BIPRIMITIVE EDGE-TRANSITIVE PENTAVALENT GRAPHS

QI CAI AND ZAI PING LU

ABSTRACT. A bipartite graph is said to be biprimitive if its bipartition preserving automorphism group acts primitively on each part of the graph. In this paper, a complete classification is given for biprimitive edge-transitive pentavalent graphs. In particular, it is proved that, up to isomorphism, there exists a unique biprimitive semisymmetric pentavalent graph, which is the incidence graph of a generalized hexagon of order $(4, 4)$.

KEYWORDS. Symmetric graph, semisymmetric graph, orbital digraph, standard double cover.

1. INTRODUCTION

In this paper, all graphs are finite without loops or parallel edges, all digraphs are finite without parallel arcs, and all groups are finite.

Let $\varGamma = (V, E)$ be a graph with vertex set V and edge set E, and denote by $\mathsf{Aut}(\varGamma)$ the (full) automorphism group of $\varGamma$. An arc in $\varGamma$ is an ordered pair of adjacent vertices. The graph $\varGamma$ is called vertex-transitive, edge-transitive or symmetric if $\mathsf{Aut}(\varGamma)$ acts transitively on V, E or the arc set of $\varGamma$, respectively. If $\varGamma$ is regular and edge-transitive but not vertex-transitive then $\varGamma$ is called a semisymmetric graph.

Let $\varGamma = (V, E)$ be a connected bipartite graph with bipartition (U, W), that is, V is partitioned into two independent sets U and W. We call each of U and W a part of the graph $\varGamma$. Denote by $\mathsf{Aut}^+(\varGamma)$ the bipartition preserving automorphism group of $\varGamma$, that is, $\mathsf{Aut}^+(\varGamma) = \{g \in \mathsf{Aut}(\varGamma) \mid U^g = U\}$. Then the graph $\varGamma$ is said to be biprimitive if $\mathsf{Aut}^+(\varGamma)$ acts primitively on both U and W.

The first classification result on biprimitive edge-transitive is given by Ivanov and Iofinova [15]. Appealing to the amalgams of edge-transitive cubic graphs obtained by Goldschmidt [14] and the classification of primitive groups with a subdegree 3 obtained by Wong [36], Ivanov and Iofinova classified biprimitive edge-transitive cubic graphs. Recently, Li and Zhang [23] classified biprimitive edge-transitive tetravalent graphs, based on their classification of finite primitive groups with solvable point-stabilizers [22]. Motivated by these works, we aim to classify biprimitive edge-transitive graphs of some special valencies. In this paper, we first classify biprimitive edge-transitive pentavalent graphs. The following is the main result of this paper.

Theorem 1.1. *Let $\varGamma$ be a connected bipartite pentavalent graph, and $G \leqslant \mathsf{Aut}^+(\varGamma)$. Assume that G acts primitively on both parts of $\varGamma$ and acts transitively on the edge set of Γ. Then one of the followings holds, where p is a prime.*

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(1) *Γ is isomorphic to the complete bipartite graph* $\mathsf{K}_{5,5}$.
(2) *Γ is isomorphic to the graph $G(2p,5)$ constructed as in* [5], $\mathsf{Aut}(\Gamma)\cong \mathrm{PGL}_2(11)$ *for $p=11$, and* $\mathsf{Aut}(\Gamma)\cong \mathbb{Z}_p{:}\mathbb{Z}_{10}$ *for $p>11$, where* $p\equiv 1\,(\mathsf{mod}\ 5)$.
(3) $\mathsf{Aut}(\Gamma)\cong(\mathbb{Z}_5^3{:}\mathrm{S}_5){:}\mathbb{Z}_2$, *and Γ is unique up to isomorphism.*
(4) *Γ is isomorphic to one of the graphs described as in Example* 7.2*; more precisely,*
   (i) $\Gamma\cong\mathsf{BCay}(\mathbb{Z}_p^2,S_{-1,b})$ *and* $\mathsf{Aut}(\Gamma)\cong(\mathbb{Z}_p^2{:}\mathrm{D}_{10}){:}\mathbb{Z}_2$, *where* $p\equiv\pm1\,(\mathsf{mod}\ 5)$ *and* $b=\frac{-1-\sqrt{5}}{2}\in\mathbb{Z}_p$*; or*
   (ii) $\Gamma\cong\mathsf{BCay}(\mathbb{Z}_p^4,S_{-1})$ *and* $\mathsf{Aut}(\Gamma)\cong(\mathbb{Z}_p^4{:}\mathrm{S}_5){:}\mathbb{Z}_2$.
(5) *G is an almost simple group,* $\mathsf{Aut}(\Gamma)\cong\mathsf{Aut}(G)$, *and Γ is isomorphic to one of the graphs described as in Example* 5.9.
(6) *G is an almost simple group, and Γ is isomorphic to one of the graphs described as in Theorem* 6.5.

In particular, Γ is semisymmetric if and only if Γ is isomorphic to the incidence graph of a generalized hexagon of order $(4,4)$, *which is included in part* (5).

2. Orbital digraphs and standard double covers

Let U be a nonempty finite set, and let Δ be a subset of $U\times U$. The pair (U,Δ) is called a digraph with vertex set U, while the elements in Δ are called arcs. (Note, loops, arcs of the form of (u,u), are allowed in the digraph (U,Δ).) Set $\Delta^*=\{(v,u)\mid(u,v)\in U\}$. Then (U,Δ^*) is also a digraph, called the paired digraph of (U,Δ).

For a digraph $\Sigma=(U,\Delta)$, the standard double cover of Σ, denoted by $\Sigma^{(2)}$, is defined as the bipartite graph with vertex set $U\times\mathbb{Z}_2$ and edge set $\{\{(u,0),(w,1)\}\mid(u,w)\in\Delta\}$. It is easy to check that each $g\in\mathsf{Aut}(\Sigma)$ induces an automorphism of $\Sigma^{(2)}$ as follows:

$$\tilde g: U\times\mathbb{Z}_2\to U\times\mathbb{Z}_2, (u,i)\mapsto(u^g,i).$$

Thus $\mathsf{Aut}(\Sigma^{(2)})$ contains a subgroup isomorphic to $\mathsf{Aut}(\Sigma)$. For convenience, we sometimes identify $\mathsf{Aut}(\Sigma)$ with a subgroup of $\mathsf{Aut}(\Sigma^{(2)})$. Define a map as follows:

$$\iota: U\times\mathbb{Z}_2\to U\times\mathbb{Z}_2, (u,i)\mapsto(u,i+1).$$

Then it is easily shown that ι is an isomorphism from $\Sigma^{(2)}$ to the standard double cover of (U,Δ^*). We have the following lemma.

Lemma 2.1. *Let $\Sigma=(U,\Delta)$ and $\Sigma_1=(U,\Delta^*)$ be paired digraphs. Then ι is an isomorphism from $\Sigma^{(2)}$ to $\Sigma_1^{(2)}$. In particular, if $\Delta=\Delta^*$ then $\iota\in\mathsf{Aut}(\Sigma^{(2)})$ and $\mathsf{Aut}(\Sigma^{(2)})\geqslant\tilde G\times\langle\iota\rangle$, where $\tilde G=\{\tilde g\mid g\in\mathsf{Aut}(\Sigma)\}$.*

For a group G and subgroups $K\leqslant H\leqslant G$, denote by $\mathbf{N}_G(K)$ the normalizer of K in G, and by $\mathsf{Aut}(G,H,K)$ the subgroup of $\mathsf{Aut}(G)$ fixing both H and K.

Lemma 2.2. *Let G be a transitive permutation group on U, $u,w\in U$, $H=G_u$ and $K=G_{uw}$. For $x\in\mathbf{N}_G(K)$, define $\Delta_x=\{(u,u^x)^g\mid g\in G\}$ and $\Sigma_x=(U,\Delta_x)$. Assume that $\delta\in\mathsf{Aut}(G,H,K)$. Then the following map*

$$\tilde\delta: U\times\mathbb{Z}_2\to U\times\mathbb{Z}_2,\ (u^g,i)\mapsto(u^{g^\delta},i)$$

is an isomorphism from $\Sigma_x^{(2)}$ to $\Sigma_{x^\delta}^{(2)}$, where $x\in\mathbf{N}_G(K)$. In particular,

(1) *if $(Kx)^\delta=Kx$ then $\tilde\delta\in\mathsf{Aut}(\Sigma_x^{(2)})$;*

(2) *if* $(Kx)^\delta = Kx^{-1}$ *then* $\tilde{\delta}\iota \in \mathsf{Aut}(\Sigma_x^{(2)})$.

Proof. Noting that $U = \{u^g \mid g \in G\}$, it is easily shown that $\tilde{\delta}$ is a bijection. For $u^{g_0}, u^{g_1} \in U$, we have

$$(u^{g_0}, u^{g_1}) \in \Delta_x \Leftrightarrow (u, u^{g_1 g_0^{-1}}) \in \Delta_x \Leftrightarrow g_1 g_0^{-1} \in HxH$$
$$\Leftrightarrow (g_1 g_0^{-1})^\delta \in Hx^\delta H \Leftrightarrow (u^{g_0^\delta}, u^{g_1^\delta}) \in \Delta_{x^\delta}.$$

It follows that $\tilde{\delta}$ is an isomorphism from $\Sigma_x^{(2)}$ to $\Sigma_{x^\delta}^{(2)}$. If $(Kx)^\delta = Kx$ then we have part (1). Noting that Σ_x and $\Sigma_{x^{-1}}$ are paired digraphs, by Lemma 2.1, we get part (2) of the lemma. This completes the proof. □

Assume that G is a transitive permutation group on U, and Δ is a G-invariant subset of $U \times U$. Then we have a G-vertex-transitive digraph (U, Δ). If Δ is a G-orbit then Δ is called an orbital of G, and the digraph (U, Δ) is called an orbital digraph. For a G-orbital Δ and $u \in U$, we have a G_u-orbit $\Delta(u) = \{w \mid (u, w) \in \Delta\}$ on U, which is called a suborbit of G at u. If $\Delta(u)$ is a suborbit then $\Delta^*(u)$ is called its paired suborbit, and $\Delta(u)$ is called self-paired if $\Delta(u) = \Delta^*(u)$, i.e. $\Delta^* = \Delta$.

Clearly, the set Δ_x defined as in Lemma 2.2 is an orbital of G, and so $\Delta_x(u) = \{u^{xh} \mid h \in H\}$ is a suborbit of G at u. For primitive permutation groups, by [11, Lemma 2.1], we have the following lemma.

Lemma 2.3. *Let G be a primitive permutation group on U, $u \in U$ and $H = G_u$. Suppose that H has a maximal subgroup K with index $k > 1$ such that $H \not\leqslant \mathbf{N}_G(K)$, and all maximal subgroups of H with index k are conjugate in H. For each $x \in \mathbf{N}_G(K)$, set $\Delta_x(u) = \{u^{xh} \mid h \in H\}$. Then, for x, $y \in \mathbf{N}_G(K) \setminus K$, the followings hold.*

(1) *$\Delta_x(u)$ is a suborbit of length k, and it is self-paired if and only if $x^2 \in K$.*
(2) *$\Delta_x(u) = \Delta_y(u)$ if and only if $yx^{-1} \in K$, i.e. $Kx = Ky$.*
(3) *$\Delta_x(u)$ and $\Delta_y(u)$ are paired suborbits if and only if $yx \in K$, i.e., $Kx^{-1} = Ky$.*

Moreover, if $\Delta(u)$ is a suborbit of length k then $\Delta(u) = \Delta_x(u)$ for some $x \in \mathbf{N}_G(K) \setminus K$.

A regular graph $\Gamma = (V, E)$ is called G-semisymmetric for some subgroup $G \leqslant \mathsf{Aut}(\Gamma)$ if G acts transitively on the edge set E but not on the vertex set V. It is well known that G has two orbits on V, which are independent sets and form a bipartition of Γ.

Let $\Sigma = (U, \Delta)$ be a G-orbital digraph, and identify G with the subgroup of $\mathsf{Aut}(\Sigma^{(2)})$ induced by G. Then $\Sigma^{(2)}$ is G-semisymmetric and, for $(u, 0), (w, 1) \in U \times \mathbb{Z}_2$, the stabilizers $G_{(u,0)}$ and $G_{(w,1)}$ are conjugate in G. Conversely, the following lemma holds.

Lemma 2.4. *Let $\Gamma = (V, E)$ be a G-semisymmetric graph of valency $k \geqslant 2$. Let $\{u, w\} \in E$, and let $U = \{u^g \mid g \in G\}$ and $W = \{w^g \mid g \in G\}$. Assume that G acts faithfully on both U and W, and the stabilizers G_u and G_w are conjugate in G. Then Γ is isomorphic to the standard double cover of some G-orbital digraph on U.*

Proof. Clearly, $V = U \cup W$ and $U \cap W = \emptyset$. Noting that $G_{u^g} = G_u^g$, since G_u and G_w are conjugate, we choose $u_0 \in U$ such that $G_{u_0} = G_w$. Noting that $|G_u : G_{uu_0}| = |G_u : G_{uw}| = |\Gamma(u)| = k \geqslant 2$, we have $u_0 \neq u$. Then u_0 lies in a G_u-orbit $\Delta(u)$ on U, and

$$|\Delta(u)| = |G_u : G_{uu_0}| = k.$$

Let Σ be the orbital digraph of G associated with $\Delta(u)$. Define

$$\phi : V \mapsto U \times \mathbb{Z}_2,\ u^g \mapsto (u^g, 0), w^g \mapsto (u_0^g, 1).$$

It is easily shown that ϕ is an isomorphism from Γ to $\Sigma^{(2)}$. Then the lemma follows. $\square$

Let R be a finite group, and S be a subset of R. Define a digraph $\mathsf{Cay}(R, S)$ with vertex set R such that (x, y) is an arc if and only if $yx^{-1} \in S$. The digraph $\mathsf{Cay}(R, S)$ is called a Cayley digraph of R, and the standard double cover of $\mathsf{Cay}(R, S)$, denoted by $\mathsf{BCay}(R, S)$, is called a bi-Cayley graph of R. Clearly, $\mathsf{BCay}(R, S)$ is of valency $|S|$. By [10, Lemmas 2.3 and 2.5] and [28, Lemma 1.3], the following lemma holds.

Lemma 2.5. *Let $\Gamma = (V, E)$ be a connected bipartite graph of valency k with bipartition (U, W). Assume that $\mathsf{Aut}(\Gamma)$ contains a subgroup R which is regular on both U and W. Then $\Gamma \cong \mathsf{BCay}(R, S)$ for some $S \subseteq R$ with $|S| = k$ and $R = \langle S\rangle$. Moreover, S may be chosen to contain the identity 1 of R. If R is abelian then $\mathsf{BCay}(R, S)$ has an automorphism $\tilde{\epsilon}\iota$, where $\epsilon \in \mathsf{Aut}(R)$ such that $x^\epsilon = x^{-1}$ for all $x \in R$; in particular, $\mathsf{Aut}(\Gamma)$ contains a regular subgroup on V.*

Lemma 2.6. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph of valency $k > 1$ with bipartition (U, W). Assume that G acts faithfully on both U and W, and G has a normal subgroup R which is regular on both U and W. Let $\{u, w\} \in E$. If G_u and G_w are conjugate in G then $\Gamma \cong \mathsf{BCay}(R, S)$, where S is a G_u-orbit on R by conjugation.*

Proof. Assume that G_u and G_w are conjugate in G. By Lemma 2.4, we may assume that $\Gamma = \Sigma^{(2)}$, where Σ is a G-orbital digraph on U. As a subgroup of $\mathsf{Aut}(\Sigma)$, the group G contains a regular normal subgroup R. Then Σ is isomorphic to a Cayley digraph of R, refer to [37, Proposition 1.2]. Up to isomorphism of digraphs, we let $\Sigma = \mathsf{Cay}(R, S)$. Let u be the vertex corresponding to the identity 1 of R. Then, by [37, Proposition 1.3], S is a G_u-orbit on R by conjugation. Thus the lemma follows. $\square$

3. On the stabilizers

In this section, we assume that $\Gamma = (V, E)$ is a connected G-semisymmetric pentavalent graph, where $G \leqslant \mathsf{Aut}\Gamma$. Let U and W be the G-orbits on V.

Since Γ has valency 5, we have $5|U| = |E| = 5|W|$, and so $|U| = |W|$. Thus, for $u \in U$ and $w \in W$, we have $|G : G_u| = |G : G_w|$, and so $|G_u| = |G_w|$. For $v \in V$, denote by $G_v^{\Gamma(v)}$ the permutation group induced by G_v on $\Gamma(v)$. Let $G_v^{[1]}$ be the kernel of G_v acting on $\Gamma(v)$. Then

$$G_v^{\Gamma(v)} \cong G_v/G_v^{[1]} \cong \mathbb{Z}_5{:}\mathbb{Z}_l,\ \ \mathrm{A}_5 \text{ or } \mathrm{S}_5,$$

where $l \in \{1, 2, 4\}$. Moreover, the following lemma is true.

Lemma 3.1. *Let $v \in V$. Then $|G_v| = 2^a 3^b 5$ for some nonnegative integers a and b. If $b \neq 0$ then G_u is insolvable for some $u \in V$.*

Proof. By [15, Lemma 3.3], we have the first part of the lemma.

Suppose that there exists $\{u, w\} \in E$ such that both G_u and G_w are solvable. Then both $G_{uw}^{\Gamma(u)}$ and $G_{uw}^{\Gamma(w)}$ are isomorphic to subgroups of $\mathbb{Z}_4$. It follows that every Sylow 3-subgroup of G_{uw} is contained in both $G_u^{[1]}$ and $G_w^{[1]}$. Let N be the subgroup of G_{uw}

generated by all Sylow 3-subgroups of G_{uw}. Then N is characteristic in both $G_u^{[1]}$ and $G_w^{[1]}$, and so N is normal in both G_u and G_w. Since Γ is connected, we have $G = \langle G_u, G_w\rangle$, refer to [33, Exercise 3.8]. Then $N \trianglelefteq G$. Clearly, N fixes the edge $\{u, w\}$. It follows from the edge-transitivity of G that N fixes E pointwise, which implies that $N = 1$. Then we have $|G_{uw}| = 2^a$, and $|G_u| = |G_w| = 2^a5$. Thus, if $b \neq 0$ then either G_u or G_w is insolvable. This completes the proof. $\square$

For a subgroup $X \leqslant G$ and a prime r, denote by $\mathbf{O}_r(X)$ the maximal normal r-subgroup of X. Note, $\mathbf{O}_r(X) = 1$ if $|X|$ is indivisible by r.

Lemma 3.2. *Let $\{u, w\}$ be an edge of Γ. Then $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_w) = \mathbf{O}_3(G_{uw}) = 1$.*

Proof. Since $\mathbf{O}_3(G_u) \trianglelefteq G_u$, all $\mathbf{O}_3(G_u)$-orbits on $\Gamma(u)$ have the same length, which is a common divisor of $|\mathbf{O}_3(G_u)|$ and $|\Gamma(u)|$. It follows that $\mathbf{O}_3(G_u)$ fixes $\Gamma(u)$ pointwise, i.e. $\mathbf{O}_3(G_u) \leqslant G_u^{[1]}$, and so $\mathbf{O}_3(G_u) \leqslant \mathbf{O}_3(G_u^{[1]})$. Noting that $\mathbf{O}_3(G_u^{[1]})$ is a characteristic subgroup of $G_u^{[1]}$, since $G_u^{[1]} \trianglelefteq G_u$, we have $\mathbf{O}_3(G_u^{[1]}) \trianglelefteq G_u$, and so $\mathbf{O}_3(G_u^{[1]}) \leqslant \mathbf{O}_3(G_u)$. Thus $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_u^{[1]})$. Noting that $G_u^{[1]} \trianglelefteq G_{uw}$, we have $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_u^{[1]}) \leqslant \mathbf{O}_3(G_{uw})$.

Recall that $G_u^{\Gamma(u)} \cong \mathbb{Z}_5{:}\mathbb{Z}_l$, A_5 or S_5, where $l \in \{1, 2, 4\}$. It is easily shown that every G_{uw}-orbit on $\Gamma(u)$ has length a divisor of 4. Considering the action of $\mathbf{O}_3(G_{uw})$ on $\Gamma(u)$, we conclude that $\mathbf{O}_3(G_{uw}) \leq G_u^{[1]}$. It follows that $\mathbf{O}_3(G_{uw}) \leqslant \mathbf{O}_3(G_u^{[1]})$. Then $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_u^{[1]}) = \mathbf{O}_3(G_{uw})$. Similarly, we have $\mathbf{O}_3(G_w) = \mathbf{O}_3(G_w^{[1]}) = \mathbf{O}_3(G_{uw})$. Thus $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_w) = \mathbf{O}_3(G_{uw}) \trianglelefteq \langle G_u, G_w\rangle$. Since Γ is connected, $G = \langle G_u, G_w\rangle$. Then $\mathbf{O}_3(G_{uw})$ is normal in G and fixes the edge $\{u, w\}$. It follows from the edge-transitivity of G on Γ that $\mathbf{O}_3(G_{uw})$ fixes E pointwise, yielding $\mathbf{O}_3(G_{uw}) = 1$. Then the lemma follows. $\square$

Lemma 3.3. *Let $\{u, w\} \in E$. If $G_u^{[1]} = 1 \neq G_w^{[1]}$ then one of the followings holds.*

(1) *$G_u \cong \mathbb{Z}_5{:}\mathbb{Z}_l \leqslant \mathrm{AGL}_1(5)$ for $l \in \{2, 4\}$, and $G_w \cong \mathbb{Z}_{10}$, $\mathbb{Z}_2.\mathrm{D}_{10}$ or $\mathbb{Z}_{20}$.*
(2) *$G_u \cong \mathrm{A}_5$, and $G_w \cong \mathrm{A}_4 \times \mathbb{Z}_5$.*
(3) *$G_u \cong \mathrm{S}_5$, and $G_w \cong \mathrm{A}_4.\mathrm{D}_{10}$ or $\mathrm{S}_4 \times \mathbb{Z}_5$.*

Proof. Assume that $G_u^{[1]} = 1 \neq G_w^{[1]}$. Then $G_w^{[1]} \trianglelefteq G_{uw} \cong G_{uw}^{\Gamma(u)}$. Recall that $|G_u| = |G_w|$. If G_u is solvable, then $G_u \cong \mathbb{Z}_5{:}\mathbb{Z}_l$ for some divisor l of 4, and so $G_w^{[1]}$ is isomorphic a subgroup of $\mathbb{Z}_l$, which yields (1) of this lemma. Thus, in the following, we assume that $G_u \cong \mathrm{A}_5$ or S_5. In particular, we have $G_w^{[1]} \trianglelefteq G_{uw} \cong \mathrm{A}_4$ or S_4, respectively.

Suppose that G_w is insolvable. Then, since $|G_u| = |G_w|$, we conclude that $G_u \cong \mathrm{S}_5$, $G_w^{\Gamma(w)} \cong \mathrm{A}_5$ and $G_w^{[1]} \cong \mathbb{Z}_2$. Note that $G_w^{[1]} \trianglelefteq G_{uw} \cong \mathrm{S}_4$. It follows that S_4 has a normal subgroup of order 2, which is impossible.

Now suppose that G_w is solvable. Then $G_w^{\Gamma(w)} \cong \mathbb{Z}_5{:}\mathbb{Z}_l$, where l is a divisor of 4. Again since $|G_u| = |G_w|$, we know that $G_w^{[1]}$ has order divisible by 3 as $1 \neq G_w^{[1]} \trianglelefteq G_{uw} \cong \mathrm{A}_4$ or S_4. It follows that $G_w^{[1]} \cong \mathrm{A}_4$ or S_4. If $G_u \cong \mathrm{A}_5$ then $G_w^{[1]} \cong \mathrm{A}_4$ and $l = 1$, which gives part (2) of the lemma. If $G_u \cong \mathrm{S}_5$ then $G_w^{[1]} \cong \mathrm{A}_4$ or S_4, and $l = 2$ or 1 respectively, and thus part (3) of this lemma holds. This completes the proof. $\square$

For an edge $\{u, w\}$ of Γ, let $G_{uw}^{[1]} = G_u^{[1]} \cap G_v^{[1]}$ and $G_u^{[2]} = \cap_{v\in\Gamma(u)} G_{uv}^{[1]}$. Then

$$G_{uw}/G_{uw}^{[1]} \lesssim (G_{uw}/G_u^{[1]}) \times (G_{uw}/G_w^{[1]}) \cong G_{uw}^{\Gamma(u)} \times G_{uw}^{\Gamma(w)} \lesssim \mathrm{S}_4^2,$$
$$G_u^{[1]}/G_u^{[2]} \lesssim \times_{v\in\Gamma(u)} (G_u^{[1]})^{\Gamma(v)} \lesssim \mathrm{S}_4^5.$$

Lemma 3.4. *Let $\{u,w\} \in E$. Assume that $G_u^{[1]} \neq 1 \neq G_w^{[1]}$. Then either $G_{uw}^{[1]}$ is a 2-group and $|G_v|$ is not divisible by 3^3, or $|G_v|$ is not divisible by 3^7, where $v \in \{u,w\}$.*

Proof. If $G_{uw}^{[1]}$ is a 2-group then, since $G_{uw}/G_{uw}^{[1]} \lesssim \mathrm{S}_4^2$ and $|G_u : G_{uw}| = 5$, the order of G_u is indivisible by 3^3, and the lemma is true.

Assume that $G_{uw}^{[1]}$ is not a 2-group. Note that $G_{uw}^{[1]}$ is a $\{2,3\}$-group and, by Lemma 3.2, $G_{uw}^{[1]}$ is not a 3-group. It follows from [2, Theorem 1.1] that, one of $G_u^{[2]}$ and $G_w^{[2]}$ say $G_u^{[2]}$ is an r-group, where $r \in \{2,3\}$. Since $G_u^{[2]} \trianglelefteq G_u^{[1]} \trianglelefteq G_u$, we have $G_u^{[2]} \leqslant \mathbf{O}_r(G_u)$. By Lemma 3.2, we conclude that $G_u^{[2]}$ is an 2-group. Recalling that $G_u^{[1]}/G_u^{[2]} \lesssim \mathrm{S}_4^5$ and $G_u/G_u^{[1]} \lesssim \mathrm{S}_5$, it follows that $|G_u|$ is indivisible by 3^7, and the lemma follows. $\square$

For normal subgroups of G, we have the following lemma, refer to [13, Lemmas 5.1 and 5.5] and [29, Lemma 3.2].

Lemma 3.5. *Let $1 \neq N \trianglelefteq G$. If $N_v \neq 1$ for some $v \in V$ then either Γ is N-semisymmetric, or N acts transitively on one of U and W and has 5 orbits on the other one. If N is intransitive on U and W then N is semiregular on V.*

4. A reduction

In this section, we assume that $\Gamma = (V,E)$ is a connected G-semisymmetric pentavalent graph, and G acts primitively on each of its orbits on V, where $G \leqslant \mathsf{Aut}\Gamma$. Let U and W be the G-orbits on V. (Note, $G \leqslant \mathsf{Aut}^+(\Gamma)$.) Recall that the socle $\mathrm{soc}(G)$ of G is generated by all minimal normal subgroups of G.

Lemma 4.1. *One of the following statements holds.*

(1) $\mathrm{soc}(G) \cong \mathbb{Z}_p^k$, *and $\mathsf{Aut}(\Gamma)$ has a regular subgroup isomorphic to $\mathbb{Z}_p^k{:}\mathbb{Z}_2$, where $1 \leqslant k \leqslant 4$ and p is a prime.*
(2) *G is almost simple, and Γ is $\mathrm{soc}(G)$-semisymmetric.*
(3) *Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{5,5}$ of order 10.*

Proof. If G is unfaithful on U then the kernel of G on U acts transitively on W, which yields that $\Gamma \cong \mathsf{K}_{5,5}$. Similarly, if G is unfaithful on W then $\Gamma \cong \mathsf{K}_{5,5}$.

Assume next that G acts faithfully on both U and W in the following. We will analyze the structure of G by using the O'Nan-Scott Theorem for finite primitive groups, refer to [9, Section 4.8, p. 137]. Let $M = \mathrm{soc}(G)$. Then $M = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, T_2, \ldots, T_k$ are isomorphic simple groups. Fix an edge $\{u,w\} \in E$ with $u \in U$ and $w \in W$, and let $v = u$ or w.

Assume that G is of Affine type on U (and hence of Affine type on W). Then $M \cong \mathbb{Z}_p^k$ for some prime p and integer $k \geqslant 1$, and M is regular on both U and W. By Lemma 2.5, Γ is isomorphic to a bi-Cayley graph of M, and M can be generated by 4 elements, yielding $k \leqslant 4$. Again by Lemma 2.5, $\mathsf{Aut}(\Gamma)$ contains an involution which inverses every element in M and interchanges U and W. Then part (1) of this lemma follows.

If M is a nonabelian simple group then part (2) of this lemma follows from Lemma 3.5. Thus the rest is to prove that G, as a primitive permutation group on U or W, is not of Regular nonabelian type, Diagonal type or Product type.

Case 1. Suppose that G has Regular nonabelian type on U or W. Recall that $|G_v| = 2^a3^b5$ and either G_v is solvable or $\mathrm{soc}(G_v^{\Gamma(v)}) \cong \mathrm{A}_5$. It follows from [9, Theorem 4.7B, p. 133] that $\mathrm{soc}(G_v) \cong \mathrm{A}_5$, and $\mathbf{N}_{G_v}(T_1)$ has a composition factor isomorphic to T_1. On the other hand, G_v acts on $\{T_1, T_2, \ldots, T_k\}$ faithfully and transitively by conjugation. This implies that $k \geqslant 5$, which forces that $\mathbf{N}_{G_v}(T_1)$ is solvable, a contradiction.

Case 2. Suppose that G has Diagonal type on U. Then $T_1 \lesssim G_u \lesssim \mathsf{Aut}(T_1) \times \mathrm{S}_k$. This implies that $T_1 \cong \mathrm{A}_5$, and $G_u^{\Gamma(u)}$ is 2-transitive on $\Gamma(u)$. By [9, Theorem 4.5A, p. 123], either $k = 2$, or G_u acts primitively on $\{T_1, T_2, \ldots, T_k\}$ by conjugation, where the kernel contains a normal subgroup isomorphic to T_1. In addition, for $k \geqslant 3$, since G_u has a unique insolvable composition factor and $|G_u|$ is indivisible by 5^2, we get $k \leqslant 4$.

By Case 1, G has Diagonal or Product type on W. If G has Diagonal type on W then a similar argument as above implies that $G_w^{\Gamma(w)}$ is 2-transitive on $\Gamma(w)$, which is impossible, refer to [13, Theorem 1.2]. Thus G is of Product type on W. By [9, Theorem 4.6A, p. 125], we conclude that either $M_w \cong T_1^d$ for some d with $1 < d < k$, or G, as a permutation group on W, is permutation isomorphic to a primitive subgroup of a wreath product $H \wr \mathrm{S}_k$ with the product action, where H is a primitive group with socle isomorphic to $T_1 \cong \mathrm{A}_5$. Noting that $|M_w|$ is indivisible by 5^2, the latter case occurs. In particular, $1 \neq M_w = (T_1)_w \times \cdots \times (T_k)_w$ and $(T_1)_w \cong \cdots \cong (T_k)_w$, and so M_w is a $\{2, 3\}$-group. In addition, G_w acts transitively on $\{T_1, T_2, \ldots, T_k\}$ by conjugation.

Let K be the kernel of G_w acting on $\{T_1, T_2, \ldots, T_k\}$. Noting that $|G_w|$ has a divisor 5, since $k \leqslant 4$, we know that $|K|$ is divisible by 5. Note that $|MK|$ is a divisor of $|G|$, and $|G|$ is a divisor of $|H \wr \mathrm{S}_k|$. Since $H \cong \mathrm{A}_5$ or S_5, it follows that $|MK|$ is a divisor of $120^k k!$. In particular, $|MK|$ is indivisible by 5^{k+1} as $k \leqslant 4$. Note that

$$|MK| = |M||K : (M \cap K)| = 60^k|K : (M \cap K)|.$$

This implies that $|M \cap K|$ is divisible by 5. Then M_w is not a $\{2, 3\}$-group as $M \cap K \leqslant M \cap G_w = M_w$, a contradiction.

Case 3. Suppose that G has Product type on U. Then, by Cases 1 and 2, G must have Product type on W. By [9, Theorem 4.6A, p. 125], either $M_v \cong T_1^d$ for some d with $1 < d < k$, or $1 \neq M_v = (T_1)_v \times \cdots \times (T_k)_v$ and $(T_1)_v \cong \cdots \cong (T_k)_v$, where $v \in \{u, w\}$. Recalling that G_v has at most one insolvable composition factor, the latter case occurs. By Lemma 3.5, Γ is M-semisymmetric. Then $|M_v|$ is divisible by 5, and hence $|M_v|$ is divisible by 5^k, which is impossible as $k > 1$. This completes the proof. □

Lemma 4.2. *Let $v \in V$ and $H = G_v$. Assume that G is almost simple and H is solvable. Then H is unique up to G-conjugacy, and (G, H) is listed in Table* 1.

Proof. Put $T = \mathrm{soc}(G)$. Choose a normal subgroup G_0 of G, which is minimal such that $H_0 := H \cap G_0$ is maximal in G_0. Then $T \leqslant G_0$ and, noting that H_0 is solvable, the pair (G_0, H_0) is included in [22, Tables 14–20]. By Lemma 4.1, Γ is G_0-semisymmetric. Then, by Lemmas 3.1 and 3.2, we have $|H_0| = 2^a3^b5$ and $\mathbf{O}_3(H_0) = 1$, where a and b are nonnegative integers. Inspecting the pairs listed in [22, Tables 14–20], we conclude that H is unique up G-conjugacy, and either the pair (G, H) is described as in Rows 1–4 of Table 1 or one of the followings holds.

(1) $G_0 = \mathrm{PSL}_2(p^f)$, and $H_0 \cong \mathrm{D}_{\frac{2(p^f+1)}{(2,p-1)}}$, where p is a prime and $p^f \notin \{7, 9\}$.

Row	G	H	
1	$\mathrm{S}_5, \mathrm{A}_5$	$\mathbb{Z}_5{:}\mathbb{Z}_4, \mathrm{D}_{10}$	
2	${}^2\mathrm{F}_4(2), {}^2\mathrm{F}_4(2)'$	$[2^9]{:}\mathbb{Z}_5{:}\mathbb{Z}_4, [2^{10}]{:}\mathbb{Z}_5{:}\mathbb{Z}_4$	
3	$\mathrm{PGL}_2(11)$	D_{20}	
4	$\mathrm{P\Gamma L}_2(9), \mathrm{PGL}_2(9), \mathrm{M}_{10}$	$\mathbb{Z}_{10}{:}\mathbb{Z}_4, \mathrm{D}_{20}, \mathbb{Z}_5{:}\mathbb{Z}_4$	
5	$\mathrm{PGL}_2(p), \mathrm{PSL}_2(p)$	$\mathrm{D}_{2(p+1)}, \mathrm{D}_{p+1}$	prime $p = 2^s 5 - 1$
6	$\mathrm{PGL}_2(p), \mathrm{PSL}_2(p)$	$\mathrm{D}_{2(p-1)}, \mathrm{D}_{p-1}$	prime $p = 2^s 5 + 1 > 11$
7	$\mathrm{PSL}_2(16).\mathbb{Z}_o$	$\mathbb{Z}_2^4{:}\mathbb{Z}_{15}{:}\mathbb{Z}_o$	$o \in \{1, 2, 4\}$
8	$\mathrm{PSU}_3(4).\mathbb{Z}_o$	$\mathbb{Z}_2^4{:}\mathbb{Z}_{15}.\mathbb{Z}_o$	$o \in \{1, 2, 4\}$

TABLE 1. Solvable stabilizers

(2) $G_0 = \mathrm{PSL}_2(p^f)$, and $H_0 \cong \mathrm{D}_{\frac{2(p^f-1)}{(2,p-1)}}$, where p is a prime and $p^f \notin \{5, 7, 9, 11\}$.

(3) $G_0 = \mathrm{PSL}_2(p^f)$, and $H_0 \cong \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$, where p is a prime.

(4) $G_0 = \mathrm{PSU}_3(2^f)$, and $H_0 \cong [p^{3f}]{:}\mathbb{Z}_{\frac{p^{2f}-1}{(3,p^f+1)}}$, where p is a prime.

Assume that (1) occurs. Then $p^f + 1$ is indivisible by 3; otherwise, $\mathbf{O}_3(H_0) \neq 1$, a contradiction. We have $p^f + 1 = 2^s 5$ for some integer $s \geqslant 0$. Since $p^f \neq 9$, we have $s \neq 1$. If $s = 0$ then $p^f = 4$ and $T \cong \mathrm{A}_5$, and so (G, H) is described as in Row 1 of Table 1. Thus, we let $s \geqslant 2$, and so $p^f \equiv -1 \,(\mathsf{mod}\ 4)$. Then f is odd. Suppose that $f > 1$. Then, since $\frac{p^f+1}{p+1}$ is odd, we have $5 = \frac{p^f+1}{p+1}$; however, $\frac{p^f+1}{p+1} > p^2 > 5$, a contradiction. Thus $f = 1$, and we get Row 5 of Table 1.

Assume that (2) occurs. Then $p^f - 1 = 2^s 5$ for some integer $s \geqslant 2$. Since $p^f - 1$ is indivisible by 3, we have $p^f \equiv -1 \,(\mathsf{mod}\ 3)$. Then f is odd. Since $\frac{p^f-1}{p-1}$ is odd, if $f > 1$ then $5 = \frac{p^f-1}{p-1} > p^2 > 5$, a contradiction. Then we have Row 6 of Table 1.

Assume that (3) occurs. Then $p^f = 5$ or $p = 2$. If $p^f = 5$ then (G, H) is described as in Row 1 of Table 1. Now let $p = 2$. Then $2^f - 1 = 3^t 5$ for some integer $t \geqslant 0$. Suppose that $f > 6$. Then, by Zsigmondy's Theorem, there is a prime r such that f is the smallest positive integer with $2^f \equiv 1 \,(\mathsf{mod}\ r)$. Noting that $r - 1$ is divisible by f, this implies that $|H_0|$ has a prime divisor no less than 7, a contradiction. Thus $f \leqslant 6$. Calculation shows that $f = 4$. Then we have Row 7 of Table 1.

Finally, for (4), by a similar argument as above, we conclude that $p^f = 4$, and Row 8 of Table 1 follows. This completes the proof. $\square$

Lemma 4.3. *Let $v \in V$ and $H = G_v$. Assume that G is almost simple with socle T, and H is insolvable. Then, up to G-conjugacy, either H is unique, or H has two choices which are listed in Table 2 up to isomorphism. In addition, if $\mathbf{O}_2(H \cap T) = 1 \neq \mathbf{O}_2(H)$ then (G, H) is listed as follows:*

(1) $G = \mathrm{S}_7$ *and* $H \cong \mathbb{Z}_2 \times \mathrm{S}_5$*;*
(2) $G = \mathrm{P\Sigma L}_2(25)$ *and* $H \cong \mathbb{Z}_2 \times \mathrm{S}_5$*;*
(3) $G = \mathrm{PSL}_2(16).\mathbb{Z}_o$ *and* $H \cong (\mathbb{Z}_2 \times \mathrm{A}_5).\mathbb{Z}_{\frac{o}{2}}$*, where* $o \in \{2, 4\}$*;*
(4) $G = \mathrm{PSL}_3(4).\mathbb{Z}_2^i \not\leqslant \mathrm{P\Gamma L}_3(4)$ *and* $H \cong 2 \times \mathrm{A}_5.\mathbb{Z}_2^{i-1}$*, where* $i \in \{1, 2\}$*.*

	G	H	
1	$\mathrm{PSL}_2(p)$	A_5	$p \equiv \pm 11, \pm 19 \pmod{40}$
2	$\mathrm{PSL}_2(p)$	A_5	$p \equiv \pm 1, \pm 9 \pmod{40}$
3	$\mathrm{PSL}_2(p^2)$	A_5	$p \equiv \pm 3 \pmod{10}$
4	$\mathrm{PSL}_2(5^2)$	S_5	
5	$\mathrm{P\Sigma L}_2(p^2)$	S_5	$p \equiv \pm 3 \pmod{10}$
6	$\mathrm{PSp}_6(p)$	S_5	$p \equiv \pm 1 \pmod 8$
7	$\mathrm{G}_2(4).\mathbb{Z}_o$	$2^{4+6}{:}(\mathrm{A}_5 \times \mathbb{Z}_3){:}\mathbb{Z}_o$	$o \in \{1, 2\}$
8	$\mathrm{P\Sigma L}_2(25)$	$\mathbb{Z}_2 \times \mathrm{S}_5$	
9	$\mathrm{PSL}_3(4).O$	$\mathbb{Z}_2^4{:}\mathrm{A}_5.O$	$\lvert O\rvert \in \{1, 2, 3, 6\}, G \leqslant \mathrm{P\Gamma L}_3(4)$
10	$\mathrm{PSp}_4(4).\mathbb{Z}_o$	$\mathbb{Z}_2^6{:}(\mathbb{Z}_3 \times \mathrm{A}_5).\mathbb{Z}_o$	$o \in \{1, 2\}$
11	$\mathrm{PSp}_4(p)$	$\mathbb{Z}_2^4.\mathrm{S}_5$	$p \equiv \pm 1 \pmod 8$

TABLE 2. Nonconjugate stabilizers

Proof. By the assumption, we have $H/G_v^{[1]} \cong \mathrm{A}_5$ or S_5. Since H is maximal in G, if $G_v^{[1]} = 1$ then, by [11, Theorem 1.3], either H is unique up to G-conjugacy, or the pair (G, H) is described as in Rows 1-6 of Table 2. Thus, in view of Lemmas 3.2 and 3.4, we assume next that $\mathbf{O}_2(H) \neq 1 = \mathbf{O}_3(H)$, and $|H|$ is indivisible by 3^7. In particular, H is a 2-local maximal subgroup of G. In addition, since $G = TH$, we have $G/T \cong H/(H \cap T)$. By the Schreier Conjecture, G/T is solvable. Since H is insolvable, $H \cap T$ is insolvable, and thus $(H \cap T)/T_v^{[1]} = T_v/T_v^{[1]} \cong \mathrm{A}_5$ or S_5.

Assume that T is an alternating group A_n, where $\geqslant 5$. For $n = 6$, by the Atlas [7], we have $H \cong \mathrm{A}_5$ or S_5, and so $\mathbf{O}_2(H) = 1$, which is not the case. Thus we let $n \neq 6$, and so $G = \mathrm{S}_n$ or A_n. Considering the natural action of G on n points, it follows from [25] that either $n \in \{7, 9\}$ and H is conjugate to the stabilizer of some $(n-5)$-set, or $n \in \{10, 20\}$ and H is conjugate to the stabilizer of some partition with equal part size $\frac{n}{5}$. Only for $G = \mathrm{S}_7$, we have $\mathbf{O}_2(H \cap T) = 1 \neq \mathbf{O}_2(H)$; in this case, $H \cong \mathbb{Z}_2 \times \mathrm{S}_5$ and $H \cap T \cong \mathrm{S}_5$. Then the lemma is true in this case.

Assume that T is one of the 26 sporadic simple groups. Meierfrankenfeld and Shpectorov [31] proved that the Atlas [7] includes the complete lists of the 2-local maximal subgroups of the Monster and the Baby Monster, see also [35, pp. 258-261, Tables 5.6 and 5.7]. Thus all 2-local maximal subgroups of sporadic almost simple groups are listed in the Atlas [7]. Inspecting these subgroups, we conclude that T is one of M_{12}, M_{22}, M_{23}, J_1, J_2, J_3, Co_2, Co_3, HS, Suz and Ru, $\mathbf{O}_2(H \cap T) \neq 1$, and H is unique up to G-conjugacy.

Assume that T is a simple exceptional group of Lie type. Suppose that T has Lie rank at least 4. Noticing the limitations on H, it follows from [6, Theorem 1] that H is either parabolic or of maximal rank. For the parabolic case, H is an extension of a 2-group by the Chevalley group determined by some subdiagram obtained from the Dynkin diagram of G by removing one node. It follows that H has an insolvable composition factor not isomorphic to A_5, which is not the case. Thus H is a subgroup of maximal rank. Inspecting the subgroups listed in [26, Tables 5.1 and 5.2], there does not exist a desired H. Therefore, T has Lie rank 1 or 2. Then all maximal subgroups of G are completely known, refer to [32] for $T = \mathrm{Sz}(q)$, [30] for $T = {}^2\mathrm{F}_4(q)$ (with $q > 2$), [19] for $T = \mathrm{Ree}(q)$, [17] for $T = {}^3\mathrm{D}_4(q)$, and [8, 19] for $T = \mathrm{G}_2(q)$, respectively. Inspecting the 2-local maximal subgroups of G, we conclude that $T = \mathrm{G}_2(4)$. By the Atlas [7], we know H has two choices up to G-conjugacy, and Row 7 of Table 2 follows.

In the following, we assume that T is a simple classical group of dimension n over a field of order $q = p^f$, where p is a prime. Noticing the isomorphisms amongst finite classical groups, we assume that T is one of the following simple groups: $\mathrm{PSL}_n(q)$ with $n \geqslant 2$, $\mathrm{PSU}_n(q)$ with $n \geqslant 3$, $\mathrm{PSp}_n(q)$ with even $n \geqslant 4$ and $(n, q) \neq (4, 2)$, $\Omega_n(q)$ with odd $n \geqslant 7$ and odd q, $\mathrm{P}\Omega_n^{\pm}(q)$ with even $n \geqslant 8$. If $T = \mathrm{PSp}_4(2^f)$ with $f > 1$ and G contains a graph automorphism of T then, by [3, p. 384, Table 8.14], we conclude that G does not contains a desired H. If $T = \mathrm{P}\Omega_8^+(q)$ and G contains a triality of T then, inspecting the maximal subgroups of G listed in [18], we conclude that $T = \mathrm{P}\Omega_8^+(4)$, and H is unique up to G-conjugacy. Thus, since H is 2-local, by Aschbacher's Theorem for maximal subgroups of classical groups, we next assume that H lies in one of the eight classes of subgroups of G, say $\mathcal{C}_i(G)$, $1 \leq i \leq 8$, which are defined as in [20].

Inspecting the members of $\mathcal{C}_i(G)$ given in [20, pp. 70-74, Tables 3.5A-3.4F], it follows that $H \in \mathcal{C}_1(G) \cup \mathcal{C}_2(G) \cup \mathcal{C}_5(G) \cup \mathcal{C}_6(G)$, H has at most two choices up to G-conjugacy, and either $n \leqslant 10$ or $T = \Omega_{15}(3)$. Then, combining with [3], we conclude that one of the followings holds.

(i) $H \in \mathcal{C}_1(G)$ if and only if T is one of the following simple groups: $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(5)$, $\mathrm{PSU}_3(5)$, $\mathrm{PSL}_4(4)$, $\mathrm{PSp}_4(4)$, $\mathrm{PSU}_5(2)$, $\mathrm{PSU}_6(2)$, $\mathrm{PSU}_7(2)$, $\mathrm{P}\Omega_8^-(2)$; in this case, H has two choices if and only if $G \leqslant \mathrm{P\Gamma L}_3(4)$ or $T = \mathrm{PSp}_4(4)$, and $\mathbf{O}_2(H \cap T) = 1$ if and only if $T = \mathrm{PSL}_3(5)$ or $G = \mathrm{PSL}_3(4).\mathbb{Z}_2^i \nleqslant \mathrm{P\Gamma L}_3(4)$, where $i \in \{1, 2\}$.

(ii) $H \in \mathcal{C}_2(G)$ if and only if T is one of the following simple groups: $\mathrm{PSp}_4(5)$, $\mathrm{PSL}_5(9)$, $\mathrm{PSL}_5(p)$ (with p a Fermat prime), $\mathrm{PSU}_5(p)$ (with p a Mersenne prime), $\mathrm{PSL}_{10}(3)$, $\mathrm{PSU}_{10}(3)$, $\mathrm{PSp}_{10}(3)$, $\mathrm{P}\Omega_{10}^+(9)$, $\mathrm{P}\Omega_{10}^+(p)$ (with p a Fermat prime), $\mathrm{P}\Omega_{10}^-(p)$ (with p a Mersenne prime), $\Omega_{15}(3)$; in this case, H is unique up to G-conjugacy.

(iii) $H \in \mathcal{C}_5(G)$ if and only if $G = \mathrm{P\Sigma L}_2(25)$ or $G = \mathrm{PSL}_2(16).\mathbb{Z}_o$ with $o \in \{2, 4\}$; in this case, $\mathbf{O}_2(H \cap T) = 1$, and H has two choices if and only if $G = \mathrm{P\Sigma L}_2(25)$ and $H \cong \mathbb{Z}_2 \times \mathrm{S}_5$.

(iv) $H \in \mathcal{C}_6(G)$ if and only if $T = \mathrm{PSp}_4(p)$ with prime $p > 3$; in this case, H has two choices if and only if $G = \mathrm{PSp}_4(p)$, $H \cong \mathbb{Z}_2^4.\mathrm{S}_5$ and $p \equiv \pm 1 \pmod 8$.

By (i)–(iv), we conclude that H is desired as in this lemma. This completes the proof. □

Theorem 4.4. *Let $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Assume that G is almost simple. Then one of the followings holds.*

(1) *G_u and G_w are conjugate in G, and Γ is isomorphic to the standard double cover of some G-orbital digraph.*

(2) *$H := G_u \cong G_w$, and the pair (G, H) is listed in Table 2.*

Proof. If G_u and G_w are conjugate in G then part (1) is true by Lemma 2.4. Assume next that G_u and G_w are not conjugate in G. By Lemma 4.2, one of G_u and G_w, say G_u is insolvable. In particular, $G_u^{\Gamma(u)} \cong G_u/G_u^{[1]} \cong \mathrm{A}_5$ or S_5, and G_{uw} is not a 2-group.

Suppose that G_w is solvable. Then, by Lemma 4.2, the pair (G, G_w) is described as in Rows 7 and 8 of Table 1. In particular, $|G_w| = 240|G : T|$. Checking the maximal subgroups of G in the Atlas [7], we conclude that G has no insolvable maximal subgroup of order $|G_u|$, a contradiction. Therefore, G_w is insolvable.

Finally, since G_u and G_w are not conjugate in G, part (2) follows from Lemma 4.3. $\square$

5. Graphs with nonconjugate stabilizers

In this section, we deal with the graphs satisfying (2) of Theorem 4.4. We first give a construction for some biprimitive graphs.

Construction 5.1. Let G be a nonregular primitive group on U, and let H_1 be a point-stabilizer. Suppose that H_2 is a core-free maximal subgroup of G such that H_2 is not conjugate to H_1 in G. Let $k = |H_1 : (H_1 \cap H_2)|$, and set

$$\mathcal{H}_1 = \{H_1^g \mid g \in G\},\ \mathcal{H}_2 = \{H_2^g \mid g \in G\}.$$

Define a bipartite graph $\Gamma(G)$ with bipartition $(\mathcal{H}_1, \mathcal{H}_2)$ such that $M_1 \in \mathcal{H}_1$ and $M_2 \in \mathcal{H}_2$ are adjacent if and only if $k = |M_1 : (M_1 \cap M_2)|$. Then $\Gamma(G)$ is a regular graph if and only if $|H_2 : (H_1 \cap H_2)| = k$, i.e., $|H_1| = |H_2|$.

It is easily shown that the inner automorphism group $\mathsf{Inn}(G)$ of G acts faithfully and primitively on both parts of $\Gamma(G)$. We always view $\mathsf{Inn}(G)$ as a subgroup of $\mathsf{Aut}(\Gamma(G))$. By the primitivity of $\mathsf{Inn}(G)$ on both parts of the graph, $\Gamma(G)$ is connected. $\square$

Lemma 5.2. *Let G, H_1, H_2, k and $\Gamma(G)$ be as in Construction 5.1. Then $\mathsf{Inn}(G) \cong G$, and every $\alpha \in \mathsf{Aut}(G, H_1, H_2)$ induces an automorphism of $\Gamma(G)$ by $(H_i^x)^\alpha = H_i^{x^\alpha}$.*

(1) *If $\delta \in \mathsf{Aut}(G)$ such that $H_1^\delta = H_2$ and $H_2^\delta = H_1$, then δ induces an automorphism of $\Gamma(G)$ by $(H_i^x)^\delta = (H_i^\delta)^{x^\delta}$, and $\Gamma(G)$ is vertex-transitive.*

(2) *The graph $\Gamma(G)$ is $\mathsf{Inn}(G)$-semisymmetric if and only if $\Gamma(G)$ has valency k, i.e., $|\{M_2 \in \mathcal{H}_2 \mid k = |H_1 : (H_1 \cap M_2)|\}| = k = |\{M_1 \in \mathcal{H}_1 \mid k = |M_1 : (M_1 \cap H_2)|\}|$.*

Proof. Since G is a nonregular primitive group, it has trivial center. Then $\mathsf{Inn}(G) \cong G$. Pick $\alpha \in \mathsf{Aut}(G, H_1, H_2)$. Then α fixes both $\mathcal{H}_1$ and $\mathcal{H}_2$ setwise. For g in G, denote by $\mathsf{Inn}(g)$ the inner automorphism of G induced by g. Then $\alpha^{-1}\mathsf{Inn}(g)\alpha = \mathsf{Inn}(g^\alpha)$. Now for $H_1^x \in \mathcal{H}_1$ and $H_2^y \in \mathcal{H}_2$, we have

$$(H_1^x \cap H_2^y)^\alpha = (H_1^{\mathsf{Inn}(x)} \cap H_2^{\mathsf{Inn}(y)})^\alpha = H_1^{\mathsf{Inn}(x^\alpha)} \cap H_2^{\mathsf{Inn}(y^\alpha)} = H_1^{x^\alpha} \cap H_2^{y^\alpha}.$$

It follows that $|H_1^x : (H_1^x \cap H_2^y)| = k$ if and only if $|H_1^{x^\alpha} : (H_1^{x^\alpha} \cap H_2^{y^\alpha})| = k$. Thus α induces an automorphism of $\Gamma(G)$.

Let $\delta \in \mathsf{Aut}(G)$ with $H_1^\delta = H_2$ and $H_2^\delta = H_1$. In particular, $|H_1| = |H_2|$, and so Γ is regular. For $H_1^x \in \mathcal{H}_1$ and $H_2^y \in \mathcal{H}_2$, we have

$$(H_1^x \cap H_2^y)^\delta = (H_1^{\mathsf{Inn}(x)} \cap H_2^{\mathsf{Inn}(y)})^\delta = H_2^{\mathsf{Inn}(x^\delta)} \cap H_1^{\mathsf{Inn}(y^\delta)} = H_2^{x^\delta} \cap H_1^{y^\delta}.$$

Then $|H_1^x : (H_1^x \cap H_2^y)| = k$ if and only if $|H_2^{x^\delta} : (H_2^{x^\delta} \cap H_1^{y^\delta})| = k$. Noting that $|H_2^{x^\delta}| = |H_1^{y^\delta}|$, we have $|H_1^{y^\delta} : (H_1^{y^\delta} \cap H_2^{x^\delta})| = k$. Thus δ induces an automorphism of $\Gamma(G)$, which interchanges $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus part (1) follows.

Next we prove part (2). Let $\Delta_1 = \{M_2 \in \mathcal{H}_2 \mid k = |H_1 : (H_1 \cap M_2)|\}$ and $\Delta_2 = \{M_1 \in \mathcal{H}_1 \mid k = |M_1 : (M_1 \cap H_2)|\}$. Then Δ_1 and Δ_2 are the neighborhoods of H_1 and H_2 in $\Gamma(G)$, respectively. Let $\{x_1, x_2, \ldots, x_k\}$ be a right transversal of $H_1 \cap H_2$ in H_1. Then $H_2^{x_i} \in \Delta_1$ for all i. Suppose that $H_2^{x_i} = H_2^{x_j}$ for some i and j. Then $x_j^{-1}x_i \in \mathbf{N}_G(H_2) = H_2$, and so $x_j^{-1}x_i \in H_1 \cap H_2$. This implies that $(H_1 \cap H_2)x_i = (H_1 \cap H_2)x_j$, yielding $i = j$. Thus, if $i \neq j$ then $H_2^{x_i}$ and $H_2^{x_i}$ are different neighbors of H_1 in $\Gamma(G)$.

Assume that $\Gamma(G)$ has valency k. Then $\Delta_1 = \{H_2^{x_i} \mid 1 \leqslant i \leqslant k\}$, and thus H_1 acts transitively on the k maximal subgroups in Δ_1 by conjugation. Recalling that $\mathsf{Inn}(G)$ acts transitively on both parts of $\Gamma(G)$, it follows that $\Gamma(G)$ is $\mathsf{Inn}(G)$-semisymmetric.

Conversely, let $\Gamma(G)$ be $\mathsf{Inn}(G)$-semisymmetric. Then $\Gamma(G)$ is regular, in particular, $|\Delta_1| = |\Delta_2|$. Noting that H_1 acts transitively on Δ_1 by conjugation, we get $\Delta_1 = \{H_2^{x_i} \mid 1 \leqslant i \leqslant k\}$, which has size k. Thus $\Gamma(G)$ has valency k. This completes the proof. □

In the following, we always assume that (G, H) is a pair described as in Table 2. Choose nonconjugate maximal subgroups H_1 and H_2 of G with $H_1 \cong H \cong H_2$ and maximal $|H_1 \cap H_2|$. Clearly, $G = \langle H_1, H_2\rangle$. Set $k = |H_1 : (H_1 \cap H_2)|$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the conjugacy classes of H_1 and H_2 in G, respectively. Let

$$\Delta_1 = \{X \in \mathcal{H}_2 \mid |H_1 : (H_1 \cap X)| = k\},\ \Delta_2 = \{X \in \mathcal{H}_1 \mid |X : (X \cap H_2)| = k\}.$$

Since $|X| = |H_2|$ for all $X \in \mathcal{H}_1$, we have $|\Delta_2| = |\{X \in \mathcal{H}_1 \mid |H_2 : (X \cap H_2)| = k\}|$.

For a subgroup $X \leqslant G$, denote by $X^{(\infty)}$ the intersection of all terms appearing the derived series of X.

Lemma 5.3. *$k \geqslant 5$.*

Proof. Suppose that $k < 5$. Let $K = H_1 \cap H_2$. Then H_1 acts unfaithfully on the set of right cosets of K in H_1 by right multiplication. Let K_1 be the kernel of this action. Then $K^{(\infty)} \leqslant H_1^{(\infty)} \leqslant K_1 \leqslant K$, and so $H_1^{(\infty)} \leqslant K^{(\infty)}$, yielding $H_1^{(\infty)} = K^{(\infty)}$. Similarly, we have $H_2^{(\infty)} = K^{(\infty)}$. Then $1 \neq K^{(\infty)} \trianglelefteq \langle H_1, H_2\rangle = G$, which is impossible. Thus $k \geqslant 5$, as desired. □

Lemma 5.4. *Let $G = \mathrm{PSL}_2(p)$ for a prime $p > 3$.*

1. *If $p \equiv \pm1, \pm9 \pmod{40}$, then $k > 5$.*
2. *If $p \equiv \pm11, \pm19 \pmod{40}$, then $|\Delta_1| = |\Delta_2| = k = 5$.*

Proof. Let $\mathcal{K}_i = \{K \leqslant H_i^g \mid K \cong \mathrm{A}_4,\ g \in G\}$, where $i = 1, 2$. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are conjugacy classes of subgroups in G. By [4, Theorem 2], G has exactly $\frac{p(p^2-1)}{24}$ subgroups isomorphic to A_4. Assume that $p \equiv \pm1, \pm9 \pmod{40}$. Then these $\frac{p(p^2-1)}{24}$ subgroups form two distinct G-conjugacy classes. It follows that $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$. In particular, $|H_1 \cap H_2| < 12$, and thus $k > 5$.

Assume that $p \equiv \pm11, \pm19 \pmod{40}$. Then G has a unique conjugacy class of subgroups isomorphic to A_4. This implies that $\mathcal{K}_1 = \mathcal{K}_2$, and thus $H_1 \cap H_2 \cong \mathrm{A}_4$. Then $k = 5$. Noting that A_5 has exactly 5 subgroups A_4, it is easily shown that every subgroup A_4 is contained in exactly one member of $\mathcal{H}_1$ and one member of $\mathcal{H}_2$. For distinct $X, Y \in \mathcal{H}_2$ with $|H_1 : (H_1 \cap X)| = 5 = |H_1 : (H_1 \cap Y)|$, we have $H_1 \cap X \cong \mathrm{A}_4 \cong H_1 \cap Y$, and so $H_1 \cap X \neq H_1 \cap Y$. This implies that $|\Delta_1| \leqslant 5$. On the other hand, noting that $\mathbf{N}_{H_1}(H_2) = H_1 \cap H_2$, we have $\Delta_1 = \{H_2^x \mid x \in H_1\}$, and $|\Delta_1| = 5$. Similarly, $\Delta_2 = \{H_1^x \mid x \in H_2\}$, and $|\Delta_2| = 5$. This completes the proof. □

Lemma 5.5. *Let $G = \mathrm{PSL}_2(p^2)$ or $\mathrm{P\Sigma L}_2(p^2)$ for a prime p with $p \equiv \pm3 \pmod{10}$. Then $k > 5$.*

Proof. Let $T = \mathrm{soc}(G)$. Then $G = TH_1 = T(H_1 \cap H_2)$. We have

$$\frac{|T||H_1|}{|H_1 \cap T|} = |G| = \frac{|T||H_1 \cap H_2|}{|H_1 \cap H_2 \cap T|},$$

yielding

$$k = |H_1 : (H_1 \cap H_2)| = |(H_1 \cap T) : (H_1 \cap H_2 \cap T)|.$$

Clearly, $H_1 \cap T$ and $H_2 \cap T$ are nonconjugate maximal subgroups of T and isomorphic to A_5. It is easily shown that T has two conjugacy classes of subgroups isomorphic to A_4. By a similar argument as in the proof of Lemma 5.4 (1), we have $k > 5$. □

Lemma 5.6. *Let $G = \mathrm{PSp}_6(p)$ for a prime p with $p \equiv \pm 1 \pmod 8$. Then $k > 5$.*

Proof. For a subgroup X of G, let $\widehat{X}$ be the preimage of X in $\mathrm{Sp}_6(p)$. Let $\mathcal{K}_i = \{\widehat{K} \mid \mathrm{S}_4 \cong K \leqslant X \in \mathcal{H}_i\}$ and $\mathcal{M} = \{\widehat{L} \times \widehat{S} \leqslant \mathrm{Sp}_6(p) \mid \widehat{L} \cong \mathrm{Sp}_2(p), \widehat{S} \cong \mathrm{Sp}_4(p)\}$, where $i = 1, 2$. Then $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{M}$ are conjugacy classes of subgroups in $\mathrm{Sp}_6(p)$. In addition, each $\widehat{K} \in \mathcal{K}_1 \cup \mathcal{K}_2$ is contained in some member of $\mathcal{M}$, refer to [11, Lemmas 5.7 and 5.8]. In particular, for each $X \in \mathcal{H}_1 \cup \mathcal{H}_2$ there is $\widehat{L}\widehat{S} \in \mathcal{M}$ with $(\widehat{X} \cap \widehat{L}\widehat{S})\widehat{L} \cong 2\mathrm{S}_4 \times \mathrm{Sp}_4(p)$.

By [3, p. 186, Proposition 4.5.21], $\mathcal{H}_1$ and $\mathcal{H}_2$ are fused by the diagonal automorphism of $\mathrm{Sp}_6(p)$, and by [3, p. 391, Table 8.28], $\mathcal{M}$ is fixed by the diagonal automorphism of $\mathrm{Sp}_6(p)$. It follows that, for $\widehat{L}\widehat{S} \in \mathcal{M}$ and $2\mathrm{S}_4 \cong \widehat{K} \leqslant \widehat{L}$, there exists $\widehat{X} \in \mathcal{H}_1 \cup \mathcal{H}_2$ such that $(\widehat{X} \cap \widehat{L}\widehat{S})\widehat{L} = \widehat{K} \times \widehat{S}$. Let

$$\mathcal{L}_i = \{\widehat{K}\widehat{S} \mid 2\mathrm{S}_4 \cong \widehat{K} \leqslant \widehat{L}, \widehat{L}\widehat{S} \in \mathcal{M}, (\widehat{X} \cap \widehat{L}\widehat{S})\widehat{L} = \widehat{K}\widehat{S}, X \in \mathcal{H}_i\},\ i = 1, 2.$$

Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\mathrm{Sp}_6(p)$-conjugacy classes, and $\mathcal{L}_1 \cup \mathcal{L}_2$ consists of all subgroups $2\mathrm{S}_4 \times \mathrm{Sp}_4(p)$ which are contained in the members of $\mathcal{M}$.

Since $p \equiv \pm 1 \pmod 8$, by [4, Theorem 2], $\mathrm{SL}_2(p)$ has $\frac{p(p^2-1)}{24}$ subgroups isomorphic to $2\mathrm{S}_4$, which form a single $\mathrm{GL}_2(p)$-conjugacy class. Then these $\frac{p(p^2-1)}{24}$ subgroups form two $\mathrm{SL}_2(p)$-conjugacy classes. Noting that $\mathrm{Sp}_2(p) \cong \mathrm{SL}_2(p)$, it follows that each $\widehat{L}\widehat{S} \in \mathcal{M}$ has exactly two conjugacy classes of subgroups isomorphic to $2\mathrm{S}_4 \times \mathrm{Sp}_4(p)$. Then $\mathcal{L}_1 \cup \mathcal{L}_2$ splits into two $\mathrm{Sp}_6(p)$-conjugacy classes, and thus $\mathcal{L}_1 \neq \mathcal{L}_2$. This implies that $|H_1 : (H_1 \cap H_2)| > 5$; otherwise, $\widehat{H_1 \cap H_2} \cong 2\mathrm{S}_4$, yielding $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$, a contradiction. Then $k > 5$, and the lemma follows. □

Lemma 5.7. *Let $G = \mathrm{PSp}_4(p)$ for a prime p with $p \equiv \pm 1 \pmod 8$. Then $k > 5$.*

Proof. Let $H = H_1$ or H_2, and let $R = 2_-^{1+4}$. Then $H \cong \mathbf{C}_{\mathsf{Aut}(R)}(\mathbf{Z}(R)) = \mathsf{Aut}(R)$, refer to [1, Theorem A(4)]. Let K be a subgroup of H with $|H : K| = 5$. Calculation with GAP [12], we conclude that K has a unique normal subgroup of order 16, and thus $\mathbf{O}_2(H)$ is this normal subgroup of K. Suppose that $|H_1 : (H_1 \cap H_2)| = 5$. Then $\mathbf{O}_2(H_1) = \mathbf{O}_2(H_2)$. This implies that $H_1 = \mathbf{N}_G(\mathbf{O}_2(H_1)) = \mathbf{N}_G(\mathbf{O}_2(H_2)) = H_2$, a contradiction. This completes the proof. □

Calculation with GAP [12], we have the following lemma.

Lemma 5.8. *Let (G, H) be as in Rows 4, 7-10 of Table 2. Then $k = 5$ if and only if $G \neq \mathrm{PSL}_2(25)$, $\mathrm{P\Sigma L}_2(25)$. In addition, if $k = 5$ then $|\Delta_1| = |\Delta_2| = 5$.*

Example 5.9. Let (G, H) be as in Rows 1, 7, 9 and 10 of Table 2. Define a bipartite graph $\Gamma(G)$ with vertex set $\mathcal{H}_1 \cup \mathcal{H}_2$ such that $X \in \mathcal{H}_1$ and $Y \in \mathcal{H}_2$ are adjacent if and only if $|X : (X \cap Y)| = 5$. Then $\Gamma(G)$ is G-semisymmetric by Lemmas 5.2 5.4 and 5.8, where G acts on $\mathcal{H}_1 \cup \mathcal{H}_2$ by conjugation. In addition, we have the following remarks.

(1) Assume that $G = \mathrm{PSL}_2(p)$ with $p \equiv \pm 11, \pm 19 \pmod{40}$. Then the conjugacy classes $\mathcal{H}_1$ and $\mathcal{H}_2$ merge into one conjugacy class in $\mathrm{PGL}_2(p)$, refer to [4, Theorem 2]. In this case, $\mathrm{PGL}_2(p)$ acts transitively on the vertex set of $\Gamma(G)$, and thus $\Gamma(G)$ is a symmetric graph.

(2) Assume that $\mathrm{soc}(G) = \mathrm{PSL}_3(4)$. Then $\Gamma(G)$ is just the point-line incidence graph of the projective plane $\mathrm{PG}(2, 4)$. The transpose-inverse automorphism of $\mathrm{PSL}_3(4)$ gives an automorphism of $\Gamma(G)$, which interchanges $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus $\Gamma(G)$ is a symmetric graph.

(3) Assume that $\mathrm{soc}(G) = \mathrm{PSp}_4(4)$. Then $\Gamma(G)$ is the incidence graph of a generalized 4-gon of order $(4, 4)$, refer to [7, p. 44]. In this case, the graph automorphism of $\mathrm{PSp}_4(4)$ interchanges $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus $\Gamma(G)$ is a symmetric graph.

(4) Assume that $\mathrm{soc}(G) = \mathrm{G}_2(4)$. Then $\Gamma(G)$ is the incidence graph of a generalized hexagon of order $(4, 4)$, and $\mathsf{Aut}(\Gamma(G))$ contains the automorphism group of this generalized hexagon, refer to [7, p. 97]. $\square$

Theorem 5.10. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric pentavalent graph, and $\{u, w\} \in E$. Assume that G is almost simple and acts primitively on both parts of Γ. Assume that G_u and G_w are not conjugate in G. Then Γ is isomorphic to one of the graphs constructed as in Example* 5.9*, and* $\mathsf{Aut}(\Gamma) \cong \mathsf{Aut}(G)$*. In particular, Γ is semisymmetric if and only if* $\mathrm{soc}(G) = \mathrm{G}_2(4)$*.*

Proof. Since G_u and G_w are not conjugate in G, by Theorem 4.4, the triple (G, G_u, G_w) is described as in Table 2. Noting that $|G_u : (G_u \cap G_w)| = 5$, by Lemmas 5.4–5.8, One of Rows 1, 7, 9 and 10 of Table 2 occurs. Define $\tau : V \to \mathcal{H}_1 \cup \mathcal{H}_2$ by $u^g \mapsto G_u^g$ and $w^g \mapsto G_w^g$. It is easily shown that τ is an isomorphism from Γ to the graph $\Gamma(G)$ defined as in Example 5.9.

Without loss of generality, we let $\Gamma = \Gamma(G)$. Thus, by the argument in Example 5.9, $\mathsf{Aut}(\Gamma)$ has a subgroup isomorphic to $\mathsf{Aut}(G)$, which acts transitively on the vertex set V of Γ unless $\mathrm{soc}(G) = \mathrm{G}_2(4)$. Let $A = \mathsf{Aut}^+(\Gamma)$. Then $|\mathsf{Aut}(\Gamma) : A| \leqslant 2$, $G \leqslant A$, Γ is A-semisymmetric, and A acts primitively (and faithfully) on both parts of Γ. It follows from Lemma 4.1 that A is an almost simple group. Suppose that A_u and A_w are conjugate in A. By Theorem 4.4, as a primitive group, A has a suborbit of length 5 on U. Then A is known by [11, Theorem 1.1], which implies that A has no subgroup isomorphic to G, a contradiction. Thus A_u and A_w are not conjugate in A. Again by Theorem 4.4, we conclude that $A \leqslant \mathsf{Aut}(G) \leqslant \mathsf{Aut}(\Gamma)$. If $\mathrm{soc}(G) \neq \mathrm{G}_2(4)$ then, since $|\mathsf{Aut}(\Gamma) : A| \leqslant 2$ and $\mathsf{Aut}(G)$ acts transitively on V, we have $\mathsf{Aut}(\Gamma) = \mathsf{Aut}(G)$.

Assume that $\mathrm{soc}(G) = \mathrm{G}_2(4)$. Without loss of generality, we may let $G = \mathrm{G}_2(4).2$, the automorphism group of a generalized hexagon of order $(4, 4)$. Then $A = G$. Suppose that $G \neq \mathsf{Aut}(\Gamma)$. Then $\mathsf{Aut}(\Gamma)$ is not almost simple and acts transitively on V. Let N be a minimal normal subgroup of $\mathsf{Aut}(\Gamma)$ with $N \neq \mathrm{soc}(G)$. Then $GN = G \times N$. Recalling that $|\mathsf{Aut}(\Gamma) : G| \leqslant 2$, we have $\mathsf{Aut}(\Gamma) = G \times N$ and $|N| = 2$. Set $N = \langle \sigma \rangle$. Choose $g \in G$ with $u^{g\sigma} = w$. We have $G_w = (G_u)^{g\sigma} = G_u^g$, which contradicts that G_u

and G_w are not conjugate in G. Thus $\mathsf{Aut}(\Gamma) = G$, and so Γ is semisymmetric. Then the theorem follows. □

6. Graphs with conjugate stabilizers

This section is to classify those graphs satisfying (1) of Theorem 4.4.

In the following, we assume that G is an almost simple primitive permutation group on a set U with a suborbit of length 5. Fix a point $u \in U$. Then the pair (G, G_u) is given as in [11, Tables 1 and 2]. Note that all subgroups of G_u with index 5 are conjugate in G_u. By lemma 2.3 and [11, Tables 4 and 5], we have the following lemma.

Lemma 6.1. *The pair (G, G_u) is listed in Table 3, where c is the number of choices of G_u up to G-conjugacy, K is a subgroup of G_u with index 5, $N = \mathbf{N}_G(K)$, r_1 and r_2 are the numbers of self paired and nonself paired suborbits of length 5 of G at u, respectively.*

	G	G_u	c	N/K	r_1	r_2	Conditions
1	$\mathrm{A}_5, \mathrm{S}_5$	$\mathrm{D}_{10}, \mathrm{AGL}_1(5)$	1	$\mathbb{Z}_2$	1	0	
2	$\mathrm{PGL}_2(9)$	D_{20}	1	$\mathbb{Z}_2$	1	0	
3	M_{10}	$\mathrm{AGL}_1(5)$	1	$\mathbb{Z}_2$	1	0	
4	$\mathrm{P\Gamma L}_2(9)$	$\mathrm{AGL}_1(5) \times \mathbb{Z}_2$	1	$\mathbb{Z}_2$	1	0	
5	$\mathrm{PGL}_2(11)$	D_{20}	1	$\mathbb{Z}_2$	1	0	
6	A_9	$(\mathrm{A}_5 \times \mathrm{A}_4){:}\mathbb{Z}_2$	1	$\mathbb{Z}_2$	1	0	
7	S_9	$\mathrm{S}_5 \times \mathrm{S}_4$	1	$\mathbb{Z}_2$	1	0	
8	$\mathrm{PSL}_2(19)$	D_{20}	1	$\mathbb{Z}_3$	0	2	
9	$\mathrm{Suz}(8)$	$\mathrm{AGL}_1(5)$	1	$\mathbb{Z}_2 \times \mathbb{Z}_2$	3	0	
10	$\mathrm{J}_3.\mathbb{Z}_o$	$\mathrm{AGL}_2(4).\mathbb{Z}_o$	1	$\mathbb{Z}_2$	1	0	$o \in \{1, 2\}$
11	Th	S_5	1	$\mathbb{Z}_2$	1	0	
12	$\mathrm{PSL}_2(p)$	A_5	2	$\mathbb{Z}_2$	1	0	$p \equiv \pm 1, \pm 9 \pmod{40}$
13	$\mathrm{PSL}_2(p^2)$	A_5	2	$\mathbb{Z}_2$	1	0	$p \equiv \pm 3 \pmod{10}$
14	$\mathrm{P\Sigma L}_2(p^2)$	S_5	2	$\mathbb{Z}_2$	1	0	$p \equiv \pm 3 \pmod{10}$
15	$\mathrm{PSp}_6(p)$	S_5	2	$\mathbb{Z}_2$	1	0	$p \equiv \pm 1 \pmod{8}$
16	$\mathrm{PSp}_6(3)$	A_5	1	$\mathbb{Z}_3$	0	2	
17	$\mathrm{PSp}_6(p)$	A_5	1	$\mathbb{Z}_{p-1}$	1	$p-3$	$p \equiv 13, 37, 43, 67 \pmod{120}$
18	$\mathrm{PSp}_6(p)$	A_5	1	$\mathbb{Z}_{p+1}$	1	$p-1$	$p \equiv 53, 77, 83, 107 \pmod{120}$
19	$\mathrm{PGSp}_6(p)$	S_5	1	$\mathbb{Z}_2$	1	0	$11 \leqslant p \equiv \pm 3 \pmod{8}$

Table 3. Almost simple primitive groups with a suborbit of length 5.

Remark 6.2. For one of Rows 12-15 in Table 3, the group G has two nonequivalent permutation representations of degree $|G : G_u|$. Nevertheless, the resulting permutation groups have isomorphic orbital digraphs. □

Lemma 6.3. *Let $G = \mathrm{PSp}_6(p)$ be as in Rows 16-18 of Table 3, and let K be a subgroup of G_u with index 5. Then $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K)/K$ is a dihedral group.*

Proof. Choose a maximal subgroup M of $\mathrm{PGSp}_6(p)$ with $G_u \leqslant M \cong \mathrm{S}_5$, refer to [3, Table 8.29]. Let $\delta \in M \setminus G_u$ be an involution. Without loss of generality, we assume that K is normalized by δ. Then $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K) = \mathbf{N}_G(K){:}\langle\delta\rangle$, yielding $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K)/K = \langle\bar{x}\rangle{:}\langle\bar{\delta}\rangle$, where $\langle\bar{x}\rangle = \mathbf{N}_G(K)/K$ and $\bar{\delta}$ is the image of δ in $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K)/K$.

Let $\bar{y} \in \mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta})$, and y be a preimage of $\bar{y}$ in $\mathbf{N}_G(K)$. Then $\delta^{-1}y^{-1}\delta y \in K$, and so $\delta^y \in K\langle\delta\rangle$. This implies that $y \in \mathbf{N}_{\mathrm{PGSp}_6(p)}(K\langle\delta\rangle)$, and so $\bar{y} \in \mathbf{N}_{\mathrm{PGSp}_6(p)}(K\langle\delta\rangle)/(K\langle\delta\rangle)$. Thus $\mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta}) \leqslant \mathbf{N}_{\mathrm{PGSp}_6(p)}(K\langle\delta\rangle)/(K\langle\delta\rangle)$. If $p = 3$ then $|\mathbf{N}_{\mathrm{PGSp}_6(p)}(K\langle\delta\rangle)/(K\langle\delta\rangle)| = 1$ by [11, Table 5 (13)], and so $\mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta}) = 1$, yielding $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K)/K = \langle\bar{x}\rangle{:}\langle\bar{\delta}\rangle \cong \mathrm{D}_6$.

Now let $p > 3$, in this case, we have $13 \leqslant p \equiv \pm 1 \pmod 8$. Noting that $K\langle\delta\rangle$ has index 5 in M, by [11, Table 5 (14)], $\mathbf{N}_{\mathrm{PGSp}_6(p)}(K\langle\delta\rangle)/(K\langle\delta\rangle) \cong \mathbb{Z}_2$, and so $|\mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta})| \leqslant 2$. It is easily shown that $\langle\bar{x}\rangle$ contains an involution which centralizes $\bar{\delta}$. Then $\mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta}) \cong \mathbb{Z}_2$.

Let n be the order of $\bar{x}$. Then $n = p+1$ or $p-1$, $n \geqslant 10$, and n is indivisible by 8 as $p \equiv \pm 3 \pmod 8$. Set $\bar{x}^{\bar{\delta}} = \bar{x}^r$, where $0 \leqslant r \leqslant n-1$. We have $\bar{x} = \bar{x}^{\bar{\delta}^2} = \bar{x}^{r^2}$, and so $r^2 \equiv 1 \pmod n$. Let d be the greatest common divisor of $r+1$ and n. Then $r-1$ is divisible by $\frac{n}{d}$. Thus $(\bar{x}^d)^{\bar{\delta}} = \bar{x}^{dr} = \bar{x}^d$, yielding $\bar{x}^d \in \mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta})$. This implies that $\bar{x}^{2d} = 1$, and so $2d \equiv 0 \pmod n$, yielding $d = n$ or $\frac{n}{2}$. In addition, since n is even and r^2-1 is divisible by n, both $r-1$ and $r+1$ are even, and hence d is even as is d the greatest common divisor of $r+1$ and n. Suppose that $d = \frac{n}{2}$. Then n is divisible by 4 but not by 8, and so d is indivisible by 4. By the choice of d, we have $r+1 \equiv 2 \pmod 4$, and then $r-1 \equiv 0 \pmod 4$. Thus $(\bar{x}^{\frac{n}{4}})^{\bar{\delta}} = \bar{x}^{\frac{n}{4}r} = \bar{x}^{\frac{n}{4}(r-1)+\frac{n}{4}} = \bar{x}^{\frac{n}{4}}$, yielding $\bar{x}^{\frac{n}{4}} \in \mathbf{C}_{\langle\bar{x}\rangle}(\bar{\delta})$. This forces that $\bar{x}^{\frac{n}{4}}$ has order 1 or 2, and so $n = 4$ or 8. By $n = p \pm 1$, we have $p < 9$, which contradicts that $p \geqslant 13$. Therefore, $d = n$. Then $\bar{x}^{\bar{\delta}} = \bar{x}^r = \bar{x}^{r+1-1} = \bar{x}^{-1}$. This says that $\langle\bar{x}\rangle{:}\langle\bar{\delta}\rangle$ is a dihedral group, and the lemma follows. $\square$

Given a subgroup K of G_u with index 5, by Lemma 2.3, every suborbit of length 5 has the form of $\Delta_x(u) := \{u^{xh} \mid h \in G_u\}$, where $x \in \mathbf{N}_G(K) \setminus K$. Denote by Σ_x the orbital digraph of G associated with $\Delta_x(u)$. In the following, we always identify G with the subgroup $\tilde{G}$ of $\mathsf{Aut}(\Sigma_x^{(2)})$ induced by G. Recall that there exists an ι isomorphism from Σ_x to $\Sigma_{x^{-1}}$, and each $\delta \in \mathsf{Aut}(G, G_u, K)$ defines an isomorphism $\tilde{\delta} : U \times \mathbb{Z}_2 \to U \times \mathbb{Z}_2$, $(u^g, i) \mapsto (u^{g^\delta}, i)$ from $\Sigma_x^{(2)}$ to $\Sigma_{x^\delta}^{(2)}$, see Section 2.

Lemma 6.4. *Let G be a primitive group in Table 3, and let $x, y \in \mathbf{N}_G(K) \setminus K$. Then*

(1) $\Sigma_x^{(2)} \cong \Sigma_{x^{-1}}^{(2)}$;

(2) $\Sigma_x^{(2)}$ *is a symmetric graph;*

(3) $\mathsf{Aut}(\Sigma_x^{(2)}) = \begin{cases} \mathsf{Aut}(\Sigma_x) \times \langle\iota\rangle, & \text{if } \Delta_x = \Delta_{x^{-1}} \\ G{:}\langle\tilde{\delta}\iota\rangle \cong \mathsf{Aut}(G) \text{ with } \delta \in \mathsf{Aut}(G, G_u, K), & \text{otherwise} \end{cases}$.

(4) $\Sigma_x^{(2)} \cong \Sigma_y^{(2)}$ *if and only if* $\Delta_x = \Delta_y$, $\Delta_{x^{-1}} = \Delta_y$ *or* $G = \mathrm{Suz}(8)$.

Proof. Part (1) and part (2) for self-paired $\Delta_x(u)$ follow directly form Lemma 2.1. In addition, if $\Delta_x(u)$ is self-paired then $\mathsf{Aut}(\Sigma_x^{(2)}) \geqslant \mathsf{Aut}(\Sigma_x) \times \langle\iota\rangle$ and, by [11, Theorem 1.2], $\mathsf{Aut}(\Sigma_x)$ is almost simple with socle $\mathrm{soc}(G)$ unless $G = \mathrm{A}_5$ or S_5 with $\mathsf{Aut}(\Sigma_x) = \mathrm{P\Sigma L}_2(9)$.

Assume that Δ_x is not self-paired. Then G is described as in Rows 8, 16-18 of Table 3. By Lemma 6.3 and calculation with GAP for $G = \mathrm{PSL}_2(19)$, we conclude that there is an involution $\delta \in \mathsf{Aut}(G, G_u, K)$ with $(Kx)^\delta = Kx^{-1}$. Then, by Lemma 2.2, $\tilde{\delta}\iota \in \mathsf{Aut}(\Sigma_x^{(2)})$. It is easy to check that $\tilde{\delta}\iota$ is an involution and interchanges $U \times \{0\}$ and $U \times \{1\}$. In particular, $\Sigma_x^{(2)}$ is a symmetric graph, and part (2) of the lemma follows. Moreover, $\tilde{\delta}\iota$ normalizes G, and $G\langle\tilde{\delta}\iota\rangle \cong \mathsf{Aut}(G)$.

Let $A = \mathsf{Aut}^+(\Sigma_x^{(2)})$. Then $G \leqslant A$, and if Δ_x is self-paired then $\mathsf{Aut}(\Sigma_x) \leqslant A$. For the self-paired case, replacing G by $\mathsf{Aut}(\Sigma_x)$ if necessary, we may choose $G = \mathsf{Aut}(\Sigma_x)$. Note that $\Sigma_x^{(2)}$ is A-semisymmetric, and A acts primitively (and faithfully) on both $U \times \{0\}$ and $U \times \{1\}$. It follows from Lemma 4.1 that A is an almost simple group. Suppose that $A_{(u,0)}$ and $A_{(u,1)}$ are not conjugate in A. Applying Theorem 4.4 to the pair $(A, A_{(u,0)})$, we conclude that either $|A : A_{(u,0)}| \neq |G : G_u|$ or A has no subgroup isomorphic to G, a contradiction. Then $A_{(u,0)}$ and $A_{(u,1)}$ are conjugate in A. Applying Theorem 4.4 and [11, Theorem 1.1] to the pair $(A, A_{(u,0)})$, we conclude that $A \leqslant G$, and so $A = G$. Since $|\mathsf{Aut}(\Sigma_x^{(2)}) : A| = 2$, part (3) of the lemma follows.

We next prove part (4) of this lemma. This is trivial if G has a unique suborbit of length 5 at u. If G has exactly two suborbits of length 5 at u, then these two suborbits are paired to each other, and (4) is true by (1). Assume that $G = \mathrm{Suz}(8)$. Then there is $\delta \in \mathsf{Aut}(G, G_u, K)$ such that $\langle\delta\rangle$ has order 3 and acts transitively on the 3-set $\{Kx \mid x \in N_G(K) \setminus K\}$, confirmed by GAP. It follows from Lemma 2.2 that the resulting standard double covers are isomorphic to every other. Thus, all that's left now is the case where G is given as in Rows 17 and 18 of Table 3.

Assume that G is given as in Row 17 or 18 of Table 3. Clearly, $\Sigma_x^{(2)} \cong \Sigma_y^{(2)}$ if $Kx = Ky$ or $Kx^{-1} = Ky$. Now let $\Sigma_x^{(2)} \cong \Sigma_y^{(2)}$, and pick an isomorphism λ from $\Sigma_y^{(2)}$ to $\Sigma_x^{(2)}$. We have $\mathsf{Aut}(\Sigma_y^{(2)}) = \lambda^{-1}\mathsf{Aut}(\Sigma_x^{(2)})\lambda$. By (3) and [11, Theorem 1.2], $\mathsf{Aut}(\Sigma_x^{(2)}) \cong \mathsf{Aut}(\Sigma_y^{(2)}) \cong \mathrm{PSp}_6(p) \times \mathbb{Z}_2$ or $\mathrm{PGSp}_6(p)$. Then $\mathsf{Aut}(\Sigma_x^{(2)})$ and $\mathsf{Aut}(\Sigma_y^{(2)})$ have a common characteristic subgroup G. Then G is normalized by λ. (Note, λ is a permutation on $U \times \mathbb{Z}_2$.) Replacing $\Sigma_x^{(2)}$ by $\Sigma_{x^{-1}}^{(2)}$, and λ by $\lambda\iota$ if necessary, we may assume that λ fixes both $U \times \{0\}$ and $U \times \{1\}$ setwise. Clearly, for each $g \in G$, we have an isomorphism λg from $\Sigma_y^{(2)}$ to $\Sigma_x^{(2)}$, which fixes both $U \times \{0\}$ and $U \times \{1\}$ setwise. Since G acts transitively on $U \times \{0\}$, replacing λ by λg for some $g \in G$, we may let $(u, 0)^\lambda = (u, 0)$.

Set $X = G\langle\lambda\rangle$. We have $G_u \trianglelefteq X_{(u,0)} = G_u\langle\lambda\rangle$, and thus we may further choose λ such that $K^\lambda = K$. Let $(u, 1)^\lambda = (w, 1)$, and choose $g \in G$ with $w = u^g$. Then

$$G_u^g = G_{(u,1)^g} = G_{(u,1)^\lambda} = G \cap X_{(u,1)^\lambda} = G \cap X_{(u,1)}^\lambda = (G \cap X_{(u,1)})^\lambda = G_u^\lambda = G_u.$$

Since G_u is a maximal subgroup of G, we have $g \in G_u$, and so $w = u$. Thus λ fixes $(u, 1)$. Consider the neighborhoods $\{(u^{yh}, 1) \mid h \in G_u\}^\lambda$ and $\{(u^{xh}, 1) \mid h \in G_u\}$ of $(u, 0)$ in $\Sigma_y^{(2)}$ and $\Sigma_x^{(2)}$, respectively. Recalling that $(u, 0)^\lambda = (u, 0)$, we have

$$\{(u^{xh}, 1) \mid h \in G_u\} = \{(u^{yh}, 1) \mid h \in G_u\}^\lambda = \{(u^{yh}, 1)^\lambda \mid h \in G_u\}.$$

For $h \in G_u$, we have

$$(u^{yh}, 1)^\lambda = (u, 1)^{yh\lambda} = (u, 1)^{(yh)^\lambda} = (u^{y^\lambda h^\lambda}, 1).$$

It follows that

$$\{(u^{xh}, 1) \mid h \in G_u\} = \{(u^{y^\lambda h^\lambda}, 1) \mid h \in G_u\} = \{(u^{y^\lambda h}, 1) \mid h \in G_u\}.$$

Then $\Delta_x(u) = \{u^{xh} \mid h \in G_u\} = \{u^{y^\lambda h} \mid h \in G_u\} = \Delta_{y^\lambda}(u)$, yielding $Kx = Ky^\lambda = (Ky)^\lambda$. Let $\bar{\lambda}$ be the automorphism of G induced by λ. Then $\bar{\lambda} \in \mathsf{Aut}(G, H, K)$. It follows from Lemma 6.3 that $(Ky)^\lambda = (Ky)^{\bar{\lambda}} = Ky$ or Ky^{-1}. Thus $Kx = Ky$ or Ky^{-1}, and part (4) of the lemma follows. This completes the proof. $\square$

	$\mathsf{Aut}(\Gamma)$	$(\mathsf{Aut}(\Gamma))_u$	G	n	Remarks
1	$\mathrm{P\Gamma L}_2(9)\times\mathbb{Z}_2$	$\mathrm{AGL}_1(5)\times\mathbb{Z}_2$	$\mathrm{PGL}_2(9)$	1	
			M_{10}		
			$\mathrm{P\Gamma L}_2(9)$		
2	$\mathrm{PGL}_2(11)\times\mathbb{Z}_2$	D_{20}	$\mathrm{PGL}_2(11)$	1	
3	$\mathrm{S}_9\times\mathbb{Z}_2$	$\mathrm{S}_5\times\mathrm{S}_4$	$\mathrm{S}_9,\mathrm{A}_9$	1	
4	$\mathrm{PGL}_2(19)$	D_{20}	$\mathrm{PSL}_2(19)$	1	
5	$\mathrm{Suz}(8)\times\mathbb{Z}_2$	$\mathrm{AGL}_1(5)$	$\mathrm{Suz}(8)$	1	
6	$\mathrm{J}_3{:}\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathrm{A\Gamma L}_2(4)$	$\mathrm{J}_3{:}\mathbb{Z}_2,\mathrm{J}_3$	1	
7	$\mathrm{Th}\times\mathbb{Z}_2$	S_5	Th	1	
8	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	A_5	$\mathrm{PSL}_2(p)$	1	$p\equiv\pm1,\pm9\ (\mathsf{mod}\ 40)$
9	$\mathrm{P\Sigma L}_2(p^2)\times\mathbb{Z}_2$	S_5	$\mathrm{P\Sigma L}_2(p^2)$	1	$3<p\equiv\pm3\ (\mathsf{mod}\ 10)$
			$\mathrm{PSL}_2(p^2)$		
			$\mathrm{A}_n,\mathrm{S}_n$		$p=3,n\in\{5,6\}$
10	$\mathrm{PSp}_6(p)\times\mathbb{Z}_2$	S_5	$\mathrm{PSp}_6(p)$	1	$p\equiv\pm1\ (\mathsf{mod}\ 8)$
11	$\mathrm{PGSp}_6(3)$	A_5	$\mathrm{PSp}_6(3)$	1	
12	$\mathrm{PGSp}_6(p)$	A_5	$\mathrm{PSp}_6(p)$	$\frac{p-3}{2}$	$p\equiv 13,37,43,67\ (\mathsf{mod}\ 120)$
13	$\mathrm{PGSp}_6(p)$	A_5	$\mathrm{PSp}_6(p)$	$\frac{p-1}{2}$	$p\equiv 53,77,83,107\ (\mathsf{mod}\ 120)$
14	$\mathrm{PGSp}_6(p)\times\mathbb{Z}_2$	S_5	$\mathrm{PGSp}_6(p)$	1	$11\leqslant p\equiv\pm3\ (\mathsf{mod}\ 8)$
			$\mathrm{PSp}_6(p)$		

TABLE 4. Examples from standard double covers.

If Σ_x is a seif-paired orbital digraph of some primitive group G described as in Table 3, then $\mathsf{Aut}(\Sigma_x)$ is known by [11, Theorem 1.2]. Thus, by Theorem 4.4, Lemmas 6.1 and 6.4, we have the following theorem.

Theorem 6.5. *Let $\Gamma=(V,E)$ be a connected G-semisymmetric pentavalent graph, and $\{u,w\}\in E$. Assume that G is almost simple and acts primitively on both parts of Γ. Assume that G_u and G_w are conjugate in G. Then Γ is a symmetric graph, and the triple $(\mathsf{Aut}(\Gamma),(\mathsf{Aut}(\Gamma))_u,G)$ is listed in Table 4, where the fifth column gives the number n of nonisomorphic graphs having the same automorphism group.*

7. Proof of Theorem 1.1

Let $\Gamma=(V,E)$ be a connected G-semisymmetric pentavalent graph, and let U and W be the G-orbits on V. Assume that G acts primitively on both U and W. If either $|U|=5$ or G acts unfaithfully on one of U and W, then Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{5,5}$, desired as in (1) of Theorem 1.1. If G is almost simple then, by Theorem 4.4, (5) and (6) of Theorem 1.1 follow from Theorems 5.10 and 6.5, respectively. Thus, by Lemma 4.1, all that's left now is to settle the case where G is an affine primitive permutation group on U (and W).

Assume G is an affine primitive permutation group on U (and W). By Lemma 4.1, $\mathrm{soc}(G)\cong\mathbb{Z}_p^k$, where $1\leqslant k\leqslant 4$ and p is a prime. By Lemma 2.5, $\mathsf{Aut}(\Gamma)$ contains a subgroup which acts regularly on V, and so Γ is symmetric. If $k=1$ then Γ has order twice a prime, and so Theorem 1.1 (2) occurs by [5, Theorem 2.4 and Table 1].

Let $k > 1$ from now on, and fix $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Let $v = u$ or w. Then, by [24, Theorem 2.3], G_v acts faithfully on $\Gamma(v)$, see also [21, Lemma 2.4]. Thus

($\circledast$) $\mathrm{soc}(G) \cong \mathbb{Z}_p^k$, and $G_v \cong \mathbb{Z}_5$, D_{10}, $\mathrm{AGL}_1(5)$, A_5 or S_5, where $2 \leqslant k \leqslant 4$.

We first deal with the case where G_u and G_w are not conjugate in G.

Lemma 7.1. *If G_u and G_w are not conjugate in G, then Theorem* 1.1 (3) *holds.*

Proof. Assume that G_u and G_w are not conjugate in G. Noticing ($\circledast$), if $p > 5$ then both G_u and G_w are complements in G of the normal Sylow p-subgroup, and so they are conjugate in G by the Schur-Zassenhaus Theorem, a contradiction. Thus $p \leqslant 5$. Dealing with G as a primitive subgroup of the symmetric group S_{p^k}, by calculation with GAP, we conclude that $p^k = 5^3$, and the following statements hold.

(i) Up to conjugacy, G is contained in a unique primitive subgroup of S_{125}, say $X \cong \mathbb{Z}_5^3{:}\mathrm{S}_5$, and $\mathbb{Z}_5^3{:}\mathrm{A}_5 \cong X' \leqslant G$, where X' is the derived subgroup of X which is also a primitive subgroup of S_{125}. In particular, $G = X$ or X', and $G_w \cong G_u \cong \mathrm{S}_5$ or A_5.

(ii) G has 5 conjugacy classes of (maximal) subgroups isomorphic to G_u.

(iii) Fix a point-stabilizer H_1 of the primitive subgroup G of S_{125}. We have $\mathbf{N}_{\mathrm{S}_{125}}(G) \cong \mathbb{Z}_5^3{:}(\mathbb{Z}_4 \times \mathrm{S}_5)$, and $\mathbf{N}_{\mathbf{N}_{\mathrm{S}_{125}}(G)}(H_1) \cong \mathbb{Z}_4 \times \mathrm{S}_5$.

(iv) There exists $H_2 \leqslant G$ such $H_2 \cong H_1$, $|H_1 : (H_1 \cap H_2)| = 5$, and H_2 is not conjugate to H_1 in G. Let $\langle \beta \rangle$ be the center of $\mathbf{N}_{\mathbf{N}_{\mathrm{S}_{125}}(G)}(H_1)$. Then H_1, H_2, H_2^{β}, $H_2^{\beta^2}$ and $H_2^{\beta^3}$ are not conjugate in G.

(v) Let $\mathcal{H}_1 = \{H_1^x \mid x \in A\}$ and $\mathcal{H}_2 = \{H_2^x \mid x \in G\}$. Then $\mathcal{H}_2$ contains exactly 5 members, each intersects H_1 at a subgroup of index 5.

Then, by Lemma 5.2, we have a biprimitive $\mathsf{Inn}(G)$-semisymmetric graph $\Gamma(G)$ of valency 5. It is easily shown that, up to isomorphism the graph $\Gamma(G)$ is independent of the choice of G. In particular, $\mathsf{Aut}^+(\Gamma(G)) \gtrsim \mathbb{Z}_5^3 : \mathrm{S}_5$.

Clearly, $\Gamma(G) \not\cong \mathsf{K}_{5,5}$, and so $\mathsf{Aut}^+(\Gamma(G))$ acts faithfully (and of course, primitively) on both parts of $\Gamma(G)$. By Lemma 4.1 and checking the order of those graphs in Theorems 5.10 and 6.5, we conclude that $\mathsf{Aut}^+(\Gamma(G))$ is an affine primitive group on each part of $\Gamma(G)$. Then ($\circledast$) holds for $\mathsf{Aut}^+(\Gamma(G))$, and thus the stabilizer of H_1 in $\mathsf{Aut}^+(\Gamma(G))$ has order a divisor of 120. It follows that $\mathsf{Aut}^+(\Gamma(G)) \cong \mathbb{Z}_5^3{:}\mathrm{S}_5$. By Lemma 2.5, $\Gamma(G)$ has an automorphism of order 2, which interchanges two parts of $\Gamma(G)$. Then $\mathsf{Aut}(\Gamma(G)) \cong (\mathbb{Z}_5^3{:}\mathrm{S}_5){:}\mathbb{Z}_2$.

Finally, without loss of generality, we may choose $G_u = H_1$ and $G_w = H_2^{\beta^i}$ for some $i \in \{0, 1, 2, 3\}$. Define a map $\theta : U \cup W \to \mathcal{H}_1 \cup \mathcal{H}_2$ by

$$u^x \mapsto H_1^{\beta^i x \beta^{-i}},\ w^y \mapsto H_2^{\beta^i y \beta^{-i}}.$$

It is easily shown that θ is an isomorphism from Γ to $\Gamma(G)$. Then the lemma follows. $\square$

Before dealing with the conjugate case, we first present a example in the following, which in fact includes all possible desired graphs. Consider $\mathbb{Z}_p^k$ as a k-dimensional row vector space over the field $\mathbb{Z}_p$, and view every matrix in $\mathrm{GL}_k(p)$ as an invertible linear transformation of $\mathbb{Z}_p^k$ by right multiplication. For $\alpha \in \mathrm{GL}_k(p)$ and $\mathbf{u} \in \mathbb{Z}_p^2$, define the affine transformation $t_{\alpha,\mathbf{u}} : \mathbb{Z}_p^k \to \mathbb{Z}_p^k$ by $t_{\alpha,\mathbf{u}} : \mathbf{v} \mapsto \mathbf{v}\alpha + \mathbf{u}$.

Example 7.2. Let p be a prime.

(1) Suppose that $k = 2$, $p \equiv \pm 1 \,(\mathsf{mod}\ 5)$, and there exist nonzero $b, c \in \mathbb{Z}_p$ with $c^2b^2 + b = -c^3$ and $b^4 + 3cb^2 = -c^2$. Let $\alpha_0 = \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix}$. We have $\alpha_0^5 = 1$. Let $\mathbf{e}_1 = (1, 0)$ and set $S_{c,b} = \{\mathbf{e}_1\alpha_0^i \mid 0 \leqslant i \leqslant 4\}$. Then $\mathsf{BCay}(\mathbb{Z}_p^2, S_{c,b})$ is connected and of valency 5. Set $X = \{t_{\alpha,\mathbf{u}} \mid \alpha \in \langle\alpha_0\rangle, \mathbf{u} \in \mathbb{Z}_p^2\}$, and identify X with the subgroup of $\mathsf{Aut}^+(\mathsf{BCay}(\mathbb{Z}_p^2, S_{c,b}))$ induced by X, see the second paragraph of Section 2. Then $\mathsf{BCay}(\mathbb{Z}_p^2, S_{c,b})$ is X-semisymmetric.

(2) Suppose that $k = 4$, $p \neq 5$ and $d \in \mathbb{Z}_p$ with $d^5 = -1$. Let

$$\alpha_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d & -d^2 & d^3 & -d^4 \end{pmatrix}.$$

and $\mathbf{e}_1 = (1, 0, 0, 0)$. Set $S_d = \{\mathbf{e}_1\alpha_0^i \mid 0 \leqslant i \leqslant 4\}$ and $X = \{t_{\alpha,\mathbf{u}} \mid \alpha \in \langle\alpha_0\rangle, \mathbf{u} \in \mathbb{Z}_p^2\}$. Identify X with the subgroup of $\mathsf{Aut}^+(\mathsf{BCay}(\mathbb{Z}_p^4, S_d))$ induced by X. Then $\mathsf{BCay}(\mathbb{Z}_p^4, S_d)$ is a connected X-semisymmetric pentavalent graph. $\square$

Let p be a prime with $p \equiv \pm 1 \,(\mathsf{mod}\ 5)$. The law of quadratic reciprocity, refer to [16, Theorem 1, p.53], asserts that 5 is a quadratic residue $(\mathsf{mod}\ p)$. Then $x^2 - 5 = 0$ has exactly two solutions in $\mathbb{Z}_p$, denoted by $\sqrt{5}$ and $-\sqrt{5}$, respectively. For a, $b \in \mathbb{Z}_p$ with $b \neq 0$, write ab^{-1} as $\frac{a}{b}$.

Lemma 7.3. *Let $S_{c,d}$ and X be as in Example 7.2 (1). Then the followings are true.*

1. *If X is a primitive subgroup of $\mathrm{AGL}_2(p)$, then $p \equiv -1 \,(\mathsf{mod}\ 5)$, $c = -1$ and $b = \frac{-1\pm\sqrt{5}}{2}$.*
2. *There exists an involution $\beta_0 \in \mathrm{GL}_k(p)$ with $\langle\alpha_0, \beta_0\rangle \cong \mathrm{D}_{10}$ and $S_{c,b}\beta_0 = S_{c,b}$ if and only if $c = -1$ and $b = \frac{-1\pm\sqrt{5}}{2}$; in this case, $\{t_{\alpha,\mathbf{u}} \mid \alpha \in \langle\alpha_0, \beta_0\rangle, \mathbf{u} \in \mathbb{Z}_p^2\}$ is a primitive subgroup of $\mathrm{AGL}_2(p)$.*

Proof. We first prove part (1) of the lemma. Assume that X is a primitive subgroup of $\mathrm{AGL}_2(p)$. Then $p \equiv -1 \,(\mathsf{mod}\ 5)$ by [11, Theorem 1.1]. This implies that $p - 1 \not\equiv 0 \,(\mathsf{mod}\ 10)$. Let $b, c \in \mathbb{Z}_p \setminus \{0\}$ with $c^2b^2 + b = -c^3$ and $b^4 + 3cb^2 = -c^2$. Put $f = \frac{b^2}{c}$. By $b^4 + 3cb^2 = -c^2$, we have $f^2 + 3f + 1 = 0$. By $c^2b^2 + b = -c^3$ and $b^2 = fc$, we have $c^3 f + b = -c^3$, i.e., $-b = c^3(f + 1)$. Then $c^6(f + 1)^2 = b^2 = fc$, and so $c^5(f + 1)^2 = f$. Since $(f + 1)^2 = f^2 + 2f + 1 = -f$, we have $-c^5 f = f$, and so $c^5 = -1$ as $f \neq 0$. Noting that $c^{10} = 1$, if $c \neq -1$ then $p - 1$ is divisible by 10, which is impossible. Thus we have $c = -1$, or equivalently, $b^2 + b = 1$. Then $b = \frac{-1\pm\sqrt{5}}{2}$, as desired.

Now we prove part (2) of the lemma. Let $\mathbf{e}_i = \mathbf{e}_1\alpha_0^{i-1}$ be as in Example 7.2 (1), where $i \in \{1, 2, 3, 4, 5\}$. If $c = -1$ and $b = \frac{-1\pm\sqrt{5}}{2}$, then $\begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}$ is one desired β_0. Conversely, suppose that $\langle\alpha_0, \beta_0\rangle \cong \mathrm{D}_{10}$ and $S_{c,b}\beta_0 = S_{c,b}$ for some $\beta_0 \in \mathrm{GL}_2(p)$. Then the permutation on $S_{c,b}$ induced by β_0 is a product of two disjoint transpositions, say $(\mathbf{e}_2, \mathbf{e}_5)(\mathbf{e}_3, \mathbf{e}_4)$ without loss of generality. By straightforward calculation, we get $c = -1$, and then $b = \frac{-1\pm\sqrt{5}}{2}$. Further calculation shows that $\langle\alpha_0, \beta_0\rangle$ does not fixes any 1-dimensional subspace of $\mathbb{Z}_p^2$. Then we have a primitive subgroup $\{t_{\alpha,\mathbf{u}} \mid \alpha \in \langle\alpha_0, \beta_0\rangle, \mathbf{u} \in \mathbb{Z}_p^2\}$ of $\mathrm{AGL}_2(p)$. This completes the proof. □

Lemma 7.4. *Let S_d and X be as in Example* 7.2 (2). *If X is a primitive subgroup of $\mathrm{AGL}_4(p)$ then $d = -1$. If $d = -1$ there exists $H \leqslant \mathrm{GL}_4(p)$ with $H \cong \mathrm{S}_5$ such that $S_d\alpha = S_d\alpha$ for all $\alpha \in H$, and $\{t_{\alpha,\mathbf{u}} \mid \alpha \in H, \mathbf{u} \in \mathbb{Z}_p^4\}$ is a primitive subgroup of $\mathrm{AGL}_4(p)$.*

Proof. Suppose that $d \neq -1$. By $d^5 = -1$, we have $d^4 - d^3 + d^2 - d + 1 = 0$. Calculation shows that 1 is an eigenvalue of α_0 in $\mathbb{Z}_p$. It follows that X is not a primitive subgroup of $\mathrm{AGL}_4(p)$. This implies the first part of the lemma.

Assume that $d = -1$. Let $\mathbf{e}_i = \mathbf{e}_1\alpha_0^{i-1}$ for $i \in \{1, 2, 3, 4, 5\}$. Noting that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ is a basis of $\mathbb{Z}_p^4$, every permutation on $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ extends naturally an invertible linear transformation of $\mathbb{Z}_p^4$, which fixes $\mathbf{e}_5 = (-1, -1, -1, -1)$. It follows that $\mathrm{GL}_4(p)$ has a subgroup $H \cong \mathrm{S}_5$ which fixes S_d. Finally, by [11, Lemma 3.4], $\{t_{\alpha,\mathbf{u}} \mid \alpha \in H, \mathbf{u} \in \mathbb{Z}_p^4\}$ is a primitive subgroup of $\mathrm{AGL}_4(p)$. This completes the proof. □

Corollary 7.5. *Let $S_{c,b}$ and S_d be as in Example* 7.2. *Then $\mathsf{Aut}^+(\mathsf{BCay}(\mathbb{Z}_p^2, S_{-1,b})) \gtrsim \mathbb{Z}_p^2{:}\mathrm{D}_{10}$, and $\mathsf{Aut}^+(\mathsf{BCay}(\mathbb{Z}_p^4, S_{-1})) \gtrsim \mathbb{Z}_p^4{:}\mathrm{S}_5$.*

Finally, the following lemma fulfills the proof of Theorem 1.1.

Lemma 7.6. *If G_u and G_w are conjugate subgroups of G, then Theorem* 1.1 (4) *holds.*

Proof. Assume that G_u and G_w are conjugate in G. By [11, Theorem 1.1], as an affine primitive group on U, the group G is explicitly known; in particular, $k = 2$ or 4, $p \equiv \pm 1 \pmod 5$ if $k = 4$, and $p \neq 5$ if $k = 4$. By Lemma 2.6, we write $\mathit{\Gamma} = \mathsf{BCay}(\mathbb{Z}_p^k, S)$, where S is an H-orbit on $\mathbb{Z}_p^k$ for some $H \leqslant \mathrm{GL}_k(p)$ with $H \cong G_u$. Thus G is the subgroup of $\mathsf{Aut}^+(\mathit{\Gamma})$ induced by $G_0 := \{t_{\alpha,\mathbf{v}} \mid \alpha \in H, \mathbf{v} \in \mathbb{Z}_p^k\}$, see the second paragraph of Section 2. Noting that the vertex set of $\mathit{\Gamma}$ is identified with $\mathbb{Z}_p^k \times \mathbb{Z}_2$, we choose $u = (\mathbf{0}, 0)$, where $\mathbf{0}$ is the zero vector of $\mathbb{Z}_p^k$.

Since $\mathit{\Gamma}$ is connected, the digraph $\mathsf{Cay}(\mathbb{Z}_p^k, S)$ is connected, and so S spans the vector space $\mathbb{Z}_p^k$. Then S contains a basis $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_k$ of $\mathbb{Z}_p^k$. For each $\alpha \in \mathrm{GL}_k(p)$, it is easily shown that $\mathsf{Cay}(\mathbb{Z}_p^k, S) \cong \mathsf{Cay}(\mathbb{Z}_p^k, S\alpha)$, and then $\mathit{\Gamma} = \mathsf{BCay}(\mathbb{Z}_p^k, S) \cong \mathsf{BCay}(\mathbb{Z}_p^k, S\alpha)$. Thus, up to isomorphism, we choose $\mathbf{e}_i$ with the ith coordinate 1 and all other coordinates 0 for $i \in \{1, 2, \ldots, k\}$, and choose $\alpha_0 \in H$ with $\alpha_0^5 = 1$ and $\mathbf{e}_i\alpha_0 = \mathbf{e}_{i+1}$ for $i \in \{1, 2, \ldots, k-1\}$. Then, by straightforward calculation, we conclude α_0 is given as in Example 7.2. We have $S = S_{c,b}$ or S_d for $k = 2$ or 4, respectively.

Let $A = \mathsf{Aut}^+(\mathit{\Gamma})$. By Lemma 2.5, $\mathit{\Gamma}$ has an automorphism τ of order 2, which interchanges two parts of $\mathit{\Gamma}$. Then $\mathsf{Aut}(\mathit{\Gamma}) = A{:}\langle\tau\rangle$. Clearly, A acts faithfully and primitively on both parts of $\mathit{\Gamma}$. By Lemma 4.1, Theorems 5.10 and 6.5, we conclude that A is an affine primitive group on each part of $\mathit{\Gamma}$. It follows from Lemma 7.1, A_u and A_w are conjugate in A, and thus A has a suborbit of length 5. By [11, Theorem 1.1], either $k = 2$ and $|A|$ is a divisor of $10p^2$, or $k = 4$ and $|A|$ is a divisor of $120p^4$. Note that ($\circledast$) holds for A, and every $\alpha \in \mathrm{GL}_k(p)$ with $S\alpha = S$ induces an element of A.

Assume that $k = 2$. Then, by Lemma 7.3, $c = -1$ and $b = \frac{-1\pm\sqrt{5}}{2}$. Recalling that $|A|$ is a divisor of $10p^2$, by Corollary 7.5, we conclude that $A \cong \mathbb{Z}_p^2{:}\mathrm{D}_{10}$, and so $\mathsf{Aut}(\mathit{\Gamma}) =$

$A{:}\langle\tau\rangle \cong (\mathbb{Z}_p^2{:}\mathrm{D}_{10}){:}\mathbb{Z}_2$. Let $b_1 = \frac{-1-\sqrt{5}}{2}$ and $b_2 = \frac{-1+\sqrt{5}}{2}$. Then $\Gamma = \mathsf{BCay}(\mathbb{Z}_p^k, S_{-1,b_1})$ or $\mathsf{BCay}(\mathbb{Z}_p^k, S_{-1,b_2})$. Take $\alpha = \begin{pmatrix} 1 & 0 \\ -1 & b_2 \end{pmatrix}$. Calculation shows that $S_{-1,b_1}\alpha = S_{-1,b_2}$, and thus $\mathsf{BCay}(\mathbb{Z}_p^2, S_{-1,b_1}) \cong \mathsf{BCay}(\mathbb{Z}_p^2, S_{-1,b_2})$. Then (i) of Theorem 1.1 (4) follows.

Now let $k = 4$. Recall that ($\circledast$) holds for A. If $|A_u|$ is odd then $A_u \cong \mathbb{Z}_5$; however, since A is a primitive subgroup of $\mathrm{AGL}_4(p)$, we have that $d = -1$ and A_u should has a subgroup isomorphic to S_5 by Lemma 7.4, a contradiction. Thus $|A_u|$ is even, and then A_u has a subgroup isomorphic to D_{10}. This implies that there exists an involution $\beta_0 \in \mathrm{GL}_k(p)$ such that $S_d\beta_0 = S_d$ and $\langle\alpha_0, \beta_0\rangle \cong \mathrm{D}_{10}$. Recall that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4 \in S_d$ and $\mathbf{e}_i\alpha_0 = \mathbf{e}_{i+1}$ for $i \in \{1, 2, 3\}$. Then $S_d = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_4\alpha_0\}$. Without loss of generality, we assume that β_0 induces the permutation $(\mathbf{e}_1, \mathbf{e}_4)(\mathbf{e}_2, \mathbf{e}_3)$ on S_d. Straightforward calculation shows that $-d^2 = d^3$, yielding $d = -1$. It follows from Lemma 7.4 that A_u has a subgroup isomorphic to S_5. Since $|A_u|$ is a divisor of 120, we have $A_u \cong \mathrm{S}_5$, and so $\mathsf{Aut}(\Gamma) = A{:}\langle\tau\rangle \cong (\mathbb{Z}_p^4{:}\mathrm{S}_5){:}\mathbb{Z}_2$, desired as in (ii) of Theorem 1.1 (4). This completes the proof. $\square$

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Q. Cai, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, China
E-mail address: `919727874@qq.com`

Z.P. Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, China
E-mail address: `lu@nankai.edu.cn`