A note on rainbow-free colorings of uniform hypergraphs

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Dedicated to Professor Xueliang Li on the occasion of his 65th birthday.

Abstract

A rainbow-free coloring of a k-uniform hypergraph H is a vertex-coloring which uses k colors but with the property that no edge of H attains all colors. Koerkamp and Živný showed that $p = (k-1)(\log n)/n$ is the threshold function for the existence of a rainbow-free coloring of the random k-uniform hypergraph $G^k(n,p)$, and presented that the case when p is close to the threshold is open. In this paper, we give an answer to the question.

Keywords: random hypergraphs; rainbow-free colorings; crossing edges

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1 Introduction

For a hypergraph H, a map $c : V(H) \to [k]$ is called a *k-coloring* of H, where $[k] := \{1, \ldots, k\}$. For a k-coloring c of a hypergraph H, we denote the color classes of c by $C_i := c^{-1}(i)$, $i \in [k]$. Given a k-uniform hypergraph H, a coloring c is called *rainbow-free* if for every edge $e = \{v_1, \ldots, v_k\} \in E(H)$ we have $c(e) = \{c(v_1), \ldots, c(v_k)\} \neq [k]$ and for every $i \in [k]$ there is a vertex $v \in V(H)$ with $c(v) = i$.

We say a k-uniform hypergraph H is *rainbow-free colorable* if there is a rainbow-free k-coloring of H. The *k-rainbow-free problem* is to determine whether a given k-uniform hypergraph is rainbow-free colorable. Particularly, for $k = 2$, a graph is rainbow-free colorable if and only if it is disconnected (cf. Remark 4 in [4]).

The k-rainbow-free problem is a special case of coloring mixed hypergraphs, which is introduced by Voloshin [6] and further extended by Král', Kratochvíl, Proskurowski and Voss [5]. A mixed hypergraph is a triple (V, C, D), where V is the vertex set and C and D are collections of subsets of V. A coloring of the vertices of a mixed hypergraph (V, C, D) is called *proper* if every edge in C contains two vertices of the same color and each edge in D contains two vertices of different colors. The *strict k-coloring problem* is to determine whether there is a proper k-coloring of a given mixed hypergraph. A mixed hypergraph (V, C, D) with $D = \emptyset$ is called *co-hypergraph*. Therefore, the strict k-coloring problem restricted to k-uniform co-hypergraphs is just the k-rainbow-free problem. Bodirsky, Kára and Martin [2] called the strict k-coloring problem of co-hypergraphs as *k-no-rainbow-coloring problem*, and stated it as an interesting case of surjective constraint satisfaction problems on a three-element domain.

In this paper, we focus on k-rainbow-free colorings of random hypergraphs. All logarithms whose base is omitted are natural. If k is clear from the context, we will call a k-coloring simply a coloring. For $n \in \mathbb{Z}$ and $p \in (0, 1)$, let $G^k(n, p)$ be a probability space consisting of k-uniform hypergraphs with n vertices, in which each element of $\binom{[n]}{k}$ occurs independently as an edge with probability p. An event occurs *with high probability* (w.h.p.) if the probability of that event approaches 1 as n tends to infinity.

Koerkamp and Živný [4] initiated the study of k-rainbow-free colorings of random hypergraphs. They showed that the function $p^* = (k-1)(\log n)/n$ is a threshold function for the property of being rainbow-free colorable [4]. More precisely, they proved the following result.

Theorem 1.1 ([4]) *For integer $k \geq 3$, w.h.p. $G^k(n, p)$ is rainbow-free colorable if $p \leq D\frac{\log n}{n}$ for $D < k - 1$. And w.h.p. $G^k(n, p)$ is not rainbow-free colorable if*

$p \geq D\frac{\log n}{n}$ *with* $D > k-1$.

Koerkamp and Živný [4] pointed out that in the case that p is close to $(k-1)(\log n)/n$, the behavior of the rainbow-free colorings of $G^k(n,p)$ is open.

In this paper, we completely determine the behavior of the rainbow-free colorings of $G^k(n,p)$ for p not covered by Theorem 1.1. Let $\mathcal{R}$ be the property of being rainbow-free colorable. We obtain the following theorem on the property $\mathcal{R}$.

Theorem 1.2 *Let* $k \geq 3$ *be an integer.*
(i) If $p = \frac{(k-1)\log n - w(n)}{n}$, *where* $w(n) = o(\log n)$ *and* $w(n) \to \infty$, *then*

$$\mathbf{Pr}[G^k(n,p) \in \mathcal{R}] \to 1.$$

(ii) If $p = \frac{(k-1)\log n + w(n)}{n}$, *where* $w(n) = o(\log n)$ *and* $w(n) \to \infty$, *then*

$$\mathbf{Pr}[G^k(n,p) \notin \mathcal{R}] \to 1.$$

(iii) If $p = \frac{(k-1)\log n + y + o(1)}{n}$, *where* y *is fixed and* $y \in \mathbb{R}$, *then*

$$\mathbf{Pr}[G^k(n,p) \in \mathcal{R}] \sim 1 - e^{-e^{-y}/(k-1)!}.$$

Moreover, the number of rainbow-free colorings has asymptotically Poisson distribution with mean $\frac{e^{-y}}{(k-1)!}$.

Remark. Our proof of Theorem 1.2 (Lemmas 2.1 and 2.2) indicates the structure of "possible" rainbow-free colorings. We obtain that only the coloring, which has only one color class of size greater than 1, could be rainbow-free with positive probability as $n \to \infty$. Indeed, for a coloring c, if there exist two color classes C_i and C_j $(i \neq j)$, such that both C_i and C_j contain at least two vertices, then Lemma 2.2 implies that the probability of c being rainbow-free tends to zero when $n \to \infty$. Combining with Lemma 2.1, we have that only if c has exactly one color class containing at least two vertices, the probability of c being rainbow-free could be positive.

The rest of the paper is organized as follows. In Section 2, we introduce some more notation and preliminaries in order to present the crux – Lemmas 2.1 and 2.2, which lead to Theorem 1.2 directly. We give the proofs of Lemmas 2.1 and 2.2 in Section 3 and Section 4, respectively. In this paper, we will always assume that n is the variable that tends to infinity.

2 Preliminaries

We use the standard notation $X_n \xrightarrow{d} X$ to denote that the sequence of variables (X_n) tends to the variable X in distribution. And denote by P_λ the Poisson distribution with mean λ. Given an integer-valued random variable X, let $\mathbf{E_r}[X]$ denote the

r-th factorial moment of X, i.e., $\mathbf{E_r}[X] = \mathbf{E}[X(X-1)\dots(X-r+1)]$. The following result on convergence in distribution will be used in our proofs.

Theorem 2.1 ([3]) *Let $\lambda = \lambda(n)$ be a non-negative bounded function on $\mathbb{N}$. Suppose that the non-negative integer-valued random variables X_1, X_2, … are such that*

$$\lim_{n\to\infty} \mathbf{E_r}[X_n] - \lambda^r = 0, \qquad r = 1, 2, \dots$$

Then

$$X_n \xrightarrow{d} P_\lambda.$$

Recall that for a coloring c of a k-uniform hypergraph H, we denote the color classes of c by $C_i := c^{-1}(i)$, $i \in [k]$. We say a k-set $e \subseteq V(H)$ is a *crossing edge* of c, if e meets all of the k classes of c.

Similar to the technique used by Koerkamp and Živný [4], we identify a coloring by the sequence $(s_1, \dots, s_k)$ where $s_i = |C_i|$ and $s_1 \le \dots \le s_k$. We divide the set of all possible sequences into four types[1]:

Type I. $(s_i)_k = (1, \dots, 1, n-k+1)$. There is one such sequence.

Type II. $(s_i)_k = (1, \dots, 1, x, n-k+2-x)$ with $x \ge 2$. This case contains $O(n)$ sequences.

Type III. $2 \le s_{k-2} \le s_{k-1}$ and $s_1 + \dots + s_{k-1} \le 6k$. This case contains $O(1)$ sequences, since k is a constant.

Type IV. $2 \le s_{k-2} \le s_{k-1}$ and $s_1 + \dots + s_{k-1} > 6k$. Note that for every i with $1 \le i \le k-1$, there are less than n choices of the value of s_i. Moreover, since $s_k = n - \sum_{i=1}^{k-1} s_i$, we have only one choice of the value of s_k when $s_1, \dots, s_{k-1}$ are fixed. Therefore this case contains $O(n^{k-1})$ sequences.

We investigate the existence of each type of colorings of $G^k(n,p)$ for p belonging to different ranges. The following two lemmas are our main results, which summarize the behavior of rainbow-free colorings of $G^k(n,p)$ when p is near $(k-1)\log n/n$. Lemma 2.1 focuses on colorings of Type I.

Lemma 2.1 *Let $k \ge 3$ be an integer, and X_I be the number of rainbow-free colorings of Type I of $G^k(n,p)$.*
(i) If $p = \frac{(k-1)\log n - w(n)}{n}$, where $w(n) = o(\log n)$ and $w(n) \to \infty$, then

$$\mathbf{Pr}[X_I > 0] \to 1.$$

[1] There are some differences with the five types of sequences defined in [4].

(ii) If $p = \frac{(k-1)\log n + w(n)}{n}$*, where* $w(n) = o(\log n)$ *and* $w(n) \to \infty$*, then*

$$\mathbf{Pr}[X_I = 0] \to 1.$$

(iii) If $p = \frac{(k-1)\log n + y + o(1)}{n}$*, where* y *is fixed and* $y \in \mathbb{R}$*, then* X_I *has asymptotically Poisson distribution with mean* $\frac{e^{-y}}{(k-1)!}$*:*

$$\mathbf{Pr}[X_I = r] \sim e^{-e^{-y}/(k-1)!} \frac{e^{-ry}}{((k-1)!)^r \, r!}.$$

Unlike Type I coloring, the following result tells us that rainbow-free colorings are not likely to occur as one of Types II, III, or IV.

Lemma 2.2 *Let* X_i *be the number of rainbow-free colorings of Type* i *of* $G^k(n,p)$ *for* $i = II, III, IV$*. If* $p = \frac{(k-1)\log n + w^*(n)}{n}$*, where* $w^*(n) = o(\log n)$*, then* $\mathbf{Pr}[X_i = 0] \to 1$ *for* $i = II, III, IV$*.*

It is easy to see that Theorem 1.2 follows immediately from Lemmas 2.1 and 2.2. We present the proof of Lemma 2.1 in Section 3, and prove Lemma 2.2 in Section 4.

3 Coloring of Type I

The standard second moment method will be used to prove Lemma 2.1. Let X be a nonnegative integer-valued random variable such that $X = \sum_{i=1}^{m} X_i$, where X_i is the indicator variable for event E_i. For indices i, j, write $i \sim j$ if $i \neq j$ and the events E_i and E_j are not independent. Let (the sum is over all ordered pairs)

$$\Delta = \sum_{i \sim j} \mathbf{Pr}[E_i \wedge E_j].$$

Theorem 3.1 (Corollary 4.3.4 in [1]) *If* $\mathbf{E}[X] \to \infty$ *and* $\Delta = o((\mathbf{E}[X])^2)$*, then*

$$\mathbf{Pr}[X > 0] \to 1.$$

Let c be a coloring of Type I of $G^k(n,p)$. It follows that $|C_i| = 1$ for $1 \leq i \leq k-1$ and $|C_k| = n - k + 1$. Thus, coloring c is rainbow-free if and only if there is no edge of $G^k(n,p)$ meeting all k color classes, i.e., any crossing edge of c should not appear in $G^k(n,p)$. There are $n - k + 1$ such crossing edges, and hence

$$\mathbf{Pr}[c \text{ is a rainbow-free coloring}] = (1-p)^{n-k+1}. \tag{3.1}$$

Enumerate all possible colorings of Type I (up to permutations of colors) by c^1 up to c^ℓ. To every coloring c^i we associate the event E_i that c^i is rainbow-free. Therefore (3.1) implies that

$$\mathbf{Pr}[E_i] = (1-p)^{n-k+1}$$

for every i.

For colorings c^i and c^j with $i \neq j$, if the number of common crossing edges of c^i and c^j is x, then

$$\mathbf{Pr}[E_i \wedge E_j] = (1-p)^{2(n-k+1)-x},$$

since the total number of crossing edges we need to forbid is $2(n-k+1)-x$. Consequently, $\mathbf{Pr}[E_i \wedge E_j] = \mathbf{Pr}[E_i] \cdot \mathbf{Pr}[E_j]$ if and only if $x = 0$, i.e., E_i and E_j are independent if and only if c^i and c^j have no common crossing edges. Recall that we write $i \sim j$ if and only if E_i and E_j are dependent, and therefore $i \sim j$ if and only if c^i and c^j have common crossing edges. For a coloring c^i, let $\mathcal{C}_F^i = \bigcup_{r\in[k-1]} C_r^i$, where C_r^i is the r-th color class of c^i. The following result tells us more about the independence of two colorings of Type I.

Proposition 3.1 *For colorings c^i and c^j of Type I with $i \neq j$, we have*
(i) $i \sim j$ if and only if $|C_k^i \cap C_k^j| = n-k$.
(ii) If $i \sim j$, then the only common crossing edge of c^i and c^j is $\mathcal{C}_F^i \cup \mathcal{C}_F^j$.

Proposition 3.1 (i) was observed by Koerkamp and Živný in [4], we rewrite the proof for completeness.

Proof. We will show that (1) if $|C_k^i \cap C_k^j| \leq n-k-1$, then there is no common crossing edge of c^i and c^j, and (2) if $|C_k^i \cap C_k^j| = n-k$, then the only common crossing edge of c^i and c^j is $\mathcal{C}_F^i \cup \mathcal{C}_F^j$. It is not difficult to obtain Proposition 3.1 from (1) and (2). Let $T = C_k^i \cap C_k^j$, $A^i = C_k^i \setminus T$ and $A^j = C_k^j \setminus T$. Set $R = V(G^k(n,p)) \setminus (C_k^i \cup C_k^j)$. Therefore A^i, A^j and R are pairwise disjoint.

(1) Assume that $|T| \leq n-k-1$. Thus, $|A^i| \geq 2$ and $|A^j| \geq 2$. Note that any crossing edge e_1 of c^i is of the form $A^j \cup R \cup \{u\}$, where $u \in C_k^i$, and any crossing edge e_2 of c^j is of the form $A^i \cup R \cup \{v\}$, where $v \in C_k^j$. Therefore, if e is a common crossing edge of both c^i and c^j, then there exist vertices $u \in C_k^i$ and $v \in C_k^j$ such that

$$e = (A^j \cup R \cup \{u\}) = (A^i \cup R \cup \{v\}). \tag{3.2}$$

Since $|A^i| \geq 2$, there is a vertex $x \in A^i \setminus \{u\}$. Note that A^i and A^j are disjoint, we have $x \notin A^j$, therefore, $x \in (A^i \cup R \cup \{v\})$ but $x \notin (A^j \cup R \cup \{u\})$. Hence, $(A^j \cup R \cup \{u\}) \neq (A^i \cup R \cup \{v\})$, which contradicts (3.2).

(2) From (1), if $i \sim j$, then $|T| = n - k$. It follows that $R \subseteq (\mathcal{C}_F^i \cup \mathcal{C}_F^j)$ with $|R| = k-2$, and $|A^i| = |A^j| = 1$. Moreover, $(\mathcal{C}_F^i \cup \mathcal{C}_F^j) = (R \cup A^i \cup A^j)$. Let $A^i = \{z\}$ and $A^j = \{w\}$. Then $w \in \mathcal{C}_F^i$ and $z \in \mathcal{C}_F^j$. Assume that e is a common crossing edge of c^i and c^j. Thus, there exist vertices $u \in C_k^i$ and $v \in C_k^j$ such that

$$e = (A^j \cup R \cup \{u\}) = (A^i \cup R \cup \{v\}).$$

By our assumption that $A^i = \{z\}$ and $A^j = \{w\}$, we have

$$z = u, \text{ and } w = v.$$

Hence, c^i and c^j have only one common crossing edge e, and $e = (\mathcal{C}_F^i \cup \mathcal{C}_F^j)$. $\square$

Proof of Lemma 2.1.

Let c be a coloring of Type I of $G^k(n,p)$. The number of colorings of Type I is $\binom{n}{n-k+1} = \Theta\left(n^{k-1}\right)$. By (3.1), the expected number of rainbow-free colorings of Type I is

$$\mathbf{E}[X_I] = \binom{n}{n-k+1}(1-p)^{n-k+1}. \tag{3.3}$$

(i) Since $\log(1+x) = x + O(x^2)$ for small x, we have

$$1 - p = e^{-p+O(p^2)}. \tag{3.4}$$

Substituting $p = \frac{(k-1)\log n - w(n)}{n}$ into (3.4), and combining with (3.3), we get

$$\begin{aligned}
\mathbf{E}[X_I] =& \Theta\left(n^{k-1}\right) e^{-(n-k+1)p+O(np^2)} \\
=& \Theta\left(e^{(k-1)\log n-(k-1)\log n+w(n)+O(((k-1)\log n-w(n))^2/n)}\right) \\
=& \Theta\left(e^{w(n)}\right).
\end{aligned}$$

Hence, $\mathbf{E}[X_I] \to \infty$ as $n \to \infty$.

Given a Type I coloring c^i, by Proposition 3.1 (i), the number of colorings c^j such that $i \sim j$, is equal to the number of colorings c^j satisfying that the largest color classes of c^i and c^j overlap in $n-k$ positions, which is $\binom{n-k+1}{n-k}(k-1) = (n-k+1)(k-1)$. Realize that the total number of crossing edges of c^i and c^j is $2(n-k+1) - 1 =$

$2n-2k+1$. Therefore,

$$\begin{aligned}
\Delta &= \sum_i (n-k+1)(k-1)(1-p)^{2n-2k+1} \\
&\leq \binom{n}{n-k+1}(n-k+1)(k-1)e^{-p(2n-2k+1)} \\
&\leq n^{k-1}n(k-1)e^{-p(2n-2k+1)} \\
&= (k-1)e^{k\log n}e^{-p(2n-2k+1)} \\
&= (k-1)e^{k\log n-2(k-1)\log n+2w(n)+(2k-1)(k-1)\log n/n-(2k-1)w(n)/n} \\
&= \Theta\left(e^{(-k+2)\log n}\right),
\end{aligned}$$

where the first inequality holds since the number of colorings of Type I is $\binom{n}{n-k+1}$, and the second inequality follows by $\binom{n}{n-k+1} \leq n^{k-1}$. Since $k \geq 3$, and $w(n) = o(\log n)$, we have $\Delta = o((\mathbf{E}[X_I])^2)$. By Theorem 3.1, we have $\mathbf{Pr}[X_I > 0] \to 1$.

(ii) From (3.3) and (3.4) we have

$$\mathbf{E}[X_I] \leq n^{k-1}e^{-(n-k+1)p+O(np^2)} \leq e^{-w(n)+o(1)} = o(1),$$

where the last equality follows by the assumption that $w(n) \to \infty$. Since $\mathbf{Pr}[X_I > 0] \leq \mathbf{E}[X_I]$, we obtain that $\mathbf{Pr}[X_I > 0] = o(1)$.

(iii) In this case, we will show that for every fixed integer $r \geq 1$, the r-th factorial moment $\mathbf{E_r}[X_I]$ of X_I is asymptotic to $(e^{-y}/(k-1)!)^r$. Recall that $\mathbf{E_r}[X_I]$ is the expected number of ordered r-tuples of colorings $(c^{i_1}, \ldots, c^{i_r})$ of Type I, such that each coloring is rainbow-free. Given an r-tuple of colorings $\mathcal{C} = (c^{i_1}, \ldots, c^{i_r})$, where every c^{i_j} is of Type I, let X_j be the indicator variable for the event that c^j is rainbow-free.

Let $S_\mathcal{C}$ be the set consisting of the crossing edges of any of $c^{i_1}, \ldots, c^{i_r}$. By Proposition 3.1 (ii), we obtain that

$$r(n-k+1) - \binom{r}{2} \leq |S_\mathcal{C}| \leq r(n-k+1). \tag{3.5}$$

In addition, we have

$$\mathbf{Pr}[X_{i_1} = 1, \ldots, X_{i_r} = 1] = (1-p)^{|S_\mathcal{C}|}. \tag{3.6}$$

Therefore, the expected number $\mathbf{E_r}[X_I]$ of ordered r-tuples of rainbow-free colorings of Type I is

$$\mathbf{E_r}[X_I] = \sum_\mathcal{C} (1-p)^{|S_\mathcal{C}|}. \tag{3.7}$$

Let N denote the number of ordered r-tuples of colorings of Type I. Since the number of colorings of Type I is $\binom{n}{n-k+1} = \binom{n}{k-1}$, we have

$$N = \binom{n}{k-1}\left(\binom{n}{k-1} - 1\right) \cdots \left(\binom{n}{k-1} - (r-1)\right) \sim \binom{n}{k-1}^r. \tag{3.8}$$

By (3.5), (3.6) and (3.7), we have

$$N(1-p)^{r(n-k+1)} \leq \mathbf{E_r}[X_I] \leq N(1-p)^{r(n-k+1)-\binom{r}{2}}. \tag{3.9}$$

For any fixed $t > 0$, we claim that $(1-p)^{r(n-k+1)} \sim (1-p)^{r(n-k+1)-t}$. Indeed, $\frac{(1-p)^{r(n-k+1)}}{(1-p)^{r(n-k+1)-t}} = (1-p)^t \to 1$ for every fixed t. Therefore, (3.9) implies that

$$\mathbf{E_r}[X_I] \sim N(1-p)^{r(n-k+1)}.$$

Then, by (3.8) and (3.3), we have that

$$\mathbf{E_r}[X_I] \sim (\mathbf{E}[X_I])^r.$$

Moreover, by (3.3), we have

$$\begin{aligned}
\mathbf{E}[X_I] &\sim \frac{n^{k-1}}{(k-1)!} e^{-(n-k+1)p+O(np^2)} \\
&= \frac{1}{(k-1)!} e^{(k-1)\log n-(k-1)\log n-y+O(((k-1)\log n+y)^2/n)} \\
&\sim \frac{e^{-y}}{(k-1)!}.
\end{aligned} \tag{3.10}$$

Applying Theorem 2.1, we obtain that X_I has asymptotically Poisson distribution with mean $\frac{e^{-y}}{(k-1)!}$.

∎

4 Colorings of Types II, III and IV

For a sequence $(s_i)_k = (s_1, \ldots, s_k)$ of a coloring c, let $\Pi = s_1 s_2 \cdots s_{k-1}$ and $\Sigma = s_1 + s_2 + \cdots + s_{k-1}$. Then $s_k = n - \Sigma \geq n/k$.

Note that c is rainbow-free if none of the $s_1 \cdots s_k$ crossing edges is present. This happens with probability

$$\mathbf{Pr}[c \text{ is rainbow-free } |(s_i)_k] = (1-p)^{s_1 \cdots s_k} \leq e^{-p s_1 \cdots s_k} = e^{-p\Pi(n-\Sigma)}. \tag{4.1}$$

Since the number of colorings with a given sequence $(s_i)_i$ is upper-bounded by $\binom{n}{s_1}\cdots\binom{n}{s_{k-1}}$, the expected number of rainbow-free colorings with a given sequence $(s_i)_i$ is bounded by

$$\mathbf{E}[\text{the number of rainbow-free colorings}|(s_i)_i] \le \binom{n}{s_1}\cdots\binom{n}{s_{k-1}} e^{-p\Pi(n-\Sigma)}. \quad (4.2)$$

In this section, we prove that if p is "close" to $\frac{(k-1)\log n}{n}$, then w.h.p. there is no rainbow-free coloring of any of Types II, III or IV of $G^k(n,p)$.

Proof of Lemma 2.2. Denote by X_i the number of rainbow-free colorings of Type i for $i = II, III, IV$. We will prove that $\mathbf{E}[X_i] = o(1)$ for every $i = II, III, IV$. Since $\mathbf{Pr}[X_i > 0] \le \mathbf{E}[X_i]$, Lemma 2.2 follows.

(1) *Type II.*

For a coloring c of Type II, recall that the sequence $(s_i)_k = (1,\ldots,1,x,n-k+2-x)$ satisfies that $2 \le x < n/2$. In this case, $\Sigma = x+k-2$ and $\Pi = x$. Let $Z(x)$ be the number of rainbow-free colorings with sequence $s(x) := (1,\ldots,1,x,n-k+2-x)$. Since $2 \le x < n/2$, the number of such sequences $s(x)$ is less than n. By (4.2), we have

$$\mathbf{E}[Z(x)] \le n^{\Sigma} e^{-px(n-(x+k-2))}.$$

Thus, we obtain that

$$\begin{aligned} n\cdot\mathbf{E}[Z(x)] \le & e^{((x+k-1)-(k-1)x)\log n - w^*(n)x + ((k-1)\log n + w^*(n))(x+k-2)x/n} \\ = & e^{((k-1)+(2-k)x)\log n - w^*(n)x + ((k-1)\log n + w^*(n))(x+k-2)x/n}. \end{aligned} \quad (4.3)$$

Clearly, if we show that the exponent of e in (4.3) tends to $-\infty$, then we have $\mathbf{E}[X_{II}] \le n\cdot\mathbf{E}[Z(x)] = o(1)$. Let $f_k(x) = ((k-1)+(2-k)x)\log n - w^*(n)x + ((k-1)\log n + w^*(n))(x+k-2)x/n$, the exponent of e in (4.3). Then $f_k'(x) = (2-k)\log n - w^*(n) + ((k-1)\log n + w^*(n))(2x+k-2)/n$. It is easy to get that $f_k'(x) \sim (2-k)\log n \to -\infty$ when $x = o(n)$. If $x = \delta n$ for some positive constant δ, then $f_k'(x) \sim ((2-k)+2\delta(k-1))\log n$. Thus, we have $f_k'(x) \ge 0$ for $\delta \ge \frac{1}{2} - \frac{1}{2(k-1)}$, and $f_k'(x) < 0$ for $0 < \delta < \frac{1}{2} - \frac{1}{2(k-1)}$. We distinguish two cases based on k.

Case 1. $k \ge 4$.

Since we want an upper bound for $f_k(x)$, it suffices to check the boundaries $x = 2$ and $x = n/2$. For $x = 2$, we get that $f_k(2) \sim (-k+3)\log n$. Since $k \ge 4$, we have $f_k(2) \to -\infty$. For $x = n/2$, note that $f_k(n/2)$ equals

$$\frac{-k+3}{4} n\log n - \frac{1}{4} n w^*(n) + \left(\frac{(k-2)(k-1)}{2} + k - 1\right)\log n + \frac{k-2}{2} w^*(n).$$

We have $f_k(n/2) \sim \frac{-k+3}{4} n \log n \to -\infty$ for $k \geq 4$. Hence, $f_k(x) \to -\infty$ for $k \geq 4$. In conclusion, we have $\mathbf{E}[X_{II}] = o(1)$ for $k \geq 4$.

Case 2. $k = 3$.

When $x = \delta n$, we have $f_k'(x) \geq 0$ for $\delta \geq \frac{1}{4}$ and $f_k'(x) < 0$ for $0 < \delta < \frac{1}{4}$. Note that $f_3(x) = (2-x)\log n - xw^*(n) + \frac{x(x+1)(2\log n + w^*(n))}{n}$. Thus $f_3(3) = -\log n - 3w^*(n) + \frac{12(2\log n + w^*(n))}{n} \to -\infty$. And it is easy to check that $f_3(n/4) \sim -\frac{n \log n}{8}$, so $f_3(n/4) \to -\infty$. Combining with (4.3), it follows that

$$\text{when } 3 \leq x \leq n/4, \text{ we have } \mathbf{E}[Z(x)] = o(n^{-1}). \tag{4.4}$$

Therefore, we need to investigate two subcases: $\frac{n}{4} < x < \frac{n}{2}$ or $x = 2$.

Subcase 2.1. $\frac{n}{4} < x < \frac{n}{2}$.

Let $x = \frac{n}{2} - \epsilon$. Thus, the sequence of coloring is $(1, \frac{n}{2} - \epsilon, \frac{n}{2} + \epsilon - 1)$. Furthermore, we have $\epsilon \geq 1/2$ since $\frac{n}{2} - \epsilon \leq \frac{n}{2} + \epsilon - 1$. Combining with our assumption that $n/4 < x < n/2$, we have

$$\frac{1}{2} \leq \epsilon < \frac{n}{4}. \tag{4.5}$$

Therefore,

$$\mathbf{E}\left[Z\left(\frac{n}{2} - \epsilon\right)\right] = n\binom{n-1}{\frac{n}{2} - \epsilon}(1-p)^{\left(\frac{n}{2}-\epsilon\right)\left(\frac{n}{2}+\epsilon-1\right)}. \tag{4.6}$$

Since $\binom{n}{s} \leq \left(\frac{ne}{s}\right)^s$ for $0 \leq s \leq n$, combining with (4.6), we have

$$\begin{aligned}
\mathbf{E}\left[Z\left(\frac{n}{2} - \epsilon\right)\right] \leq & n\left(\frac{(n-1)e}{\frac{n}{2} - \epsilon}\right)^{\frac{n}{2}-\epsilon} e^{-p\left(\frac{n^2}{4} - \frac{n}{2} - \epsilon(\epsilon-1)\right)} \\
\leq & n\left(\frac{ne}{\frac{n}{2} - \epsilon}\right)^{\frac{n}{2}} e^{-p\left(\frac{n^2}{4} - \frac{n}{2} - \epsilon(\epsilon-1)\right)} \\
= & e^{-\frac{n}{2}\log(n-2\epsilon) - \frac{nw^*(n)}{4} + \frac{n}{2}(1+\log 2) + 2\log n + \frac{w^*(n)}{2} + o(1)}.
\end{aligned} \tag{4.7}$$

Since $\epsilon < n/4$ by (4.5), we have $n - 2\epsilon > n/2$. Substituting this into (4.7), we obtain that

$$\mathbf{E}\left[Z\left(\frac{n}{2} - \epsilon\right)\right] < e^{-\frac{n}{2}\log n - \frac{nw^*(n)}{4} + \frac{n}{2}(1+2\log 2) + 2\log n + \frac{w^*(n)}{2} + o(1)}.$$

Recall that $w^*(n) = o(\log n)$, we have

$$\mathbf{E}\left[Z\left(\frac{n}{2} - \epsilon\right)\right] = e^{-\frac{n}{2}\log n(1+o(1))} = o(n^{-1}).$$

Namely,

$$\text{when } n/4 < x < n/2, \text{ we have } \mathbf{E}[Z(x)] = o(n^{-1}). \tag{4.8}$$

Subcase 2.2. $x = 2$.

Now the sequence of coloring is $(1, 2, n-3)$, and the number of such colorings is $\binom{n}{1}\binom{n-1}{2} = n\binom{n-1}{2}$. Therefore,

$$\begin{aligned}\mathbf{E}[Z(2)] = n\binom{n-1}{2}(1-p)^{2(n-3)} &\leq \frac{1}{2}n^3(1-p)^{2(n-3)} \\ &\leq \frac{1}{2}e^{-\log n - 2w^*(n) + \frac{12\log n + 6w^*(n)}{n}} = \Theta(n^{-1}). \end{aligned} \tag{4.9}$$

Let $Z(3+)$ be the number of rainbow-free colorings of types $(1, x, n-1-x)$ for $3 \leq x < \frac{n}{2}$. Then we have that when $k = 3$,

$$\mathbf{E}[X_{II}] = \mathbf{E}[Z(3+)] + \mathbf{E}[Z(2)]. \tag{4.10}$$

By (4.4) and (4.8), we have

$$\mathbf{E}[Z(3+)] = o(1). \tag{4.11}$$

Therefore, from (4.9), (4.10) and (4.11), we obtain that $\mathbf{E}[X_{II}] = o(1)$ for $k = 3$.

(2) *Type III.*

Given a coloring sequence $(s_i)_k$, since

$$\binom{n}{s_1}\cdots\binom{n}{s_{k-1}} \leq n^{\Sigma},$$

by (4.2), we have

$$\begin{aligned}&\mathbf{E}[\text{the number of rainbow-free colorings}|(s_i)_k] \\ \leq& e^{(\Sigma - \Pi(k-1))\log n - w^*(n)\Pi + ((k-1)\log n + w^*(n))\Sigma\Pi/n} \\ =& \exp\left(\left(\Sigma - (k-1)\Pi\left(1 - \frac{\Sigma}{n}\right)\right)\log n - w^*(n)\Pi\left(1 - \frac{\Sigma}{n}\right)\right). \end{aligned} \tag{4.12}$$

We need to estimate the terms in the exponent of e in (4.12). We first bound the term $\left(\Sigma - (k-1)\Pi\left(1 - \frac{\Sigma}{n}\right)\right)\log n$. For any Type III coloring, we have $s_{k-2} \geq 2$. Observe that the minimal value of Π is reached when $s_1 = \cdots = s_{k-3} = 1$, $s_{k-2} = 2$ and $s_{k-1} = \Sigma - (k-3) - 2 = \Sigma - k + 1$. It follows that $\Pi \geq 2(\Sigma - k + 1)$. Further notice that $1 - \frac{\Sigma}{n} \to 1$ since Σ is bounded, we have

$$\begin{aligned}\Sigma - (k-1)\Pi\left(1 - \frac{\Sigma}{n}\right) \leq& \Sigma - (k-1)2(\Sigma - k + 1)\left(1 - \frac{\Sigma}{n}\right) \\ \sim& \Sigma - 2(k-1)(\Sigma - k + 1) = (-2k+3)\Sigma + 2(k-1)^2. \end{aligned}$$

Since $k \geq 3$, we have $-2k + 3 \leq -3 < 0$. Note that $\Sigma \geq k + 1$ since Σ is a sum of $k - 1$ positive terms, of which the last two are at least 2. Hence,

$$(-2k+3)\Sigma + 2(k-1)^2 \leq (-2k+3)(k+1) + 2(k-1)^2 = -3k + 5.$$

Therefore, we have

$$\left(\Sigma - (k-1)\Pi\left(1-\frac{\Sigma}{n}\right)\right)\log n \leq (-3k+5)\log n. \tag{4.13}$$

Now we estimate the term $-w^*(n)\Pi\left(1-\frac{\Sigma}{n}\right)$ in (4.12). Since $\Sigma \leq 6k$, we have $\Pi < (6k)^{k-1}$. Note that $\left(1-\frac{\Sigma}{n}\right) < 1$, we have

$$-w^*(n)\Pi\left(1-\frac{\Sigma}{n}\right) \leq |w^*(n)|(6k)^{k-1}. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.12), we obtain that

$$\begin{aligned}&\mathbf{E}[\text{the number of rainbow-free colorings}|(s_i)_k]\\ &\quad < \exp\left((-3k+5)\log n + o(\log n)\right) = \Theta(n^{-3k+5}).\end{aligned}$$

Since the number of sequences $(s_i)_k$ of Type III is bounded by some constant, we obtain that $\mathbf{E}[X_{III}] = o(1)$ for $k \geq 3$.

(4) *Type IV.*

In this case, (4.12) still holds. The exponent of e in (4.12) is

$$\begin{aligned}&\left(\Sigma - \Pi(k-1)\left(1-\frac{\Sigma}{n}\right)\right)\log n - w^*(n)\Pi\left(1-\frac{\Sigma}{n}\right)\\ \leq&\left(\Sigma - \Pi(k-1)\left(1-\frac{\Sigma}{n}\right)\right)\log n + |w^*(n)|\Pi\left(1-\frac{\Sigma}{n}\right)\\ =&\left(\Sigma - \Pi\left(1-\frac{\Sigma}{n}\right)\left(k-1+\frac{|w^*(n)|}{\log n}\right)\right)\log n\\ =&\left(\Sigma - \Pi\left(1-\frac{\Sigma}{n}\right)(k-1+o(1))\right)\log n, \end{aligned} \tag{4.15}$$

where the last equality holds since $w^*(n) = o(\log n)$.

We claim that

$$\Sigma - \Pi\left(1-\frac{\Sigma}{n}\right)(k-1+o(1)) < -k+1.$$

Note that $1-\frac{\Sigma}{n} \geq \frac{1}{k}$ since $s_k = n - \Sigma \geq \frac{n}{k}$. Considering the difference between the two sides of the above inequality, we have

$$\begin{aligned}&\Sigma - \Pi\left(1-\frac{\Sigma}{n}\right)(k-1+o(1)) + k - 1\\ \leq&\Sigma - \Pi\left(\frac{k-1}{k}+o(1)\right) + k - 1.\end{aligned}$$

Since $\Pi \geq 2(\Sigma - k + 1)$ holds as before,

$$\begin{aligned}&\Sigma - \Pi\left(\frac{k-1}{k} + o(1)\right) + k - 1\\ \leq &\left(\frac{-k+2}{k} + o(1)\right)\Sigma + \frac{2(k-1)^2}{k} + k - 1 + o(1).\end{aligned}$$

Realize that $\frac{-k+2}{k} < 0$ for $k \geq 3$, combining with the assumption that $\Sigma > 6k$, we have

$$\begin{aligned}&\left(\frac{-k+2}{k} + o(1)\right)\Sigma + \frac{2(k-1)^2}{k} + k - 1 + o(1)\\ < &\left(\frac{-k+2}{k} + o(1)\right)6k + \frac{2(k-1)^2}{k} + k - 1 + o(1)\\ = &-3k + 7 + \frac{2}{k} + o(1),\end{aligned}$$

which is less than 0 for $k \geq 3$. Thus, we have proved

$$\Sigma - \Pi\left(1 - \frac{\Sigma}{n}\right)(k - 1 + o(1)) < -k + 1.$$

Substituting this into (4.15), we have

$$\mathbf{E}[\text{the number of rainbow-free colorings}|(s_i)_k] = o(n^{-k+1}).$$

Since the number of sequences $(s_i)_k$ of Type IV is bounded by n^{k-1}, we obtain that $\mathbf{E}[X_{IV}] = o(1)$. ∎

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