Planar Turán numbers of cubic graphs and disjoint union of cycles

Yongxin Lan* Yongtang Shi† Zi-Xia Song‡

Abstract

The planar Turán number of a graph H, denoted by $ex_{\mathcal{P}}(n, H)$, is the maximum number of edges in a planar graph on n vertices without containing H as a subgraph. This notion was introduced by Dowden in 2016 and has attracted quite some attention since then; those work mainly focus on finding $ex_{\mathcal{P}}(n, H)$ when H is a cycle or Theta graph or H has maximum degree at least four. In this paper, we first completely determine the exact values of $ex_{\mathcal{P}}(n, H)$ when H is a cubic graph. We then prove that $ex_{\mathcal{P}}(n, 2C_3) = \lceil 5n/2 \rceil - 5$ for all $n \geq 6$, and obtain the lower bounds of $ex_{\mathcal{P}}(n, 2C_k)$ for all $n \geq 2k \geq 8$. Finally, we also completely determine the exact values of $ex_{\mathcal{P}}(n, K_{2,t})$ for all $t \geq 3$ and $n \geq t + 2$.

Key words. Turán number, extremal planar graph, planar triangulation
AMS subject classifications. 05C10, 05C35

1 Introduction

All graphs considered in this paper are finite and simple. We use K_n, C_n and P_n to denote the complete graph, cycle and path on n vertices, respectively. Given a graph G, we use $|G|$ to denote the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree. For a vertex $v \in V(G)$, we will use $N_G(v)$ to denote the set of vertices in G which are adjacent to v. We define $N_G[v] := N_G(v) \cup \{v\}$. For any $S, S' \subseteq V(G)$, we use $e_G(S, S')$ to denote the size of edge set $\{xy \in E(G) \mid x \in S \text{ and } y \in S'\}$. For any set $S \subset V(G)$, the subgraph of G induced on S, denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) \mid x, y \in S\}$. We denote by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$. If $S = \{v\}$, then we simply write $G \setminus v$. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex-disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \,|\, x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). For a positive integer t and a graph H, we use tH to denote the disjoint union of t copies of H. Let T_n denote a plane triangulation on $n \geq 3$ vertices. We use T_n^- and K_n^- to denote a graph obtained from T_n and K_n with one edge removed, respectively. Given two isomorphic graphs

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G and H, we may (with a slight but common abuse of notation) write $G = H$. For any positive integer k, we define $[k] := \{1, 2, \ldots, k\}$.

Given a graph H, a graph is H-free if it does not contain H as a subgraph. One of the best known results in extremal graph theory is Turán's Theorem [12], which gives the maximum number of edges that a K_t-free graph on n vertices can have. The celebrated Erdős-Stone Theorem [3] then extends this to the case when K_t is replaced by an arbitrary graph H with at least one edge, showing that the maximum number of edges possible is $(1 + o(1))\binom{n}{2}\left(\frac{\chi(H)-2}{\chi(H)-1}\right)$, where $\chi(H)$ denotes the chromatic number of H.

In this paper, we continue to study the topic of "extremal" planar graphs, that is, how many edges can an H-free planar graph on n vertices have? We define $ex_{\mathcal{P}}(n, H)$ to be the maximum number of edges in an H-free planar graph on n vertices. Dowden [2] initiated the study of $ex_{\mathcal{P}}(n, H)$ and proved the following result.

Theorem 1.1 (Dowden [2]) *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, C_3) = 2n - 4$ *for all* $n \geq 3$.

(b) $ex_{\mathcal{P}}(n, K_4) = 3n - 6$ *for all* $n \geq 4$.

(c) $ex_{\mathcal{P}}(n, C_4) \leq 15(n-2)/7$ *for all* $n \geq 4$, *with equality when* $n \equiv 30 \pmod{70}$.

(d) $ex_{\mathcal{P}}(n, C_5) \leq 12(n-2)/5$ *for all* $n \geq 5$.

(e) $ex_{\mathcal{P}}(n, C_5) \leq (12n - 33)/5$ *for all* $n \geq 11$. *Equality holds for infinity many* n.

This topic has attracted quite some attention since then. We refer the reader to a recent survey [11] of the present authors for more information. Let Θ_k denote the family of Theta graphs on $k \geq 4$ vertices, that is, graphs obtained from C_k by adding an additional edge joining two non-consecutive vertices. The present authors [10] obtained tight upper bounds for $ex_{\mathcal{P}}(n, \Theta_k)$ for $k \in \{4, 5\}$ and an upper bound for $ex_{\mathcal{P}}(n, \Theta_6)$.

Theorem 1.2 (Lan, Shi and Song [10]) *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, \Theta_4) \leq 12(n-2)/5$ *for all* $n \geq 4$, *with equality when* $n \equiv 12 \pmod{20}$.

(b) $ex_{\mathcal{P}}(n, \Theta_5) \leq 5(n-2)/2$ *for all* $n \geq 5$, *with equality when* $n \equiv 50 \pmod{120}$.

(c) $ex_{\mathcal{P}}(n, C_6) \leq ex_{\mathcal{P}}(n, \Theta_6) \leq 18(n-2)/7$ *for all* $n \geq 6$.

Theorem 1.2(c) has been strengthened by the authors in [5, 6] with tight upper bounds.

Theorem 1.3 (Ghosh et al. [5, 6]) *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, C_6) \leq (5n - 14)/2$ *for all* $n \geq 18$, *with equality when* $n \equiv 10 \pmod{18}$.

(b) $ex_{\mathcal{P}}(n, \Theta_6) \leq (18n - 48)/7$ *for all* $n \geq 14$. *Equality holds for infinitely many* n.

As observed in [2], for all $n \geq 6$, the planar triangulation $2K_1 + C_{n-2}$ is K_4-free. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all graphs H which contains K_4 as a subgraph and $n \geq 6$. The present authors [9] also investigated a variety of sufficient conditions on K_4-free planar graphs H such that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$.

Theorem 1.4 (Lan, Shi and Song [9]) *Let* H *be a* K_4*-free planar graph and let* $n \geq |H|$ *be an integer. Then* $ex_{\mathcal{P}}(n, H) = 3n - 6$ *if one of the following holds, where* $n_k(H)$ *denotes the number of vertices of degree* k *in* H *for a positive integer* k.

(a) $\chi(H) = 4$ *and* $n \geq |H| + 2$.

(b) $\Delta(H) \geq 7$.

(c) $\Delta(H) = 6$ *and either* $n_6(H) + n_5(H) \geq 2$ *or* $n_6(H) + n_5(H) = 1$ *and* $n_4(H) \geq 5$.

(d) $\Delta(H) = 5$ *and either* H *has at least three* 5*-vertices or* H *has exactly two adjacent* 5*-vertices.*

(e) $\Delta(H) = 4$ *and* $n_4(H) \geq 7$.

(f) H *is* 3*-regular with* $|H| \geq 9$ *or* H *has at least three vertex-disjoint cycles or* H *has exactly one vertex* u *of degree* $\Delta(H) \in \{4, 5, 6\}$ *such that* $\Delta(H[N_H(u)]) \geq 3$.

(g) $\delta(H) \geq 4$ *or* H *has exactly one vertex of degree at most* 3.

In the same paper, the present authors [9] also determined the values of $ex_{\mathcal{P}}(n, H)$ when H is a star, or wheel or (t, r)-fan. Ghosh, Győri, Paulos and Xiao [7] recently determined the values of $ex_{\mathcal{P}}(n, H)$ when H is a double star. Theorem 1.4 implies that $ex_{\mathcal{P}}(n, H)$ remains wide open when H is subcubic. In particular, it seems quite non-trivial to determine $ex_{\mathcal{P}}(n, C_k)$ for all $k \geq 7$. Very recently, Cranston, Lidický, Liu and Shantanam [1] proved that for each $k \geq 11$ and n sufficiently large (as a function of k),

$$ex_{\mathcal{P}}(n, C_k) > \left(3 - \frac{3}{k}\right) n - 6 - \frac{6}{k}.$$

They further proposed the following conjecture.

Conjecture 1.5 (Cranston, Lidický, Liu and Shantanam [1]) *There exists a constant* D *such that for all* k *and for all sufficiently large* n, *we have*

$$ex_{\mathcal{P}}(n, C_k) \leq \left(3 - \frac{3}{Dk^{lg_2^3}}\right) n.$$

In this paper, we continue to study the planar Turán numbers of k-regular graphs, disjoint union of cycles and complete bipartite graphs. We prove the following main results.

Theorem 1.6 *Let H be a k-regular planar graph with $k \geq 3$ and let $n \geq |H|$ be an integer. Then*

$$ex_{\mathcal{P}}(n, H) = \begin{cases} 3n-6 & \text{if } |H| \geq 8, \text{ or } |H| = 6 \text{ and } n \geq 10; \\ 3n-7 & \text{if } |H| = 6 \text{ and } n \leq 9. \end{cases}$$

Theorem 1.7 *For integers n and t with $n \geq 3t \geq 3$, we have*

$$ex_{\mathcal{P}}(n, tC_3) = \begin{cases} 3n-6 & \text{if } t \geq 3; \\ \lceil \frac{5n}{2} \rceil - 5 & \text{if } t = 2; \\ 2n-4 & \text{if } t = 1. \end{cases}$$

Theorem 1.8 *Let n and k be positive integers.*

(a) Suppose $n \geq 2k \geq 8$, and r is the remainder of $n-3$ when divided by $k-2$. Then

$$ex_{\mathcal{P}}(n, 2C_k) \geq \left(3 - \frac{1}{k-2}\right) n + \frac{3+r}{k-2} - 5 + \max\{1-r, 0\}.$$

(b) Suppose $n \geq 2k \geq 14$, and ε_1 and ε_2 are the remainder of $n-(2k-1)$ when divided by $k-4+\frac{k-1}{2}$ (k is odd) and $k-6+\frac{k}{2}$ (k is even), respectively.

(b1) If k is odd, then $ex_{\mathcal{P}}(n, 2C_k) = 3n-6$ for all $n \leq 3k-4$, and

$$ex_{\mathcal{P}}(n, 2C_k) \geq \left(3 - \frac{1}{k-4+\lfloor k/2 \rfloor}\right) n + \frac{5+\varepsilon_1}{k-4+\lfloor k/2 \rfloor} - \frac{17}{3} + \max\{1-\varepsilon_1, 0\} \text{ for all } n \geq 3k-3.$$

(b2) If k is even, then $ex_{\mathcal{P}}(n, 2C_k) = 3n-6$ for all $n \leq 3k-7$, and

$$ex_{\mathcal{P}}(n, 2C_k) \geq \left(3 - \frac{1}{k-6+k/2}\right) n + \frac{7+\varepsilon_2}{k-6+k/2} - \frac{17}{3} + \max\{1-\varepsilon_2, 0\} \text{ for all } n \geq 3k-6.$$

Theorem 1.9 *For integers $t \geq 3$ and $n \geq t+2$, we have*

$$ex_{\mathcal{P}}(n, K_{2,t}) = \begin{cases} 3n-6 & \text{if } t \geq 5 \text{ and } n \geq t+2, \text{ or } t = 4 \text{ and } n \geq 9, \text{ or } t = 3 \text{ and } n \geq 12; \\ 3n-7 & \text{if } t = 4 \text{ and } n \leq 8; \\ 3n-8 & \text{if } t = 3 \text{ and } n \leq 11. \end{cases}$$

The remainder of the paper is organized as follows: we prove Theorem 1.6 in Section 2, Theorems 1.7 and 1.8 in Section 3, and Theorem 1.9 in Section 4.

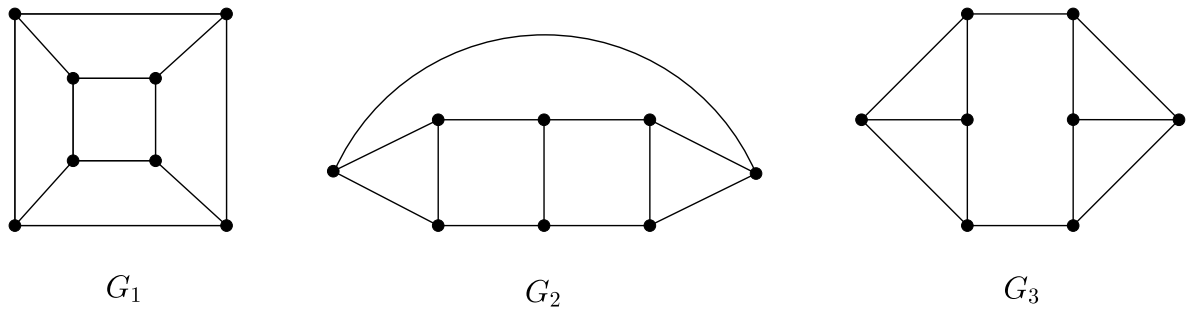


Figure 1: Graphs G_1, G_2 and G_3

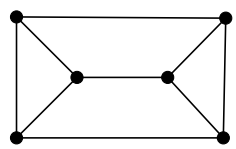


Figure 2: The unique cubic planar graph H on 6 vertices.

2 Planar Turán number of regular graphs

We begin this section with two lemmas that will be essential in determining the planar Turán numbers of regular planar graphs. Proofs of Lemmas 2.1 and 2.2 are given in the Appendix.

Lemma 2.1 *Let G be a cubic planar graph on 8 vertices. If G is K_4-free, then $G \in \{G_1, G_2, G_3\}$, where graphs G_1, G_2, G_3 are depicted in Figure 1.*

Lemma 2.2 *Let G be a cubic planar graph on 6 vertices. Then G is the graph given in Figure 2.*

Proof of Theorem 1.6: Let H and n be given as in the statement. By Theorem 1.1(b) and Theorem 1.4(f, g), we see that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$ if H contains a copy of K_4, or $k = 3$ and $|H| \geq 9$, or $k \geq 4$. For the remainder of the proof, let H be a K_4-free cubic planar graph with $|H| \leq 8$. Assume first that $|H| = 8$. By Lemma 2.1, we see that $H \in \{G_1, G_2, G_3\}$. Then the planar triangulation $2K_1 + C_{n-2}$ is H-free when $n > |H|$, and the planar triangulation $K_2 + P_{n-2}$ is H-free when $n = |H|$. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$ and $|H| = 8$.

It remains to consider the case when $|H| = 6$. Then H is the graph given in Figure 2. We next show that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq 10$. Let $n := 4k + 2 + \ell$ for some $\ell \in \{0, 1, 2, 3\}$ and integer $k \geq 2$. Let Q_k be a plane triangulation on $n = 4k + 2$ vertices constructed as follows: for each $i \in [k]$, let C^i be a cycle with vertices $u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}$ in order, let Q_k be the plane triangulation obtained from disjoint union of $C^1, \ldots, C^k$ by adding edges $u_{i,j}u_{i+1,j}$ and $u_{i,j}u_{i+1,j+1}$ for all $i \in [k-1]$ and $j \in [4]$, where all arithmetic on the index $j + 1$ here is done modulo 4, and finally adding two new nonadjacent vertices u and v such that u is adjacent to all vertices of C^1

and v is adjacent to all vertices of C^k. The graph Q_k when $k = 3$ is depicted in Figure 3. Let $Q_k^\ell = Q_k$ if $\ell = 0$. For $\ell \in \{1, 2, 3\}$, let F_j be the face of Q_k with vertices $u_{k-1,j}, u_{k,j}, u_{k,j+1}$ for each $j \in [\ell]$, and let Q_k^ℓ be the plane triangulation on n vertices obtained from Q_k by adding one new vertex, say x_j, adjacent to the three vertices on the boundary of F_j for each $j \in [\ell]$. It can be checked that Q_k^ℓ is H-free. Therefore, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq 10$.

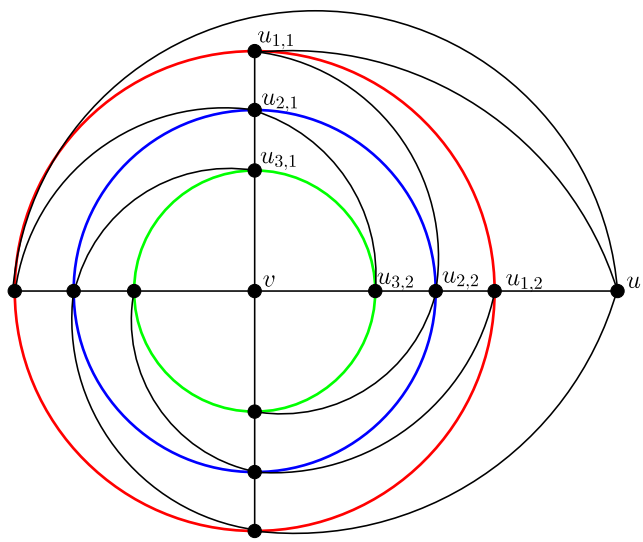


Figure 3: The plane triangulation Q_k when $k = 3$, where C^1, C^2, C^3 are in red, blue and green, respectively.

We next show that $ex_{\mathcal{P}}(n, H) = 3n - 7$ for each $n \in \{6, 7, 8, 9\}$. To obtain the desired upper bound, it suffices to show that every plane triangulation T on $n \in \{6, 7, 8, 9\}$ vertices contains a copy of H. Suppose not. Let T be an H-free plane triangulation such that $|T| \in \{6, 7, 8, 9\}$ is minimum. Note that every plane triangulation on 6 vertices contains a copy of H. Thus $|T| \in \{7, 8, 9\}$. Let $x \in V(T)$ with $d_T(x) = \delta(T)$. Then $d_T(x) \in \{3, 4\}$. If $d_T(x) = 3$, then $T - x$ is a plane triangulation and contains a copy of H by the minimality of $|T|$, a contradiction. Thus $d_T(x) = 4$. Clearly, $T[N[x]]$ is a wheel on 5 vertices. Let $Y = V(T) \setminus N[x]$. Note that $|Y| \leq 4$. Then $d_{N(x)}(y) \leq 2$ for every $y \in Y$, else T contains a copy of H. It follows that $|Y| = 4$ and so $|T| = 9$, else $e(T) = e(T[N[x]]) + e(N(x), Y) + e(T[Y]) \leq 8 + 2|Y| + 2|Y| - 3 < 3(5 + |Y|) - 6$. Since $\delta(T) = 4$, we see that $e(T[Y]) \leq 5$. Then $e(T) = e(T[N[x]]) + e(N(x), Y) + e(T[Y]) \leq 8 + 2|Y| + 5 = 21$, which yields $T \cong Q_2 - u + u_{1,1}u_{1,3}$ and so T contains a copy of H, a contradiction. Thus, every plane triangulation T on $n \in \{6, 7, 8, 9\}$ vertices contains a copy of H. On the other hand, for each $n \in \{6, 7, 8\}$, the planar graph $K_2 + (P_3 \cup P_{n-5})$ is H-free with $3n - 7$ edges because every induced subgraph of $K_2 + (P_3 \cup P_{n-5})$ on 6 vertices contains a vertex of degree two or is isomorphic to $K_2 + 2P_2$; for $n = 9$, the planar graph $Q_2 \setminus u$ is H-free with $3n - 7$ edges because every induced subgraph of $Q_2 \setminus u$ on 6 vertices either contains a vertex of degree two or is isomorphic to a wheel

on 6 vertices. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 7$ for each $n \in \{6, 7, 8, 9\}$, as desired. ■

3 Planar Turán number of disjoint union of cycles

Given a plane graph G and an integer $i \geq 3$, an i-face in G is a face of order i. Let f_i and $f(G)$ denote the number of i-faces and all faces in G, respectively. In this section we study the planar Turán number of disjoint union of cycles. We first consider tC_k, the t vertex-disjoint copies of C_k, and give a tight bound for $ex_{\mathcal{P}}(n, tC_3)$ for all $n \geq 3t \geq 3$. It is worth noting that $ex_{\mathcal{P}}(n, H) = 3n-6$ if H has three vertex-disjoint cycles, due to Theorem 1.4(f).

Proof of Theorem 1.7: By Theorem 1.1(a) and Theorem 1.4(f), $ex_{\mathcal{P}}(n, tC_3) = 2n - 4$ if $t = 1$, and $ex_{\mathcal{P}}(n, tC_3) = 3n - 6$ if $t \geq 3$. We may assume that $t = 2$. Then $n \geq 6$. We first show that $ex_{\mathcal{P}}(n, 2C_3) \geq \lceil 5n/2 \rceil - 5$. Let P be a path on $n-2$ vertices and S be a maximum independent set of P containing the two ends of P. Let G be the planar graph on n vertices obtained from P by adding two new adjacent vertices u and v such that u is joined to every vertex in $V(P)$ and v is joined to every vertex in S. Then G is $2C_3$-free with $|G| = n$ and $e(G) = (n - 3) + (n - 1) + \lceil (n - 2)/2 \rceil = \lceil 5n/2 \rceil - 5$. Hence, $ex_{\mathcal{P}}(n, 2C_3) \geq e(G) = \lceil 5n/2 \rceil - 5$.

We next show that $ex_{\mathcal{P}}(n, 2C_3) \leq \lceil 5n/2 \rceil - 5$. It can be easily checked that every plane graph T_n^- contains a copy of $2C_3$, where $n \in \{6, 7\}$. Hence, $ex_{\mathcal{P}}(n, 2C_3) \leq e(T_n^-) - 1 = 3n - 8 = \lceil 5n/2 \rceil - 5$ when $n \in \{6, 7\}$. We may assume that $n \geq 8$. Let G be a $2C_3$-free plane graph on $n \geq 8$ vertices. We first prove that

$(*)$ $f_3 \leq n - 1$.

To prove $(*)$, suppose $f_3 \geq n \geq 8$. Let $\mathcal{F}$ be the set of all 3-faces of G. Then $|\mathcal{F}| = f_3$. For each $v \in V(G)$, let $\mathcal{F}(v) := \{F \in \mathcal{F} \mid v \in V(F)\}$. Then $|\mathcal{F}(v)| \leq n - 1$ and so $\mathcal{F} \setminus \mathcal{F}(v) \neq \emptyset$. Since G is $2C_3$-free, we see that $V(F) \cap V(F') \neq \emptyset$ for every pair $F, F' \in \mathcal{F}$. Since $f_3 \geq n \geq 8$, there exist $F', F'' \in \mathcal{F}$ such that $|V(F') \cap V(F'')| = 1$. We may assume that $V(F') = \{x, y, z\}$ and $V(F'') = \{x, u, w\}$, where x, y, z, u, w are pairwise distinct. It follows that for every $F \in \mathcal{F} \setminus \mathcal{F}(x)$, we have $|V(F) \cap \{y, z\}| \geq 1$ and $|V(F) \cap \{u, w\}| \geq 1$.

Suppose $|V(F) \cap \{y, z\}| = 1$ and $|V(F) \cap \{u, w\}| = 1$ for every $F \in \mathcal{F} \setminus \mathcal{F}(x)$. Then $|\mathcal{F} \setminus \mathcal{F}(x)| \leq 4$. In addition, if $|\mathcal{F} \setminus \mathcal{F}(x)| = 1$, then $|\mathcal{F}(x)| \leq 6$ (see Figure 4(a) when $|\mathcal{F}(x)| = 6$); if $|\mathcal{F} \setminus \mathcal{F}(x)| = 2$, then $|\mathcal{F}(x)| \leq 4$ (see Figure 4(b) when $|\mathcal{F}(x)| = 4$, where vertices a, b, c are not necessary distinct); if $3 \leq |\mathcal{F} \setminus \mathcal{F}(x)| \leq 4$, then $|\mathcal{F}(x)| \leq 3$ (see Figure 4(c, d) when $|\mathcal{F}(x)| = 3$, where vertices a and z in Figure (c) are not necessary distinct). Thus $f_3 = |\mathcal{F} \setminus \mathcal{F}(x)| + |\mathcal{F}(x)| \leq 7$, contrary to the assumption that $f_3 \geq n \geq 8$.

Suppose for some $F^* \in \mathcal{F} \setminus \mathcal{F}(x)$, we have $y, z \in V(F^*)$ or $u, w \in V(F^*)$, say the former. Recall that for every $F \in \mathcal{F} \setminus \mathcal{F}(x)$, we have $|V(F) \cap \{y, z\}| \geq 1$ and $|V(F) \cap \{u, w\}| \geq 1$. We may further assume that $u \in V(F^*)$. Then $|\mathcal{F} \setminus \mathcal{F}(x)| \leq 3$. In addition, if $|\mathcal{F} \setminus \mathcal{F}(x)| = 1$, then $|\mathcal{F}(x)| \leq 5$ (see Figure 5(a) when $|\mathcal{F}(x)| = 5$, where vertices a and w, or b and c are not necessary distinct); if $2 \leq |\mathcal{F} \setminus \mathcal{F}(x)| \leq 3$, then $|\mathcal{F}(x)| \leq 4$ (see Figure 5(b, c), where vertices a, b, c in Figure (b) and a, z

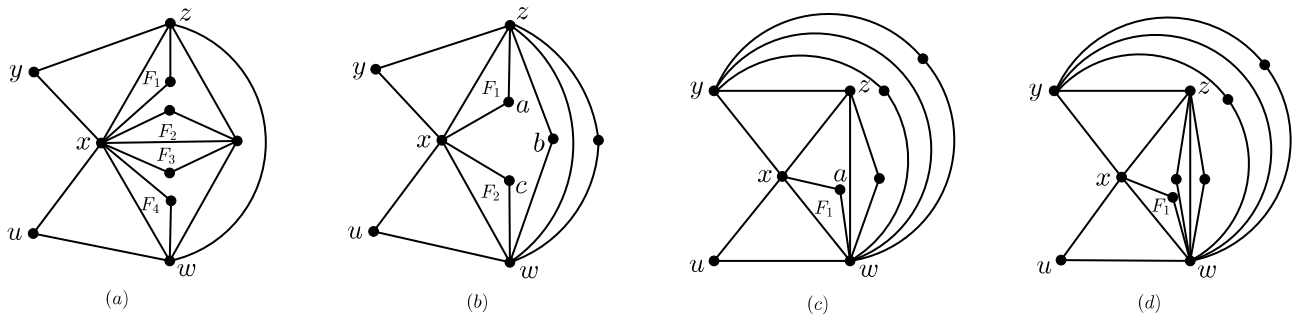


Figure 4: (a) $\mathcal{F}(x) = \{F', F'', F_1, F_2, F_3, F_4\}$; (b) $\mathcal{F}(x) = \{F', F'', F_1, F_2\}$; (c, d) $\mathcal{F}(x) = \{F', F'', F_1\}$.

in Figure (c) are not necessary distinct). It follows that $f_3 = |\mathcal{F} \setminus \mathcal{F}(x)| + |\mathcal{F}(x)| \leq 7$, contrary to the assumption that $f_3 \geq n \geq 8$. This completes the proof of $(*)$.

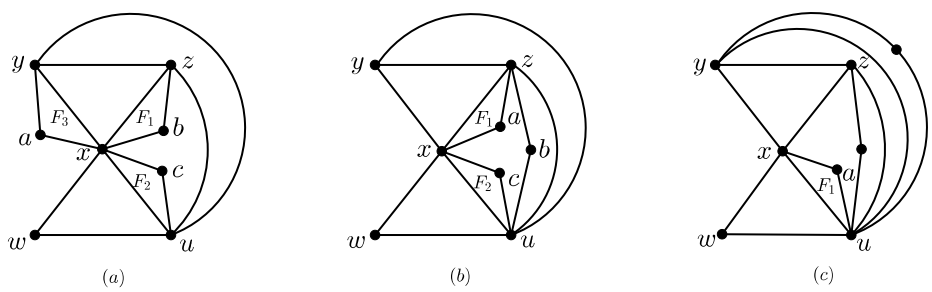


Figure 5: (a) $\mathcal{F}(x) = \{F', F'', F_1, F_2, F_3\}$; (b) $\mathcal{F}(x) = \{F', F'', F_1, F_2\}$; (c) $\mathcal{F}(x) = \{F', F'', F_1\}$.

By $(*)$, we see that

$$2e(G) = 3f_3 + \sum_{i \geq 4} i f_i \geq 3f_3 + 4(f(G) - f_3) = 4f(G) - f_3 \geq 4f(G) - (n-1),$$

which implies that $f(G) \leq (2e(G) + n - 1)/4$. By Euler's formula,

$$n - 2 = e(G) - f(G) \geq e(G)/2 - (n-1)/4.$$

Hence, $e(G) \leq \lceil 5n/2 \rceil - 5$, as desired. This completes the proof of Theorem 1.7. ■

For general $k \geq 4$, we give lower bound constructions for $ex_{\mathcal{P}}(n, 2C_k)$. It would be interesting to know whether the bounds in Theorem 1.8(b) are desired tight upper bounds for $ex_{\mathcal{P}}(n, 2C_k)$ for all $k \geq 7$ and n sufficiently large.

Proof of Theorem 1.8: (a) Let n, k, r be given as in the statement. Let $t \geq 0$ be an integer satisfying

$$(2k-1) + t(k-2) + r = n.$$

Let $P_1, P_2, \ldots, P_{t+1}$ be vertex-disjoint paths with $|P_{t+1}| = 2k-3$ and $|P_i| = k-2$ for each $i \in [t]$. Let $H = P_1 \cup \cdots \cup P_{t+1}$. Then $|H| = (2k-3) + t(k-2) = n-2-r$ and

$$e(H) = 2k-4+t(k-3) = 2k-4+(n-2k+1-r)\left(1-\frac{1}{k-2}\right).$$

Let Q be null graph when $r=0$ and a path on r vertices such that $V(Q) \cap V(H) = \emptyset$ when $r \geq 1$. Let G be the planar graph obtained from $H \cup Q$ by adding two new adjacent vertices such that each new vertex is joined to each vertex of $H \cup Q$. Clearly, G is $2C_k$-free with $|G| = |H| + |Q| + 2 = (n-2-r)+r+2 = n$ and

$$\begin{aligned} e(G) &= 2n-3+e(H)+e(Q) \\ &= 2n-3+2k-4+(n-2k+1-r)\left(1-\frac{1}{k-2}\right)+\max\{r-1,0\} \\ &= \left(3-\frac{1}{k-2}\right)n+\frac{3+r}{k-2}-4-r+\max\{r-1,0\} \\ &= \left(3-\frac{1}{k-2}\right)n+\frac{3+r}{k-2}-5+\max\{1-r,0\}. \end{aligned}$$

Hence, $ex_{\mathcal{P}}(n, 2C_k) \geq e(G') = \left(3-\frac{1}{k-2}\right)n+\frac{3+r}{k-2}-5+\max\{1-r,0\}$, as desired. This completes the proof of Theorem 1.8(a). ■

(b) Let $n, k, \varepsilon_1, \varepsilon_2$ be given as in the statement. Throughout the proof, let $\mathcal{T}_m := K_2 + P_{m-2}$ be the plane triangulation on $m \geq 2$ vertices, with x and y on the outer face of $\mathcal{T}_m$, where x and y are the two adjacent vertices of degree $m-1$ in $\mathcal{T}_m$. Note that we allow m to be 2 here for a simpler proof later on; $\mathcal{T}_m$ has exactly $2m-4$ 3-faces. For each integer s satisfying $m \leq s \leq 2m-4$, let $\mathcal{T}_s^m$ denote a plane triangulation on s vertices obtained from $\mathcal{T}_m$ by adding $s-m$ new vertices (no two new vertices are added to the same face): each to a 3-face F of $\mathcal{T}_m$ and then joining it to all vertices on the boundary of F.

To prove ($b1$), let $k := 2p+1$, where $p \geq 3$ is an integer. We first show that $ex_{\mathcal{P}}(n, 2C_k) = 3n-6$ for all $n \leq 3k-4$. Note that $\mathcal{T}_{2p+1}$ has exactly $2(2p+1)-4 = 4p-2$ 3-faces. For each $n \leq 3k-4 = 6p-2 = (2p+1)+(4p-2)$, $\mathcal{T}_n^{2p+1}$ is $2C_k$-free because each C_k in $\mathcal{T}_n^{2p+1}$ must contain at least $p+1$ vertices of $\mathcal{T}_{2p+1}$. Hence, $ex_{\mathcal{P}}(n, 2C_k) = 3n-6$ for all $n \leq 3k-4$. We next consider the case $n \geq 3k-3$. Let $t \geq 0$ be an integer satisfying

$$t(3p-3)+\varepsilon_1 = n-(2k-1).$$

Let $H_i := \mathcal{T}_{3p-1}^{p+1}$ for each $i \in [t]$ when $t \geq 1$; $H_{t+1} := \mathcal{T}_{\varepsilon_1+2}$ when $\varepsilon_1+2 \leq k-1$ and the plane triangulation $\mathcal{T}_{\varepsilon_1+2}^{p+1}$ when $\varepsilon_1+2 \geq k$; and $H_{t+2} := \mathcal{T}_{2k-1}$. For each $j \in [t+1]$, it is easy to check that each cycle of H_j has length at most k; each cycle of length exactly k must contain the vertices x and y; H_j is $2C_k$-free. Note that H_{t+2} is $2C_k$-free, and each cycle of length exactly k in H_{t+2} must contain either x or y (or possibly both). Finally, let G be the planar graph obtained from disjoint

copies of $H_1, H_2, \ldots, H_{t+2}$ by pasting along the subgraph K_2 induced by x and y. It follows that G is $2C_k$-free and $|G| = t(3p-3) + (2k-1) + \varepsilon_1 = n$. Note that $|H_{t+1}| \geq 2$ with equality when $\varepsilon_1 = 0$. Therefore,

$$\begin{aligned}
ex_{\mathcal{P}}(n, 2C_k) \geq e(G) &= \sum_{i=1}^{t}(e(H_i)-1) + (e(H_{t+1})-1) + (e(H_{t+2})-1) + 1 \\
&= t[3(3p-1)-7] + \max\{3(\varepsilon_1+2)-7, 0\} + (3(2k-1)-7) + 1 \\
&= 3n - t - 7 + \max\{1-\varepsilon_1, 0\} \\
&= 3n - \frac{n-(2k-1)-\varepsilon_1}{3p-3} - 7 + \max\{1-\varepsilon_1, 0\} \\
&= \left(3 - \frac{1}{3p-3}\right)n + \frac{2k-1+\varepsilon_1}{3p-3} - 7 + \max\{1-\varepsilon_1, 0\} \\
&= \left(3 - \frac{1}{k-4+\frac{k-1}{2}}\right)n + \frac{2k-1+\varepsilon_1}{k-4+\frac{k-1}{2}} - 7 + \max\{1-\varepsilon_1, 0\} \\
&= \left(3 - \frac{1}{k-4+\lfloor k/2 \rfloor}\right)n + \frac{5+\varepsilon_1}{k-4+\lfloor k/2 \rfloor} - \frac{17}{3} + \max\{1-\varepsilon_1, 0\}.
\end{aligned}$$

It remains to prove $(b2)$. Let $k := 2p$, where $p \geq 4$ is an integer. Similar to the proof of (I), we see that $\mathcal{T}_n^{2p-1}$ is $2C_k$-free for each $n \leq 3k-7$, because each C_k in $\mathcal{T}_n^{2p-1}$ must contain at least p vertices of $\mathcal{T}_{2p-1}$. Hence, $ex_{\mathcal{P}}(n, 2C_k) = 3n-6$ for all $n \leq 3k-7$. We next consider the case $n \geq 3k-6$. Let $t \geq 0$ be an integer satisfying

$$t(3p-6) + \varepsilon_2 = n - (2k-1).$$

Let $H_i := \mathcal{T}_{3p-4}^p$ for each $i \in [t]$ when $t \geq 1$; $H_{t+1} := \mathcal{T}_{\varepsilon_2+2}$ when $\varepsilon_2 + 2 \leq k-1$ and the plane triangulation $\mathcal{T}_{\varepsilon_2+2}^p$ when $\varepsilon_2+2 \geq k$; and $H_{t+2} := \mathcal{T}_{2k-1}$. For each $j \in [t+1]$, it is easy to check that each cycle of H_j has length at most k; each cycle of length exactly k must contain the vertices x and y; H_j is $2C_k$-free. Note that H_{t+2} is $2C_k$-free, and each cycle of length exactly k in H_{t+2} must contain either x or y (or possibly both). Finally, let G be the planar graph obtained from disjoint copies of $H_1, H_2, \ldots, H_{t+2}$ by pasting along the subgraph K_2 induced by x and y. It follows that G is $2C_k$-free and $|G| = t(3p-6) + (2k-1) + \varepsilon_2 = n$. Note that $|H_{t+1}| \geq 2$ with equality when

$\varepsilon_2 = 0$. Therefore,

$$\begin{aligned}
ex_{\mathcal{P}}(n, 2C_k) \geq e(G) &= \sum_{i=1}^{t}(e(H_i) - 1) + (e(H_{t+1}) - 1) + (e(H_{t+2}) - 1) + 1 \\
&= t[3(3p-4) - 7] + \max\{3(\varepsilon_2 + 2) - 7, 0\} + (3(2k-1) - 7) + 1 \\
&= 3n - t - 7 + \max\{1 - \varepsilon_2, 0\} \\
&= 3n - \frac{n - (2k-1) - \varepsilon_2}{3p-6} - 7 + \max\{1 - \varepsilon_2, 0\} \\
&= \left(3 - \frac{1}{3p-6}\right) n + \frac{2k - 1 + \varepsilon_2}{3p-6} - 7 + \max\{1 - \varepsilon_2, 0\} \\
&= \left(3 - \frac{1}{k-6+k/2}\right) n + \frac{2k - 1 + \varepsilon_2}{k-6+k/2} - 7 + \max\{1 - \varepsilon_2, 0\} \\
&= \left(3 - \frac{1}{k-6+k/2}\right) n + \frac{7 + \varepsilon_2}{k-6+k/2} - \frac{17}{3} + \max\{1 - \varepsilon_2, 0\},
\end{aligned}$$

as desired. This completes the proof of Theorem 1.8(b). ■

We end this section with a result showing that $ex_{\mathcal{P}}(n, C_3^+) = ex_{\mathcal{P}}(n, C_3)$ for all $n \geq 4$, where C_k^+ denotes the graph on $k+1$ vertices obtained from C_k by adding a pendant edge.

Proposition 3.1 *For all $n \geq 4$, $ex_{\mathcal{P}}(n, C_3^+) = ex_{\mathcal{P}}(n, C_3)$.*

Since C_3 is a subgraph of C_3^+, we have $ex_{\mathcal{P}}(n, C_3^+) \geq ex_{\mathcal{P}}(n, C_3)$. Let G be a C_3^+-free planar graph with at least four vertices. If G is connected, then G is also C_3-free; if G is disconnected, then we may prove $e(G) \leq ex_{\mathcal{P}}(n, C_3)$ by induction on n, which is left to readers.

4 Planar Turán number of complete bipartite graphs

Finally, we study the planar Turán number of $K_{m,t}$, where $t \geq m \geq 1$. Note that $ex_{\mathcal{P}}(n, K_{m,t}) = 3n - 6$ when $m \geq 3$; Theorem 6 in [9] completely determines the values of $ex_{\mathcal{P}}(n, K_{m,t})$ when $m = 1$; Theorem 1.1(c) settles $ex_{\mathcal{P}}(n, K_{m,t})$ for the case $m = t = 2$. We prove the remaining cases for $ex_{\mathcal{P}}(n, K_{2,t})$. Let O_n denote the unique outerplane graph with $2n - 3$ edges, maximum degree 4, and the outer face of order n. The graph O_n when $n = 12$ is depicted in Figure 6. It is easy to see that O_n is $K_{1,5}$-free because the maximum degree of O_n is four.

Proof of Theorem 1.9: Let n and t be given as in the statement. Note that O_{n-1} is $K_{2,3}$-free because O_{n-1} is an outerplane graph. It follows that $K_1 + O_{n-1}$ is $K_{2,5}$-free. Hence, $ex_{\mathcal{P}}(n, K_{2,t}) = 3n - 6$ for all $t \geq 5$ and $n \geq t + 2$. We may assume that $t \in \{3, 4\}$. We first consider the case $n \geq 12$. Let H_ℓ be the plane graph on 6ℓ vertices constructed as follows: for each $i \in [\ell]$, let C^i be a cycle with vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,6}$ in order, let H_ℓ be the plane graph obtained from disjoint union of $C^1, \ldots, C^\ell$ by adding edges $u_{i,j}u_{i+1,j}$ and $u_{i,j}u_{i+1,j+1}$ for all $i \in [\ell - 1]$ and $j \in [6]$, where all arithmetic on the index $j + 1$ here is done modulo 6. For $i \in \{1, \ell\}$, let W_5^i be a wheel on 6

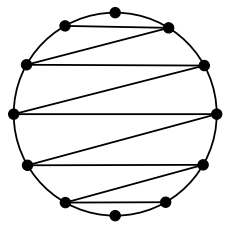


Figure 6: The graph O_{12}.

vertices $v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,5}$, where $v_{i,0}$ is the center vertex of W_5^i; K_p^i be a complete graph with vertices $w_{i,1}, w_{i,2}, \ldots, w_{i,p}$.

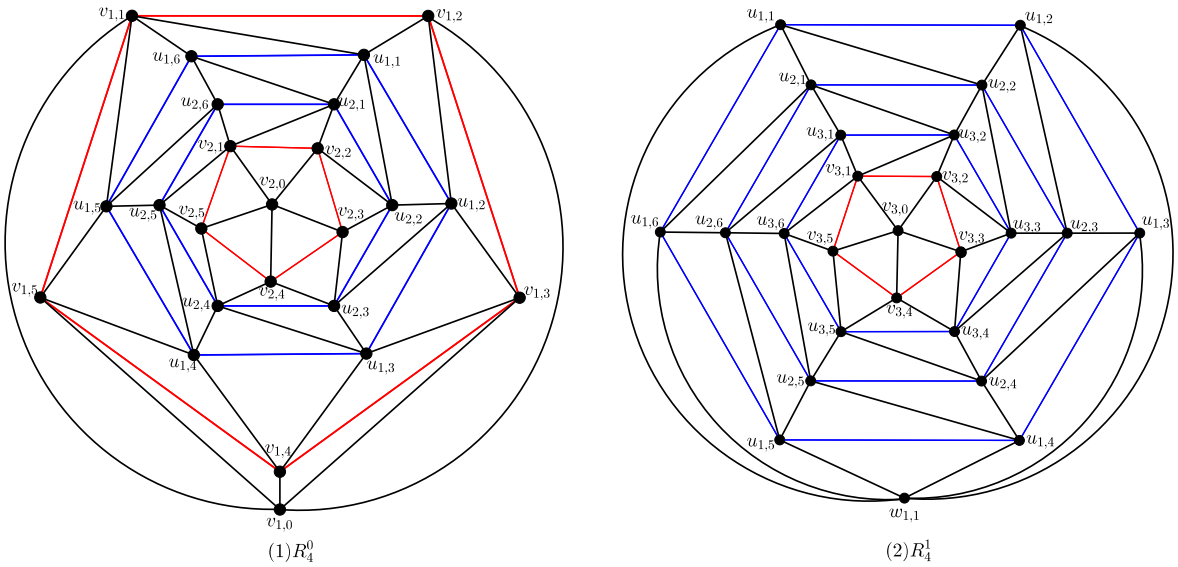
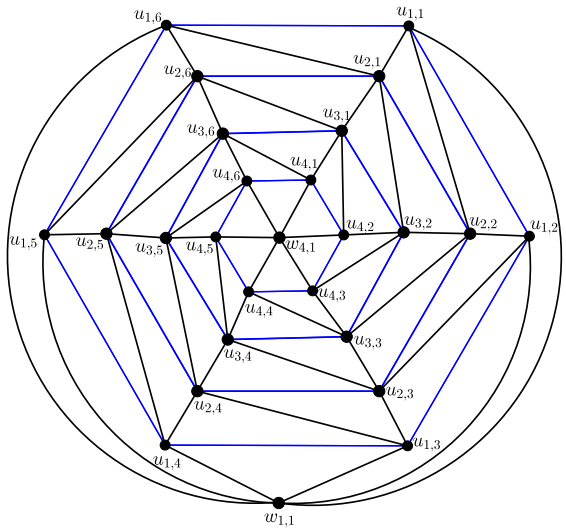
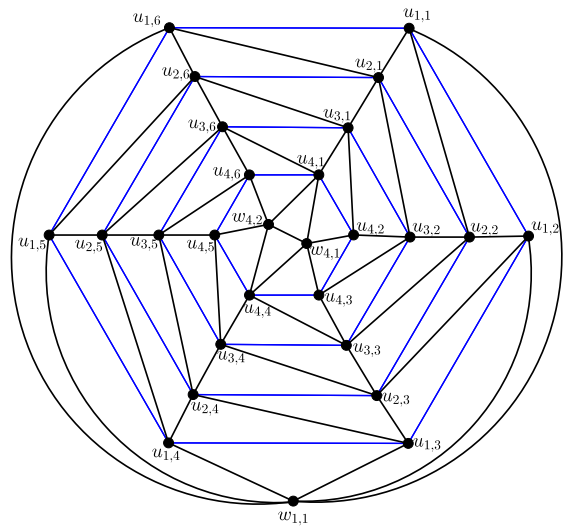


Figure 7: The plane triangulations R_k^0 and R_k^1 when $k = 4$, where C^1, C^2, C^3 are in blue.

Let $n := 6k + r$ for some $r \in \{0, 1, \ldots, 5\}$ and integer $k \geq 2$. When $r = 0$, let R_k^0 be the plane triangulation on n vertices obtained from $H_{k-2} \cup W_5^1 \cup W_5^{k-2}$ by adding edges $u_{i,j}v_{i,j}$, $u_{i,j}v_{i,j+1}$ and $u_{i,6}v_{i,1}$ for all $i \in \{1, k-2\}$ and $j \in [5]$, where all arithmetic on the index $j+1$ here and henceforth is done modulo 5. When $r = 1$, let R_k^1 be the plane triangulation on n vertices obtained from $H_{k-1} \cup W_5^{k-1} \cup K_1^1$ by adding $w_{1,1}u_{1,m}$ for any $m \in [6]$ and adding edges $v_{k-1,j}u_{k-1,j}$, $v_{k-1,j}u_{k-1,j+1}$ and $u_{k-1,6}v_{k-1,1}$ for all $j \in [5]$, where all arithmetic on the index $j+1$ here is done modulo 5. When $r = 2$, let R_k^2 be the plane triangulation on n vertices obtained from $H_k \cup K_1^1 \cup K_1^k$ by adding edges $u_{i,j}w_{i,1}$ for all $i \in \{1, k\}$ and $j \in [6]$. When $r = 3$, let R_k^3 be the plane triangulation on n vertices obtained from $H_k \cup K_1^1 \cup K_2^k$ by adding edges $u_{1,i}w_{1,1}$ and $u_{k,j}w_{k,1}$ and $u_{k,s}w_{k,2}$ for all $i \in [6]$, $j \in [4]$ and $s \in \{1, 4, 5, 6\}$. When $r = 4$, let R_k^4 be the plane triangulation on n vertices obtained from $H_k \cup K_2^1 \cup K_2^k$ by adding edges $u_{i,j}w_{i,1}$ and $u_{i,s}w_{i,2}$ for all $i \in \{1, k\}$, $j \in [4]$ and $s \in \{1, 4, 5, 6\}$. When $r = 5$, let R_k^5 be the plane triangulation on n vertices obtained from $H_k \cup K_2^1 \cup K_3^k$ by first adding edges $u_{1,j}w_{1,1}$ and $u_{1,s}w_{1,2}$ for all $j \in [4]$ and $s \in \{1, 4, 5, 6\}$, and then

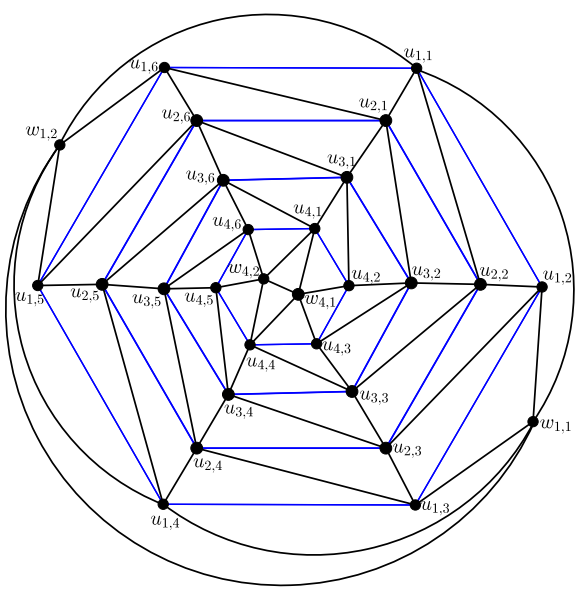


(3)R_4^2

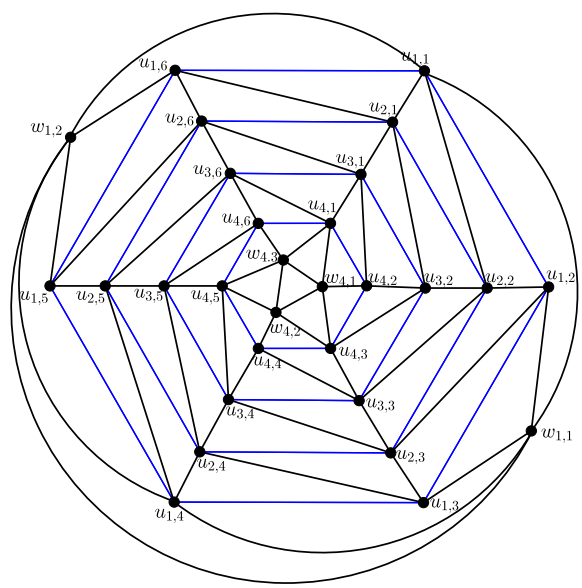


(4)R_4^3

Figure 8: The plane triangulations R_k^2 and R_k^3 when $k = 4$, where C^1, C^2, C^3, C^4 are in blue.



(5)R_4^4



(6)R_4^5

Figure 9: The plane triangulations R_k^4 and R_k^5 when $k = 4$, where C^1, C^2, C^3, C^4 are in blue.

joining $w_{k,1}$ to $u_{k,1}, u_{k,2}, u_{k,3}$; $w_{k,2}$ to $u_{k,3}, u_{k,4}, u_{k,5}$; $w_{k,3}$ to $u_{k,1}, u_{k,5}, u_{k,6}$. For all $r \in \{0, 1, \ldots, 5\}$, the graphs R_k^r when $k = 4$ are depicted in Figures 7-9. One can check that R_k^r is $K_{2,t}$-free and so $ex_{\mathcal{P}}(n, K_{2,t}) = 3n - 6$ for all $t \in \{3, 4\}$ and $n \geq 12$.

We then consider the case $t = 4$ and $6 \leq n \leq 11$. Let Q_2 be the plane graph defined in the proof of Theorem 1.6. Let Q_2' be the plane triangulation on 9 vertices obtained from $Q_2 \setminus v$ by adding edge $u_{2,1}u_{2,3}$; Q_2'' be the plane triangulation on 11 vertices obtained from Q_2 by adding one vertex w joining to $v, u_{2,1}, u_{2,2}$. Then Q_2', Q_2, Q_2'' are $K_{2,4}$-free. Hence, $ex_{\mathcal{P}}(n, K_{2,4}) = 3n - 6$ for all $9 \leq n \leq 11$. Let G_1 and G_2 be the unique plane graphs with degree sequence $5, 4, 4, 3, 3, 3$ and $6, 4, 4, 4, 4, 3, 3$ respectively. Then the plane graphs $G_1, G_2, Q_2' \setminus u$ are $K_{2,4}$-free. Hence, $ex_{\mathcal{P}}(n, K_{2,4}) \geq 3n - 7$ for all $n \in \{6, 7, 8\}$. Clearly, for $n \in \{6, 7\}$, every plane triangulation on n vertices is not $K_{2,4}$-free. Hence, $ex_{\mathcal{P}}(n, K_{2,4}) = 3n - 7$ for all $n \in \{6, 7\}$. We next prove that $ex_{\mathcal{P}}(n, K_{2,4}) \leq 3n - 7$ when $n = 8$. Suppose not. Let G be a $K_{2,4}$-free plane triangulation on n vertices. Let $v \in V(G)$ with $d_G(v) = \delta(G)$. Then $\delta(G) = 4$, else $e(G) \geq 5n/2 > 3n - 6$ because $n = 8$ when $\delta(G) \geq 5$, or $e(G) = e(G \setminus v) + d_G(v) \leq 3(n-1) - 7 + 3 = 3n - 7$ when $\delta(G) \leq 3$. Then there exists $u \in V(G) \setminus N_G[v]$ such that $N_G(u) \cap N_G(v) \geq 3$. But then G contains a $K_{2,4}$ as a subgraph, a contradiction, as desired. Hence, $ex_{\mathcal{P}}(n, K_{2,4}) = 3n - 7$ when $n = 8$.

It remains to consider the case $t = 3$ and $5 \leq n \leq 11$. Since T_5^- has a copy of $K_{2,3}$ and $K_1 + P_4$ is $K_{2,3}$-free, we have $ex_{\mathcal{P}}(n, K_{2,3}) = 3n - 8$ when $n = 5$. Let O_7' be the near 4-regular plane graph obtained from O_7 by adding edges between vertices of degree i, where $i \in \{2, 3\}$; O_8' be the 4-regular plane graph obtained from O_8 by adding edges between vertices of degree at most 3. Let J be the plane graph given in Figure 10. Let J' be the plane graph obtained from $J \setminus x_2$ by adding edge x_1x_3, J'' be the plane graph obtained from $J \setminus \{x_1, x_3\}$ by joining x_2 to x_4, x_5. Then the plane graphs $K_1 + C_5, O_7', O_8', J'', J', J$ are $K_{2,3}$-free. Hence, $ex_{\mathcal{P}}(n, K_{2,3}) \geq 3n - 8$ for all $6 \leq n \leq 11$. We shall show that $ex_{\mathcal{P}}(n, K_{2,3}) \leq 3n - 8$ for all $6 \leq n \leq 11$. Suppose this is not true. Let G be a $K_{2,3}$-free plane graph on n vertices with $e(G) \geq 3n - 7$, where $6 \leq n \leq 11$. We choose such a G with n minimum. Let $v \in V(G)$ with $d_G(v) = \delta(G)$. Then $\delta(G) \leq 4$, else $e(G) \geq 5n/2 > 3n - 6$ because $n \leq 11$, a contradiction. Next, if $\delta(G) \leq 3$, then $e(G \setminus v) \leq 3(n-1) - 8$ by the minimality of n and the fact that $ex_{\mathcal{P}}(n, K_{2,3}) \leq 3n - 8$ when $n = 5$. Thus, $e(G) = e(G \setminus v) + d_G(v) \leq 3(n-1) - 8 + 3 = 3n - 8$, a contradiction. This proves that $\delta(G) = 4$. We see G is not plane triangulation because G is $K_{2,3}$-free. So G has at least three vertices of degree four, else $e(G) \geq \lceil (5n-2)/2 \rceil \geq 3n - 6$ because $n \leq 11$, a contradiction. This implies that G contains $K_1 + C_4$ as a subgraph and so G is not $K_{2,3}$-free, a contradiction. Therefore, $ex_{\mathcal{P}}(n, K_{2,3}) = 3n - 8$ for all $6 \leq n \leq 11$.

This completes the proof of Theorem 1.9.

Data Availability There is no additional data associated with this paper.

Declaration

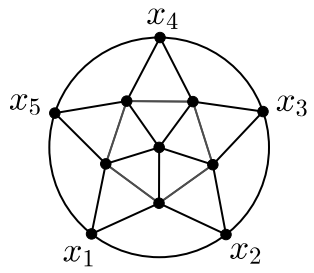


Figure 10: The graph J.

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix

Proof of Lemma 2.1: Let G be given as in the statement. To establish the desired result, we first assume that there exists a vertex, say $v_1 \in V(G)$, such that v_1 belongs to no triangle in G. Then $e(G[N_G[v_1]]) = 3$. Let $N_G(v_1) = \{v_2, v_3, v_4\}$ and $A = V(G) \setminus N_G[v_1]$. Then $e_G(N_G(v_1), A) = 6$ because G is a cubic graph. Hence, $e(G[A]) = e(G) - e(G[N_G[v_1]]) - e_G(N_G(v_1), A) = 3$, which implies that $G[A] \in \{K_{1,3}, P_4, K_3 \cup K_1\}$. Since G is cubic, we see $G[A] \neq K_3 \cup K_1$ by the planarity of G. That is either $G[A] = K_{1,3}$ or $G[A] = P_4$. If $G[A] = K_{1,3}$ and $x \in A$ is the center of $K_{1,3}$, then $G \setminus \{v_1, x\} = C_6$ and so $G = G_1$. So we next assume that $G[A] = P_4$. Let $G[A]$ be a path with vertices v_5, v_6, v_7, v_8 in order. We claim that v_6 and v_7 have no common neighbour in G. Suppose v_6 and v_7 have a common neighbour, say v_2. Then $N_G(v_3) = N_G(v_4) = \{v_1, v_5, v_8\}$ because G is a cubic graph. But then G has a $K_{3,3}$-minor with one part $\{v_2, v_3, v_4\}$ and the other part $\{v_1, \{v_5, v_6\}, \{v_7, v_8\}\}$, a contradiction. Thus, v_6 and v_7 have no common neighbour in G. Without loss of generality, we may assume that $v_6v_2, v_7v_3 \in E(G)$. Notice that $N_G(v_4) = \{v_1, v_5, v_8\}$ because G is a cubic graph. Then we see that either $v_5v_2 \in E(G)$ or $v_5v_3 \in E(G)$. If $v_5v_3 \in E(G)$, then $v_8v_2 \in E(G)$ and so G contains a $K_{3,3}$-minor with one part $\{v_2, v_3, v_4\}$ and the other part $\{v_1, \{v_5, v_6\}, \{v_7, v_8\}\}$, a contradiction. Hence, $v_5v_2 \in E(G)$ and so $v_8v_3 \in E(G)$, which implies that $G = G_2$.

Next we assume that every vertex in G belongs to at least one triangle. We claim that there must exist a vertex such that it belongs to two triangles. Suppose that every vertex in G belongs to exactly one triangle. Let $v_1 \in V(G)$. Then $G[N_G[v_1]] = K_1 + (K_2 \cup K_1)$. Let $A = V(G) \setminus N_G[v_1]$. Then $e_G(N_G(v_1), A) = 4$ because G is a cubic graph. Hence, $e(G[A]) = e(G) - e(G[N_G[v_1]]) - e_G(N_G(v_1), A) = 4$, which implies that either $G[A] = C_4$ or $G[A] = K_1 + (K_2 \cup K_1)$. If $G[A] = C_4$, there exist two vertices in A which does not belong to any triangle, a contradiction. If $G[A] = K_1 + (K_2 \cup K_1)$, then there exists one vertex such that it belongs to either two triangles or no triangle, a contradiction. Thus, there must exist a vertex belonging to two triangles. Assume that v_1 belongs to two triangles in G. Since G is K_4-free, we have $G[N_G[v_1]] = K_4^-$. Let $A = V(G) \setminus N_G[v_1]$. Then $e_G(N_G(v_1), A) = 2$ because G is a cubic graph. Hence, $e(G[A]) = e(G) - e(G[N_G[v_1]]) - e_G(N_G(v_1), A) = 5$, which implies that $G[A] = K_4^-$. Thus, $G = G_3$. ■

Proof of Lemma 2.2: Let G be given as in the statement. Let v be an arbitrary vertex in G and $V(G) \setminus N_G[v] = \{v_1, v_2\}$. Then $v_1v_2 \in E(G)$, else $N_G(v_i) = N_G(v)$ for any $i \in [2]$, which yields that $G \cong K_{3,3}$. Since G is a cubic graph, we see that $|N_{N_G(v)(v_i)}| = 2$ for any $i \in [2]$, and $N_{N_G(v)}(v_1) \notin N_{N_G(v)}(v_2)$. Hence, $G[N_G(v)] \cong K_2 \cup K_1$, which implies that G is the graph given in Figure 2.