Homomorphisms to small negative even cycles

Jiaao Li[1], Yongtang Shi[2], Zhouningxin Wang[1], Chunyan Wei[2]

[1]School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
[2]Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China
Emails: {lijiaao, shi, wangzhou}@nankai.edu.cn; yan1307015@163.com

Abstract

A strengthening of Jaeger's circular flow conjecture, restricted to planar graphs, asserts that every planar graph of odd girth at least $4k+1$ admits a homomorphism to the odd cycle C_{2k+1}, and the first case is verified and known as the famous Grötzsch theorem. In this paper, we prove analogous results for signed planar graphs: For $k \in \{2,3,4\}$ every signed bipartite planar graph of negative girth at least $6k-4$ admits a homomorphism to C_{-2k}. Here the negative girth is the length of a shortest cycle with an odd number of negative edges. Note that the $k=2$ case was previously obtained in [J. Combin. Theory Ser. B, 153 (2022) 81–104] through a coloring method.

Considering the duality between circular colorings and circular flows of planar graphs, our approach is based on the tools developed in the study of flows and group connectivity, and a potential method is applied in handling orientations with special boundaries for planar graphs. Furthermore, our results have several implications for the circular chromatic numbers of signed planar graphs with given girth conditions.

Keywords: Orientation, strong $\mathbb{Z}_{2k}$-connectivity, circular flow, circular coloring, signed planar graphs.

1 Introduction

1.1 Homomorphism to cycles

The *odd girth* of a graph G is the length of a shortest odd cycle of G. The dual version of Zhang's strengthening [34] of the well-known Jaeger's circular flow conjecture [7], when restricted to planar graphs, is as follows:

Conjecture 1.1. [Jaeger-Zhang conjecture] *Every planar graph of odd girth at least* $4k+1$ *admits a homomorphism to* C_{2k+1}.

A considerable amount of work has been done related to Jaeger's circular flow conjecture and Conjecture 1.1, see [1, 6, 9, 16, 27, 35] and references therein. In particular, a result of Lovász, Thomassen, Wu, and Zhang [16] in 2013 implies the following theorem, which is currently the best general result towards Conjecture 1.1.

Theorem 1.2. [16] *Every planar graph of odd girth at least* $6k+1$ *admits a homomorphism to* C_{2k+1}.

Even though Jaeger's circular flow conjecture (for $k \geqslant 3$) was disproved in [6], all those counterexamples found so far are non-planar graphs, and thus Conjecture 1.1 still remains open. For small k, compared with the odd-girth condition of $6k+1$ shown in Theorem 1.2, there are some improvements towards Conjecture 1.1: For $k=1$, it is the famous Grötzsch theorem; for $k=2$, it follows from the results of [4] and [3] independently that every planar graph of odd girth at least 11 admits a homomorphism to C_5; similarly, for $k=3$, the odd-girth condition 17 is sufficient, implied from the results of [3, 24] independently.

It is well-known that a graph admits a homomorphism to C_{2k+1} if and only if it has a circular $\frac{2k+1}{k}$-coloring[1]. A natural circular coloring conjecture asserts that every planar graph of odd girth at least $2t+1$ admits a circular $\frac{2t+2}{t}$-coloring and Conjecture 1.1 is a part of this conjecture when $t=2k$. For an odd value t, when $t=1$ it is the 4-color theorem, and more generally when $t=2k-1$ it has been proved in [15] that the odd-girth condition of $6k-1$ is sufficient for a planar graph to admit a circular $\frac{4k}{2k-1}$-coloring.

In this paper, we consider analogs of Conjecture 1.1 and the general circular coloring problems for signed planar graphs. A *signed graph* (G,σ) is a graph G together with a signature $\sigma : E(G) \to \{+,-\}$. We simply write $\widehat{G}$ if the signature is clear from the context. We call an edge with $+$ ($-$, respectively) a *positive* edge (a *negative* edge, respectively). A cycle of length ℓ with an odd number of negative edges is called a *negative cycle*, denoted by $C_{-\ell}$, while a cycle of length ℓ with an even number of negative edges is called a *positive cycle*, denoted by C_ℓ. The *negative girth* of a signed graph is the length of a shortest negative cycle in it. A *homomorphism* of a signed graph (G,σ) to another signed graph (H,π) is a mapping $\varphi : V(G) \to V(H)$ that preserves the adjacencies and the signs of all the closed walks. Naturally, considering signed planar graphs, there is an analogous question of Conjecture 1.1:

Question 1.3. *For a positive integer* k*, what is the smallest integer* g_k *such that all signed bipartite planar graphs of negative girth at least* g_k *admit homomorphisms to* C_{-2k}*?*

This question has been proposed and investigated in [2, 19]. Note that the $k=1$ case is trivial (with negative girth at least 2) and the $k=2$ case was shown in [18] as follows.

Theorem 1.4. [18] *Every signed bipartite planar graph of negative girth at least* 8 *admits a homomorphism to* C_{-4}*. Moreover, this negative-girth condition is best possible.*

[1]The notion of circular colorings was first introduced by Vince [32]: Given positive integers p and q, a circular $\frac{p}{q}$-coloring of a graph G is a mapping $f : V(G) \to \{1,2,\dots,p\}$ such that for each edge $uv \in E(G)$, $q \leqslant |f(u)-f(v)| \leqslant p-q$.

In this work, we make some progress on the next two cases ($k = 3, 4$) of Question 1.3 and prove the following results.

Theorem 1.5. *(1) Every signed bipartite planar graph of negative girth at least* 14 *admits a homomorphism to* C_{-6}*.*

(2) Every signed bipartite planar graph of negative girth at least 20 *admits a homomorphism to* C_{-8}*.*

Although the above results may suggest that the negative girth increases 6 each time as k increases, we have no evidence to show whether each of the negative-girth conditions for Theorem 1.5 is tight.

Similar to the equivalence between homomorphisms of graphs to C_{2k+1} and circular $\frac{2k+1}{k}$-colorings of graphs, a homomorphism of a signed bipartite graph to C_{-2k} is proved [21] to be equivalent to a circular $\frac{4k}{2k-1}$-coloring of the signed graph. The notion of the circular coloring of signed graphs was introduced in [22]: Given positive integers p and q where p is even and $p \geqslant 2q$, a *circular* $\frac{p}{q}$*-coloring* of a signed graph (G, σ) is a mapping $f : V(G) \to \{1, 2, \ldots, p\}$ such that for each positive edge uv, $q \leqslant |f(u) - f(v)| \leqslant p - q$ and for each negative edge uv, either $|f(u) - f(v)| \leqslant \frac{p}{2} - q$ or $|f(u) - f(v)| \geqslant \frac{p}{2} + q$.[2] The *circular chromatic number* of a signed graph (G, σ) is defined to be $\chi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-coloring}\}$.

A natural signed graph analog of the circular coloring conjecture has been studied. Kardoš and Narboni recently showed [8] that there is a signed planar graph that is not circular 4-colorable, disproving a conjecture of [17]. Generalizing this example, Naserasr, Wang, and Zhu [22] gave lower and upper bounds on the supremum of the circular chromatic numbers of signed planar graphs, which are $4 + \frac{2}{3}$ and 6, respectively. In this direction, we provide some upper bounds on the circular chromatic numbers of signed planar graphs with given girth conditions.

Theorem 1.6. *Let G be a planar graph of girth g and let σ be a signature on G.*

(1) If $g \geqslant 4$, then $\chi_c(G, \sigma) \leqslant 4$.

(2) If $g \geqslant 7$, then $\chi_c(G, \sigma) \leqslant 3$.

(3) If $g \geqslant 10$, then $\chi_c(G, \sigma) \leqslant \frac{8}{3}$.

Note that (1) and (2) of Theorem 1.6 are already known in [17, 20, 22], and our methods provide alternative (and unified) proofs of those two results. In particular, the proof of Theorem 1.6 (2) is shorter than that of [20] using the notion of homomorphisms of signed graphs. Moreover, the result in (3) provides a better upper bound on the circular chromatic number of signed planar graphs of girth at least 10.

[2] Intuitively, we may view p points placed on a circle with equal distance, the images of two vertices joined with a positive edge are at circular distance at least q while the images of two vertices joined with a negative edge are at circular distance at most $\frac{p}{2} - q$.

1.2 Strongly $\mathbb{Z}_\ell$-connected graphs

Our approach to Theorems 1.5 and 1.6 relies on some stronger orientation results, which are motivated from a newly-developed duality theorem from [14]: A signed bipartite plane graph admits a homomorphism to C_{-2k} if and only if its dual signed Eulerian plane graph admits a mod $2k$-orientation. Given a signed plane graph (G, σ), its *dual signed plane graph*, denoted by (G^*, σ^*), is defined as follows: G^* is the dual of the underlying graph G and $\sigma^*(e^*) = \sigma(e)$ for each edge $e^* \in E(G^*)$ which is the dual edge of $e \in E(G)$. To prove our main results, we study the related notions of orientations and flows with boundaries and develop some tools in this direction. In this paper, $\mathbb{Z}_\ell$, for a positive integer ℓ, denotes the group that consists of the elements $\{0, 1, 2, \ldots, \ell-1\}$ with addition modulo ℓ as the operation. Sometimes, we may view the elements in $\mathbb{Z}_\ell$ as integers for convenience.

Definition 1.7. *Given an integer $\ell \geqslant 2$, a graph G is called* strongly $\mathbb{Z}_\ell$-connected *if for any mapping $\theta : V(G) \to \mathbb{Z}_\ell$ with $\sum_{v \in V(G)} \theta(v) \equiv |E(G)| \pmod{\ell}$, there exists an orientation D on G such that for each $v \in V(G)$, $d_D^+(v) \equiv \theta(v) \pmod{\ell}$ where $d_D^+(v)$ is the out-degree of v under D.*

It was observed in [27] that a graph G is strongly $\mathbb{Z}_2$-connected if and only if it is connected. For general ℓ, it has been shown that every $(3\ell-3)$-edge-connected graph is strongly $\mathbb{Z}_\ell$-connected in [16, 33]. Those results are also related to the tree decomposition and factorization of graphs as studied in [11, 28, 29]. For planar graphs, it is proved in [12, 25] that every 5-edge-connected planar graph is strongly $\mathbb{Z}_3$-connected and it is shown in [3] that every 11-edge-connected (or 17-edge-connected) planar graph is strongly $\mathbb{Z}_5$-connected (respectively, strongly $\mathbb{Z}_7$-connected).

Motivated by these known results, we prove the following main result in this paper, which improves the general result of [16] for the cases when $\ell = 4, 6, 8$.

Theorem 1.8. *(1) Every 8-edge-connected planar graph is strongly $\mathbb{Z}_4$-connected.*

(2) Every 14-edge-connected planar graph is strongly $\mathbb{Z}_6$-connected.

(3) Every 20-edge-connected planar graph is strongly $\mathbb{Z}_8$-connected.

Using Theorem 1.8, we provide an alternative proof of Theorem 1.4 and, furthermore, prove Theorems 1.5 and 1.6.

The proof of Theorem 1.8 is based on a modified potential method, which can be traced back to [3, 10]. In fact, a stronger technical result (Theorem 4.6) is established on the strong $\mathbb{Z}_{2k}$-connectivity which implies Theorem 1.8. The main idea in the proof of Theorem 4.6 is as follows: We introduce a weight function in Formula (5) so that the studied graph class is closed under contraction and each element inside has enough density. With the setting of orientation with boundaries, a potential method is applied, which allows the usage of lifting and contraction operations frequently to find sufficiently many reducible configurations.

The rest of the paper is organized as follows. In the next section, we give basic preliminaries. In Section 3, serving for induction bases of Theorem 4.6, we provide some sufficient conditions for small graphs (on at most four vertices) to be strongly $\mathbb{Z}_\ell$-connected for $\ell \geqslant 2$. In Section 4, we study the properties of the minimum counterexample to Theorem 4.6. Based on the weight function that we introduce there (applied in the potential method), using lifting and contraction operations, we first provide some forbidden configurations for every even value ℓ in Subsection 4.1. Then we discuss each of the cases $\ell \in \{4, 6, 8\}$ and conclude with separate discharging phases in Subsections 4.2, 4.3, and 4.4, finishing the proof Theorem 4.6 and thus Theorem 1.8. In Section 5, as applications of our group connectivity results, we provide the proofs of Theorems 1.4, 1.5 and 1.6. In Section 6, we conclude by constructing a signed bipartite planar graph whose dual does not admit a homomorphism to C_{-2k}.

2 Preliminaries

All graphs considered in this work are allowed to have parallel edges but no loops. Let $G = (V, E)$ be a graph. For a connected pair uv of G, we denote by $\mu_G(uv)$ the number of parallel edges joining u and v. The *multiplicity* of G, denoted by $\mu(G)$, is the maximum value of $\mu_G(uv)$, taken over all connected pairs uv of G. The number of vertices of G is denoted by $v(G)$ and the number of edges of G is denoted by $e(G)$. Moreover, $d_G(v)$ denotes the degree of a vertex v in G, $\delta(G)$ denotes the minimum degree of G, and $\Delta(G)$ denotes the maximum degree of G. A vertex of odd (or even) degree is called an *odd vertex* (respectively, *even vertex*). Given a subset X of $V(G)$, we denote by $G[X]$ the subgraph induced by X, denote by $[X, X^c]$ the *edge-cut* between X and $X^c := V(G) \setminus X$, and let $d_G(X) = |[X, X^c]|$ (or $d(X)$ instead if the graph G is clear from the context).

We use the following notations to denote three types of graphs on at most 4 vertices:

(1) αK_2 is the graph on two vertices with α parallel edges in between;

(2) $T_{a,b,c}$ is the multi-triangle on the vertex set $\{x, y, z\}$ with $\mu(xy) = a, \mu(yz) = b$ and $\mu(zx) = c$;

(3) $Q_{a,b,c,d}$ is the multi-cycle on 4 vertices x, y, z, w such that $\mu(xy) = a, \mu(yz) = b, \mu(zw) = c$ and $\mu(wx) = d$.

See Figure 1 for an illustration.

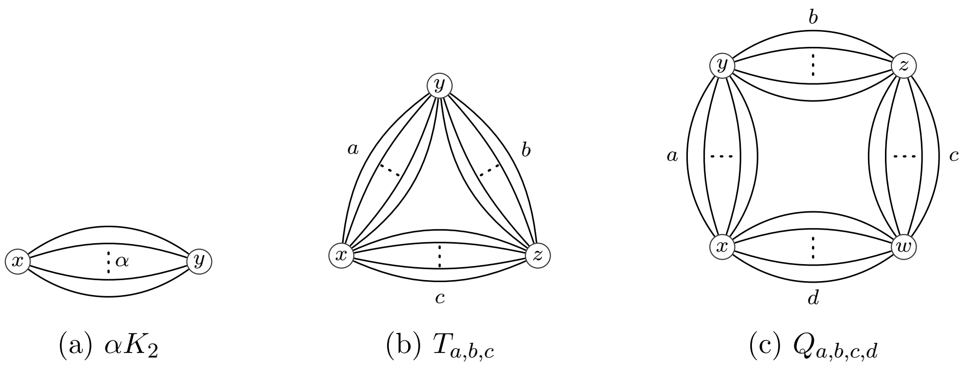


Figure 1: The graphs αK_2, $T_{a,b,c}$, and $Q_{a,b,c,d}$.

Given a planar graph G with a planar embedding, let $F(G)$ denote the set of faces and let $\ell(f)$ denote the length of the boundary cycle of face $f \in F(G)$. A face f with $\ell(f) = k$ (or $\ell(f) \geqslant k$) is called a *k-face* (respectively, *k^+-face*). Two faces are *adjacent* if their boundaries share a common edge. Let $f_1 f_2 \cdots f_s$ be a *face chain* of length $s-1$ if f_i and f_{i+1} are adjacent for $i \in [s-1]$. Moreover, we say two faces f and f' are *weakly adjacent* if there is a face chain $f f_1 \cdots f_t f'$, where f_i is a 2-face for each $i \in [t]$. Here the length of $f f_1 \cdots f_t f'$ can be 1, in which case f and f' are adjacent.

Given an orientation on G, we use (u, v) to denote a directed edge oriented from u to v. Given an orientation D and a vertex $v \in V(G)$, we denote the set of edges oriented from v (i.e., out-edges) by $E_D^+(v)$ and the set of edges oriented towards v (i.e., in-edges) by $E_D^-(v)$, and denote the *out-degree* and *in-degree* of v on G by $d_D^+(v)$ and $d_D^-(v)$, respectively. Given a graph G, an orientation D on G and a mapping $f : E(G) \to \mathbb{Z}$, let

$$\partial_D f(v) := \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e)$$

for each vertex $v \in V(G)$.

Rather than using out-degrees, it is much more convenient to use the differences between in-degree and out-degree in certain orientations. With this notation, an equivalent definition of strongly $\mathbb{Z}_\ell$-connected graphs (Definition 1.7), stated below, will be used frequently in our proofs.

Definition 2.1. [15] *(1) Given a graph G, a* $(2\ell, \beta)$-boundary *of G is a mapping $\beta : V(G) \to \{0, \pm 1, \ldots, \pm \ell\}$ such that for each vertex $v \in V(G)$, $\beta(v) \equiv d(v) \pmod 2$ and $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2\ell}$.*

(2) Given a $(2\ell, \beta)$-boundary of G, an orientation D on G is a $(2\ell, \beta)$-orientation *if it satisfies that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{2\ell}$ for each vertex $v \in V(G)$.*

Proposition 2.2. [13] *A graph G is strongly $\mathbb{Z}_\ell$-connected if and only if for any $(2\ell, \beta)$-boundary of G, it admits a $(2\ell, \beta)$-orientation.*

Observation 2.3. *Let $G = (V, E)$ be a graph and let E' be a subset of E. Let G' be a graph obtained from G by deleting E'. If G' is strongly $\mathbb{Z}_\ell$-connected, then G is also strongly $\mathbb{Z}_\ell$-connected.*

Proof. Let $\theta : V(G) \to \mathbb{Z}_\ell$ with $\sum_{v \in V(G)} \theta(v) \equiv |E(G)| \pmod{\ell}$ and let D' be an orientation on the edges in E'. We define a new mapping $\theta' : V(G) \to \mathbb{Z}_\ell$ as follows: $\theta'(v) := \theta(v) - d_{D'}^+(v)$. Clearly, the mapping θ' satisfies that $\sum_{v \in V(G')} \theta'(v) \equiv |E(G')| \pmod{\ell}$. Since G' is strongly $\mathbb{Z}_\ell$-connected, by Definition 1.7 there is an orientation D'' on G' such that $d_{D''}^+(v) \equiv \theta'(v) \pmod{\ell}$. Note that the orientation $D = D' \cup D''$ is an orientation on G such that $d_D^+(v) \equiv \theta(v) \pmod{\ell}$ and thus G is strongly $\mathbb{Z}_\ell$-connected. $\square$

The following result provides us a necessary density condition for graphs to be strongly $\mathbb{Z}_\ell$-connected.

Proposition 2.4. [13] *If a graph G is strongly $\mathbb{Z}_\ell$-connected, then it contains $\ell - 1$ edge-disjoint spanning trees, and particularly, $|E(G)| \geqslant (\ell - 1)(|V(G)| - 1)$.*

Thus the minimum degree of a strongly $\mathbb{Z}_\ell$-connected graph is at least $\ell-1$. The next lemma shows that for αK_2 this necessary condition is also sufficient.

Lemma 2.5. *The graph αK_2 is strongly $\mathbb{Z}_\ell$-connected if and only if $\alpha \geqslant \ell - 1$.*

Proof. Let αK_2 be a graph on two vertices v_1 and v_2 with α parallel edges. In one direction, assuming that αK_2 is strongly $\mathbb{Z}_\ell$-connected, it follows from Proposition 2.4 that $\alpha \geqslant \ell-1$. In the other direction, assuming $\alpha \geqslant \ell - 1$, by Observation 2.3 it suffices to show that $(\ell-1)K_2$ is strongly $\mathbb{Z}_\ell$-connected. For any mapping $\theta : \{v_1, v_2\} \to \mathbb{Z}_\ell$ satisfying that $\theta(v_1) + \theta(v_2) \equiv \ell - 1 \pmod{\ell}$, let k be an integer with $0 \leqslant k \leqslant \ell-1$ and $k \equiv \theta(v_1) \pmod{\ell}$. We can orient k edges from v_1 to v_2 and $\ell-1-k$ edges from v_2 to v_1, and such an orientation is called D. Note that $d^+_D(v_1) + d^+_D(v_2) = \ell - 1$ and for each $i \in \{1,2\}$, $d^+_D(v_i) \equiv \theta(v_i) \pmod{\ell}$. Therefore, $(\ell-1)K_2$ is strongly $\mathbb{Z}_\ell$-connected. □

2.1 Contraction and lifting

To *contract* an edge uv of a graph is an operation to identify the endpoints u and v, and then delete the resulting loop. Moreover, to contract a connected subgraph H of G is an operation to contract all the edges of H, and we denote the new graph by G/H.

Observation 2.6. *Given a graph G and its connected subgraph H, let $G' = G/H$ and let w denote the new vertex obtained from contracting H. For any $(2\ell, \beta)$-boundary of G, a mapping $\beta' : V(G') \to \{0, \pm 1, \ldots, \pm \ell\}$ is defined as follows: $\beta'(w) \equiv \sum_{v \in V(H)} \beta(v) \pmod{2\ell}$ and for any vertex $v \in V(G') \setminus \{w\}$, $\beta'(v) = \beta(v)$. Then it is a $(2\ell, \beta')$-boundary of G'.*

Proof. Since $\sum_{v\in V(G')} \beta'(v) \equiv \sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2\ell}$,

$$\beta'(w) \equiv \sum_{v\in V(H)} \beta(v) \equiv \sum_{v \in V(H)} d_G(v) = d_{G'}(w) + \sum_{v\in V(H)} d_H(v) \equiv d_{G'}(w) \pmod{2},$$

and $\beta'(v) \equiv d_G(v) = d_{G'}(v) \pmod 2$ for $v \in V(G') \setminus \{w\}$, the mapping β' is a $(2\ell, \beta')$-boundary of G'. □

When it is clear from the context, we say that it is a *corresponding* $(2\ell, \beta')$-boundary of G/H with respect to the $(2\ell, \beta)$-boundary of G.

As contraction is a useful operation in the study of flows and orientations, we generalize Lemma 1.6 of [3] and obtain the next lemma which is a key fact in our later proofs. Note that the proof of the following lemma is an analog of the proof of Lemma 1.6 of [3] and thus we leave the details to the reader.

Lemma 2.7. *Given a graph G with a $(2\ell, \beta)$-boundary and its connected subgraph H, let $G' = G/H$. For any given $(2\ell, \beta)$-boundary of G, assume that G' has a corresponding $(2\ell, \beta')$-boundary as defined above. If H is strongly $\mathbb{Z}_\ell$-connected, then every $(2\ell, \beta')$-orientation on G' can be extended to a $(2\ell, \beta)$-orientation on G. In particular, if H and G' are both strongly $\mathbb{Z}_\ell$-connected, then G is also strongly $\mathbb{Z}_\ell$-connected.*

Given a vertex v of G, we call two adjacent edges uv, vw an *edge pair* at v. To *lift* at v is an operation to delete some edge pairs at v and for each edge pair uv, vw (being deleted), add one edge connecting u and w (allowing parallel edges) to the graph G. Sometimes, we say *to lift an edge triple* wx, xy, yz if we recursively lift edge pairs wx, xy and then wy, yz, noting that wy is created by first lifting the edge pair wx, xy, and this operation creates an edge wz in the end. Here we slightly abuse the notion of "edge" when there is no confusion about which edge we are referring to.

The lifting operation also plays an important role in the study of strongly $\mathbb{Z}_\ell$-connected graphs. Let G' be a graph obtained from G by lifting some edge pairs at a vertex v. Observe that for any $(2\ell, \beta)$-boundary of G, any $(2\ell, \beta)$-orientation on G' can be extended naturally to a $(2\ell, \beta)$-orientation on G by orienting those lifted edge pairs. We generalize a result of [3] as follows.

Proposition 2.8. [3] *Given a graph G and a vertex v of G, let G' be a graph obtained from G by lifting some edge pairs at v and let G'' be a graph obtained from G' by deleting the vertex v. Then the following statements hold.*

(1) Given a connected subgraph H of G', if H and G'/H are both strongly $\mathbb{Z}_\ell$-connected, then G is also strongly $\mathbb{Z}_\ell$-connected.

(2) If $d_{G'}(v) \geqslant \ell - 1$ and G'' is strongly $\mathbb{Z}_\ell$-connected, then G' and G are also strongly $\mathbb{Z}_\ell$-connected.

Proof. (1). By Lemma 2.7, G' is strongly $\mathbb{Z}_\ell$-connected. By the above observation, G is also strongly $\mathbb{Z}_\ell$-connected.

(2). Let $G_0 = G'/G''$. Note that G_0 is αK_2 with $\alpha \geqslant \ell - 1$. It follows from Lemma 2.5 that G_0 is strongly $\mathbb{Z}_\ell$-connected. Thus, by Lemma 2.7, G' is also strongly $\mathbb{Z}_\ell$-connected, so is G. $\square$

2.2 Vertex partition

Given a graph G, a collection of subsets of $V(G)$, denoted by $\mathcal{P} = \{P_1, \ldots, P_t\}$, is called a *partition* of $V(G)$ if it satisfies that for any distinct $i, j \in [t]$, $P_i \cap P_j = \emptyset$ and $\bigcup_{i\in[t]} P_i = V(G)$. Each P_i is called a *part* of $\mathcal{P}$. Given a graph G and a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(G)$, let $G/\mathcal{P}$ denote the graph obtained from G by identifying all the vertices of P_i for $i \in [t]$ and deleting the resulting loops.

Definition 2.9. *Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be a partition of $V(G)$.*

- *A partition $\mathcal{P}$ is said to be* trivial *if each part P_i consists of only one single vertex.*
- *A partition $\mathcal{P}$ is said to be* almost trivial *if there is one part P_j satisfying that $|P_j| = 2$ and all the other parts have exactly one vertex in each of them.*
- *If a partition is neither trivial nor almost trivial, then we call it* normal.

Both almost trivial partitions and normal partitions are called *nontrivial* partitions. In this paper, we exclude the partition with a single part $\{V(G)\}$.

Definition 2.10. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ and $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_s\}$ be two partitions of $V(G)$, we say $\mathcal{P}'$ is a* refinement *of $\mathcal{P}$, if $\mathcal{P}'$ is obtained by partitioning some P_i of $\mathcal{P}$ into smaller sets. More precisely, we have $s \geqslant t$, and for every $P'_i \in \mathcal{P}'$ there exists $P_j \in \mathcal{P}$ such that $P'_i \subseteq P_j$.*

Given a graph G and a connected subgraph H of G, let $G' = G/H$ and let x denote the vertex of G' obtained by contraction. For any partition $\mathcal{P}' = \{P_1, P_2, \ldots, P_t\}$ of $V(G')$, assuming that $x \in P_1$, we define $\mathcal{P}$ to be the partition of $V(G)$ *corresponding to* $\mathcal{P}'$ of $V(G/H)$ as follows:

$$\mathcal{P} = \{P_1 \cup V(H) \setminus \{x\}, P_2, \ldots, P_t\}.$$

Observation 2.11. *Given a graph G and its connected subgraph H, let $G' = G/H$. For any partition $\mathcal{P}'$ of $V(G')$ and the partition $\mathcal{P}$ of $V(G)$ corresponding to $\mathcal{P}'$ of $V(G')$, we know $G/\mathcal{P} = G'/\mathcal{P}'$.*

In Section 3, we aim to find some necessary conditions for small graphs to be strongly $\mathbb{Z}_\ell$-connected. Our method somehow relies on the existence of sufficiently many edge-disjoint spanning trees on the given graph. In this direction, we need the well-known Nash-Williams–Tutte Theorem.

Theorem 2.12. [Nash-Williams–Tutte Theorem] [23, 31] *A graph G contains t edge-disjoint spanning trees if and only if for any partition $\mathcal{P} = \{P_1, P_2, \ldots, P_s\}$ of $V(G)$, there are at least $t(s-1)$ edges connecting the parts of $\mathcal{P}$.*

2.3 The γ-function and Hakimi's orientation theorem

To study the $(\mathbb{Z}_\ell, \beta)$-orientation of a graph, in [16], a tool called τ-function has been introduced. Similarly, in the study of $(2\ell, \beta)$-boundary and $(2\ell, \beta)$-orientation of a graph, we find a corresponding "γ-function" with some properties.

Lemma 2.13. *For any $(2\ell, \beta)$-boundary of G, there exists an integer-valued function $\gamma : V(G) \to \{0, \pm 1, \ldots, \pm(2\ell - 1)\}$ such that*

(1) for each vertex $v \in V(G)$, $\gamma(v) \equiv \beta(v) \pmod{2\ell}$ and $\gamma(v) \equiv d(v) \pmod 2$.

(2) $\sum_{v \in V(G)} \gamma(v) = 0$.

(3) $\max_{v \in V(G)} \{\gamma(v)\} - \min_{v \in V(G)} \{\gamma(v)\} \leqslant 2\ell$.

Proof. First of all, given a $(2\ell, \beta)$-boundary of G, we can easily find a mapping $\gamma : V(G) \to \{0, \pm 1, \ldots, \pm(2\ell - 1)\}$ satisfying that $\gamma(v) \equiv \beta(v) \pmod{2\ell}$ and $\gamma(v) \equiv d(v) \pmod 2$ for each vertex $v \in V(G)$.

For the second condition, we assume to the contrary that $\sum_{v\in V(G)}\gamma(v)\neq 0$ and choose $|\sum_{v\in V(G)}\gamma(v)|$ to be minimized. By symmetry, suppose that $\sum_{v\in V(G)}\gamma(v)>0$, and thus there exists at least one vertex, which is called v^*, such that $\gamma(v^*)>0$. Furthermore, as $\gamma(v)\equiv\beta(v) \pmod{2\ell}$, $\sum_{v\in V(G)}\gamma(v)$ must be a multiple of 2ℓ and so $\sum_{v\in V(G)}\gamma(v)\geqslant 2\ell$. We define a new mapping $\gamma':V(G)\to\{0,\pm1,\ldots,\pm(2\ell-1)\}$ such that $\gamma'(v^*)=\gamma(v^*)-2\ell$ and $\gamma'(v)=\gamma(v)$ for every $v\neq v^*$. Note that such γ' satisfies Condition (1). However,

$$|\sum_{v\in V(G)}\gamma'(v)|=|\sum_{v\neq v^*}\gamma(v)+(\gamma(v^*)-2\ell)|=|\sum_{v\in V(G)}\gamma(v)-2\ell|<|\sum_{v\in V(G)}\gamma(v)|,$$

a contradiction.

For the last one, assume that there exists a function γ satisfying Conditions (1) and (2) but not (3), which means $\max_{v\in V(G)}\{\gamma(v)\}-\min_{v\in V(G)}\{\gamma(v)\}>2\ell$. We choose such a counterexample to satisfy the following conditions in this order of priority:

(i) $|\max\limits_{v\in V(G)}\{\gamma(v)\}-\min\limits_{v\in V(G)}\{\gamma(v)\}|$ is minimized;

(ii) $|\{x:\gamma(x)=\max\limits_{v\in V(G)}\{\gamma(v)\}\}|+|\{y:\gamma(y)=\min\limits_{v\in V(G)}\{\gamma(v)\}\}|$ is as small as possible.

Let x_1 and x_2 be two vertices satisfying that $\gamma(x_1)=\max_{v\in V(G)}\{\gamma(v)\}$ and $\gamma(x_2)=\min_{v\in V(G)}\{\gamma(v)\}$. So we have that $\gamma(x_1)-\gamma(x_2)>2\ell$. We define a new mapping $\gamma':V(G)\to\{0,\pm1,\ldots,\pm(2\ell-1)\}$ such that

$$\gamma'(x_1)=\gamma(x_1)-2\ell,\ \gamma'(x_2)=\gamma(x_2)+2\ell,\ \text{and}\ \gamma'(v)=\gamma(v)\ \text{for all the other}\ v\in V(G).$$

Clearly, γ' also satisfies Conditions (1) and (2). Moreover, since

$$\gamma(x_1)=\max_{v\in V(G)}\{\gamma(v)\}>\gamma(x_2)+2\ell=\gamma'(x_2)\ \text{and}\ \gamma(x_2)=\min_{v\in V(G)}\{\gamma(v)\}<\gamma(x_1)-2\ell=\gamma'(x_1),$$

we have $x_1\notin\{u:\gamma'(u)=\min\limits_{v\in V(G)}\{\gamma'(v)\}\}$ and $x_2\notin\{u:\gamma'(u)=\max\limits_{v\in V(G)}\{\gamma'(v)\}\}$, otherwise we obtain a contradiction to Condition (i). In this case, $|\{x:\gamma'(x)=\max_{v\in V(G)}\{\gamma'(v)\}\}|+|\{y:\gamma'(y)=\min_{v\in V(G)}\{\gamma'(v)\}\}|$ is smaller, contradicting Condition (ii). □

It is easy to see that any γ-orientation with a function γ defined in Lemma 2.13 is indeed a $(2\ell,\beta)$-orientation on G. To determine the existence of the "γ-orientation" and thus $(2\ell,\beta)$-orientation of G, we also need the following Hakimi's orientation theorem.

Theorem 2.14. [Hakimi's orientation theorem][5] *Let G be a graph and γ be a mapping $\gamma:V(G)\to\mathbb{Z}$ satisfying that $\gamma(v)\equiv d(v) \pmod 2$ for each vertex $v\in V(G)$ and $\sum_{v\in V(G)}\gamma(v)=0$. The following statements are equivalent.*

(i) There exists an orientation D on G such that $d^+(v)-d^-(v)=\gamma(v)$ for $v\in V(G)$.

(ii) *Every subset S of $V(G)$ satisfies* $|\sum\limits_{v\in S}\gamma(v)| \leqslant d(S)$.

We may define those vertex subsets not satisfying the second statement of Theorem 2.14 to be "bad", which is formally stated as follows.

Definition 2.15. *Given a graph G and a mapping $\gamma : V(G) \to \{0, \pm 1, \ldots, \pm(2\ell - 1)\}$ with $\gamma(v) \equiv d(v) \pmod 2$ for each $v \in V(G)$ and $\sum_{v\in V(G)}\gamma(v) = 0$, a vertex subset $S \subset V(G)$ is called* bad *with respect to γ if $|\sum_{v\in S}\gamma(v)| > d(S)$.*

Note that for any subset $S \subset V(G)$, $|\sum_{v\in S^c}\gamma(v)| = |-\sum_{v\in S}\gamma(v)|$. Since $d(S) = d(S^c)$, if S is a bad set, then its complement S^c is also a bad set. For example, for graphs on three vertices $T_{a,b,c}$, we may only consider one type of bad sets (i.e., singleton vertex sets).

3 Small strongly $\mathbb{Z}_\ell$-connected graphs

In this section, for mathematical induction bases, we shall provide some sufficient conditions, for example, high edge-connectivity and minimum degree, under which some small graphs (on at most 4 vertices) are proved to be strongly $\mathbb{Z}_\ell$-connected for $\ell \geqslant 3$. We have seen in Lemma 2.5 that αK_2 is strongly $\mathbb{Z}_\ell$-connected if and only if $\alpha \geqslant \ell - 1$. Now we consider the graphs on three vertices.

Lemma 3.1. *Let $T_{a,b,c}$ be a multi-triangle on the vertex set $\{v_1, v_2, v_3\}$ with $\mu(v_1v_2) = a$, $\mu(v_2v_3) = b$, and $\mu(v_3v_1) = c$. If $a + b + c \geqslant 2\ell - 2$ and $\delta(T_{a,b,c}) \geqslant \ell - 1$, then $T_{a,b,c}$ is strongly $\mathbb{Z}_\ell$-connected.*

Proof. By Observation 2.3, it is enough to prove the claim with the assumption $a+b+c = 2\ell - 2$ and $\delta(G) \geqslant \ell - 1$. Suppose to the contrary that G is a $T_{a,b,c}$ with $a+b+c = 2\ell-2$, $\delta(G) \geqslant \ell - 1$ but not strongly $\mathbb{Z}_\ell$-connected. To obtain a contradiction, by Hakimi's orientation theorem (Theorem 2.14), it suffices to prove the next claim.

Claim. *Given a $(2\ell, \beta)$-boundary of G, assume that G has no $(2\ell, \beta)$-orientation. Then for any function $\gamma : V(G) \to \{0, \pm 1, \ldots, \pm(2\ell - 1)\}$ with $\gamma(v_i) \equiv \beta(v_i) \pmod{2\ell}$, $\gamma(v_i) \equiv d(v_i) \pmod 2$ for every $i \in \{1, 2, 3\}$, we have*

$$|\gamma(v_i)| \leqslant d(v_i),\ \forall i \in \{1, 2, 3\}.$$

Suppose to the contrary, without loss of generality, G has a bad set $S = \{v_1\}$ with $|\gamma(v_1)| > d(v_1)$. There are two possibilities that either $\gamma(v_1) > d(v_1)$ or $-\gamma(v_1) > d(v_1)$.

We first assume that $\gamma(v_1) > d(v_1)$. Then we have

$$2\ell - 1 \geqslant \gamma(v_1) \geqslant d(v_1) + 2 \geqslant \ell + 1, \tag{1}$$

and

$$-d(v_1) \leqslant -\ell + 1 \leqslant \gamma(v_1) - 2\ell \leqslant 0. \tag{2}$$

Since $a+b+c = 2\ell-2$ and $\delta(G) \geqslant \ell-1$, by Nash-Williams–Tutte Theorem (Theorem 2.12) it is easy to verify that G contains $\ell - 1$ edge-disjoint spanning trees. Let $\mathcal{T}$ be the set of those $\ell - 1$ edge-disjoint spanning trees. Note that $\bigcup_{T\in\mathcal{T}} E(T) = E(G)$. At the vertex v_1, assume that there are s edge-disjoint spanning trees T of $\mathcal{T}$ such that $d_T(v_1) = 2$ (denoted by T^2) and t edge-disjoint spanning trees T such that $d_T(v_1) = 1$ (denoted by T^1). Considering the relation $2s+t = d(v_1)$ and $s+t = \ell-1$, we have that $s = d(v_1)-\ell+1$ and $t = 2\ell - d(v_1) - 2$. Those two values are both non-negative following from Condition (1). Now we lift $d(v_1) - \ell + 1$ pairs of edges from all those T^2's at the vertex v_1. Note that there are $2\ell - d(v_1) - 2$ edges left at vertex v_1. By Conditions (1) and (2), we can orient $2\ell - \gamma(v_1) \geqslant 0$ edges into v_1 and the rest $\gamma(v_1) - d(v_1) - 2 \geqslant 0$ edges one-in-one-out in pairs. Therefore, $d^+(v_1) - d^-(v_1) = \gamma(v_1) - 2\ell \equiv \beta(v_1) \pmod{2\ell}$.

We now assume that $-\gamma(v_1) > d(v_1)$ and in this case we have

$$2\ell - 1 \geqslant -\gamma(v_1) \geqslant d(v_1) + 2 \geqslant \ell + 1, \tag{3}$$

and

$$-d(v_1) \leqslant -\ell + 1 \leqslant -\gamma(v_1) - 2\ell \leqslant 0. \tag{4}$$

Similarly, let $\mathcal{T}$ be the set of $\ell - 1$ edge-disjoint spanning trees. we can also compute the number of edge-disjoint spanning trees T^2 of $\mathcal{T}$ such that $d_{T^2}(v_1) = 2$ is $d(v_1) - \ell + 1$ and the number of edge-disjoint spanning trees T^1 such that $d_{T^1}(v_1) = 1$ is $2\ell - d(v_1) - 2$ as in the previous case. Both of these two values are non-negative by Condition (3). Thus we can lift $d(v_1) - \ell + 1$ pairs of edges from all those T^2's at the vertex v_1 and there are $2\ell - d(v_1) - 2$ edges left at vertex v_1. By Conditions (3) and (4), we are able to orient $2\ell + \gamma(v_1)$ edges out from v_1 and the left $-(\gamma(v_1) + d(v_1)) - 2$ edges one-in-one-out in pairs. Therefore, we have $d^+(v_1) - d^-(v_1) \equiv \beta(v_1) \pmod{2\ell}$.

In both cases, we have achieved the boundary at v_1. We denote by G' the resulting graph obtained from G by lifting those edge pairs (of T^2's) and deleting v_1. Note that G' has two vertices and $2\ell - 2 - d(v_1) + (d(v_1) - \ell + 1) = \ell - 1$ edges. By Lemma 2.5, G' is strongly $\mathbb{Z}_\ell$-connected. Thus, by Proposition 2.8, for any $(2\ell, \beta)$-boundary, G has a $(2\ell, \beta)$-orientation, a contradiction. $\square$

By Nash-Williams–Tutte Theorem, a multi-triangle $T_{a,b,c}$ contains $\ell-1$ edge-disjoint spanning trees if and only if $a + b + c \geqslant 2\ell - 2$ and $\delta(G) \geqslant \ell - 1$. Thus following from Proposition 2.4 and Lemma 3.1, we have the next corollary.

Corollary 3.2. *A multi-triangle $T_{a,b,c}$ contains $\ell - 1$ edge-disjoint spanning trees if and only if $T_{a,b,c}$ is strongly $\mathbb{Z}_\ell$-connected.*

Note that there is an alternate proof of Lemma 3.1 and here we give this relatively complex proof for expressing the proof ideas of the following lemma on four vertices.

Lemma 3.3. *Let G be a graph on four vertices. If G contains $\ell-1$ edge-disjoint spanning trees and $e(G) \geqslant 3\ell-2$, then G is strongly $\mathbb{Z}_\ell$-connected.*

Proof. By Observation 2.3, it suffices to prove the claim assuming that G contains $\ell-1$ edge-disjoint spanning trees and $e(G) = 3\ell-2$. Assume to the contrary that G is not strongly $\mathbb{Z}_\ell$-connected. Let $V(G) = \{v_1, v_2, v_3, v_4\}$. As G contains $\ell-1$ edge-disjoint spanning trees, for any proper subset $S \subset V(G)$, $d(S) \geqslant \ell-1$.

Claim 3.3.1. *G is $(\ell+1)$-edge-connected. In particular, $\delta(G) \geqslant \ell+1$.*

Proof. Assume to the contrary that G is not $(\ell+1)$-edge-connected and we discuss two cases: (1). There is a vertex v such that $d(v) = \delta(G) \leqslant \ell$; (2). There is a subset S of size 2 such that $d(S) \leqslant \ell$. In the first case, noting that $e(G) = 3\ell-2$, $G-v$ has at least $2\ell-2$ edges and contains $\ell-1$ edge-disjoint spanning trees. By Corollary 3.2, $G-v$ is strongly $\mathbb{Z}_\ell$-connected. It follows from Proposition 2.8 (2) that G is strongly $\mathbb{Z}_\ell$-connected, a contradiction. In the second case, without loss of generality, we assume that $S = \{v_1, v_2\}$. As there are at least $2\ell-2$ edges in $E(G)\setminus[S, S^c]$, there are at least $\ell-1$ edges in either $G[S]$ or $G[S^c]$, without loss of generality, say $G[S]$. It implies that $G[S]$ is strongly $\mathbb{Z}_\ell$-connected. Let $H = G/G[S]$. Noting that H contains $\ell-1$ edge-disjoint spanning trees, by Corollary 3.2 H is strongly $\mathbb{Z}_\ell$-connected. Therefore, by Lemma 2.7, G is strongly $\mathbb{Z}_\ell$-connected, a contradiction. $\Diamond$

Claim 3.3.2. *The maximum degree $\Delta(G)$ is at most $3\ell-7$.*

Proof. Suppose to the contrary that $\Delta(G) \geqslant 3\ell-6$. Let v_1 be the vertex of degree $\Delta(G)$. Since $e(G) = 3\ell-2$, there are at most 4 edges among the remaining three vertices. Thus the average degree among vertices v_2, v_3 and v_4 is $\frac{3\ell-6+4\times 2}{3} = \ell + \frac{2}{3} < \ell+1$, which contradicts the fact that $\delta(G) \geqslant \ell+1$. $\Diamond$

We next prove a similar claim as in Lemma 3.1.

Claim 3.3.3. *Given a $(2\ell, \beta)$-boundary of G, if G has no $(2\ell, \beta)$-orientation, then for any function $\gamma : V(G) \to \{0, \pm 1, \ldots, \pm(2\ell-1)\}$ with $\gamma(v_i) \equiv \beta(v_i) \pmod{2\ell}$, $\gamma(v_i) \equiv d(v_i) \pmod 2$ for any $i \in [4]$, we have*

$$|\gamma(v_i)| \leqslant d(v_i), \ \forall i \in [4].$$

Proof. Without loss of generality, suppose that G has a bad set $S = \{v_1\}$ and thus $|\gamma(v_1)| > d(v_1)$. Here, we only consider the case $\gamma(v_1) > d(v_1)$. The other case $-\gamma(v_1) > d(v_1)$ can be discussed similarly, so we omit the details. Note that in this case,

$$2\ell-1 \geqslant \gamma(v_1) > d(v_1) \geqslant \ell+1 \text{ and } -d(v_1) \leqslant -(\ell+1) < \gamma(v_1) - (2\ell-1) \leqslant 0.$$

In particular, $d(v_1) \leqslant 2\ell-2$.

Let $\mathcal{T}$ be a set of the $\ell - 1$ edge-disjoint spanning trees of G. Since $e(G) = 3\ell - 2$ and every spanning tree has exactly three edges, we have $3\ell - 2 - 3(\ell - 1) = 1$ edge not contained in any spanning tree of $\mathcal{T}$. We assume that at the vertex v_1, there are a spanning trees having degree one, b spanning trees having degree two, and c spanning trees having degree three. We have $a + b + c = \ell - 1$ and $a + 2b + 3c \in \{d(v_1), d(v_1) - 1\}$, solving which we know $a \geqslant c$.

When some spanning tree T_i has degree two at vertex v_1, we can directly lift this pair of edges and obtain a spanning tree T_i' of $G - v_1$; for any T_i having degree three at vertex v_1, as $a \geqslant c$, we can always find another spanning tree T_j with degree one at vertex v_1. Applying some proper switching edges of T_i and T_j, we can obtain two new spanning trees T_i' and T_j' both of which have degree two at v_1. Then we lift these two pairs of edges of T_i' and T_j'.

So we can lift $2b + 4c \in \{2d(v_1) - (2\ell - 2), 2d(v_1) - 2\ell\}$ edges of G and orient the remaining either $(2\ell - 2) - d(v_1)$ or $2\ell - d(v_1)$ edges. More precisely, we orient $2\ell - \gamma(v_1)$ edges into v_1 and the left edges choose one-in-one-out in pairs, and thus we achieve the $(2\ell, \beta)$-boundary $\beta(v_1)$. Deleting v_1 from G, the resulting graph G' (on three vertices) has either $2\ell - 1$ or $2\ell - 2$ many edges (followed from $3\ell - 2 - d(v_1) + k$, where $k \in \{d(v_1) - (\ell - 1), d(v_1) - \ell\}$, and thus by Lemma 3.1 it is strongly $\mathbb{Z}_\ell$-connected. By Proposition 2.8, G has a $(2\ell, \beta)$-orientation, a contradiction. $\Diamond$

By Claim 3.3.3, we know the set containing only one vertex cannot be a bad set corresponding to a given γ-function and thus a given $(2\ell, \beta)$-boundary. So the bad set must be in the form of $\{v_i, v_j\}$ for $i \neq j$. Given a $(2\ell, \beta)$-boundary, assume that there is a function γ satisfying the conditions in Lemma 2.13. Without loss of generality, we may also assume that

$$\max_{v \in V(G)} \{\gamma(v)\} = \gamma(v_1) \geqslant \gamma(v_2) \geqslant \gamma(v_3) \geqslant \gamma(v_4) = \min_{v \in V(G)} \{\gamma(v)\}.$$

So $\gamma(v_1) - \gamma(v_4) \leqslant 2\ell$. By symmetry, suppose that $\max_{v \in V(G)}\{\gamma(v)\} \geqslant -\min_{v \in V(G)}\{\gamma(v)\}$. In this case, we claim that $-\ell \leqslant \gamma(v_4) \leqslant \gamma(v_3) < 0$. Otherwise, if $\gamma(v_3) \geqslant 0$, then $\gamma(v_1) \leqslant \ell$. As G is $(\ell + 1)$-edge-connected, any two-vertex set $\{v_i, v_j\}$ satisfies that $|\gamma(v_i) + \gamma(v_j)| \leqslant \ell < \ell + 1 \leqslant d(\{v_i, v_j\})$ and thus it is not a bad set, a contradiction.

Claim 3.3.4. *For $\{v_i, v_j\} \subseteq V(G)$ with $\gamma(v_i) + \gamma(v_j) \geqslant 0$, we have*

$$\textit{either} \;\; \gamma(v_i) + \gamma(v_j) \leqslant d(\{v_i, v_j\}) \;\; \textit{or} \;\; |\gamma(v_i) + \gamma(v_j) - 2\ell| \leqslant d(\{v_i, v_j\}).$$

Proof. We prove this claim by contradiction, that is to say, there exists a bad set $S = \{v_i, v_j\}$ such that $\gamma(v_i) + \gamma(v_j) > d(\{v_i, v_j\})$ and $|\gamma(v_i) + \gamma(v_j) - 2\ell| > d(\{v_i, v_j\})$. We claim that $\gamma(v_i) + \gamma(v_j) - 2\ell < 0$, as if not, $\gamma(v_i) + \gamma(v_j) - 2\ell > d(\{v_i, v_j\})$ and thus $2\min_{v \in V(G)}\{\gamma(v)\} \leqslant \sum_{v \in S^c} \gamma(v) = -(\gamma(v_i) + \gamma(v_j)) < -d(\{v_i, v_j\}) - 2\ell < -2\ell$, and it implies that $\gamma(v_4) = \min_{v\in V(G)}\{\gamma(v)\} < -\ell$, a contradiction. Thus

$$\gamma(v_i) + \gamma(v_j) > d(\{v_i, v_j\}) \text{ and } -(\gamma(v_i) + \gamma(v_j) - 2\ell) > d(\{v_i, v_j\}),$$

solving which we obtain that $d(\{v_i, v_j\}) < \ell$, contradicting the fact that G is $(\ell+1)$-edge-connected. $\Diamond$

Claim 3.3.5. *For $\{v_i, v_j\} \subseteq V(G)$ with $\gamma(v_i) + \gamma(v_j) \geqslant 0$, we have $\gamma(v_i) + \gamma(v_j) \leqslant 2\ell$. Moreover, if $d(\{v_i, v_j\}) \geqslant 2\ell - 1$, then*

$$\gamma(v_i) + \gamma(v_j) \leqslant d(\{v_i, v_j\}) \quad \textit{and} \quad |\gamma(v_i) + \gamma(v_j) - 2\ell| \leqslant d(\{v_i, v_j\}).$$

Proof. Let $S = \{v_i, v_j\}$. For $\min_{v\in V(G)}\{\gamma(v)\} \geqslant -\ell$, we have

$$\gamma(v_i) + \gamma(v_j) = -\sum_{v\in S^c} \gamma(v) \leqslant -2 \min_{v\in V(G)}\{\gamma(v)\} \leqslant 2\ell.$$

Moreover, if $d(\{v_i, v_j\}) \geqslant 2\ell - 1$, by the fact that the two values $\gamma(v_i) + \gamma(v_j)$ and $d(\{v_i, v_j\})$ have the same parity, then $\gamma(v_i) + \gamma(v_j) \leqslant d(\{v_i, v_j\})$. Since $\gamma(v_i) + \gamma(v_j) \geqslant 0$ and $\gamma(v_i) + \gamma(v_j) \leqslant 2\ell$, we have $|\gamma(v_i) + \gamma(v_j) - 2\ell| \leqslant 2\ell$. As $d(\{v_i, v_j\}) \geqslant 2\ell - 1$, by the parity $|\gamma(v_i) + \gamma(v_j) - 2\ell| \leqslant d(\{v_i, v_j\})$. $\Diamond$

Claim 3.3.6. *With respect to γ, the only bad set is $\{v_1, v_2\}$ (equivalently, $\{v_3, v_4\}$).*

Proof. We discuss two cases based on the value of $\gamma(v_2)$.

Case 1: If $\gamma(v_2) \leqslant 0$, then $\gamma(v_1) > 0 \geqslant \gamma(v_2) \geqslant \gamma(v_3) \geqslant \gamma(v_4)$. Since $\sum_{i=1}^{4} \gamma(v_i) = 0$ and $\gamma(v_1) - \gamma(v_4) \leqslant 2\ell$, we have

$$0 \leqslant \gamma(v_1)+\gamma(v_4) \leqslant \gamma(v_1)+\gamma(v_3) \leqslant \frac{1}{2}(\gamma(v_1)+\gamma(v_3))+\frac{1}{2}(\gamma(v_1)+\gamma(v_2)) = \frac{1}{2}(\gamma(v_1)-\gamma(v_4)) \leqslant \ell.$$

Thus $\{v_1, v_3\}$ and $\{v_1, v_4\}$ are not bad sets and the only possible bad set is $\{v_1, v_2\}$.

Case 2: If $\gamma(v_2) > 0$, then $\gamma(v_1) \geqslant \gamma(v_2) > 0 \geqslant \gamma(v_3) \geqslant \gamma(v_4)$. We first assume that $S = \{v_i, v_j\}$ satisfies that $\gamma(v_i) > 0$ and $\gamma(v_j) \leqslant 0$, i.e., $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Then

$$-\ell \leqslant \min_{v\in V(G)}\{\gamma(v)\} \leqslant \gamma(v_i) + \gamma(v_j) = -\sum_{v\in S^c} \gamma(v) \leqslant -\min_{v\in V(G)}\{\gamma(v)\} \leqslant \ell.$$

As G is $(\ell+1)$-edge-connected, such $S = \{v_i, v_j\}$ is not a bad set. The only possible bad set is $\{v_1, v_2\}$. $\Diamond$

It follows from Claim 3.3.6 that $\gamma(v_1)+\gamma(v_2) > d(\{v_1, v_2\})$. Moreover, by Claim 3.3.5, $d(\{v_1, v_2\}) < 2\ell-1$ (i.e., $d(\{v_1, v_2\}) \leqslant 2\ell-2$). Since $d(\{v_1, v_2\})+d(\{v_1, v_3\})+d(\{v_1, v_4\}) = 2e(G) = 6\ell - 4 > 3(2\ell - 2)$, we have either $d(\{v_1, v_3\}) \geqslant 2\ell - 1$ or $d(\{v_1, v_4\}) \geqslant 2\ell - 1$.

- If $d(\{v_1, v_3\}) \geqslant 2\ell - 1$, we define a new γ'-function as follows:

$$\gamma'(v_1) = \gamma(v_1) - 2\ell,\ \gamma'(v_4) = \gamma(v_4) + 2\ell,\ \text{and } \gamma'(v_i) = \gamma(v_i) \text{ for } i \in \{2, 3\}.$$

  Now we know that, with respect to γ', $\{v_1, v_2\}$ is not a bad set by Claim 3.3.4 and $\{v_1, v_3\}$ is not a bad set by Claim 3.3.5. Meanwhile, as $|\gamma'(v_1) + \gamma'(v_4)| = |\gamma(v_1) - 2\ell + \gamma(v_4) + 2\ell| \leqslant d(\{v_1, v_4\})$, $\{v_1, v_4\}$ is not a bad set. By Theorem 2.14 (Hakimi's orientation theorem), G admits a γ'-orientation and thus a $(2\ell, \beta)$-orientation, a contradiction.

- If $d(\{v_1, v_4\}) \geqslant 2\ell - 1$, similarly we define

$$\gamma'(v_1) = \gamma(v_1) - 2\ell,\ \gamma'(v_3) = \gamma(v_3) + 2\ell,\ \text{and } \gamma'(v_i) = \gamma(v_i) \text{ for } \{2, 4\}.$$

  Thus, with respect to γ', $\{v_1, v_2\}$ is not a bad set by Claim 3.3.4 and $\{v_1, v_4\}$ is not a bad set by Claim 3.3.5. Meanwhile, $|\gamma'(v_1) + \gamma'(v_3)| = |\gamma(v_1) - 2\ell + \gamma(v_3) + 2\ell| \leqslant d(\{v_1, v_3\})$, thus $\{v_1, v_3\}$ is not a bad set. It leads to a contradiction by Theorem 2.14.

In conclusion, G has no bad set with respect to γ', thus by Hakimi's orientation theorem, G is strongly $\mathbb{Z}_\ell$-connected, a contradiction. It completes the proof of the lemma. $\square$

We remark that the condition $e(G) \geqslant 3\ell - 2$ in Lemma 3.3 is somehow necessary as $(2t-1)K_4$ (a multi-K_4 with $\mu(uv) = 2t - 1$ for each pair uv) is not strongly $\mathbb{Z}_{4t-1}$-connected. Applying Nash-Williams–Tutte Theorem (Theorem 2.12) and Lemma 3.3, we have the next result.

Lemma 3.4. *If G is a graph with $v(G) = 4$, $e(G) \geqslant 3\ell - 2$, $\mu(G) \leqslant \ell - 2$, and $\delta(G) \geqslant \ell - 1$, then G is strongly $\mathbb{Z}_\ell$-connected.*

Proof. For any graph G on four vertices satisfying $\mu(G) \leqslant \ell - 2$ and $\delta(G) \geqslant \ell - 1$, it can be readily verified that G satisfies the conditions in the Nash-Williams–Tutte Theorem and thus G has $\ell - 1$ edge-disjoint spanning trees. Since $e(G) \geqslant 3\ell - 2$, Lemma 3.3 implies that G is strongly $\mathbb{Z}_\ell$-connected. $\square$

4 Strongly $\mathbb{Z}_{2k}$-connected graphs

In this section, we will prove a stronger result (Theorem 4.6 below), which implies Theorem 1.8, using the notion of weight functions introduced in [3].

Given a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ of $V(G)$ and a positive integer k, we define the *k-weight function* of $\mathcal{P}$ as follows:

$$\omega_G^k(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - (6k - 4)t + (12k - 12) \tag{5}$$

and $\omega^k(G) = \min_{\mathcal{P}} \{\omega_G^k(\mathcal{P})\}$.

Lemma 4.1. *Let k be a positive integer. Given a graph G and a connected subgraph H of G, $\omega^k(G/H) \geqslant \omega^k(G)$.*

Proof. Let $G' = G/H$. Assume that $\mathcal{P}_0'$ is a partition of $V(G')$ satisfying that $\omega^k(G') = \omega_{G'}^k(\mathcal{P}_0')$ and let $\mathcal{P}_0$ be the partition of $V(G)$ corresponding to the partition $\mathcal{P}_0'$ of $V(G')$. We have that $\omega^k(G') = \omega_{G'}^k(\mathcal{P}_0') = \omega_G^k(\mathcal{P}_0) \geqslant \min_{\mathcal{P}}\{\omega_G^k(\mathcal{P})\} = \omega^k(G)$. □

Proposition 4.2. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$ with $|P_1| \geqslant 2$, let $H = G[P_1]$ and let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_s\}$ be a partition of $V(H)$. Then the partition $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ of $V(G)$ satisfies that*

$$\omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) = \omega_G^k(\mathcal{P}) + \omega_H^k(\mathcal{Q}) - (6k-8).$$

Proof. By the definition, $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a refinement of $\mathcal{P}$, thus

$$\begin{aligned}
\omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) &= \sum_{i=1}^{s} d_G(Q_i) + \sum_{i=2}^{t} d_G(P_i) - (6k-4)(t-1+s) + (12k-12) \\
&= \sum_{i=1}^{s} d_H(Q_i) + d_G(P_1) + \sum_{i=2}^{t} d_G(P_i) - (6k-4)(t-1+s) + (12k-12) \\
&= [\sum_{i=1}^{s} d_H(Q_i) - (6k-4)s + (12k-12)] \\
&\quad + [\sum_{i=1}^{t} d_G(P_i) - (6k-4)t + (12k-12)] + (6k-4) - (12k-12) \\
&= \omega_H^k(\mathcal{Q}) + \omega_G^k(\mathcal{P}) - (6k-8).
\end{aligned}$$

□

Definition 4.3. *Let k be an integer with $k \geqslant 2$. Let*

$$\mathcal{N}_k := \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\} \cup \{T_{a,b,c} : 3k-1 \leqslant a+b+c \leqslant 4k-3, \delta(T_{a,b,c}) \geqslant 2k-1\}.$$

A graph G has a troublesome partition *with respect to strong $\mathbb{Z}_{2k}$-connectivity if there is a partition $\mathcal{P}$ of $V(G)$ such that $G/\mathcal{P} \in \mathcal{N}_k$.*

Note that each graph in $\mathcal{N}_k$ is not strongly $\mathbb{Z}_{2k}$-connected.

Observation 4.4. *For $\alpha K_2, T_{a,b,c} \in \mathcal{N}_k$, $\omega^k(\alpha K_2) \leqslant 4k-8$ and $\omega^k(T_{a,b,c}) \leqslant 2k-6$.*

Let $\mathcal{S}_k := \{(2k-1)K_2\} \cup \{T_{a,b,c} : a+b+c = 4k-2, \delta(T_{a,b,c}) \geqslant 2k-1\}$. The next result follows from Lemma 2.5 and Lemma 3.1 when $\ell = 2k$.

Proposition 4.5. *Each graph in $\mathcal{S}_k$ is strongly $\mathbb{Z}_{2k}$-connected.*

Based on the notions of weight functions and troublesome partitions, we state the main theorem as follows.

Theorem 4.6. *Given a planar graph G and an integer k with $2 \leqslant k \leqslant 4$, if $\omega^k(G) \geqslant 0$, then either G is strongly $\mathbb{Z}_{2k}$-connected or G has a troublesome partition.*

Next we use Theorem 4.6 to prove the following result (Theorem 4.7), which implies the main theorem (Theorem 1.8). The detailed proof of Theorem 4.6 is organized as follows: in Subsection 4.1, we provide some forbidden configurations in a minimum counterexample to the theorem for general k, and in Subsections 4.2, 4.3, and 4.4, we prove that for $k = 2, 3, 4$ the minimum counterexample must contain one element from the forbidden configuration set and obtain a contradiction.

Theorem 4.7. *Given an integer k with $2 \leqslant k \leqslant 4$, let G be a planar graph and x be a vertex of G. Assume that $2k-1 \leqslant d_G(x) \leqslant 8k-5$ and except $[\{x\}, V(G) \setminus \{x\}]$ every other cut $[X, X^c]$ for $X \subseteq V(G)$ has size at least $6k-4$. Then both $G - x$ and G are strongly $\mathbb{Z}_{2k}$-connected.*

Proof. By Lemma 2.5, this theorem trivially holds when $|V(G)| = 2$, and so assume $|V(G)| \geq 3$. Let $H = G - x$. Note that $|V(H)| \geqslant 2$. Moreover, H is connected. Assume not and let H_1 and H_2 denote two components of H. Since $|[V(H_1), V(H_1)^c]| \geqslant 6k-4$ and $|[V(H_2), V(H_2)^c]| \geqslant 6k-4$, we have $d_G(x) \geqslant 12k-8$, contradicting the assumption that $d_G(x) \leqslant 8k-5$.

Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be an arbitrary partition of $V(H)$. Note that $d_G(P_i) \geqslant 6k-4$ for each $i \in \{1, \ldots, t\}$ and $d_G(x) \leqslant 8k-5$. We have that

$$\omega_H^k(\mathcal{P}) = \sum_{i=1}^{t} d_H(P_i) - (6k-4)t + (12k-12) = \sum_{i=1}^{t} d_G(P_i) - d_G(x) - (6k-4)t + (12k-12) \geqslant 4k-7.$$

It implies that $\omega^k(H) \geqslant 4k-7 \geqslant 0$. Since by Observation 4.4 each graph of $\mathcal{N}_k$ has its weight value at most $4k-8$, H cannot have a troublesome partition. By Theorem 4.6, H is strongly $\mathbb{Z}_{2k}$-connected.

Moreover, note that G/H is αK_2 where $\alpha \geqslant 2k-1$, which is strongly $\mathbb{Z}_{2k}$-connected. By Lemma 2.7, G is also strongly $\mathbb{Z}_{2k}$-connected. □

Let G be a $(6k-4)$-edge-connected planar graph. If G contains a vertex of degree at most $8k-5$, then we can directly apply Theorem 4.7 to conclude that G is strongly $\mathbb{Z}_{2k}$-connected. Otherwise, we may add one vertex x and add $2k-1$ edges connecting it to the vertices of $V(G)$ (preserving the planarity). Note that now the resulting graph $G+x$ satisfies the conditions in Theorem 4.7, and thus G is strongly $\mathbb{Z}_{2k}$-connected. Hence, Theorem 1.8 (restated below) is a particular case of Theorem 4.7.

Theorem 1.8. *Given an integer k with $2 \leqslant k \leqslant 4$, every $(6k-4)$-edge-connected planar graph is strongly $\mathbb{Z}_{2k}$-connected.*

4.1 Properties of the minimum counterexample to Theorem 4.6

In the sequel, let G be a minimum counterexample to Theorem 4.6 with respect to $v(G) + e(G)$. That is to say, $\omega^k(G) \geqslant 0$, G is not strongly $\mathbb{Z}_{2k}$-connected and for any partition $\mathcal{P}$ of $V(G)$, $G/\mathcal{P} \notin \mathcal{N}_k$. Moreover, by the minimality of G, for any planar graph H such that $v(H) + e(H) < v(G) + e(G)$ and $\omega^k(H) \geqslant 0$, either H is strongly $\mathbb{Z}_{2k}$-connected or there exists a partition $\mathcal{P}$ of $V(H)$ such that $H/\mathcal{P} \in \mathcal{N}_k$.

Lemma 4.8. *Let k be an integer with $k \geqslant 2$. Let H be a planar graph such that $v(H) + e(H) < v(G) + e(G)$ and $\omega^k(H) \geqslant 0$. The following statements hold.*

(1) If $\omega^k_H(\mathcal{P}) \geqslant 4k-7$ for any nontrivial partition $\mathcal{P}$ of $V(H)$, then either H is strongly $\mathbb{Z}_{2k}$-connected or $H \in \mathcal{N}_k$.

(2) If $\omega^k(H) \geqslant 4k-7$, then H is strongly $\mathbb{Z}_{2k}$-connected.

(3) Assume that H is $(2k-1)$-edge-connected and let $k \geqslant 3$. If $\omega^k_H(\mathcal{P}) \geqslant 2k-5$ for any nontrivial partition $\mathcal{P}$ of $V(H)$, then either H is strongly $\mathbb{Z}_{2k}$-connected or $H \in \mathcal{N}_k \setminus \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$.

Proof. (1). Since H satisfies that $e(H) + v(H) < e(G) + v(G)$ and $\omega^k(H) \geqslant 0$, either H is strongly $\mathbb{Z}_{2k}$-connected or H has a troublesome partition. Assuming that H has a partition $\mathcal{P}_0$ such that $H/\mathcal{P}_0 \in \mathcal{N}_k$, as $\omega^k_H(\mathcal{P}) \geqslant 4k-7$ for any nontrivial partition $\mathcal{P}$, we obtain that $\mathcal{P}_0$ is a trivial partition and $H \in \mathcal{N}_k$.

(2). In this case, for any partition $\mathcal{P}$ of $V(H)$ (including the trivial partition), $\omega^k_H(\mathcal{P}) \geqslant 4k-7$. Thus $H \notin \mathcal{N}_k$ and it follows from Case (1) that H is strongly $\mathbb{Z}_{2k}$-connected.

(3). Since H is $(2k-1)$-edge-connected, we have $H \notin \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$. Since each value of the weight function of graphs in $\mathcal{N}_k \setminus \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$ is less than $2k-6$, and $\omega^k_H(\mathcal{P}) \geqslant 2k-5$ for any nontrivial partition $\mathcal{P}$ of $V(H)$, we know that either H is strongly $\mathbb{Z}_{2k}$-connected or $H \in \mathcal{N}_k \setminus \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$. □

We provide some structural properties of G in the following.

Lemma 4.9. *The graph G contains no strongly $\mathbb{Z}_{2k}$-connected subgraph H with $v(H) \geqslant 2$.*

Proof. Assume that there is a strongly $\mathbb{Z}_{2k}$-connected subgraph H of G with $v(H) \geqslant 2$. Let $G' = G/H$. By Lemma 4.1, $\omega^k(G') \geqslant \omega^k(G) \geqslant 0$. Noting that $v(G')+e(G') < v(G)+e(G)$, by the minimality of G, we know either G' is strongly $\mathbb{Z}_{2k}$-connected or it has a troublesome partition. Noting that G has no troublesome partition, by Observation 2.11 we have $G'/\mathcal{P}' \notin \mathcal{N}_k$ for any partition $\mathcal{P}'$ of $V(G')$. Therefore, G' is strongly $\mathbb{Z}_{2k}$-connected. By Lemma 2.7 G is strongly $\mathbb{Z}_{2k}$-connected, a contradiction. □

Lemma 4.10. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$ and k be an integer with $k \geqslant 2$.*

(1) If $\mathcal{P}$ is nontrivial, then $\omega^k_G(\mathcal{P}) \geqslant 2k$.

(2) If $\mathcal{P}$ is normal, then $\omega^k_G(\mathcal{P}) \geqslant 4k-3$.

(3) If $|P_1| \geqslant 2$ and $|P_2| \geqslant 3$, then $\omega_G^k(\mathcal{P}) \geqslant 6k-3$.

Proof. Recall that for a subgraph H of G and a partition $\mathcal{Q}$ of $V(H)$, by Proposition 4.2, we have that

$$\omega_H^k(\mathcal{Q}) = \omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) - \omega_G^k(\mathcal{P}) + (6k-8). \tag{6}$$

For (1), assume to the contrary that $\omega_G^k(\mathcal{P}) \leqslant 2k-1$ for a nontrivial partition $\mathcal{P}$. As $\mathcal{P}$ is nontrivial, without loss of generality, we may assume that $|P_1| \geqslant 2$. Let $H = G[P_1]$. For any partition $\mathcal{Q}$ of $V(H)$, since $\omega^k(G) \geqslant 0$ and thus $\omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geqslant 0$, by Formula (6) $\omega_H^k(\mathcal{Q}) \geqslant 0-(2k-1)+(6k-8) \geqslant 4k-7$. Therefore, $\omega^k(H) \geqslant 4k-7$ and H is strongly $\mathbb{Z}_{2k}$-connected by Lemma 4.8 (2), which is a contradiction.

For (2), let $\mathcal{P}$ be a normal partition of $V(G)$ and assume to the contrary that $\omega_G^k(\mathcal{P}) \leqslant 4k-4$. We consider the following two possibilities.

Case (a): Assume that $\mathcal{P}$ has two nontrivial parts $\{P_1, P_2\}$ with $|P_1| \geqslant 2$ and $|P_2| \geqslant 2$. Let $H = G[P_1]$. For any partition $\mathcal{Q}$ of $V(H)$, $\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}$ is a nontrivial partition of $V(G)$, thus $\omega_G^k(\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}) \geqslant 2k$ by Case (1). By Formula (6), $\omega_H^k(\mathcal{Q}) \geqslant 2k-(4k-4)+(6k-8) = 4k-4$. Therefore, $\omega^k(H) \geqslant 4k-4$ and H is strongly $\mathbb{Z}_{2k}$-connected by Lemma 4.8 (2), a contradiction.

Case (b): Without loss of generality, assume that $\mathcal{P}$ contains a unique nontrivial part P_1 with $|P_1| \geqslant 3$. Let $H = G[P_1]$ and $\mathcal{Q}$ be a partition of P_1. If $\mathcal{Q}$ is a trivial partition, noting that $\omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geqslant 0$, then by Formula (6) $\omega_H^k(\mathcal{Q}) \geqslant 0-(4k-4)+(6k-8) = 2k-4$; If $\mathcal{Q}$ is a nontrivial partition, then $\omega_G^k(\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})) \geqslant 2k$ by Case (1), thus by Formula (6) $\omega_H^k(\mathcal{Q}) \geqslant 2k-(4k-4)+(6k-8) = 4k-4$. We conclude that for any nontrivial partition $\mathcal{Q}$ of H, $\omega_H^k(\mathcal{Q}) \geqslant 4k-4$ and $\omega^k(H) \geqslant 2k-4 \geqslant 0$. Since $|H| = |P_1| \geqslant 3$ and by Observation 4.4, $H \notin \mathcal{N}_k$. Thus, H is strongly $\mathbb{Z}_{2k}$-connected by Lemma 4.8 (1), a contradiction.

For (3), assume to the contrary that $\omega_G^k(\mathcal{P}) \leqslant 6k-4$. Let $H = G[P_1]$. Note that for any partition $\mathcal{Q}$ of $V(H)$, $\mathcal{Q} \cup (\mathcal{P} \setminus \{P_1\})$ is a normal partition of $V(G)$. By Case (2) and Formula (6), $\omega_H^k(\mathcal{Q}) \geqslant (4k-3)-(6k-4)+(6k-8) = 4k-7$ and thus $\omega^k(H) \geqslant 4k-7$. By Lemma 4.8 (2), H is strongly $\mathbb{Z}_{2k}$-connected, a contradiction. □

Next, we show that the minimum counterexample G must have at least five vertices.

Lemma 4.11. *We have $v(G) \geqslant 5$.*

Proof. Assume to the contrary that $v(G) \leqslant 4$. It is trivial that $v(G) \neq 1$. That $v(G) \neq 2$ follows from the fact that a graph on two vertices is either in $\mathcal{N}_k$ or strongly $\mathbb{Z}_{2k}$-connected.

Assume that $v(G) = 3$ and $V(G) = \{x, y, z\}$. As $\omega^k(G) \geqslant 0$, based on the trivial partition of $V(G)$, we have that $2e(G) - (6k-4) \times 3 + (12k-12) \geqslant 0$ and thus $e(G) \geqslant 3k$. We first claim that $G = T_{a,b,c}$. If not, without loss of generality, we may assume that G is a path xyz. Since $G/\mathcal{P} \notin \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$ for any partition $\mathcal{P}$ of $V(G)$, $\min\{\mu(xy), \mu(yz)\} \geqslant 2k-1$. Thus G contains $(2k-1)K_2$ as a subgraph, contradicting

Lemma 4.9. Since $G = T_{a,b,c}$ has no troublesome partition (in particular, for the trivial partition $\mathcal{P}_0$ of $V(G)$, $G/\mathcal{P}_0 \notin T_{a,b,c} : 3k-1 \leqslant a+b+c \leqslant 4k-3\}$), $a+b+c \geqslant 4k-2$. Moreover, as for any subset $X \subset V(G)$ with $|X| = 2$, $G/X \notin \{\alpha K_2 : 2 \leqslant \alpha \leqslant 2k-2\}$, $\delta(G) \geqslant 2k-1$. Such $T_{a,b,c}$ is in $\mathcal{S}_k$, a contradiction.

Now we assume that $v(G) = 4$. Since $\omega^k(G) \geqslant 0$, we have $e(G) \geqslant 6k-2$. Moreover, by Lemma 2.5 and Lemma 4.9, G has no copy of $(2k-1)K_2$ and thus $\mu(G) \leqslant 2k-2$. We claim that $\delta(G) \geqslant 2k-1$. Assume not, and let v be the vertex with $d(v) = \delta(G) \leqslant 2k-2$ and H be the graph obtained from G by deleting v. Since H is a subgraph of G, $\mu(H) \leqslant \mu(G) \leqslant 2k-2$. As H has three vertices and at least $(6k-2)-(2k-2) = 4k$ edges, H is a multi-triangle. Moreover, $\delta(H)+\mu(H) \geqslant e(H) \geqslant 4k$, i.e., $\delta(H) \geqslant 2k+2$. By Lemma 3.1, H is a strongly $\mathbb{Z}_{2k}$-connected graph, contradicting Lemma 4.9. This completes the proof of the claim. Since $\delta(G) \geqslant 2k-1$, G satisfies all the conditions in Lemma 3.4 and thus it is strongly $\mathbb{Z}_{2k}$-connected, a contradiction. □

Lemma 4.12. *G contains no $T_{1,1,2k-2}$ as a subgraph.*

Proof. Let $T_{1,1,2k-2}$ be a multi-triangle with the vertex set $\{x, y, z\}$ satisfying that $d(x) = 2$ and $d(y) = d(z) = 2k-1$. Assume to the contrary that G contains $T_{1,1,2k-2}$ as a subgraph. We lift the edge pair yx, xz to obtain a new edge yz, then contract the subgraph $(2k-1)K_2$ between y and z, and denote the resulting graph by G'.

For the trivial partition $\mathcal{P}'$ of $V(G')$, we have $\omega^k_{G'}(\mathcal{P}') \geqslant \omega^k(G) - 2 \times 2k + (6k-4) \geqslant 0 - 4k + (6k-4) = 2k-4$. For any nontrivial partition $\mathcal{P}'$ of $V(G')$, there is a normal partition $\mathcal{P}$ of $V(G)$ corresponding to $\mathcal{P}'$ of $V(G')$. By Lemma 4.10 (2), $\omega^k_G(\mathcal{P}) \geqslant 4k-3$. Recall that the vertices y and z are in the same part of $\mathcal{P}$. Observe that xy and xz are the edges which may be counted in $\omega^k_G(\mathcal{P})$ but not in $\omega^k_{G'}(\mathcal{P}')$. Hence, $\omega^k_{G'}(\mathcal{P}') \geqslant \omega^k_G(\mathcal{P}) - 2 \times 2 = 4k-7$. Altogether, $\omega^k(G') \geqslant \min\{2k-4, 4k-7\} \geqslant 0$. By Lemma 4.8 (1), G' is either strongly $\mathbb{Z}_{2k}$-connected or in $\mathcal{N}_k$. The latter case is impossible since $v(G') = v(G) - 1 \geqslant 4$ by Lemma 4.11. It then follows from Proposition 2.8 (1) that G is also strongly $\mathbb{Z}_{2k}$-connected, a contradiction. □

Using Lemmas 4.10 and 4.11, we give a lower bound on the edge-connectivity of the minimum counterexample G in the next lemma.

Lemma 4.13. *The graph G is $(2k+1)$-edge-connected. Moreover, if $[X, X^c]$ is an edge-cut of G with $|X| \geqslant 2$ and $|X^c| \geqslant 3$, then $d(X) \geqslant 3k+1$.*

Proof. For any $X \subset V(G)$, let $\mathcal{P} = \{X, X^c\}$ be a partition of $V(G)$. Note that $[X, X^c]$ is an edge-cut of G. Since $v(G) \geqslant 5$ by Lemma 4.11, such a partition $\mathcal{P}$ is always a normal partition. By Lemma 4.10 (2), $\omega^k_G(\mathcal{P}) \geqslant 4k-3$. It follows from the definition that $\omega^k_G(\mathcal{P}) = 2d(X) - (6k-4) \times 2 + (12k-12) \geqslant 4k-3$, solving which we have that $d(X) \geqslant 2k + \frac{1}{2}$ and thus $d(X) \geqslant 2k+1$. Hence, G is $(2k+1)$-edge-connected.

For the moreover part, let $\mathcal{P} = \{X, X^c\}$ with $|X| \geqslant 2$ and $|X^c| \geqslant 3$. By Lemma 4.10 (3), $\omega_G^k(\mathcal{P}) \geqslant 6k-3$. Again by solving $\omega_G^k(\mathcal{P}) = 2d(X) - (6k-4) \times 2 + 12k - 12 \geqslant 6k-3$, we have that $d(X) \geqslant 3k + \frac{1}{2}$ and thus $d(X) \geqslant 3k+1$. □

To obtain more reducible configurations, from now on we assume that $k \geqslant 3$.

Lemma 4.14. *G contains no $Q_{1,1,1,2k-2}$ as a subgraph.*

Proof. Let $Q_{1,1,1,2k-2}$ be a multi-cycle with vertices x, y, z, and w satisfying that $d(y) = d(z) = 2$ and $d(x) = d(w) = 2k-1$. Suppose to the contrary that G contains $Q_{1,1,1,2k-2}$. We lift an edge triple xy, yz, zw to obtain a new (parallel) edge xw, and then contract the resulting subgraph $(2k-1)K_2$. The resulting graph is denoted by G'. By Lemma 4.13, $d_{G'}(y) \geqslant 2k-1$ and $d_{G'}(z) \geqslant 2k-1$, and by the moreover part of Lemma 4.13, $d_{G'}(u^*) \geqslant (3k+1)-2 \geqslant 3k-1 \geqslant 2k-1$ where u^* is the new vertex of $V(G')$ obtained by the contraction. Hence, G' is $(2k-1)$-edge-connected.

For the trivial partition $\mathcal{P}_0'$ of $V(G')$, we have that $\omega_{G'}^k(\mathcal{P}_0') \geqslant \omega^k(G) - 2 \times (2k+1) + 6k - 4 \geqslant 2k-6 \geqslant 0$. Similar to the proof of Lemma 4.12, for any nontrivial partition $\mathcal{P}'$ of $V(G')$, $\omega_{G'}^k(\mathcal{P}') \geqslant \omega_G^k(\mathcal{P}) - 2 \times 3 \geqslant 4k-3-6 = 4k-9 \geqslant 2k-5$. Since $v(G') \geqslant 4$ by Lemma 4.11, $G' \notin \mathcal{N}_k$. So G' is strongly $\mathbb{Z}_{2k}$-connected by Lemma 4.8 (3). Hence, G is strongly $\mathbb{Z}_{2k}$-connected by Proposition 2.8 (1), which is a contradiction. □

We now improve the edge-connectivity of G and then obtain another reducible configuration.

Lemma 4.15. *The minimum degree of G is at least $2k+3$. Moreover, G is $(2k+3)$-edge-connected.*

Proof. Assume to the contrary that $\delta(G) \leqslant 2k+2$. As G is $(2k+1)$-edge-connected by Lemma 4.13, $\delta(G) \in \{2k+1, 2k+2\}$. Thus there is a vertex, say z, whose degree is either $2k+1$ or $2k+2$, and assume that there are two edges xz and zy incident with z. We lift the edge pair xz, zy and then delete the vertex z. The resulting graph is denoted by G'. Since G has no copy of $T_{1,1,2k-2}$ by Lemma 4.12, $\mu_{G'}(xy) \leqslant 2k-2$. Towards a contradiction, by Proposition 2.8 (2), it suffices to show that G' is strongly $\mathbb{Z}_{2k}$-connected.

For the trivial partition $\mathcal{P}_0'$ of $V(G')$, we know that $\omega_{G'}^k(\mathcal{P}_0') \geqslant \omega^k(G) - 2 \times (2k+2-1) + (6k-4) \geqslant 2k-6 \geqslant 0$. For the almost trivial partition $\mathcal{P}_1' = \{P_1', P_2', \ldots, P_t'\}$ of $V(G')$ with $P_1' = \{x, y\}$ (noting that $t = v(G') - 1$), as $\mu_{G'}(xy) \leqslant 2k-2$, $\omega_{G'}^k(\mathcal{P}_1') \geqslant \omega_{G'}^k(\mathcal{P}_0') - 2 \times (2k-2) + 6k - 4 \geqslant 4k-6$. For any normal partition $\mathcal{P}'$ of $V(G')$, noting that $\mathcal{P}' \cup \{z\}$ is a normal partition of $V(G)$, by Lemma 4.10 (2), $\omega_{G'}^k(\mathcal{P}') \geqslant \omega_G^k(\mathcal{P}' \cup \{z\}) - 2 \times (2k+2) + 6k - 4 \geqslant 6k-11$. Hence, for any nontrivial partition $\mathcal{P}$ of $V(G')$, $\omega_{G'}^k(\mathcal{P}) \geqslant 4k-6$. By Lemma 4.11, $v(G') = v(G) - 1 \geqslant 4$, so $G' \notin \mathcal{N}_k$. Applying Lemma 4.8 (1), we conclude that G' is strongly $\mathbb{Z}_{2k}$-connected. We complete the proof of the first part.

Moreover, since $v(G) \geqslant 5$, for any edge-cut $[X, X^c]$ of G with $|X| \geqslant 2$, by the moreover part of Lemma 4.13, $d(X) \geqslant 3k+1 \geqslant 2k+3$ (as $k \geqslant 3$). Together with the fact that $\delta(G) \geqslant 2k+3$, G is $(2k+3)$-edge-connected. □

Lemma 4.16. *G contains no $T_{2,2,2k-3}$ as a subgraph.*

Proof. Let $T_{2,2,2k-3}$ be a multi-triangle with vertices x, y, and z satisfying that $d(x) = 4$ and $d(y) = d(z) = 2k-1$. Assume that G contains $T_{2,2,2k-3}$ as a subgraph. We lift two edge pairs at x to obtain two new parallel edges connecting y and z, and contract the resulting $(2k-1)K_2$. The resulting graph is denoted by G'. By Lemma 4.15, $d_{G'}(x) \geqslant 2k-1$, and by the moreover part of Lemma 4.13, $d_{G'}(u^*) \geqslant (3k+1) - 4 \geqslant 3k-3 \geqslant 2k-1$ where u^* is the new vertex obtained by the contraction. Hence, G' is $(2k-1)$-edge-connected.

The trivial partition $\mathcal{P}'_0$ of $V(G')$ satisfies $\omega^k_{G'}(\mathcal{P}'_0) \geqslant \omega^k(G) - 2\times(2k+1) + 6k-4 \geqslant 2k-6 \geqslant 0$. Since for any normal partition $\mathcal{P}$ of $V(G)$, $\omega^k_G(\mathcal{P}) \geqslant 4k-3$ by Lemma 4.10 (2), any nontrivial partition $\mathcal{P}'$ of $V(G')$ satisfies that $\omega^k_{G'}(\mathcal{P}') \geqslant \omega^k_G(\mathcal{P}) - 2\times 4 \geqslant 4k-11 \geqslant 2k-5$ as $k \geqslant 3$. By Lemma 4.11, $v(G') \geqslant 4$, thus $G' \notin \mathcal{N}_k$. It follows from Lemma 4.8 (3) that G' is strongly $\mathbb{Z}_{2k}$-connected, so G is by Proposition 2.8 (1), a contradiction. □

4.2 Strongly $\mathbb{Z}_4$-connected graphs

In this subsection, for $k = 2$, we consider the weight function $\omega^2_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 8t + 12$ and $\omega^2(G) = \min_{\mathcal{P}}\{\omega^2_G(\mathcal{P})\}$.We shall prove the following claim:

A planar graph G with $\omega^2(G) \geqslant 0$ either is strongly $\mathbb{Z}_4$-connected or has a troublesome partition.

By Proposition 4.5 together with Lemma 4.9, and Lemma 4.12, we know that a minimum counterexample to this claim contains no configuration from $\mathcal{F}_2 = \{3K_2, T_{1,1,2}\}$. Since a graph G with $\omega^2(G) \geqslant 0$ satisfies $e(G) \geqslant 4v(G) - 6$, to this end, it suffices to prove the following lemma.

Lemma 4.17. *Given a planar graph G with $e(G) \geqslant 4v(G) - 6$, if G has no troublesome partition, then G contains at least one configuration of $\mathcal{F}_2 = \{3K_2, T_{1,1,2}\}$.*

Proof. Assume that G contains no configurations of $\mathcal{F}_2$. As $e(G) \geqslant 4v(G) - 6$ (i.e., $2e(G) - 8v(G) + 12 \geqslant 0$), by Euler's formula that $v(G) + f(G) - e(G) = 2$, we have that $2e(G) - 8 \times \big(2 - f(G) + e(G)\big) + 12 \geqslant 0$, thus

$$\sum_{f \in F(G)} \ell(f) = 2e(G) \leqslant \frac{8}{3} f(G) - \frac{4}{3}.$$

We assign to each face f an initial charge $c(f) = \ell(f)$, and thus the total charge is strictly smaller than $\frac{8}{3} f(G)$. We then apply the following discharging rule.

Rule. *Each 2-face receives $\frac{1}{3}$ from each of its weakly adjacent 3^+-faces.*

We shall prove that each face ends with a charge at least $\frac{8}{3}$ after discharging, which is a contradiction. Every 2-face receives $\frac{1}{3}$ from each of its two weakly adjacent 3^+-faces, thus it has charge at least $2 + \frac{2}{3} \geqslant \frac{8}{3}$. For a 3-face f (viewed as the inner face of $T_{a,b,c}$), since G does not contain $T_{1,1,2}$, we have $a + b + c \leqslant 3$ and hence, f always ends with a charge of at least 3. Since G contains no $3K_2$, after the discharging, each 4^+-face f has charge $c'(f) \geqslant \ell(f) - \frac{1}{3}\ell(f) = \frac{2}{3}\ell(f) \geqslant \frac{8}{3}$. Therefore, every face ends with a charge at least $\frac{8}{3}$ and this completes the proof. □

4.3 Strongly $\mathbb{Z}_6$-connected graphs

In this subsection, for $k = 3$, we consider the weight function $\omega^3_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 14t + 24$ and $\omega^3(G) = \min_{\mathcal{P}}\{\omega^3_G(\mathcal{P})\}$. Recall that $\mathcal{N}_3 = \{\alpha K_2 : 2 \leqslant \alpha \leqslant 4\} \cup \{T_{a,b,c} : a + b + c \in \{8, 9\}\}$ and $\mathcal{S}_3 = \{5K_2, T_{2,4,4}, T_{3,3,4}\}$. We shall prove the following claim:

Given a planar graph G, if $\omega^3(G) \geqslant 0$, then either G is strongly $\mathbb{Z}_6$-connected or G has a troublesome partition.

By Proposition 4.5 together with Lemma 4.9, and Lemmas 4.12, 4.14, and 4.16, we obtain the forbidden configurations set $\mathcal{F}_3 = \{5K_2, T_{1,1,4}, T_{2,2,3}, Q_{1,1,1,4}\}$ of the minimum counterexample G to the above claim. Since a graph G with $\omega^3(G) \geqslant 0$ has $e(G) \geqslant 7v(G) - 12$, similarly, we shall prove Lemma 4.18 to finish the proof of the above claim.

Lemma 4.18. *Given a planar graph G with $e(G) \geqslant 7v(G) - 12$, if G has no troublesome partition, then G contains at least one configuration of $\mathcal{F}_3 = \{5K_2, T_{1,1,4}, T_{2,2,3}, Q_{1,1,1,4}\}$.*

Proof. Assume that G is a counterexample of this lemma. As $e(G) \geqslant 7v(G) - 12$ (i.e., $2e(G) - 14v(G) + 24 \geqslant 0$), by Euler's formula, we have $2e(G) - 14 \times \big(2 - f(G) + e(G)\big) + 24 \geqslant 0$, thus

$$\sum_{f \in F(G)} \ell(f) = 2e(G) \leqslant \frac{7}{3}f(G) - \frac{2}{3}.$$

We assign to each face f the initial charge $\ell(f)$. The total charge is strictly smaller than $\frac{7}{3}f(G)$. We then apply the following discharging rule.

Rule. *Each 2-face receives $\frac{1}{6}$ from each of its weakly adjacent 3^+-faces.*

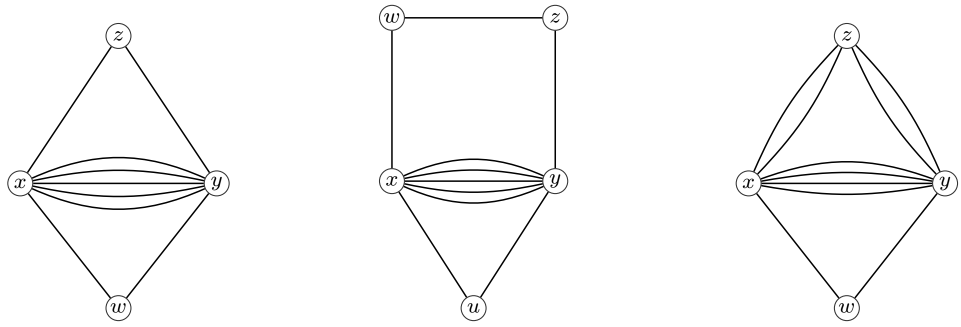
As every 2-face has exactly two weakly adjacent 3^+-faces, its charge is increased to $\frac{7}{3}$. Since G has no $5K_2$, each face f has at most $3\ell(f)$ weakly adjacent 2-faces, and moreover, when $\ell(f) \geqslant 5$, we have that $c'(f) \geqslant \ell(f) - 3\ell(f) \times \frac{1}{6} \geqslant \frac{5}{2} > \frac{7}{3}$. The remaining cases are 3-faces and 4-faces. Since G contains no $T_{1,1,4}$ and $T_{2,2,3}$, every 3-face f has at most 4 weakly adjacent 2-faces. Hence, f ends with a charge $c'(f) \geqslant 3 - 4 \times \frac{1}{6} = \frac{7}{3}$. Since G contains no $Q_{1,1,1,4}$, every 4-face f (viewed as the inner face of $Q_{a,b,c,d}$) has at most 8 weakly adjacent 2-faces and thus $c'(f) \geqslant 4 - 8 \times \frac{1}{6} = \frac{8}{3}$. Every face ends with a charge at least $\frac{7}{3}$, a contradiction. □

4.4 Strongly $\mathbb{Z}_8$-connected graphs

For $k = 4$, we consider the weight function $\omega^4_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 20t + 36$ and $\omega^4(G) = \min_{\mathcal{P}}\{\omega^4_G(\mathcal{P})\}$. Recall that $\mathcal{N}_4 = \{\alpha K_2 : 2 \leqslant \alpha \leqslant 6\} \cup \{T_{a,b,c} : a + b + c \in \{11, 12, 13\}\}$ and a graph G has a troublesome partition with respect to strong $\mathbb{Z}_8$-connectivity if it has a partition $\mathcal{P}$ such that $G/\mathcal{P} \in \mathcal{N}_4$. Each graph of $\mathcal{S}_4 = \{7K_2, T_{2,6,6}, T_{3,5,6}, T_{4,4,6}, T_{4,5,5}\}$ is strongly $\mathbb{Z}_8$-connected. We shall conclude the following claim:

Given a planar graph G, if $\omega^4(G) \geqslant 0$, then either G is strongly $\mathbb{Z}_8$-connected or G has a troublesome partition.

To this end, we need first to show that graphs $T^o_{1,1,6}, T^o_{2,2,5}$ and $Q^o_{1,1,1,6}$, depicted in Figure 2, are forbidden configurations in the minimum counterexample G of the above claim.



(a) $T^o_{1,1,6}$ (b) $Q^o_{1,1,1,6}$ (c) $T^o_{2,2,5}$

Figure 2: The graphs $T^o_{1,1,6}$, $Q^o_{1,1,1,6}$, and $T^o_{2,2,5}$

Lemma 4.19. *G contains no $T^o_{1,1,6}$ as a subgraph.*

Proof. Let $T^o_{1,1,6}$ be a multi-graph with the vertex set $\{x, y, w, z\}$ satisfying that $d(x) = d(y) = 7$ and $d(w) = d(z) = 2$, and see Figure 2(a). Suppose that G contains $T^o_{1,1,6}$ as a subgraph. We lift two edge pairs xw, wy and xz, zy to obtain two new edges connecting x and y, contract the resulting $7K_2$, and denote the final graph by G'. Clearly, G' is 9-edge-connected by Lemma 4.15 and the moreover part of Lemma 4.13. Observe that $G' \notin \mathcal{N}_4$ as $v(G') \geqslant 4$. For the trivial partition $\mathcal{P}'_0$ of $V(G')$, we have $\omega^4_{G'}(\mathcal{P}'_0) \geqslant \omega^4(G) - 2 \times 9 + 20 \geqslant 2$. For any nontrivial partition $\mathcal{P}'$ of $V(G')$, $\omega^4_{G'}(\mathcal{P}') \geqslant \omega^4_G(\mathcal{P}) - 2 \times 4 \geqslant 13 - 8 = 5$, where $\omega^4_G(\mathcal{P}) \geqslant 13$ follows from Lemma 4.10 (2). Thus by Lemma 4.8 (3), G' is strongly $\mathbb{Z}_8$-connected, so G is by Proposition 2.8 (1), a contradiction. $\square$

Lemma 4.20. *G contains no $Q^o_{1,1,1,6}$ as a subgraph.*

Proof. Let $Q^o_{1,1,1,6}$ be a multi-graph with vertices x, y, w, z, u satisfying $d(x) = d(y) = 7$ and $d(u) = d(w) = d(z) = 2$, and see Figure 2(b). Assume that G contains $Q^o_{1,1,1,6}$. Similarly, we first lift an edge pair xu, yu and also lift a 2-path xw, wz, zy to become two new parallel edges connecting x and y. We then contract the newly obtained $7K_2$

and denote the resulting graph by G'. Note that G' is 9-edge-connected and $G' \notin \mathcal{N}_4$. For the trivial partition $\mathcal{P}'_0$ of $V(G')$, $\omega^4_{G'}(\mathcal{P}'_0) \geqslant \omega^4(G) - 2 \times 10 + 20 \geqslant 0$, and for any nontrivial partition $\mathcal{P}'$ of $V(G')$, $\omega^4_{G'}(\mathcal{P}') \geqslant \omega^4_G(\mathcal{P}) - 2 \times 5 \geqslant 13 - 10 = 3$. Therefore, G' is strongly $\mathbb{Z}_8$-connected by Lemma 4.8 (3) and G is also strongly $\mathbb{Z}_8$-connected by Proposition 2.8 (1), a contradiction. □

Next, to obtain the last forbidden configuration depicted in Figure 2(c), we need to improve the bound of the weight function of normal partitions of $V(G)$.

Lemma 4.21. *If $\mathcal{P}$ is a normal partition of $V(G)$, then $\omega^4_G(\mathcal{P}) \geqslant 15$.*

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_s\}$ be a normal partition of $V(G)$ with $|P_1| \geqslant 2$ and assume to the contrary that $\omega^4_G(\mathcal{P}) \leqslant 14$. Let $H = G[P_1]$ and let $\mathcal{Q}$ be a partition of $V(H)$. By Proposition 4.2, $\omega^4_H(\mathcal{Q}) = \omega^4_G(\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}) - \omega^4_G(\mathcal{P}) + 16 \geqslant \omega^4_G(\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}) - 14 + 16 \geqslant \omega^4_G(\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}) + 2$. If for any partition $\mathcal{Q}$ of $V(H)$, $\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}$ is a nontrivial partition of $V(G)$, then by Lemma 4.10 (1), $\omega^4(H) \geqslant 8 + 2 = 10$. By Lemma 4.8 (2), H is strongly $\mathbb{Z}_8$-connected, a contradiction to Lemma 4.9. Thus we assume that there is a partition $\mathcal{Q}$ of $V(H)$ such that $\mathcal{Q} \cup \mathcal{P} \setminus \{P_1\}$ is the trivial partition of $V(G)$ and thus $\omega^4(H) \geqslant 2$. In this case, since $\mathcal{P}$ is a normal partition, we have $|P_1| \geqslant 3$, so $H \notin \{\alpha K_2 : 2 \leqslant \alpha \leqslant 6\}$. Moreover, since $\omega^4(T_{a,b,c}) < 2$ when $a + b + c \in \{11, 12\}$, $H \notin \{T_{a,b,c} : 11 \leqslant a + b + c \leqslant 12\}$. By Lemmas 4.12 and 4.16, G has no copy of $T_{1,1,6}$ and $T_{2,2,5}$, so $H \notin \{T_{a,b,c} : a + b + c = 13\}$. Thus, we conclude that $H \notin \mathcal{N}_4$. Note that $\omega^4_H(\mathcal{Q}') \geqslant 10 > 9$ for any nontrivial partition $\mathcal{Q}'$ of $V(H)$, so H is strongly $\mathbb{Z}_8$-connected by Lemma 4.8 (1), again contradicting Lemma 4.9. □

Lemma 4.22. *G contains no $T^o_{2,2,5}$ as a subgraph.*

Proof. Let $T^o_{2,2,5}$ be a multi-graph with vertices x, y, w, z satisfying that $d(x) = d(y) = 7, d(w) = 2$ and $d(z) = 4$, and see Figure 2(c). Assume G contains $T^o_{2,2,5}$. We lift two edge pairs at z and an edge pair xw, wy to obtain three new parallel edges connecting x and y, then contract the resulting subgraph $7K_2$ and denote the final graph by G'. Since G is 11-edge-connected by Lemma 4.15, by the moreover part of Lemma 4.13, G' is 7-edge-connected.

For the trivial partition $\mathcal{P}'_0$ of $V(G')$, $\omega^4_{G'}(\mathcal{P}'_0) \geqslant \omega^4(G) - 2 \times 10 + 20 \geqslant 0$. For any nontrivial partition $\mathcal{P}'$, there is a partition $\mathcal{P}$ of $V(G)$ corresponding to $\mathcal{P}'$ of $V(G')$. Note that $\mathcal{P}$ is a normal partition and thus $\omega^4_G(\mathcal{P}) \geqslant 15$ by Lemma 4.21. We have that $\omega^4_{G'}(\mathcal{P}') \geqslant \omega^4_G(\mathcal{P}) - 2 \times 6 \geqslant 15 - 12 = 3$. Again since $v(G') \geqslant 4$, $G' \notin \mathcal{N}_4$. By Lemma 4.8 (3), G' is strongly $\mathbb{Z}_8$-connected, so G is by Proposition 2.8 (1), a contradiction. □

By Proposition 4.5 together with Lemma 4.9, and Lemmas 4.12, 4.14, 4.16, 4.19, 4.20, and 4.22, we have the forbidden configurations set

$$\mathcal{F}_4 = \{7K_2, T_{1,1,6}, T_{2,2,5}, Q_{1,1,1,6}, T^o_{1,1,6}, T^o_{2,2,5}, Q^o_{1,1,1,6}\}.$$

Similarly, since a graph G with $\omega^4(G) \geqslant 0$ satisfies $e(G) \geqslant 10v(G) - 18$, we only need to prove the following lemma to end the proof.

Lemma 4.23. *Given a planar graph G with $e(G) \geqslant 10v(G) - 18$, if G has no troublesome partition, then G contains one configuration of $\mathcal{F}_4$.*

Proof. We assume that G is a counterexample. As $e(G) \geqslant 10v(G) - 18$, $2e(G) - 20v(G) + 36 \geqslant 0$. By Euler's formula, we have $2e(G) - 20 \times \big(2 - f(G) + e(G)\big) + 36 \geqslant 0$, then

$$\sum_{f \in F(G)} \ell(f) = 2e(G) \leqslant \frac{20}{9} f(G) - \frac{4}{9}.$$

We assign to each face f the initial charge $\ell(f)$ and the total charge is strictly smaller than $\frac{20}{9} f(G)$. To obtain a contradiction, we redistribute the charge by the following rules:

Rule (i). *Each 2-face receives charge $\frac{1}{9}$ from each of its weakly adjacent 3^+-faces.*

Rule (ii). *Each 3-face receives charge $\frac{1}{9}$ from each of its weakly adjacent 4^+-faces.*

We shall show each face ends with a charge of at least $\frac{20}{9}$, which is a contradiction.

By **Rule (i)**, every 2-face ends with $2 + 2 \times \frac{1}{9} = \frac{20}{9}$.

We first consider a 5^+-face f. Since G contain no $7K_2$, f has at most $5\ell(f)$ weakly adjacent 2-faces. Moreover, G contains no copy of $T_{1,1,6}$. Hence, f sends in total at most $\frac{5}{9}\ell(f)$ to its weakly adjacent 2-faces and 3-faces by **Rule (i)** and **Rule (ii)**, and thus f ends with a charge of at least $\ell(f) - \frac{5}{9}\ell(f) = \frac{4}{9}\ell(f) \geqslant \frac{20}{9}$.

We then consider a 4-face f. Since G has no copy of $Q_{1,1,1,6}$, f has at most 16 weakly adjacent 2-faces. Moreover, since G contains no $Q^o_{1,1,1,6}$, f sends charge in total at most $\frac{16}{9}$ to its weakly adjacent 2-faces and 3-faces by **Rule (i)** and **Rule (ii)**. It ends with a charge of at least $\frac{20}{9}$.

Finally, we consider all the 3-faces. Let f be the inner 3-face of the subgraph $T_{a,b,c}$ (of G). When $a+b+c \leqslant 10$, f ends with a charge of at least $3 - 7 \times \frac{1}{9} = \frac{20}{9}$ by **Rule (i)**. When $a + b + c \geqslant 11$, since G has no copy of $T_{1,1,6}$ and $T_{2,2,5}$, we only need to consider the following three possibilities: $T_{1,5,5}$, $T_{3,4,4}$, and $T_{4,4,4}$.

- If f is the inner face of subgraph $T_{1,5,5}$, then each face weakly adjacent to f through an edge of multiplicity 5 is neither a 3-face nor a 4-face by Lemmas 4.19 and 4.20. Thus, f ends with a charge of at least $3 - (11 - 3) \times \frac{1}{9} + 2 \times \frac{1}{9} = \frac{21}{9}$ by **Rule (i)** and **Rule (ii)**.

- If f is the inner face of $T_{3,4,4}$, then each face weakly adjacent to f through an edge of multiplicity 4 is not a 3-face by Lemma 4.22. So f ends with a charge of at least $3 - (11 - 3) \times \frac{1}{9} + 2 \times \frac{1}{9} = \frac{21}{9}$ by **Rule (i)** and **Rule (ii)**.

- The last case is when f is the inner face of $T_{4,4,4}$. Clearly, each face weakly adjacent to f is not a 3-face by Lemma 4.22, so f ends with a charge of at least $3-(12-3)\times\frac{1}{9}+3\times\frac{1}{9}=\frac{21}{9}$ by **Rule (i)** and **Rule (ii)**.

We are done. $\square$

5 Homomorphisms and circular colorings of signed graphs

In this section, we first give a negative-girth condition for a signed bipartite planar graph to admit a homomorphism to C_{-2k}. To prove this result, we need the definition of circular $\frac{p}{q}$-flow in signed graphs and the duality theorem between circular flows and circular colorings.

Definition 5.1. [14] *Given positive integers p and q with p even, a circular $\frac{p}{q}$-flow in a signed graph (G,σ) is a pair (D,f) where D is an orientation on G and $f: E(G)\to\mathbb{Z}$ satisfies the following conditions.*

- *For each positive edge e, $|f(e)|\in\{q,\dots,p-q\}$;*
- *For each negative edge e, $|f(e)|\in\{0,\dots,\frac{p}{2}-q\}\cup\{\frac{p}{2}+q,\dots,p-1\}$;*
- *For each vertex v, $\partial_D f(v):=\sum\limits_{(u,v)\in D} f(uv)-\sum\limits_{(v,w)\in D} f(vw)=0$.*

Note that using Tutte's theorem[3] [30], a signed graph admits a circular $\frac{p}{q}$-flow if and only if it admits a modular $\frac{p}{q}$-flow, i.e., a pair (D,f) by replacing the last condition of Definition 5.1 with $\partial_D f(v)\equiv 0 \pmod p$.

Proposition 5.2. [14] *For a signed plane graph (G,σ) and its dual signed graph (G^*,σ^*), (G,σ) admits a circular $\frac{p}{q}$-coloring if and only if (G^*,σ^*) admits a circular $\frac{p}{q}$-flow.*

When restricted to signed Eulerian graphs, the next theorem provides us with a necessary and sufficient condition to admit a circular $\frac{4k}{2k-1}$-flow. Let $p_{\widehat{G}}(v)$ denote the number of positive edges incident with v in the signed graph $\widehat{G}$. It is not difficult to observe that by adding $2k$ to the flow value of each positive edge of the signed graph, we may view the obtained flow with only values $\{-1,+1\}$ (taken modulo $4k$) as a special orientation achieving certain boundary $2k\cdot p_{\widehat{G}}(v)$ at each vertex. Note that the flow value 0 can be ignored because the graph is Eulerian. For the sake of completeness, we provide its proof here.

Theorem 5.3. [14] *Given a positive integer k, a signed Eulerian graph $\widehat{G}$ admits a circular $\frac{4k}{2k-1}$-flow if and only if $\widehat{G}$ admits a $(4k,\beta)$-orientation with $\beta(v)\equiv 2k\cdot p_{\widehat{G}}(v) \pmod{4k}$ for each vertex $v\in V(G)$.*

[3]If a graph admits a modular k-flow (D,f), then it admits an integer k-flow (D,f') such that $f'(e)\equiv f(e) \pmod k$ for every edge e.

Proof. Since $\widehat{G}$ is Eulerian, by handshaking lemma $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$ is a $(4k, \beta)$-boundary. Assume that there is a $(4k, \beta)$-orientation D_1 on $\widehat{G}$. Let D be an orientation on G. We first define a mapping $f_1 : E(G) \to \{1, -1\}$ such that $f_1(e) = 1$ if e is oriented in D the same as in D_1 and $f_1(e) = -1$ otherwise. Note that $\partial_D f_1(v) = d^+_{D_1}(v) - d^-_{D_1}(v) \equiv \beta(v) \pmod{4k}$. We define another mapping $f_2 : E(G) \to \{0, 2k\}$ satisfying that $f_2(e) = 2k$ for each positive edge e and $f_2(e) = 0$ for each negative edge e. We have that $\partial_D f_2(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$. Let $f = f_1 + f_2$. It is easily observed that $f(e) \in \{2k-1, 2k+1\}$ for each positive edge e and $f(e) \in \{-1, 1\}$ for each negative edge e. Moreover, $\partial_D f(v) = \partial_D f_1(v) + \partial_D f_2(v) \equiv \beta(v) + 2k \cdot p_{\widehat{G}}(v) \equiv 0 \pmod{4k}$. Therefore, (D, f) is a circular $\frac{4k}{2k-1}$-flow in $\widehat{G}$.

Conversely, assume that (D, f) is a circular $\frac{4k}{2k-1}$-flow in $\widehat{G}$. We first define a mapping $f_1 : E(G) \to \mathbb{Z}_{4k}$ such that $f_1(e) = 2k$ for each positive edge e and $f_1(e) = 0$ for each negative edge e. It follows that $\partial_D f_1(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$. Then, we define another mapping $f_2(e) := f_1(e) - f(e)$ for $e \in E(G)$ and note that for each edge e, $f_2(e) \in \{-1, 0, 1\}$. Let X be the subset of edges e such that $f_2(e) = 0$. As G is Eulerian, it is easy to see that $X = \{e \in E(G) \mid f(e) = 0 \text{ or } f(e) = 2k\}$ and thus every vertex of $G[X]$ is of even-degree. Based on the orientation D and the mapping f_2, we define a new orientation D_1 on $\widehat{G}$ as follows: If $f_2(e) = 1$, then e is oriented in D_1 the same as in D and if $f_2(e) = -1$, then e is oriented oppositely in D_1 and D. For edges in $G[X]$, we orient them at each vertex one-in-one-out. Under D_1, for each vertex $v \in V(G)$,

$$d^+_{D_1}(v) - d^-_{D_1}(v) = \partial_D f_2(v) = \partial_D f_1(v) - \partial_D f(v) \equiv 2k \cdot p_{\widehat{G}}(v) \equiv \beta(v) \pmod{4k}.$$

Therefore, D_1 is a $(4k, \beta)$-orientation with $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$ for each vertex $v \in V(G)$. □

The next corollary follows directly from Theorem 5.3.

Corollary 5.4. *Let G be an Eulerian graph. If G is strongly $\mathbb{Z}_{2k}$-connected, then for any signature σ on G, (G, σ) admits a circular $\frac{4k}{2k-1}$-flow.*

Combining Theorems 1.8 and 5.4, we have the following result.

Corollary 5.5. *Given an integer k with $2 \leqslant k \leqslant 4$, every $(6k-4)$-edge-connected signed Eulerian planar graph admits a circular $\frac{4k}{2k-1}$-flow. Equivalently, every signed bipartite planar graph of girth at least $6k-4$ is circular $\frac{4k}{2k-1}$-colorable.*

We furthermore improve this result by replacing the girth condition with the same negative-girth condition. We first prove Theorem 5.6, similar to a result of [11]: Given a graph G and its connected subgraph H which is strongly $\mathbb{Z}_{2k+1}$-connected, G admits a circular $\frac{2k+1}{k}$-flow if and only if G/H does.

Theorem 5.6. *Given an Eulerian graph G and its connected subgraph H, let $G' = G/H$. Assume that H is strongly $\mathbb{Z}_{2k}$-connected. Then for any signature σ on G, (G,σ) admits a circular $\frac{4k}{2k-1}$-flow if and only if $(G',\sigma|_{G'})$ does.*

Proof. Let $\widehat{G} := (G,\sigma)$ and $\widehat{G}' := (G',\sigma|_{G'})$. Clearly, G' is also Eulerian. Let w denote the new vertex obtained after the contraction in G'. For any $(4k,\beta)$-boundary of G with $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$, the corresponding $(4k,\beta')$-boundary of G' (as defined in Observation 2.6) satisfies that

$$\beta'(w) \equiv \sum_{v\in V(H)} \beta(v) \equiv \sum_{v\in V(H)} 2k\cdot p_{\widehat{G}}(v) = 2k\cdot(p_{\widehat{G}'}(w)+2|E^+_\sigma(H)|) \equiv 2k\cdot p_{\widehat{G}'}(w) \pmod{4k},$$

where $E^+_\sigma(H)$ denotes the set of positive edges of the signed subgraph of (G,σ) induced by $V(H)$, and $\beta'(v) \equiv 2k \cdot p_{\widehat{G}'}(v) \pmod{4k}$ for each vertex $v \in V(G') \setminus \{w\}$.

For one direction, by Theorem 5.3, we assume that $\widehat{G}$ admits a $(4k,\beta)$-orientation D with $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$. Let D' be the restriction of D on $\widehat{G}'$. Considering the orientation D', we have $d^+_{D'}(w) - d^-_{D'}(w) \equiv \sum_{v\in V(H)} \beta(v) \equiv \beta'(w) \pmod{4k}$ and $d^+_{D'}(v) - d^-_{D'}(v) \equiv \beta(v) = \beta'(v) \pmod{4k}$ for any other vertex $v \neq w$. Thus, the orientation D' is a $(4k,\beta')$-orientation on $\widehat{G}'$ with $\beta'(v) \equiv 2k \cdot p_{\widehat{G}'}(v) \pmod{4k}$. Noting that G' is an Eulerian graph, by Theorem 5.3 $\widehat{G}'$ admits a circular $\frac{4k}{2k-1}$-flow.

For the other direction, since G' is Eulerian and $\widehat{G}'$ admits a circular $\frac{4k}{2k-1}$-flow, by Theorem 5.3 the signed graph $\widehat{G}'$ has a $(4k,\beta')$-orientation on $\widehat{G}$ with $\beta'(v) \equiv 2k \cdot p_{\widehat{G}'}(v) \pmod{4k}$. As H is strongly $\mathbb{Z}_{2k}$-connected, by Lemma 2.7 we can obtain a $(4k,\beta)$-orientation with $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$ on G. Therefore, $\widehat{G}$ admits a circular $\frac{4k}{2k-1}$-flow by Theorem 5.3. This completes the proof. □

Lemma 5.7. [Bipartite folding lemma][26] *Let $\widehat{G}$ be a signed bipartite plane graph of negative girth $2k$ and let $C = v_1v_2\cdots v_t$ be a facial cycle (of $\widehat{G}$) which is not a negative $2k$-cycle. Then there is an integer $i \in \{1,\dots,t\}$ such that the signed graph $\widehat{G}'$, obtained from $\widehat{G}$ by identifying two vertices v_i and v_{i+2} (index is taken modulo t) after a possible switching at one of the two vertices, still has negative girth $2k$.*

By applying this lemma repeatedly, we get a homomorphic image of $\widehat{G}$ which is also a signed bipartite plane graph in which every facial cycle is a negative cycle of length exactly $2k$. Now we are ready to prove Theorems 1.4 and 1.5 together, unified in the following theorem.

Theorem 5.8. *Given an integer k with $2 \leqslant k \leqslant 4$, every signed bipartite planar graph of negative girth at least $6k-4$ is circular $\frac{4k}{2k-1}$-colorable, i.e., it admits a homomorphism to C_{-2k}.*

Proof. By Lemma 5.7, we may assume that (G,σ) is a minimum counterexample together with a planar embedding such that each facial cycle of (G,σ) is a negative even cycle of length $6k-4$. Then its dual signed graph (G^*, σ^*) is a $(6k-4)$-regular signed Eulerian plane graph. If G^* is $(6k-4)$-edge-connected, then we are done by Corollary 5.5. Since (G, σ) is bipartite and has negative girth at least $6k-4$, every negative even edge-cut of (G^*, σ^*) is of size at least $6k-4$. Thus we may assume that (G^*, σ^*) has a small positive even edge-cut and we choose the edge-cut $[X, X^c]$ with $|X|$ being minimized among all the possible choices. First observe that $|X| \geq 2$ since G^* is $(6k-4)$-regular. Note that $d_{G^*}(X) \leqslant 6k-6$ but for any proper subset $Y \subsetneq X$, $d_{G^*}(Y) \geqslant 6k-4$.

Let $H_1 = G^*[X]$ and let G_0^* be the graph obtained from G^* by identifying all the vertices of X^c into a new vertex x. Note that $d_{G_0^*}(x) \leqslant 6k-6$ and except $[\{x\}, V(G_0^*) \backslash \{x\}]$ every other cut of G_0^* has size at least $6k-4$. By Theorem 4.7, H_1 is strongly $\mathbb{Z}_\ell$-connected. Moreover, H_1 is connected by the minimality of X.

Let $H_2 = G^*/H_1$. Clearly, $(H_2, \sigma^*|_{H_2})$ is a signed Eulerian planar graph with $v(H_2) < v(G^*)$. Thus $(H_2, \sigma^*|_{H_2})$ admits a circular $\frac{4k}{2k-1}$-flow, as its dual graph is a proper subgraph of the minimum counterexample (G, σ). Since H_1 is strongly $\mathbb{Z}_\ell$-connected and G^*/H_1 admits a circular $\frac{4k}{2k-1}$-flow, (G^*, σ^*) admits a circular $\frac{4k}{2k-1}$-flow by Theorem 5.6. Hence, (G, σ) is circular $\frac{4k}{2k-1}$-colorable by duality, a contradiction. □

Another natural implication of Theorem 1.8 is Theorem 1.6, to obtain which we need the next result. We denote by $2G$ the graph obtained from G by replacing each edge with 2 parallel edges.

Theorem 5.9. *Given a graph G and an integer k with $k \geqslant 2$, if $2G$ is strongly $\mathbb{Z}_{2k}$-connected, then for any signature σ on G, the signed graph (G, σ) admits a circular $\frac{2k}{k-1}$-flow.*

Proof. For each edge $e \in E(G)$, we denote the parallel edges in $E(2G)$ corresponding to e by e_i, $i \in [2]$. Let $\widehat{G} := (G, \sigma)$. Since $2G$ is strongly $\mathbb{Z}_{2k}$-connected, for any $(4k, \beta)$-boundary with $\beta(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$ of $2G$, there is a $(4k, \beta)$-orientation D_1 on $2G$.

Let D be an orientation on G. Let I be a mapping from $E(2G)$ to $\{1, -1\}$ satisfying that $I(e_i) = 1$ if e_i in D_1 has the same orientation as e in D and $I(e_i) = -1$ otherwise. We define a mapping $f_1(e) := I(e_1) + I(e_2)$ for each $e \in E(G)$. Note that $f_1(e) \in \{-2, 0, 2\}$ for each edge $e \in E(G)$ and $\partial_D f_1(v) \equiv \beta(v) \pmod{4k}$ for each vertex $v \in V(G)$.

We then define another mapping $f_2 : E(G) \to \{0, 2k\}$ satisfying that $f_2(e) = 2k$ for each positive edge e and $f_2(e) = 0$ for each negative edge e. Thus $\partial_D f_2(v) \equiv 2k \cdot p_{\widehat{G}}(v) \pmod{4k}$ for each vertex $v \in V(G)$.

Let $f = f_1 + f_2$. Note that $f(e) \in \{2k-2, 2k, 2k+2\}$ for each positive edge e, $f(e) \in \{-2, 0, 2\}$ for each negative edge e and $\partial_D f(v) = \partial_D f_1(v) + \partial_D f_2(v) \equiv 0 \pmod{4k}$ for each vertex $v \in V(G)$. Hence, (D, f) is a circular $\frac{4k}{2k-2}$-flow, which is equivalent to a circular $\frac{2k}{k-1}$-flow. □

Last, we give the proof of Theorem 1.6 on the chromatic numbers of signed planar graphs with given girth conditions.

Theorem 1.6. *Given an integer k with $2 \leqslant k \leqslant 4$, every signed planar graph of girth at least $3k-2$ is circular $\frac{2k}{k-1}$-colorable.*

Proof. Given an integer k with $2 \leqslant k \leqslant 4$, let (G,σ) be a signed plane graph of girth at least $3k-2$ and (G^*,σ^*) be its dual signed graph. Thus the underlying graph G^* is $(3k-2)$-edge-connected. Thus $2G^*$ is $(6k-4)$-edge-connected, which is strongly $\mathbb{Z}_{2k}$-connected by Theorem 1.8. By Theorem 5.9, (G^*,σ^*) admits a circular $\frac{2k}{k-1}$-flow and by duality, (G,σ) is circular $\frac{2k}{k-1}$-colorable. □

6 Concluding Remarks

This paper proves that every signed bipartite planar graph of negative girth at least $6k-4$ admits a homomorphism to C_{-2k} for $k \in \{2,3,4\}$, noting that the case when $k=1$ is trivial. This negative-girth bound is shown to be tight when $k=2$ in [18], but we do not know whether it is tight for $k=3,4$. From the duality, by Corollary 5.4, Theorem 1.4 also indicates that there exist 6-edge-connected Eulerian planar graphs which are not strongly $\mathbb{Z}_4$-connected. But it is still open whether every $(4k-2)$-edge-connected planar graph is strongly $\mathbb{Z}_{2k}$-connected for $k \geqslant 3$. Note that, by Proposition 2.4, every $(4k-4)$-regular graph G with $v(G) > 2k-1$ is not strongly $\mathbb{Z}_{2k}$-connected. With a special signature, we construct below a $(4k-4)$-edge-connected $(4k-4)$-regular signed Eulerian planar graph that does not admit a circular $\frac{4k}{2k-1}$-flow. Thus its dual provides a signed bipartite planar graph of girth $4k-4$ which does not admit a homomorphism to C_{-2k}.

Given an integer $k \geqslant 2$, let $(\widetilde{W}_{4k-4},\sigma)$ be a signed multi-wheel defined as follows: the vertex set is $\{v_1,\ldots,v_{4k-4},w\}$, the edge set is $\{v_1v_2,\ldots,v_iv_{i+1},\ldots,v_{4k-4}v_1,wv_1,\ldots,wv_{4k-4}\}$, and $\mu(wv_i)=1$, $\mu(v_iv_{i+1})=2k-3$ and $\mu(v_{i+1}v_{i+2})=2k-2$ for $i \in \{1,3,\ldots,4k-5\}$; for each pair of vertices $v_{2i}v_{2i+1}$ (indices are taken modulo $4k-4$), we assign a negative sign to one edge and positive signs to the others, and for the vertex w, assign a negative sign to wv_1 and positive signs to all the other edges incident to w. See Figure 3 for $(\widetilde{W}_8,\sigma)$ as an example.

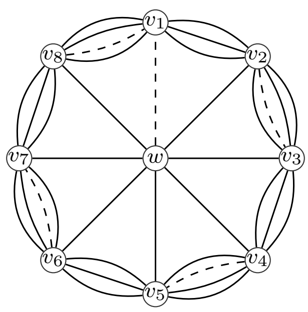


Figure 3: The graph $(\widetilde{W}_8,\sigma)$.

We claim that the signed graph $\widehat{W}_{4k-4} := (\widetilde{W}_{4k-4},\sigma)$ admits no circular $\frac{4k}{2k-1}$-flow.

Note that $\widehat{W}_{4k-4}$ is $(4k-4)$-edge-connected. Given a $(4k,\beta)$-boundary of $\widehat{W}_{4k-4}$ with $\beta(v_1)=0$ and $\beta(v_2)=\cdots=\beta(v_{4k-4})=\beta(w)=2k$, let $\gamma : V(\widehat{W}_{4k-4}) \to \{0,\pm 2k\}$ be the mapping satisfying the conditions (1), (2), (3) of Lemma 2.13 with respect to β. Since $\sum_{v\in V(\widehat{W}_{4k-4})}\gamma(v)=0$, we have $|\gamma^{-1}(2k)|=|\gamma^{-1}(-2k)|=2k-2$ and it implies a natural partition of the vertex set $\{v_2,v_3,\dots,v_{4k-4},w\}$ into V_1 and V_2 such that $\gamma(v)=2k$ for $v\in V_1$ and $\gamma(v)=-2k$ for $v\in V_2$.

Recall that $\gamma(v_1)\equiv 0 \pmod{4k}$ and $\gamma(v)\equiv 2k \pmod{4k}$ for $v\in V_1\cup V_2$, and $\max\{\gamma(v)\}-\min\{\gamma(v)\}\leqslant 4k$. Next, we shall show that there is no γ-orientation for any possible γ, by Lemma 2.13 which implies that there is no $(4k,\beta)$-orientation on $\widehat{W}_{4k-4}$. By Theorem 2.14, it suffices to prove that with respect to γ , there is a bad set in $\widehat{W}_{4k-4}$. We need to consider the following two cases.

- If for some $i\in\{2,\dots,4k-5\}$, $\gamma(v_i)=\gamma(v_{i+1})$, then $|\gamma(v_i)+\gamma(v_{i+1})|=4k$. Moreover, $d(\{v_i,v_{i+1}\})\leqslant 2+2\times(2k-2)=4k-2$. Noting that $|\gamma(v_i)+\gamma(v_{i+1})|>d(\{v_i,v_{i+1}\})$, $\{v_i,v_{i+1}\}$ is a bad set.
- Assume that for any $i\in\{2,\dots,4k-4\}$, $\gamma(v_i)\neq\gamma(v_{i+1})$. By alternating the values $2k$ and $-2k$ on the vertex of the path $v_2v_3\cdots v_{4k-4}$ which is of an even length, we have $\gamma(v_2)=\gamma(v_{4k-4})$. Let $S=\{v_1,v_2,v_{4k-4}\}$. Note that $|\sum_{v\in S}\gamma(v)|=4k$ and $d(S)=3+(2k-2)+(2k-3)=4k-2$. Thus, S is a bad set.

Hence, by Theorem 5.3, $(\widetilde{W}_{4k-4},\sigma)$ admits no circular $\frac{4k}{2k-1}$-flow.

Acknowledgments

The authors would like to thank Daniel W. Cranston for numerous helpful suggestions that improved the presentation of our manuscript. Jiaao Li is partially supported by National Key Research and Development Program of China (No. 2022YFA1006400), National Natural Science Foundation of China (Nos. 12222108, 12131013), Natural Science Foundation of Tianjin (No. 22JCYBJC01520), and the Fundamental Research Funds for the Central Universities, Nankai University. Yongtang Shi and Chunyan Wei are partially supported by the National Natural Science Foundation of China (No. 12161141006), the Natural Science Foundation of Tianjin (No. 20JCJQJC00090) and the Fundamental Research Funds for the Central Universities, Nankai University. Zhouningxin Wang is partially supported by National Natural Science Foundation of China (Nos. 12301444) and the Fundamental Research Funds for the Central Universities, Nankai University.

References

[1] O.V. Borodin, S.-J. Kim, A.V. Kostochka and D. West, Homomorphisms from sparse graphs with large girth, J. Combin. Theory Ser. B, 90 (2004) 147–159.

[2] C. Charpentier, R. Naserasr and E. Sopena, Homomorphisms of sparse signed graphs, Electron. J. Comb., 27 (2020) 3–6.

[3] D.W. Cranston and J. Li, Circular flows in planar graphs, SIAM J. Discrete Math., 34 (2020) 497–519.

[4] Z. Dvořák and L. Postle, Density of 5/2-critical graphs, Combinatorica, 37 (2017) 863–886.

[5] S.L. Hakimi, On the degrees of the vertices of a directed graph, J. Franklin Inst., 279 (1965) 290–308.
[6] M. Han, J. Li, Y. Wu and C.-Q. Zhang, Counterexamples to Jaeger's circular flow conjecture, J. Combin. Theory Ser. B, 131 (2018) 1–11.
[7] F. Jaeger, Nowhere-zero flow problems, Selected Topics in Graph Theory, 3 (1988) 71–95.
[8] F. Kardoš and J. Narboni, On the 4-color theorem for signed graphs, European J. Combin., 91 (2021) 103215.
[9] A. Kostochka and M. Yancey, Ore's conjecture for $k = 4$ and Grötzsch's theorem, Combinatorica, 34 (2014) 323–329.
[10] A. Kostochka and M. Yancey, Ore's conjecture on color-critical graphs is almost true, J. Combin. Theory Ser. B, 109 (2014) 73–101.
[11] H.-J. Lai, Mod $(2p + 1)$-orientations and $K_{1,2p+1}$-decompositions, SIAM J. Discrete Math., 21 (2007) 844–850.
[12] H.-J. Lai and X. Li, Group chromatic number of planar graphs of girth at least 4, J. Graph Theory, 52 (2006) 51–72.
[13] J. Li, H.-J. Lai and R. Luo, Group connectivity, strongly $\mathbb{Z}_m$-connectivity, and edge disjoint spanning trees, SIAM J. Discrete Math., 31 (2017) 1909–1922.
[14] J. Li, R. Naserasr, Z. Wang and X. Zhu, Circular flows in mono-directed signed graphs, (2022) arXiv 2212.10757.
[15] J. Li, Y. Wu and C.-Q. Zhang, Circular flows via extended Tutte orientations, J. Combin. Theory Ser. B, 145 (2020) 307–322.
[16] L.M. Lovász, C. Thomassen, Y. Wu and C.-Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, J. Combin. Theory Ser. B, 103 (2013) 587–598.
[17] E. Máčajová, A. Raspaud and M. Škoviera, The chromatic number of a signed graph, Electron. J. Combin., 23 (2016), no. 1, Paper 1.14, 10 pp.
[18] R. Naserasr, L.A. Pham and Z. Wang, Density of C_{-4}-critical signed graphs, J. Combin. Theory Ser. B, 153 (2022) 81–104.
[19] R. Naserasr, E. Rollová and E. Sopena, Homomorphisms of signed graphs, J. Graph Theory, 79 (2015), no. 3, 178–212.
[20] R. Naserasr, R. Škrekovski, Z. Wang and R. Xu, Mapping sparse signed graphs to (K_{2k}, M), J. Graph Theory, 104 (2023), no. 3, 611–644.
[21] R. Naserasr and Z. Wang, Signed bipartite circular cliques and a bipartite analogue of Grötzsch's theorem, Discrete Math., 346 (2023), 113604.
[22] R. Naserasr, Z. Wang and X. Zhu, Circular chromatic number of signed graphs, Electron. J. Combin., 28 (2021), no. 2, Paper No. 2.44, 40 pp.
[23] C.S.J. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc., 36 (1961), 445–450.
[24] L. Postle and E. Smith-Roberge, On the Density of C_7-Critical Graphs, Combinatorica, 42 (2022), no. 2, 253–300.
[25] R.B. Richter, C. Thomassen and D.H. Younger, Group-colouring, group-connectivity, claw-decompositions, and orientations in 5-edge-connected planar graphs, J. Comb., 7 (2016), no. 2-3, 219–232.
[26] E. Sopena, E. Rollová and R. Naserasr, Homomorphisms of planar signed graphs to signed projective cubes, Discrete Math. Theor. Comput. Sci., 15 (2013), no. 3, 1–11.
[27] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, J.

Combin. Theory Ser. B, 102 (2012) 521–529.
[28] C. Thomassen, Graph factors modulo k, J. Combin. Theory Ser. B, 106 (2014) 174–177.
[29] C. Thomassen, Factorizing regular graphs, J. Combin. Theory Ser. B, 141 (2020) 343–351.
[30] W.T. Tutte, A contribution to the theory of chromatic polynomials. Canad. J. Math., 6 (1954), 80–91.
[31] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. the London Math. Society, 1 (1961) 221–230.
[32] A. Vince, Star chromatic number, J. Graph Theory, 12 (1988), no. 4, 551–559.
[33] Y. Wu, Integer flows and modulo orientations, PhD Dissertation, West Virginia University, (2012).
[34] C.-Q. Zhang, Circular flows of nearly eulerian graphs and vertex-splitting, J. Graph Theory, 40 (2002), no. 3, 147–161.
[35] X. Zhu, Circular chromatic number of planar graphs of large odd girth, Electron. J. Combin., 8 (2001), no. 1, Research Paper 25, 11 pp.