Rainbow pancyclicity in a collection of graphs under the Dirac-type condition

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Abstract

Let $\mathbf{G} = \{G_i : i \in [n]\}$ be a collection of not necessarily distinct n-vertex graphs with the same vertex set V, where $\mathbf{G}$ can be seen as an edge-colored (multi)graph and each G_i is the set of edges with color i. A graph F on V is called *rainbow* if any two edges of F come from different G_is'. We say that $\mathbf{G}$ is *rainbow pancyclic* if there is a rainbow cycle C_ℓ of length ℓ in $\mathbf{G}$ for each integer $\ell \in [3, n]$. In 2020, Joos and Kim proved a rainbow version of Dirac's theorem: If $\delta(G_i) \geq \frac{n}{2}$ for each $i \in [n]$, then there is a rainbow Hamiltonian cycle in $\mathbf{G}$. In this paper, under the same condition, we show that $\mathbf{G}$ is rainbow pancyclic except that n is even and $\mathbf{G}$ consists of n copies of $K_{\frac{n}{2},\frac{n}{2}}$. This result supports the famous meta-conjecture posed by Bondy.

Keywords: rainbow; Hamiltonian cycle; rainbow pancyclic; meta-conjecture

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1 Introduction

Recently, some classic results in graph theory have been considered for a collection of graphs, which can be seen as the rainbow versions of these results. We first give an exact definition of the rainbow structures in a collection of graphs. Let $\mathbf{G} = \{G_i : i \in [m]\}$ be a collection of not necessarily distinct n-vertex graphs with the same vertex set V, where $\mathbf{G}$ can be seen as an edge-colored (multi)graph and each G_i can be seen as a set of edges with color i. A graph F on V is called *rainbow* if any two edges of F come from different G_is', in other words, any

two edges of F have distinct colors. In 2020, Aharoni, DeVos, de la Maza, Montejano and Šámal [2] gave the rainbow version of Mantel's Theorem as follows.

Theorem 1.1. [2] *Suppose* $\mathbf{G} = \{G_1, G_2, G_3\}$ *is a collection of not necessarily distinct* n*-vertex graphs with the same vertex set such that* $|E(G_i)| > \frac{1+\tau^2}{4}n^2$ *for all* $1 \leq i \leq 3$*, where* $\tau = \frac{4-\sqrt{7}}{9}$*. Then there exists a rainbow triangle in* $\mathbf{G}$*.*

The authors also constructed an example to show that the bounds on the numbers of edges are sharp. What's surprising is that each graph G_i $(1 \leq i \leq 3)$ has to have more than $\frac{1+\tau^2}{4}n^2$ edges in order to guarantee the existence of rainbow triangles (while the Mantel's Theorem says that an n-vertex graph with more than only $n^2/4$ edges has a triangle). In the same paper, the authors posed a conjecture on the rainbow version of Dirac's Theorem, and the conjecture was solved by Cheng, Wang and Zhao [9] asymptotically, and by Joos and Kim [11] completely.

Theorem 1.2. [11] *Let* $\mathbf{G} = \{G_i : i \in [n]\}$ *be a collection of not necessarily distinct* n*-vertex graphs with the same vertex set. If* $\delta(G_i) \geq \frac{n}{2}$ *for each* $i \in [n]$*, then there exists a rainbow Hamiltonian cycle in* $\mathbf{G}$*.*

Unlike the rainbow version of Mantel's Theorem, the minimum degree condition of each G_i $(i \in [n])$ in Theorem 1.2 agrees with that in Dirac's Theorem (Dirac's Theorem says that any n-vertex graph with minimum degree at least $n/2$ contains a Hamiltonian cycle). The above two theorems motivate us to consider an interesting problem: for a known result that any n-vertex graph satisfying a property $\mathcal{P}$ contains H as a subgraph, is it true that any collection $\mathbf{G} = \{G_i : i \in [e(H)]\}$ of n-vertex graphs on the same vertex set contains a rainbow H if each G_i satisfies the property $\mathcal{P}$? It is obvious that the answer is "Yes" for Dirac's Theorem but is "No" for Mantel' Theorem.

In 1971, Bondy [4] proved that every n-vertex graph is pancyclic under the same degree condition of Dirac's Theorem expect for $K_{\frac{n}{2},\frac{n}{2}}$, and posed the following meta-conjecture in [5]: Almost any nontrivial sufficient conditions for the Hamiltonicity of graphs can also guarantee the pancyclicity of graphs expect for maybe a simple family of exceptional graphs. Inspired by the meta-conjecture, we consider the rainbow pancyclicity of a collection of graphs under the Dirac-type condition. The main result of this paper indicates that the answer of the above problem is "Yes" for the Bondy's Theorem, and it is also supports the meta-conjecture.

Before the proof of our main result, some notions and known results are needed. For any two integers $i \leq j$, we use $[i, j]$ to denote the set $\{i, i+1, \ldots, j\}$ of integers. In particular, the set $[1, n]$ of integers is denoted by $[n]$. Given a vertex v of a graph G and a subgraph F of G, we use $N_G(v, F)$ to denote the set of neighbours of v in F. Set $d_G(v, F) = |N_G(v, F)|$. Let P_k and C_k denote the path and cycle on k vertices, respectively. For a cycle $C = v_1v_2 \cdots v_\ell v_1$ and two integers $1 \leq i < j \leq \ell$, we use $v_iC^+v_j$ and $v_jC^-v_i$ to denote the paths $v_iv_{i+1} \cdots v_j$

and $v_j v_{i-1} \cdots v_i$, respectively. For a path $P = v_1 v_2 \cdots v_\ell$ and two integers $1 \leq i < j \leq \ell$, we define $v_i P^+ v_j$ and $v_j P^- v_i$, similarly. We say that $\mathbf{G} = \{G_i : i \in [t]\}$ consists of t *copies* of G if $G_1 = G_2 = \ldots = G_t = G$.

For convenience, in what follows we always use $\mathbf{G}$ to denote a collection $\{G_i : i \in [n]\}$ of not necessarily distinct n-vertex graphs with the same vertex set V. We define $\delta(\mathbf{G}) = \min\{\delta(G_i) : i \in [n]\}$, where $\delta(G)$ denotes the minimum degree of a graph G. We say that $\mathbf{G}$ is *rainbow pancyclic* if $\mathbf{G}$ has a rainbow cycle C_ℓ of length ℓ for every integer ℓ with $3 \leq \ell \leq n$, and $\mathbf{G}$ is *rainbow vertex-pancyclic* if each vertex of V is contained in a rainbow cycle of length ℓ for every integer ℓ with $3 \leq \ell \leq n$. In 2023, Li, Li and Li [12] showed the rainbow vertex-pancyclicity of $\mathbf{G}$ by improving the value of minimum degree in Theorem 1.2.

Theorem 1.3. [12] *If $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, then $\mathbf{G}$ is rainbow vertex-pancyclic.*

The following lemmas can be found in [12] and [6], respectively, which will be used in our proof.

Lemma 1.1. [12] *If $\delta(\mathbf{G}) \geq \frac{n}{2}$, then $\mathbf{G}$ has a rainbow C_{n-1} or $\mathbf{G}$ consists of n copies of $K_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 1.2. [6] *For an integer n and a cyclic group $(\mathbb{Z}, +)$ of n elements, let $k \in [n-1]$ and $I \subseteq \mathbb{Z}_n$. If $J = (I + k) \cup (I - k)$ and $|I| = |J|$, then $I = I + 2k$.*

For more results on this topic, please see [1, 3, 7, 8, 10, 13].

2 Main result

Now we give our main result.

Theorem 2.1. *If $\delta(\mathbf{G}) \geq \frac{n}{2}$, then either $\mathbf{G}$ is rainbow pancyclic or $\mathbf{G}$ consists of n copies of $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. From Theorem 1.3, the result follows when n is odd. Hence, we assume that n is even below. Combining Theorem 1.2 and Lemma 1.1, we only need to prove that $\mathbf{G}$ has a rainbow C_ℓ for each $3 \leq \ell \leq n-2$.

Claim 1. $\mathbf{G}$ *has either a rainbow C_{n-3} or a rainbow C_{n-2}.*

Proof. Suppose to the contrary that $\mathbf{G}$ has no rainbow C_{n-2} or C_{n-3}. By Theorem 1.2, we can choose a rainbow path $P_{n-3} = v_1 v_2 \cdots v_{n-3}$. Without loss of generality, suppose that $v_i v_{i+1} \in E(G_i)$ for each $i \in [n-4]$ and $V \setminus V(P) = \{x_1, x_2, x_3\}$. Since there is no

rainbow C_{n-3} or C_{n-2}, we have that $v_1v_{n-3} \notin E(G_{n-1}) \cup E(G_n)$ and either $v_1x_i \notin E(G_{n-1})$ or $v_{n-3}x_i \notin E(G_n)$ for each $i \in [3]$. Hence,

$$d_{G_{n-1}}(v_1, \{x_1, x_2, x_3\}) + d_{G_n}(v_{n-3}, \{x_1, x_2, x_3\}) \leq 3,$$

which implies that $d_{G_{n-1}}(v_1, V(P)) + d_{G_n}(v_{n-3}, V(P)) \geq n-3$. Now we define the following two sets:

$$I_1 = \{i \in [n-5] : v_1v_{i+1} \in E(G_{n-1})\} \text{ and } I_{n-3} = \{i \in [2, n-4] : v_{n-3}v_i \in E(G_n)\}.$$

Note that $|I_1| + |I_{n-3}| = d_{G_{n-1}}(v_1, V(P)) + d_{G_n}(v_{n-3}, V(P)) \geq n-3$ and $I_1 \cup I_{n-3} \subseteq [n-4]$. This implies that $I_1 \cap I_{n-3} \neq \emptyset$. Choose an integer $i \in I_1 \cap I_{n-3}$, and then we get that $v_1P^+v_iv_{n-3}P^-v_{i+1}v_1$ is a rainbow C_{n-3} with $v_iv_{n-3} \in E(G_n)$ and $v_{i+1}v_1 \in E(G_{n-1})$, a contradiction. The claim thus follows. □

Claim 2. **G** *has a rainbow* C_{n-2}.

Proof. Suppose to the contrary that **G** has no rainbow C_{n-2}. From Claim 1, **G** has a rainbow C_{n-3}. Assume that $C = v_1v_2 \ldots v_{n-3}v_1$ is a rainbow cycle with $v_iv_{i+1} \in E(G_i)$ for each $i \in [n-3]$ and $V \setminus V(C) = \{x_1, x_2, x_3\}$, where $v_{n-2} = v_1$. Set

$$I_{n-1} = \{i \in [n-3] : x_1v_{i+1} \in E(G_{n-1})\} \text{ and } I_n = \{i \in [n-3] : x_1v_i \in E(G_n)\}.$$

The hypothesis that **G** has no rainbow C_{n-2} implies that $I_{n-1} \cap I_n = \emptyset$. Since $I_{n-1}, I_n \subseteq [n-3]$, we have $|I_{n-1}| + |I_n| \leq n-3$. Note that $|I_{n-1}| \geq d_{G_{n-1}}(x_1) - 2 \geq \frac{n}{2} - 2$ and $|I_n| \geq d_{G_n}(x_1) - 2 \geq \frac{n}{2} - 2$. Then, $n-4 \leq |I_{n-1}| + |I_n| \leq n-3$. We distinguish the following cases to proceed.

Case 1. $|I_{n-1}| + |I_n| = n-3$.

Then, either $|I_{n-1}| = \frac{n}{2} - 1$ and $|I_n| = \frac{n}{2} - 2$ or $|I_{n-1}| = \frac{n}{2} - 2$ and $|I_n| = \frac{n}{2} - 1$. By symmetry, we can assume that the former holds. Note that $I_{n-1} \cup I_n = [n-3]$ and $I_{n-1} \cap I_n = \emptyset$. Choose a maximum set $\{i, i+1, \ldots, i+k\} = A$ such that $A \subseteq I_{n-1}$ and $i-1, i+k+1 \notin I_{n-1}$. It follows from $\frac{n}{2} - 1 > \frac{n-3}{2}$ that $|A| \geq 2$, and hence $i-1 \in I_n$ and $i+k+1 \in I_n$. Therefore, $x_1v_{i+k+1}C^+v_{i+k}x_1$ is a rainbow C_{n-2} with $x_1v_{i+k} \in E(G_{n-1})$ and $x_1v_{i+k+1} \in E(G_n)$, a contradiction.

Case 2. $|I_{n-1}| + |I_n| = n-4$.

Then, $|I_{n-1}| = |I_n| = \frac{n}{2} - 2$ and $x_1x_2x_3x_1$ is a triangle both in G_{n-1} and G_n. By symmetry of G_{n-2} and G_{n-1}, $x_1x_2x_3x_1$ is also a triangle in G_{n-2}. Recall that $I_{n-1} \cup I_n \subseteq [n-3]$ and $I_{n-1} \cap I_n = \emptyset$. Without loss of generality, set $I_{n-1} \cup I_n = [n-4]$.

We first prove that $N_{G_{n-1}}(x_1, C) = N_{G_n}(x_1, C) = \{v_2, v_4, \ldots, v_{n-4}\}$. Let $A = \{a, a+1, \ldots, a+k\}$ be a maximum subset of I_{n-1} such that $a-1 \notin I_{n-1}$ and $a+k+1 \notin I_{n-1}$. If $|A| \geq 2$, then it follows from the definition of I_{n-1} that $x_1v_{a+k} \in E(G_{n-1})$ and $x_1v_{a+k+1} \in$

$E(G_{n-1})$. Since neither $v_{a+k}x_1v_{a+k+1}C^+v_{a+k}$ nor $v_{a+k+1}x_1v_{a+k+2}C^+v_{a+k+1}$ is a rainbow C_{n-2}, we have $a+k+1 \notin I_n$ and $a+k+2 \notin I_n$, respectively. Recall that $a+k+1 \notin I_{n-1}$ and $I_{n-1} \cup I_n = [n-4]$. Then, $a+k+1 = n-3$, $a+k+2 = 1 \in I_{n-1}$ and $a+k = n-4 \in I_{n-1}$. Recall that $n-5 = s+k-1 \in I_{n-1}$. Therefore, $I_n \subseteq [2, n-6]$. Since $|I_n| = \frac{n}{2}-2$, it follows that there exists a maximal set $B = \{b, b+1, \ldots, b+k'\} \subseteq I_n$ such that $b-1, b+k'+1 \notin I_n$ and $|B| \geq 2$. Since $1, n-5 \in I_{n-1}$ and $I_{n-1} \cup I_n = [n-4]$, we have that $b-1, b+k'+1 \in I_{n-1}$. Therefore, $v_bx_1v_{b+1}C^+v_b$ is a rainbow C_{n-2}, where $v_bx_1 \in E(G_{n-1})$ and $v_{b+1}x_1 \in E(G_n)$, a contradiction. Thus, we have $|A| = 1$, which means that all the elements of I_{n-1} and all the elements of I_n alternate in $[n-4]$. If $1 \in I_n$, then $n-4 \in I_{n-1}$, and hence $v_{n-3}x_1v_1C^+v_{n-3}$ is a rainbow C_{n-3} with $x_1v_{n-3} \in E(G_{n-1})$ and $x_1v_1 \in E(G_n)$, a contradiction. Hence, $I_{n-1} = \{1, 3, \ldots, n-5\}$ and $I_n = \{2, 4, \ldots, n-4\}$. Consequently, $N_{G_{n-1}}(x_1, C) = N_{G_n}(x_1, C) = \{v_2, v_4, \ldots, v_{n-4}\}$.

By the symmetry of $n-2, n-1, n$, we have that

$$N_{G_{n-2}}(x_1, C) = N_{G_{n-1}}(x_1, C) = N_{G_n}(x_1, C) = \{v_2, v_4, \ldots, v_{n-4}\}.$$

More generally, by the symmetry of x_1, x_2, x_3 and a similar discussion, we have that for each $j \in \{2, 3\}$, either

$$N_{G_{n-2}}(x_j, C) = N_{G_{n-1}}(x_j, C) = N_{G_n}(x_j, C) = \{v_2, v_4, \ldots, v_{n-4}\}$$

or

$$N_{G_{n-2}}(x_j, C) = N_{G_{n-1}}(x_j, C) = N_{G_n}(x_j, C) = \{v_1, v_3, \ldots, v_{n-3}\} \setminus \{v_t\}$$

for some odd integer $t \in [n-3]$. Let $U_j = \{i : v_i \in N_{G_{n-2}}(x_j, C)\}$ for $j \in [3]$. Recall that $U_1 = \{2, 4, \ldots, n-4\}$. If one of U_2, U_3 is the even integer set (say $U_2 = \{2, 4, \ldots, n-4\}$), recalling that $x_1x_2x_3x_1$ is a triangle in G_i for $i = n-2, n-1, n$, then $x_1x_2v_4C^+v_2x_1$ is a rainbow C_{n-2} with $x_1x_2 \in E(G_n)$, $x_2v_4 \in E(G_{n-2})$ and $x_1v_2 \in E(G_{n-1})$, a contradiction. Thus, suppose $U_2 = \{1, 3, \ldots, n-3\} \setminus \{a\}$ and $U_3 = \{1, 3, \ldots, n-3\} \setminus \{b\}$, where a, b are odd integers of $[n-3]$. If $a \neq b$ or $a = b$ and $n \geq 8$, then there are two consecutive odd integers $c, c+2 \in [n-3]$ such that either $c \in U_2$ and $c+2 \in U_3$ or $c \in U_3$ and $c+2 \in U_2$. Without loss of generality, suppose $c \in U_2$ and $c+2 \in U_3$. Then $x_2x_3v_{c+2}C^+v_cx_1$ is a rainbow C_{n-2} with $x_2x_3 \in E(G_{n-2})$, $x_3v_{c+2} \in E(G_{n-1})$ and $x_2v_c \in E(G_n)$, a contradiction. If $a = b$ and $n \leq 6$, then we have $n = 6$. Without loss of generality, suppose $U_1 = U_2 = \{1\}$. Then, $x_1x_3v_1v_2x_1$ is a rainbow C_4 with $x_1x_3 \in E(G_{n-2})$, $x_3v_1 \in E(G_{n-1})$, $v_1v_2 \in E(G_1)$ and $E(v_2x_1) \in E(G_n)$, a contradiction. $\square$

From Claim 2, we assume that $C = v_1v_2 \cdots v_{n-2}v_1$ is a rainbow cycle with $v_iv_{i+1} \in E(G_i)$ for each $i \in [n-2]$ and $V \setminus V(C) = \{x_1, x_2\}$, where $v_{n-1} = v_1$. Next we need to find a rainbow C_ℓ for each $3 \leq \ell \leq n-3$ in $\mathbf{G}$. Suppose to the contrary that $\mathbf{G}$ has no rainbow C_ℓ for some integer $\ell \in [3, n-3]$. Set

$$I_{n-1} = \{i \in [n-2] : x_1v_i \in E(G_{n-1})\} \text{ and } I_n = \{i \in [n-2] : x_1v_{i+\ell-2} \in E(G_n)\}.$$

If $I_{n-1} \cap I_n \neq \emptyset$, choosing an integer $i \in I_{n-1} \cap I_n$, then $x_1 v_i C^+ v_{i+\ell-2} x_1$ is a rainbow C_ℓ with $x_1 v_i \in E(G_{n-1})$ and $x_1 v_{i+\ell-2} \in E(G_n)$, a contradiction. So, $I_{n-1} \cap I_n = \emptyset$. Note that $|I_{n-1}| \geq d_{G_{n-1}}(x_1) - 1 = \frac{n}{2} - 1$ and $|I_n| \geq d_{G_n}(x_1) - 1 = \frac{n}{2} - 1$, which means that $|I_{n-1}| + |I_n| \geq n-2$. It follows from $I_{n-1} \cap I_n = \emptyset$ and $I_{n-1} \cup I_n \subseteq [n-2]$ that $|I_{n-1}| = |I_n| = \frac{n}{2} - 1$ and $x_1 x_2 \in E(G_{n-1}) \cap E(G_n)$. Then, $d_{G_{n-1}}(x_1, C) = d_{G_n}(x_1, C) = \frac{n}{2} - 1$. By symmetry of x_1 and x_2, we have that $d_{G_{n-1}}(x_2, C) = d_{G_n}(x_2, C) = \frac{n}{2} - 1$.

Claim 3. *For each integer $i \in [n-2]$, we have $x_1 x_2 \in E(G_{i-1}) \cup E(G_i) \cup E(G_{i+1})$.*

Proof. Without loss of generality, we only need to prove $x_1 x_2 \in E(G_1) \cup E(G_2) \cup E(G_3)$.

We first assert that one of $v_1 v_2, v_2 v_3, v_3 v_4$ belongs to $E(G_{n-1}) \cup E(G_n)$. Assume to the contrary that $E(G_{n-1}) \cup E(G_n)$ does not contain any edge of $v_1 v_2, v_2 v_3, v_3 v_4$. For each integer $i \in [4, n-2]$, we pair $v_2 v_i$ and $v_3 v_{i+1}$, and there are $n-5$ such pairs. For each pair, if $v_2 v_i \in E(G_{n-1})$ and $v_3 v_{i+1} \in E(G_n)$, then $v_2 v_i C^- v_3 v_{i+1} C^+ v_2$ is a rainbow C_{n-2} such that this rainbow cycle contains no edge of G_2. By a similar discussion, we can deduce that $x_1 x_2 \in E(G_2)$, and the claim thus follows. Then we assume that either $v_2 v_i \notin E(G_{n-1})$ or $v_3 v_{i+1} \notin E(G_n)$ for each $i \in [4, n-2]$. Hence, on the one hand, we have

$$|N_{G_{n-1}}(v_2) \cap V(C)| + |N_{G_n}(v_3) \cap V(C)| \leq n-5.$$

On the other hand,

$$|N_{G_{n-1}}(v_2) \cap V(C)| + |N_{G_n}(v_3) \cap V(C)| \geq 2(\frac{n}{2} - 2) = n-4,$$

a contradiction. Therefore, one of $v_1 v_2, v_2 v_3, v_3 v_4$ belongs to $E(G_{n-1}) \cup E(G_n)$.

Suppose $v_i v_{i+1} \in E(G_{n-1}) \cup E(G_n)$ for a fixed integer $i \in [3]$ (say $v_i v_{i+1} \in E(G_{n-1})$. Then there is a new rainbow cycle $v_1 v_2 \ldots v_{n-2}$ with $v_i v_{i+1} \in E(G_{n-1})$ and $v_j v_{j+1} \in E(G_i)$ for each $i \in [n-2] - \{i\}$, where $v_{n-1} = v_1$. Note that this rainbow cycle contains no edge of G_i. By a similar discussion, we have $x_1 x_2 \in E(G_i)$, the claim thus follows. □

Recall that $I_{n-1} = \{i \in [n-2] : x_1 v_i \in E(G_{n-1})\}$. Now we can regard I_{n-1} as a subset of the cyclic group $(\mathbb{Z}_{n-2}, +)$ and $I_{n-1} + k = \{i + k : i \in I_{n-1}\}$. Set

$$J = (I_{n-1} + (\ell - 2)) \cup (I_{n-1} - (\ell - 2));$$

$$J' = (I_{n-1} + (\ell - 3)) \cup (I_{n-1} - (\ell - 3)).$$

If there exists an integer $a \in J$ such that $x_1 v_a \in E(G_n)$, then the definition of J implies that there is an integer $b \in I_{n-1}$ such that either $a = b + (\ell - 2)$ or $a = b - (\ell - 2)$. Without loss of generality, set $a = b - (\ell - 2)$. It follows from the definition of I_{n-1} that $x_1 v_a C^+ v_b x_1$ is a rainbow C_ℓ with $x_1 v_a \in E(G_n)$ and $x_1 v_b \in E(G_{n-1})$, a contradiction. Therefore, $|J| \leq$

$n-2-d_{G_n}(x_1,C)=n-2-|I_n|=|I_{n-1}|$. On the other hand, we have $|J|\geq|I_{n-1}|$. Then, $|J|=|I_{n-1}|$. By Lemma 1.2, we have $I_{n-1}=I_{n-1}+2(\ell-2)$.

If there exists an integer $a\in J'$ such that $x_2v_a\in E(G_n)$, then the definition of J' implies that there is an integer $b\in I_{n-1}$ such that $a=b+(\ell-3)$ or $a=b-(\ell-3)$. Without loss of generality, set $a=b-(\ell-3)$. Note that $v_{b+3}\in V(v_bC^+v_a)$ since $\ell\leq n-3$. By Claim 3, we have $x_1x_2\in E(G_b)\cup E(G_{b+1})\cup E(G_{b+2})$. Without loss of generality, set $x_1x_2\in E(G_b)$. It follows from the definition of I_{n-1} that $x_1x_2v_a\overrightarrow{C}v_bx_1$ is a rainbow C_ℓ with $x_1x_2\in E(G_b)$, $x_2v_a\in E(G_n)$ and $x_1v_b\in E(G_{n-1})$, a contradiction. Therefore, $|J'|\leq n-2-d_{G_n}(x_2,C)=n-2-|I_n|=|I_{n-1}|$. On the other hand, we have $|J'|\geq|I_{n-1}|$. Then, $|J'|=|I_{n-1}|$. By Lemma 1.2, we have $I_{n-1}=I_{n-1}+2(\ell-3)$.

Combining $I_{n-1}=I_{n-1}+2(\ell-2)$ with $I_{n-1}=I_{n-1}+2(\ell-3)$, we can get $I_{n-1}=I_{n-1}+2$. It follows that either $I_{n-1}=\{1,3,\ldots,n-3\}$ or $I_{n-1}=\{2,4,\ldots,n-2\}$, which implies that either $N_{G_{n-1}}(x_1)=\{v_1,v_3,\ldots,v_{n-3},x_2\}$ or $N_{G_{n-1}}(x_1)=\{v_2,v_4,\ldots,v_{n-2},x_2\}$. By symmetry, we can deduce that $N_{G_i}(x_j)=\{v_1,v_3,\ldots,v_{n-3},x_{3-j}\}$ or $N_{G_i}(x_j)=\{v_2,v_4,\ldots,v_{n-2},x_{3-j}\}$ for each $i\in\{n-1,n\}$ and $j\in[2]$. Without loss of generality, set $I_{n-1}=\{1,3,\ldots,n-3\}$. So, $N_{G_{n-1}}(x_1)=\{v_1,v_3,\ldots,v_{n-3},x_2\}$. Recall that $I_{n-1}\cap I_n=\emptyset$ and $I_{n-1}\cup I_n\subseteq[n-2]$. Then, $I_n=\{2,4,\ldots,n-2\}$.

Claim 4. *ℓ is odd.*

Proof. Suppose to the contrary that ℓ is even. If $N_{G_n}(x_1)=\{v_1,v_3,\ldots,v_{n-3},x_2\}$, then $x_1v_1C^+v_{\ell-1}x_1$ is a rainbow C_ℓ with $x_1v_1\in E(G_{n-1})$ and $v_{\ell-1}x_1\in E(G_n)$, a contradiction. Then, $N_{G_n}(x_1)=\{v_2,v_4,\ldots,v_{n-2},x_2\}$.

From the proof of Claim 3, we know that one of v_1v_2,v_2v_3,v_3v_4 is in $E(G_{n-1})\cup E(G_n)$. Without loss of generality, we assume $v_1v_2\in E(G_n)$. Let $C^*=v_1v_2\cdots v_{n-2}v_1$ be a rainbow C_{n-2} with $v_1v_2\in E(G_n)$ and $v_iv_{i+1}\in E(G_i)$ for each $i\in[2,n-2]$, where $v_{n-1}=v_1$. Repeating the above discussion, we have $N_{G_1}(x_1)=N_{G_n}(x_1)=\{v_2,v_4,\ldots,v_{n-2},x_2\}$. Then, $x_1v_2C^+v_\ell x_1$ is a rainbow C_ℓ with $x_1v_2\in E(G_1)$ and $v_\ell x_1\in E(G_n)$, a contradiction. The claim thus follows. □

From Claim 4, we always assume that ℓ is odd. Recall that $N_{G_{n-1}}(x_1)=\{v_1,v_3,\ldots,v_{n-4},x_2\}$. If $N_{G_n}(x_1)=\{v_2,v_4,\ldots,v_{n-3},x_2\}$, then $x_1v_1C^+v_{\ell-1}x_1$ is a rainbow C_ℓ with $x_1v_1\in E(G_{n-1})$ and $v_{\ell-1}x_1\in E(G_n)$, a contradiction. Hence, $N_{G_n}(x_1)=N_{G_{n-1}}(x_1)=\{v_1,v_3,\ldots,v_{n-3},x_2\}$.

If $N_{G_{n-1}}(x_2)=\{v_1,v_3,\ldots,v_{n-3},x_1\}$, then $x_2x_1v_1C^+v_{\ell-2}x_2$ is a rainbow C_ℓ with $x_1v_1\in E(G_n)$, $v_{\ell-2}x_2\in E(G_{n-1})$ and $x_2x_1\in E(G_{\ell-2})\cup E(G_{\ell-1})\cup E(G_\ell)$ (by Claim 3), a contradiction. Then, $N_{G_{n-1}}(x_2)=\{v_2,v_4,\ldots,v_{n-2},x_1\}$. Similarly, we can deduce that $N_{G_n}(x_2)=N_{G_{n-1}}(x_2)=\{v_2,v_4,\ldots,v_{n-2},x_1\}$.

For each odd integer $a\in[n-2]$, let $C^a=v_{a-1}x_2v_{a+1}C^+v_{a-1}$ be a rainbow C_{n-2} with $v_{a-1}x_2\in E(G_{n-1})$ and $x_2v_{a+1}\in E(G_n)$. Note that C^a does not use any edge of

G_a or G_{a-1}. Repeating the above discussions, we can get that $N_{G_a}(x_1) = N_{G_{a-1}}(x_1) = \{v_1, v_3, \ldots, v_{n-3}, x_2\}$. Then, $N_{G_i}(x_1) = \{v_1, v_3, \ldots, v_{n-3}, x_2\}$ for each $i \in [n]$. Similarly, we can deduce that $N_{G_i}(x_2) = \{v_2, v_4, \ldots, v_{n-2}, x_1\}$ for each $i \in [n]$. Note that C^a contains no v_a. Similarly, we have $N_{G_i}(v_a) = \{v_2, v_4, \ldots, v_{n-2}, x_1\}$ for each $i \in [n]$.

For each even integer $b \in [n-2]$, let $C^b = v_{b-1}x_1v_{b+1}C^+v_{b-1}$ be a rainbow C_{n-2} with $v_{b-1}x_1 \in E(G_{n-1})$ and $x_1v_{b+1} \in E(G_n)$. Repeating the above discussions again, we can get that $N_{G_i}(v_b) = \{v_1, v_3, \ldots, v_{n-3}, x_2\}$ for each $i \in [n]$.

In summary, it is not difficult to see that G_i is a bipartite graph with bipartition

$$(\{v_1, v_3, \ldots, v_{n-3}, x_2\}, \{v_2, v_4, \ldots, v_{n-2}, x_1\})$$

for each $i \in [n]$. This implies that $\mathbf{G}$ consists of n copies of $K_{\frac{n}{2},\frac{n}{2}}$, and Theorem 2.1 thus follows. □

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References

[1] R. Aharoni, E. Berger, M. Chudnovsky, D. Howard, P. Seymour, Large rainbow matchings in general graphs, Europ. J. Combin. **79** (2019), 222–227.

[2] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, R. Šámal, A rainbow version of Mantel's theorem, Adv. Comb. 2(2020), 12 pp.

[3] R. Aharoni, D. Howard, A rainbow r-partite version of the Erdös-Ko-Rado theorem, Combin. Probab. Comput. **26** (2017), 321–337.

[4] J.A. Bondy, Pancyclic graphs I, J. Comb. Theory, Ser. B, 11(1)(1971), 80-84.

[5] J.A. Bondy, Pancyclic graphs, in: Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing, Louisiana State University, Baton Rouge, LA. (1971), 167-172.

[6] P. Bradshaw, Transversals and Bipancyclicity in Bipartite Graph Families, Electron. J. Combin. 28(4):#P4.25, 2021

[7] P. Bradshaw, K. Halasz, L. Stacho, From one to many rainbow Hamiltonian cycles, Graphs & Comb. 38(6)(2022), 1–21.

[8] Y. Cheng, J. Han, B. Wang, G. Wang, Rainbow spanning structures in graph and hypergraph system, arXiv:2105.10219.

[9] Y. Cheng, G. Wang, Y. Zhao, Rainbow pancyclicity in graph systems, Electron. J. Combin. 28(3): #P3.24, 2021.

[10] A. Ferber, J. Han, D. Mao, Dirac-type problem of rainbow matchings and Hamilton cycles in random graphs, arXiv: 2211.05477.

[11] F. Joos, J. Kim, On a rainbow version of Dirac's theorem, Bull. London Math. Soc. 52(3)(2020), 498–504.

[12] L. Li, P. Li, X. Li, Rainbow structures in a collection of graphs with degree conditions, J. Graph. Theory. (2023), 1–19. https://doi.org/10.1002/jgt.22966

[13] R. Montgomery, A. Müyesser, Y. Pehova, Transversal factors and spanning trees, Adv. Comb. **3** (2020), 25 pp.