+1}) = v_{i+1}(a_i\ a_{i+1}).$$

The proof of the following lemma is the same as that of Lemma 3.9, and so is omitted.

Lemma 3.10. *For $1 \leq i \leq m$, v_i is the maximum element of $[u, v]^{(i)}$.*

Theorem 3.11. *For $1 \leq i \leq m$, the block $[u, v]^{(i)}$ is the Bruhat interval $[u_i, v_i]$.*

Proof. By Lemmas 3.9 and 3.10, we have

$$[u, v]^{(i)} \subseteq [u_i, v_i].$$

The reverse inclusion is explained as follows. For $w \in [u_i, v_i]$, by (3.5) and the fact $u_i^{-1}(k) = v_i^{-1}(k) = a_i$, Proposition 2.1 forces that

$$w^{-1}(s) = u_i^{-1}(s), \quad \text{for } 1 \leq s \leq k,$$

which in particular leads to $w^{-1}(k) = a_i$, and so $w \in [u, v]^{(i)}$, as desired. ∎

4 Proof of Theorem 1.3

We are now ready to give a proof of Theorem 1.3. Let us start with the observation that the Poincaré polynomial of an odd diagram class $[u, v]$ is the product of the Poincaré polynomial of the Bruhat interval $[u_m, v]$ and the factor $1 + t + \cdots + t^{m-1}$.

Lemma 4.1. *With the notation as in Section 3, we have*

$$P_{u,v}(t) = (1 + t + \cdots + t^{m-1})P_{u_m,v}(t). \tag{4.1}$$

Proof. By Proposition 3.6 and Theorem 3.11, for $1 \le i \le m-1$, ϕ_i is a bijection from the interval $[u_i, v_i]$ to the interval $[u_{i+1}, v_{i+1}]$. Along with Proposition 3.8 and the fact that $\ell(u_{i+1}) = \ell(u_i) + 1$, it is easily checked that

$$P_{u_i,v_i}(t) = P_{u_{i+1},v_{i+1}}(t),$$

from which we deduce that

$$\begin{aligned} P_{u,v}(t) &= \sum_{i=1}^{m} t^{\ell(u_i)-\ell(u_1)} P_{u_i,v_i}(t) \\ &= (1 + t + \cdots + t^{m-1}) P_{u_m,v}(t), \end{aligned}$$

as required. ∎

Proof of Theorem 1.3. We proceed to consider the Bruhat interval $[u_m, v]$. Carry out the same procedure as in Section 3 by replacing $[u, v]$ with $[u_m, v]$ to find a uniform partition of the latter. This is sketched as follows. For convenience, write $w = u_m$. Set $d(w, v) = k'$, and $w^{-1}(k') = a'$ and $v^{-1}(k') = b'$. Let

$$\{a' = a'_1 < a'_2 < \cdots < a'_{m'} = b'\} = \{a' \le i \le b' \colon w(a') \le w(i) \le w(b')\}.$$

For $1 \le i \le m'$, set

$$[w, v]^{(i)} = \{\pi \in [w, v] \colon \pi^{-1}(k') = a'_i\}.$$

For $1 \le i < m'$, we define a map

$$\phi'_i \colon [w, v]^{(i)} \longrightarrow [w, v]^{(i+1)}$$

by letting

$$\phi'_i(\pi) = \pi\,(a'_i\ a'_{i+1}), \quad \text{for } \pi \in [w, v]^{(i)}.$$

As in Proposition 3.6, we can show that each ϕ'_i is a bijection, and so the blocks $[w, v]^{(i)}$ form a uniform partition of the interval $[w, v]$.

Set $w_1 = w$, and $w_{i+1} = \phi'_i(w_i)$ for $i = 1, 2, \ldots, m' - 1$. Similarly to Lemma 4.1, we then deduce that

$$P_{w,v}(t) = (1 + t + \cdots + t^{m'-1}) P_{w_{m'},v}(t).$$

We can iterate this reasoning and apply it to find a partition of the interval $[w_{m'}, v]$. The procedure eventually terminates, and we reach a proof of Theorem 1.3. ∎

Remark. For any odd diagram class $[u, v]$, the proof of Theorem 1.3 gives an explicit algorithm to determine a sequence of positive integers $m_1, \ldots, m_h$ such that

$$P_{u,v}(t) = \prod_{i=1}^{h} (1 + t + \cdots + t^{m_i - 1}).$$

5 Concluding remarks

This section is devoted to some observations and problems concerning odd diagram classes. As mentioned in the introduction, when a lower interval is rank-symmetric, its Poincaré polynomial factors into polynomials of the form $1 + t + \cdots + t^m$. Theorem 1.3 tells us that the Poincaré polynomial of an odd diagram class satisfies a similar factorization. It is natural to ask if odd diagram classes share more properties satisfied by rank-symmetric lower intervals.

5.1 Self-dual odd diagram classes

The "top-heavy" phenomenon of a lower Bruhat interval $[e, w]$ was established by Björner and Ekedahl [3]. For $0 \le k \le \ell(w)$, Let

$$P_k^w = \{u \le w \colon \ell(u) = k\}$$

denote the rank k component of $[e, w]$.

Theorem 5.1 (Björner–Ekedahl [3]). *For $w \in S_n$ and $0 \le k \le \ell(w)/2$,*

$$\#P_k^w \le \#P_{\ell(w)-k}^w. \tag{5.1}$$

It should be pointed out that Theorem 5.1 holds in general for parabolic quotients of Weyl groups.

When the equality in (5.1) holds, $[e, w]$ is rank-symmetric. In general, a lower interval is not self-dual. Gaetz and Gao [9] found that the self-duality of $[e, w]$ is determined by local information of $[e, w]$. Let Γ_w (resp., Γ^w) denote the bipartite graph on $P_1^w \cup P_2^w$ (resp., $P_{\ell(w)-1}^w \cup P_{\ell(w)-2}^w$) with edges given by the covering relations in the Bruhat order.

Theorem 5.2 (Gaetz–Gao [9, Theorem 4]). *The interval $[e, w]$ in the symmetric group is self-dual if and only if the bipartite graphs Γ_w and Γ^w are isomorphic.*

Note that there are two other criteria for the self-duality of $[e, w]$ in [9, Theorem 4].

As noticed by Brenti, Carnevale and Tenner [7], odd diagram classes are not self-dual in general. For example, the following odd diagram class

$$[654172839, 958172634]$$

is not self-dual. In fact, all odd diagram classes in S_n for $n \le 8$ are self-dual, and there are 8 and 118 non-self-dual odd diagram classes in S_9 and S_{10}, respectively.

We can analogously define bipartite graphs in the same spirit for any Bruhat interval. We checked that the odd diagrams classes in S_n for $n \le 10$ satisfy the bipartite graph criterion for the self-duality, as given in Theorem 5.2.

Question 5.3. *Is the criterion in Theorem 5.2 true for odd diagram classes?*

5.2 Kazhdan-Lusztig polynomials

Let us start with a brief overview of the Kazhdan-Lusztig polynomials introduced by Kazhdan and Lusztig [13]. See [2, Chapter 5] or [12, Chapter 7] for further information. Let (W, S) be a Coxeter system, and $\leq$ denote the Bruhat order on W. For $w \in W$, a generator $s \in S$ is a (right) descent if $\ell(ws) < \ell(w)$. As usual, we use $D(w)$ to denote the set of descents of w.

For $x, y \in W$, the R-polynomial $R_{x,y}(q)$ can be defined in a recursive way:

(i) $R_{x,y}(q) = 0$ if $x \not\leq y$;

(ii) $R_{x,y}(q) = 1$ if $x = y$;

(iii) If $x < y$ and $s \in D(y)$, then

$$R_{x,y}(q) = \begin{cases} R_{xs,\,ys}(q), & \text{if } s \in D(x), \\ qR_{xs,\,ys}(q) + (q-1)R_{x,\,ys}(q), & \text{if } s \notin D(x). \end{cases}$$

The Kazhdan-Lusztig polynomials $P_{x,y}(q)$ are the unique family of polynomials determined by the following conditions:

(i) $P_{x,y}(q) = 0$ if $x \not\leq y$;

(ii) $P_{x,y}(q) = 1$ if $x = y$;

(iii) if $x \leq y$, then

$$\deg(P_{x,y}(q)) \leq \lfloor (\ell(y) - \ell(x) - 1)/2 \rfloor$$

and

$$q^{\ell(y)-\ell(x)} P_{x,y}\left(\frac{1}{q}\right) = \sum_{x \leq z \leq y} R_{x,z}(q) P_{z,y}(q).$$

From the context, no confusion should be caused by the similarity of the notation of the Kazhdan-Lusztig polynomial $P_{x,y}(q)$ and the Poincaré polynomial $P_{u,v}(t)$.

Recall from Introduction that $[e, w]$ is rank-symmetric if and only if $P_{e,w}(q) = 1$. We computed that $P_{u,v}(q) = 1$ for all odd diagram classes $[u, v]$ in S_n for $n \leq 10$.

Conjecture 5.4. *The Kazhdan-Lusztig polynomial associated to any odd diagram class is equal to 1.*

Remark. In the lower interval case, $P_{e,w}(q) = 1$ if and only if $[e, w]$ is rank-symmetric. However, this is not true for a general interval. For example, let $w_0 = n \cdots 21$ be the longest permutation in S_n. For any $w \in S_n$, we have $P_{w,w_0}(q) = 1$ [4, (21)], but not every interval $[w, w_0]$ is rank-symmetric. The following conditions are equivalent for a general Kazhdan-Lusztig polynomial equal to 1.

Theorem 5.5 (Carrell [8, Theorem C])**.** *Let (W, S) be a Coxeter system, and $T = \cup_{w \in W} wSw^{-1}$ be the set of reflections. Then the following are equivalent:*

(i) $P_{x,y}(q) = 1$*;*

(ii) $P_{w,y}(q) = 1$ *for all* $w \in [x, y]$*;*

(iii) for all $w \in [x, y]$,
$$\#\{t \in T \colon w < tw \leq y\} = \ell(y) - \ell(w).$$

Acknowledgments. We wish to thank the referees for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (11971250, 12071320) and Sichuan Science and Technology Program (Grant No. 2020YJ0006).

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