Extremal spectral results of planar graphs without vertex-disjoint cycles*

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Abstract Given a planar graph family $\mathscr{F}$, let $\mathrm{ex}_{\mathscr{P}}(n,\mathscr{F})$ and $\mathrm{spex}_{\mathscr{P}}(n,\mathscr{F})$ be the maximum size and maximum spectral radius over all n-vertex $\mathscr{F}$-free planar graphs, respectively. Let tC_k be the disjoint union of t copies of k-cycles, and $t\mathscr{C}$ be the family of t vertex-disjoint cycles without length restriction. Tait and Tobin [Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B 126 (2017) 137–161] determined that K_2+P_{n-2} is the extremal graph among all planar graphs with sufficiently large order n, which give answers to $spex_{\mathscr{P}}(n,tC_\ell)$ and $spex_{\mathscr{P}}(n,t\mathscr{C})$ for $t\geq 3$. In this paper, we first determine $spex_{\mathscr{P}}(n,tC_\ell)$ and $spex_{\mathscr{P}}(n,t\mathscr{C})$ and characterize the unique extremal graph for $1\leq t\leq 2$, $\ell\geq 3$ and sufficiently large n. Secondly, we obtain the exact values of $\mathrm{ex}_{\mathscr{P}}(n,2C_4)$ and $\mathrm{ex}_{\mathscr{P}}(n,2\mathscr{C})$, which answers a conjecture of Li [Planar Turán number of disjoint union of C_3 and C_4, arxiv:2212.12751v1 (2022)]. These present a new exploration of approaches and tools to investigate extremal problems of planar graphs.

Keywords: Turán number; Planar graph; Vertex-disjoint cycles; Quadrilateral

AMS Classification: 05C35; 05C50

1 Introduction

Given a graph family $\mathscr{F}$, a graph is said to be *$\mathscr{F}$-free* if it does not contain any $F\in\mathscr{F}$ as a subgraph. When $\mathscr{F}=\{F\}$, we write F-free instead of $\mathscr{F}$-free. One of the earliest results in extremal graph theory is the Turán's theorem, which gives the maximum number of edges in an n-vertex K_k-free graph. The *Turán number* $\mathrm{ex}(n,\mathscr{F})$ is the maximum number of edges in an $\mathscr{F}$-free graph on n vertices. Füredi and Gunderson [10] determined $\mathrm{ex}(n,C_{2k+1})$ for all n and k. However, the exact value of $\mathrm{ex}(n,C_{2k})$ is still open. Erdős [6] determined $\mathrm{ex}(n,tC_3)$ for $n\geq 400(t-1)^2$, and the unique extremal graph is characterized. Subsequently, Moon [18] showed that Erdős's result is still valid whenever $n>\frac{9}{2}t-12$. Erdős and Pósa [7] showed that $\mathrm{ex}(n,t\mathscr{C})=(2t-1)(n-t)$ for $t\geq 2$ and $n\geq 24t$. For more results on Turán-type problem, we refer the readers to the survey paper [11].

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One extension of the classical Turán number is to study extremal spectral radius in a planar graph with a forbidden structure. The planar spectral extremal value of a given graph family $\mathscr{F}$, denoted by $spex_{\mathscr{P}}(n,\mathscr{F})$, is the maximum spectral radius over all n-vertex $\mathscr{F}$-free planar graphs. An $\mathscr{F}$-free planar graph on n vertices with maximum spectral radius is called an *extremal graph* to $spex_{\mathscr{P}}(n,\mathscr{F})$. Boots and Royle [2] and independently Cao and Vince [3] conjectured that K_2+P_{n-2} is the unique planar graph with the maximum spectral radius. The conjecture was finally proved by Tait and Tobin [22] for sufficiently large n.

In order to study the spectral extremal problems on planar graphs, we first give the following useful theorem.

Theorem 1.1. *Let $n \geq 2.16\times 10^{17}$ and $\mathscr{F}$ be a graph family such that $|V(F)| \leq \frac{n}{2}$ for any graph $F \in \mathscr{F}$. If $K_{2,n-2}$ is $\mathscr{F}$-free, then the extremal graph to $spex_{\mathscr{P}}(n,\mathscr{F})$ contains a copy of $K_{2,n-2}$.*

Let tC_k be the disjoint union of t copies of k-cycles, and $t\mathscr{C}$ be the family of t vertex-disjoint cycles without length restriction. We use J_n to denote the graph obtained from $K_1+(n-1)K_1$ by embedding a maximum matching within its maximum independent set. For $t\geq 3$, it is easy to check that K_2+P_{n-2} is tC_ℓ-free and $t\mathscr{C}$-free. Theorem 1.1 implies that K_2+P_{n-2} is the extremal graph to $spex_{\mathscr{P}}(n,tC_\ell)$ and $spex_{\mathscr{P}}(n,t\mathscr{C})$ for $t\geq 3$ and sufficiently large n. For three positive integers n,ℓ_1,ℓ_2 with $n\geq \ell_1$, let

$$H(\ell_1,\ell_2) := \begin{cases} P_{\ell_1}\cup \frac{n-2-\ell_1}{\ell_2}P_{\ell_2} & \text{if } \ell_2 \mid (n-2-\ell_1), \\ P_{\ell_1}\cup \lfloor\frac{n-2-\ell_1}{\ell_2}\rfloor P_{\ell_2}\cup P_{n-2-\ell_1-\lfloor\frac{n-2-\ell_1}{\ell_2}\rfloor\ell_2} & \text{otherwise.} \end{cases}$$

In the paper, we give answers to $\mathrm{spex}_{\mathscr{P}}(n,tC_\ell)$ for $t\in\{1,2\}$ and $\mathrm{spex}_{\mathscr{P}}(n,2\mathscr{C})$?? as follows.

Theorem 1.2. *For integers $\ell\geq 3$ and $n\geq \max\{2.16\times 10^{17}, 9\times 2^{\ell-1}+3\}$, the graph $K_2+H(2\ell-3,\ell-2)$ is the extremal graph to $spex_{\mathscr{P}}(n,2C_\ell)$.*

Theorem 1.3. *For $n\geq 2.16\times 10^{17}$, $K_2+H(3,1)$ is the extremal graph to $spex_{\mathscr{P}}(n,2\mathscr{C})$.*

Theorem 1.4. *For integers $\ell\geq 3$ and $n\geq \max\{2.16\times 10^{17}, 9\times 2^{\lfloor\frac{\ell-1}{2}\rfloor}+3, \frac{625}{32}\lfloor\frac{\ell-3}{2}\rfloor^2+2\}$,*
(i) $K_{2,n-2}$ is the unique extremal graph to $spex_{\mathscr{P}}(n,C_3)$;
(ii) J_n is the unique extremal graph to $spex_{\mathscr{P}}(n,C_4)$;
(iii) $K_2+H(\lceil\frac{\ell-3}{2}\rceil,\lfloor\frac{\ell-3}{2}\rfloor)$ is the unique extremal graph to $spex_{\mathscr{P}}(n,C_\ell)$ for $\ell\geq 5$.

To prove our main results, we need to study another extension of the classical Turán number, i.e., the planar Turán number. Dowden [5] initiated the following problem: what is the maximum number of edges in an n-vertex $\mathscr{F}$-free planar graph? This extremal number is called planar Turán number of $\mathscr{F}$ and denoted by $\mathrm{ex}_{\mathscr{P}}(n,\mathscr{F})$. The planar Turán number for short cycles are studied in [5, 12, 13], but $\mathrm{ex}_{\mathscr{P}}(n,C_k)$ is still open for general k. For more results on planar Turán-type problem, we refer the readers to a survey of Lan, Shi and Song [15]. It is easy to see that $\mathrm{ex}_{\mathscr{P}}(n,t\mathscr{C})=n-1$ for $t=1$. Lan, Shi and Song [14] showed that $\mathrm{ex}_{\mathscr{P}}(n,t\mathscr{C})=3n-6$ for $t\geq 3$, and the double wheel $2K_1+C_{n-2}$ is the unique extremal graph.We prove the case of $t=2$, which will be used to prove our main theorems.

Theorem 1.5. $\mathrm{ex}_{\mathscr{P}}(n,2\mathscr{C})=2n-1$ *for $n\geq 5$. The extremal graphs are obtained from $2K_1+C_3$ and an independent set of size $n-5$ by joining each vertex of the independent set to any two vertices of the triangle.*

Moreover, Lan, Shi and Song [14] also proved that $\mathrm{ex}_{\mathscr{P}}(n,tC_k)=3n-6$ for all $k,t\geq 3$. They [16] further showed that $\mathrm{ex}_{\mathscr{P}}(n,2C_3)=\lceil\frac{5n}{2}\rceil-5$ and obtained lower bounds of $\mathrm{ex}_{\mathscr{P}}(n,2C_k)$ for $k\geq 4$, which was improved by Li [17] for sufficient large n recently. Li [17] also conjectured that $\mathrm{ex}_{\mathscr{P}}(n,2C_4)\leq\frac{19}{7}(n-2)$ for $n\geq 23$, and the bound is sharp for $14\mid(n-2)$. In this paper, we determine the exact value of $\mathrm{ex}_{\mathscr{P}}(n,2C_4)$ for large n.

Theorem 1.6. *For $n\geq 2661$,*

$$\mathrm{ex}_{\mathscr{P}}(n,2C_4)=\begin{cases}\frac{19n}{7}-6 & \text{if } 7\mid n,\\ \lfloor\frac{19n-34}{7}\rfloor & \text{otherwise.}\end{cases}$$

2 Proof of Theorem 1.5

Above all, we shall introduce the *Jordan Curve Theorem*: any simple closed curve C in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets (see [1], P. 244). The corresponding two open sets are called the interior and the exterior of C. We denote them by $int(C)$ and $ext(C)$, and their closures by $Int(C)$ and $Ext(C)$, respectively. A *plane graph* is a planar embedding of a planar graph. The Jordan Curve Theorem gives the following lemma.

Lemma 2.1. *Let C be a cycle of a plane graph G, and let x,y be two vertices of G with $x\in int(C)$ and $y\in ext(C)$, then $xy\notin E(G)$.*

Let G be a plane graph. A face in G of size i is called an i-face. Let $f_i(G)$ denote the number of i-faces in G, and let $f(G)$ denote $\sum_i f_i(G)$.

Lemma 2.2. (Proposition 2.5 of [1], P. 250) *Let G be a planar graph, and let f be an arbitrary face in some planar embedding of G. Then G admits a planar embedding whose outer face has the same boundary as f.*

Let $\delta(G)$ be the minimum degree of a graph G. It is well known that every graph G with $\delta(G)\geq 2$ contains a cycle. In the following, we give a more delicate characterization on planar graphs, which contains an important structural information of the extremal graphs in Theorem 1.5.

Lemma 2.3. *Let G be a plane graph on n vertices with $\delta(G)\geq 3$. Then G contains two vertex-disjoint cycles unless $G\in\{2K_1+C_3,K_1+C_{n-1}\}$.*

Proof. We first deal with some trivial cases. Since $\delta(G)\geq 3$, we have $n\geq 1+\delta(G)\geq 4$. If $n=4$, then $G\cong K_1+C_3$. If $n=5$, then $2e(G)=\sum_{v\in V(G)}d_G(v)\geq 3\times 5=15$, and so $e(G)\geq 8$. On the other hand, $e(G)\leq 3n-6=9$, since G is planar. Thus, $e(G)\in\{8,9\}$. It is not hard to verify that $G\cong 2K_1+C_3$ when $e(G)=9$ and $G\cong K_1+C_4$ when $e(G)$=8, as desired. If G is not connected, then G contains at least two components G_1 and G_2 with $\delta(G_i)\geq 3$ for $i\in\{1,2\}$, which implies that each G_i contains a cycle. Thus, G contains two vertex-disjoint cycles, as desired. If G has a cut vertex v, then $G-\{v\}$ has at least two components G_3 and G_4. Since $\delta(G)\geq 3$, we have $\delta(G_i)\geq 2$ for $i\in\{3,4\}$, which implies that both G_3 and G_4 contain a cycle. Thus, G also contains two vertex-disjoint cycles.

Next, we only need to consider the case that G is a 2-connected graph of order $n\geq 6$. Since G is 2-connected, each face of G is a cycle. Let C be a face of G with minimum size g. By Lemma 2.2, we may assume without loss of generality that C is the outer face of G. Let $G_1=G-V(C)$.

If G_1 contains a cycle, then G contains two vertex-disjoint cycles, as desired. Now assume that G_1 is acyclic. Since $\delta(G) \geq 3$, we have $2e(G) = \sum_{v \in V(G)} d_G(v) \geq 3n$. This, together with Euler's formula $n-2 = e(G) - f(G)$, gives $e(G) \leq 3f(G) - 6$. On the other hand,

$$2e(G) = \sum_{i \geq g} i f_i(G) \geq g \sum_{i \geq g} f_i(G) = g f(G).$$

Hence, $gf(G) \leq 2e(G) \leq 6f(G) - 12$, yielding $g \leq \frac{6f(G)-12}{f(G)} < 6$. Subsequently, we shall give several claims.

Claim 2.1. *We have* $g = 3$.

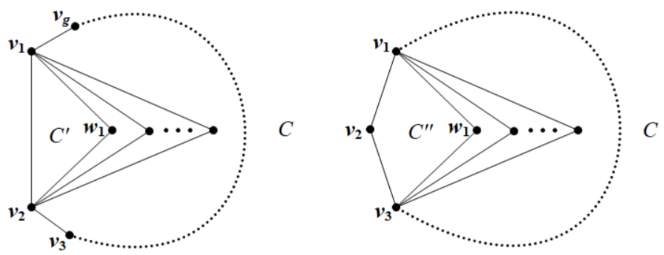

Figure 1: Two possible local structures of G.

Proof. Suppose to the contrary that $g \in \{4,5\}$, and let $C = v_1 v_2 \ldots v_g v_1$. We first consider the case that there exists a vertex of G_1 adjacent to two consecutive vertices of C. Without loss of generality, let $w_1 \in V(G_1)$ and $\{w_1, v_1, v_2\}$ induces a triangle C'. More generally, we define $A = \{w \in V(G_1) \mid v_1, v_2 \in N_C(w)\}$. Clearly, $w_1 \in A$. We can select a vertex, say w_1, in A such that $A \subseteq Ext(C')$ (see Fig. 1). Notice that C' is not a face of G, as $g \in \{4,5\}$. Then, $int(C') \neq \varnothing$. By Lemma 2.1, every vertex in $int(C')$ has no neighbors in $ext(C')$. Moreover, by the definitions of A and w_1, every vertex in $int(C')$ has at most one neighbor in $\{v_1, v_2\}$. It follows that every vertex in $int(C')$ has at least one neighbor in $int(C')$, as $\delta(G) \geq 3$. Thus, $G[int(C')]$ is nonempty, that is, $G_1[int(C')]$ is nonempty. Recall that G_1 is acyclic. Then $G_1[int(C')]$ contains at least two pendant vertices, one of which (say w_2) is not adjacent to w_1. Hence, w_2 is also a pendant vertex of G_1, as w_2 has no neighbors in $ext(C')$. On the other hand, w_2 has at most one neighbor in $\{v_1, v_2\}$, and so $d_C(w_2) \leq 1$. Therefore, $d_G(w_2) = d_{G_1}(w_2) + d_C(w_2) \leq 2$, contradicting $\delta(G) \geq 3$.

Now it remains the case that each vertex of G_1 is not adjacent to two consecutive vertices of C. Note that $\delta(G) \geq 3$ and G_1 is acyclic. Then G_1 contains a vertex w_0 with $d_{G_1}(w_0) \leq 1$, and thus $d_C(w_0) = d_G(w_0) - d_{G_1}(w_0) \geq 2$. Now, since $g \in \{4,5\}$, we may assume without loss of generality that $v_1, v_3 \in N_C(w_0)$. Let $A' = \{w \in V(G_1) \mid v_1, v_3 \in N_C(w)\}$. Clearly, $w_0 \in A'$ and $v_2 \notin N_C(w)$ for each $w \in A'$. Now, we can select a vertex, say w_1, in A' such that $A' \subseteq Ext(C'')$, where $C'' = w_1 v_1 v_2 v_3 w_1$ (see Figure 1). We can see that $int(C'') \neq \varnothing$ (otherwise, $d_G(v_2) = |\{v_1, v_3\}| = 2$, a contradiction). By the definition of w_1, we have $int(C'') \cap A' = \varnothing$. Furthermore, every vertex in $int(C'')$ has no neighbors in $ext(C'')$ and has at most one neighbor in $\{v_1, v_2, v_3\}$. Thus, every vertex in $int(C'')$ has at least one neighbor in $int(C'')$. By a similar argument as above, we can find a vertex $w_2 \in int(C'')$ with $d_G(w_2) = d_{G_1}(w_2) + d_C(w_2) \leq 2$, which contradicts $\delta(G) \geq 3$. $\square$

By Claim 2.1, the outer face of G is a triangle $C = v_1v_2v_3v_1$. In the following, we denote $B_i = \{w \in V(G_1) \mid d_C(w) = i\}$ for $i \leq 3$. Since $\delta(G) \geq 3$, we have $w \in B_3$ for each isolated vertex w of G_1, and $w \in B_2 \cup B_3$ for each pendant vertex w of G_1.

Claim 2.2. $|B_3| \leq 1$ *and* $|B_2| + |B_3| \geq 2$.

Proof. Since C is the outer face of G, every vertex of G_1 lies in $int(C)$. Furthermore, since G is planar, it is easy to see that $|B_3| \leq 1$. This implies that G_1 contains at most one isolated vertex. Recall that $|G_1| = n-3 \geq 3$ and G_1 is acyclic. Then G_1 contains at least two pendant vertices w_1 and w_2. Therefore, $|B_2| + |B_3| \geq |\{w_1, w_2\}| = 2$. □

Claim 2.3. *Let w_0, w_1 be two vertices in $V(G_1)$ such that $N_C(w_0) \supseteq \{v_3\}$ and $N_C(w_1) \supseteq \{v_1, v_2\}$ (see Fig. 2). Then*
(i) $v_3, w_0 \in ext(C''')$, where $C''' = w_1v_1v_2w_1$;
(ii) if $w_0 \notin B_3$, then G_1 contains a pendant vertex in $ext(C''')$.

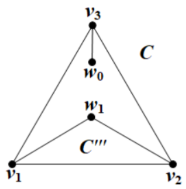

Figure 2: A local structure in Claim 2.3.

Proof. (i) Since C is the outer face and $v_3 \in V(C) \setminus V(C''')$, we have $v_3 \in ext(C''')$. Furthermore, using $w_0v_3 \in E(G)$ and Lemma 2.1 gives $w_0 \in ext(C''')$.

(ii) Since $w_0 \notin B_3$, we have $d_C(w_0) \leq 2$, and so $d_{G_1}(w_0) = d_G(w_0) - d_C(w_0) \geq 1$. By (i), we know that $w_0 \in ext(C''')$. If $d_{G_1}(w_0) = 1$, then w_0 is a desired pendant vertex. It remains the case that $d_{G_1}(w_0) \geq 2$. Now, whether w_1 is a neighbor of w_0 or not, w_0 has at least one neighbor in $V(G_1) \cap ext(C''')$. Thus, $G_1[ext(C''')]$ is nonempty. Recall that G_1 is acyclic. Then $G_1[ext(C''')]$ contains at least two pendant vertices, one of which (say w_2) is not adjacent to w_1. Hence, w_2 is also a pendant vertex of G_1, as w_2 has no neighbors in $int(C''')$. □

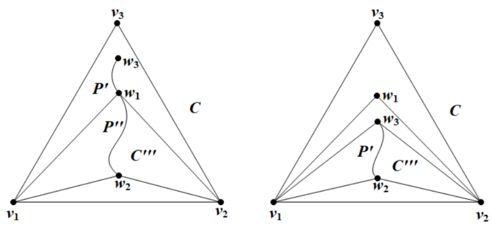

Figure 3: Two possible local structures in Claim 2.4.

Claim 2.4. *Let $w_1, w_2 \in V(G_1)$ with $N_C(w_1) \cap N_C(w_2) \supseteq \{v_1, v_2\}$. Assume that $C''' = w_1v_1v_2w_1$ and $w_2 \in int(C''')$ (see Fig. 3). Then G contains a cycle C_{v_i} such that $V(C_{v_i}) \subseteq Int(C''')$ and $V(C_{v_i}) \cap V(C) = \{v_i\}$ for each $i \in \{1, 2\}$.*

Proof. We first claim that $N_C(w_2) = \{v_1, v_2\}$. By Claim 2.3, we know that $v_3 \in ext(C''')$. Now, since $w_2 \in int(C''')$, we have $w_2v_3 \notin E(G)$ by Lemma 2.1, and so $N_C(w_2) = \{v_1, v_2\}$. Furthermore, we have $d_{G_1}(w_2) \geq 1$. Then G_1 contains a path P with endpoints w_2 and w_3, where w_3 is a pendant vertex of G_1. If $V(P) \not\subseteq int(C''')$, then by $w_2 \in int(C''')$ and Lemma 2.1, we have $V(P) \cap V(C''') = \{w_1\}$ as $v_1, v_2 \notin V(G_1)$. Now let P' be the subpath of P with endpoints w_2 and w_1. Then $V(P') \setminus \{w_1\} \subseteq int(C''')$, and G contains a cycle $C(v_i) = v_iw_1P'w_2v_i$ for each $i \in \{1,2\}$, as desired. Next, assume that $V(P) \subseteq int(C''')$. Then, $w_3 \in int(C''')$. By $v_3 \in ext(C''')$ and Lemma 2.1, we get that $w_3v_3 \notin E(G)$, and so $w_3 \notin B_3$. Moreover, $d_{G_1}(w_3) = 1$ and $\delta(G) \geq 3$ give $w_3 \in B_2$. Thus, $N_C(w_3) = \{v_1, v_2\}$. Therefore, G contains a cycle $C_{v_i} = v_iw_2Pw_3v_i$ for each $i \in \{1,2\}$, as desired. $\square$

Having above four claims, we are ready to give the final proof of Lemma 2.3. By Claim 2.2, we have $|B_3| \leq 1$ and $|B_2| \geq 1$. We may without loss of generality that $w_1 \in B_2$ and $N_C(w_1) = \{v_1, v_2\}$. For each $i \in \{1,2\}$, let $\bar{i} \in \{1,2\} \setminus \{i\}$. Since $d_C(w_1) = 2$, we have $d_{G_1}(w_1) \geq 1$. Hence, G_1 is nonempty, and so G_1 contains at least two pendant vertices. According to the size of B_3, we now distinguish two cases to complete the proof.

Case 1. $|B_3| = 1$.

Assume that $B_3 = \{w_0\}$. Then $N_C(w_0) = \{v_1, v_2, v_3\}$ (see Fig. 4). We then consider two subcases according to the size of B_2.

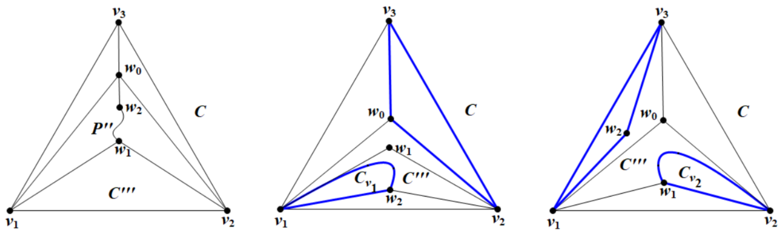

Figure 4: Three possible structures in Case 1.

Subcase 1.1. $|B_2| = 1$, that is, $B_2 = \{w_1\}$.

For each pendant vertex w of G_1, we have $d_C(w) = d_G(w) - d_{G_1}(w) \geq 2$, consequently, $w \in B_2 \cup B_3 = \{w_1, w_0\}$. This indicates that G_1 contains exactly two pendant vertices w_1 and w_0. Furthermore, we can see that G_1 contains no isolated vertices (otherwise, every isolated vertex of G_1 has at least three neighbors in $V(C)$ and so belongs to B_3, while the unique vertex $w_0 \in B_3$ is a pendant vertex of G_1). Therefore, G_1 is a path of order $n - |C|$ with endpoints w_1 and w_0.

Now we know that G_1 is a path with $|G_1| = n - 3 \geq 3$. Let $N_{G_1}(w_0) = \{w_2\}$ and $P'' = G_1 - \{w_0\}$. Then P'' is a path with endpoints w_1 and w_2. Since $d_{G_1}(w_2) = 2$, we have $d_C(w_2) \geq 1$. If $w_2v_3 \in E(G)$, then G contains two vertex-disjoint cycles $v_3w_0w_2v_3$ and $w_1v_1v_2w_1$, as desired. If $w_2v_i \in E(G)$ for some $i \in \{1,2\}$, then G contains two vertex-disjoint cycles $v_iw_1P''w_2v_i$ and $w_0v_{\bar{i}}v_3w_0$, as desired.

Subcase 1.2. $|B_2| \geq 2$.

Let $w_2 \in B_2 \setminus \{w_1\}$. If $N_C(w_1) = N_C(w_2)$, then we may assume that $w_2 \in int(C''')$ by the symmetry of w_1 and w_2, where $C''' = w_1v_1v_2w_1$. By Claim 2.4, G contains a cycle C_{v_1} such that $V(C_{v_1}) \subseteq Int(C''')$ and $V(C_{v_1}) \cap V(C) = \{v_1\}$. On the other hand, Claim 2.3 implies that $w_0 \in ext(C''')$. Hence, $w_0 \notin V(C_{v_1})$. Therefore, G contains two vertex-disjoint cycles C_{v_1} and $w_0v_2v_3w_0$, as desired.

It remains the case that $N_C(w_1) \neq N_C(w_2)$. Now $N_C(w_2) = \{v_i, v_3\}$ for some $i \in \{1,2\}$. We define $C''' = w_0v_1v_2w_0$ instead of the original one in Claim 2.4. Then $w_1 \in int(C''')$. Moreover, $w_2 \in ext(C''')$ as $w_2v_3 \in E(G)$. By Claim 2.4, there exists a cycle $C_{v_{\bar{i}}}$ such that $V(C_{v_{\bar{i}}}) \subseteq Int(C''')$ and $V(C_{v_{\bar{i}}}) \cap V(C) = \{v_{\bar{i}}\}$. Therefore, G contains two vertex-disjoint cycles $C_{v_{\bar{i}}}$ and $w_2v_iv_3w_2$, as desired.

Case 2. $|B_3| = 0$.

Recall that $A = \{w \in V(G_1) \mid v_1, v_2 \in N_C(w)\}$. Since $|B_3| = 0$, we can see that $N_C(w) = N_C(w_1) = \{v_1, v_2\}$ for each $w \in A$. We may assume without loss of generality that $A \subseteq Int(C''')$ by the symmetry of vertices in A. By Claim 2.3, there exists a pendant vertex w_3 of G_1 in $ext(C''')$, which implies that $d_C(w_3) \geq 2$. Since $|B_3| = 0$, we have $w_3 \in B_2$, and thus $B_2 \supseteq \{w_1, w_3\}$. Moreover, $w_3 \notin A$ as $A \subseteq Int(C''')$. Assume without loss of generality that $N_C(w_3) = \{v_1, v_3\}$ (see Fig. 5). We also consider two subcases according to $|B_2|$.

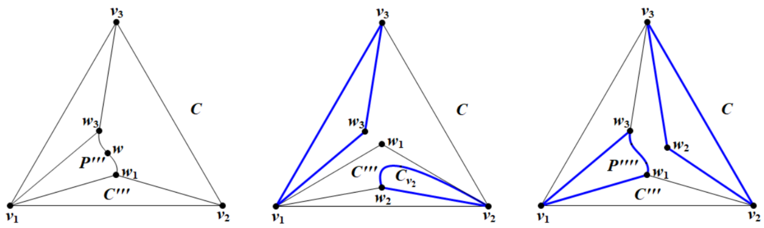

Figure 5: Three possible structures in Case 2.

Subcase 2.1. $|B_2| = 2$, that is, $B_2 = \{w_1, w_3\}$.

Since w_3 is a pendant vertex of G_1, which implies that G_1 is non-empty and has at least two pendant vertices. On the other hand, since $\delta(G) \geq 3$ while $B_3 = \varnothing$, we can see that G_1 contains no isolated vertices, and $w \in B_2 = \{w_1, w_3\}$ for each pendant vertex w of G_1. Therefore, G_1 contains exactly two pendant vertices w_1 and w_3, more precisely, G_1 is a path with endpoints w_1 and w_3. Let w be an arbitrary vertex in $V(G_1) \setminus \{w_1, w_3\}$. Then, $d_C(w) = d_G(w) - d_{G_1}(w) = d_G(w) - 2 \geq 1$.

If $wv_2 \in E(G)$, then G contains two vertex-disjoint cycles $v_2w_1P'''wv_2$ and $v_1w_3v_3v_1$, where P''' is the subpath of G_1 from w_1 to w (see Fig. 5(a)). If $wv_3 \in E(G)$, then G contains two vertex-disjoint cycles $v_3w_3P'''wv_3$ and $v_1w_1v_2v_1$, where P''' is the subpath of G_1 from w_3 to w (see Fig. 5(a)). If $N_C(w) = \{v_1\}$ for each $w \in V(G_1) \setminus \{w_1, w_3\}$, then $G \cong K_1 + C_{n-1}$, as desired.

Subcase 2.2. $|B_2| \geq 3$.

For each vertex $w \in B_2$, it is clear that $N_C(w)$ is one of $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$. We first consider the case that there exist two vertices in B_2 which have the same neighbors in C. Without loss of generality, assume that we can find a vertex $w_2 \in B_2$ with $N_C(w_2) = N_C(w_1) = \{v_1, v_2\}$. Then $w_2 \in A$. Recall that $A \subseteq Int(C''')$ and $C''' = w_1v_1v_2w_1$ (see Fig. 5(b)). Then, we can further get that $w_2 \in int(C''')$. By Claim 2.4, there exists a cycle C_{v_2} such that $V(C_{v_2}) \subset Int(C''')$ and $V(C_{v_2}) \cap V(C) = \{v_2\}$. On the other hand, Claim 2.3 implies that $w_3 \in ext(C''')$. Hence, $w_3 \notin V(C_{v_2})$. Therefore, G contains two vertex-disjoint cycles C_{v_2} and $w_3v_1v_3w_3$, as desired.

Now it remains the case that any two vertices in B_2 have different neighborhoods in C. This implies that $|B_2| = 3$ and we can find a vertex $w_2 \in B_2$ with $N_C(w_2) = \{v_2, v_3\}$. Now we have $B_2 = \{w_1, w_2, w_3\}$. Furthermore, since $\delta(G) \geq 3$ and $B_3 = \varnothing$, we have $d_{G_1}(w) \geq 1$ for each $w \in V(G_1)$, and if $d_{G_1}(w) = 1$, then $w \in B_2$. Since $|B_2| = 3$, we can see that G_1 has only one connected component, that is, G_1 is a tree and some w_i, say w_2, is a pendant vertex of G_1. Now, $G_1 - \{w_2\}$ contains a subpath P'''' with endpoints w_1 and w_3 (see Fig. 5(c)). Then G contains two vertex-disjoint cycles $v_1w_1P''''w_3v_1$ and $w_2v_2v_3w_2$, as desired.

This completes the proof of Lemma 2.3. $\square$

Let $\mathscr{G}_n^*$ be the family of graphs obtained from $2K_1 + C_3$ and an independent set of size $n-5$ by joining each vertex of the independent set to arbitrary two vertices of the triangle (see Fig. 6). Clearly, every graph in $\mathscr{G}_n^*$ is planar. Now, let $\mathscr{G}_n$ be the family of planar graphs obtained from $2K_1 + C_3$ by iteratively adding vertices of degree 2 until the resulting graph has n vertices. Then $\mathscr{G}_n^* \subseteq \mathscr{G}_n$.

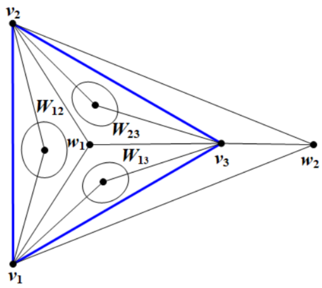

Figure 6: An extremal graph in $\mathscr{G}_n^*$.

Lemma 2.4. *For any graph $G \in \mathscr{G}_n$, G is $2\mathscr{C}$-free if and only if $G \in \mathscr{G}_n^*$.*

Proof. Let $V_1 := \{v_1, v_2, v_3\}$ be the set of vertices of degree 4 and $V_2 := \{w_1, w_2\}$ be the set of vertices of degree 3 in $2K_1 + C_3$, respectively. Then V_1 induces a triangle. We first show that every graph G in $\mathscr{G}_n^*$ is $2\mathscr{C}$-free. It suffices to prove that every cycle of G contains at least two vertices in V_1. Let C be an arbitrary cycle of G. If $V(C) \subseteq V_1$, then there is nothing to prove. It remains the case that there exists a vertex $w \in V(C) \setminus V_1$. By the definition of $\mathscr{G}_n^*$, we can see that $N_C(w) \subseteq N_G(w) \subseteq V_1$. Note that $|N_C(w)| \geq 2$. Hence, C contains at least two vertices in V_1.

In the following, we will show that every graph $G \in \mathscr{G}_n \setminus \mathscr{G}_n^*$ contains two vertex-disjoint cycles. By the definition of $\mathscr{G}_n$, G is obtained from $2K_1 + C_3$ by iteratively adding $n-5$ vertices $u_1, u_2, \ldots, u_{n-5}$ of degree 2. Now, let $G_{n-5} = G$, and $G_{i-1} = G_i - \{u_i\}$ for $i \in \{1, 2, \ldots, n-5\}$. Then $G_0 \cong 2K_1 + C_3$. Moreover, $|G_i| = i + 5$ and $d_{G_i}(u_i) = 2$ for each $i \in \{1, 2, \ldots, n-5\}$. Now let

$$i^* = \max\{i \mid 0 \leq i \leq n-5,\ G_i \in \mathscr{G}_{i+5}^*\}.$$

Since $G_0 = 2K_1 + C_3 \in \mathscr{G}_5^*$ and $G_{n-5} \notin \mathscr{G}_n^*$, we have $0 \leq i^* \leq n-6$. By the choice of i^*, we know that $G_{i^*} \in \mathscr{G}_{i^*+5}^*$ and $G_{i^*+1} \notin \mathscr{G}_{i^*+6}^*$, which implies that $N_{G_{i^*}}(u_i^*) \subseteq V_1$ and $N_{G_{i^*+1}}(u_{i^*+1}) \nsubseteq V_1$.

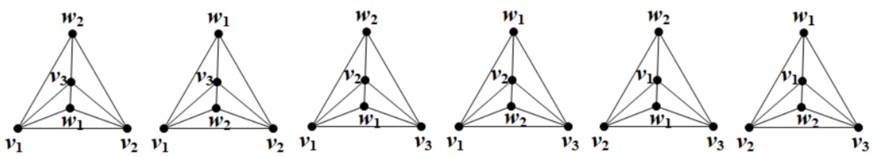

Figure 7: An extremal graph in $\mathscr{G}_n^*$.

Now we may assume that G_{n-5} is a planar embedding of G, and G_0 is a plane subgraph of G_{n-5}. Observe that $2K_1+C_3$ has six planar embeddings (see Fig. 7). Without loss of generality, assume that G_0 is the leftmost graph in Fig. 7. Then, u_{i^*+1} lies in one of the following regions (see Fig. 7):

$$ext(w_2v_1v_2w_2), int(w_2v_1v_3w_2), int(w_2v_2v_3w_2),$$

$$int(w_1v_1v_2w_1), int(w_1v_1v_3w_1), int(w_1v_2v_3w_1).$$

By Lemma 2.2, we can assume that u_{i^*+1} lies in the outer face, that is, $u_{i^*+1} \in ext(w_2v_1v_2w_2)$. For simplify, we denote $C' = w_2v_1v_2w_2$. Let u be an arbitrary vertex with $uv_3 \in E(G_{i^*+1})$. Then by Lemma 2.1 and $v_3 \in int(C')$, we have $u \in int(C')$, and thus $uu_{i^*+1} \notin E(G_{i^*+1})$. This implies that $N_{G_{i^*+1}}(u_{i^*+1}) \subseteq V(C') \cup W_{12}$, where $W_{12} = \{u \mid u \in V(G_{i^*+1}),\ N_{G_{i^*+1}}(u) = \{v_1, v_2\}\}$. Recall that $d_{G_{i^*+1}}(u_{i^*+1}) = 2$ and $N_{G_{i^*+1}}(u_{i^*+1}) \nsubseteq V_1$. Then, $|N_{G_{i^*+1}}(u_{i^*+1}) \cap \{v_1, v_2\}| \leq 1$. If $|N_{G_{i^*+1}}(u_{i^*+1}) \cap \{v_1, v_2\}| = 1$, then we may assume without loss of generality that $v_1 \in N_{G_{i^*+1}}(u_{i^*+1})$, and $u' \in N_{G_{i^*+1}}(u_{i^*+1}) \setminus \{v_1\}$. Since $N_{G_{i^*+1}}(u_{i^*+1}) \subseteq V(C') \cup W_{12}$, we have $u' \in \{w_2\} \cup W_{12}$, and so $u'v_1 \in E(G_{i^*+1})$. Thus, G_{n-5} contains two vertex-disjoint cycles $u_{i^*+1}v_1u'u_{i^*+1}$ and $w_1v_2v_3w_1$, as desired. Now consider the case that $|N_{G_{i^*+1}}(u_{i^*+1}) \cap \{v_1, v_2\}| = 0$. This implies that $N_{G_{i^*+1}}(u_{i^*+1}) \subseteq \{w_2\} \cup W_{12}$. Let $N_{G_{i^*+1}}(u_{i^*+1}) = \{u', u''\}$. Then $u'v_1, u''v_1 \in E(G_{i^*+1})$. Therefore, G_{n-5} contains two vertex-disjoint cycles $u_{i^*+1}u'v_1u''u_{i^*+1}$ and $w_1v_2v_3w_1$. □

Given a graph G, let $\widetilde{G}$ be the largest induced subgraph of G with minimal degree at least 3. It is easy to see that $\widetilde{G}$ can be obtained from G by iteratively removing the vertices of degree at most 2 until the resulting graph has minimum degree at least 3 or is empty. It is well known that $\widetilde{G}$ is unique and does not depend on the order of vertex deletion (see [21]).

In the following, we give the proof of Theorem 1.5.

Proof. Let $n \geq 5$ and G be an extremal graph corresponding to $\mathrm{ex}_{\mathscr{P}}(n, 2\mathscr{C})$. Observe that $K_2 + (P_3 \cup (n-5)K_1)$ is a planar graph which contains no two vertex-disjoint cycles (see Fig. 6). Thus, $e(G) \geq e(K_2 + (P_3 \cup (n-5)K_1)) = 2n-1$.

If $\widetilde{G}$ is empty, then $e(G) \leq 2(n-1)$ because G is simple, a contradiction.

Now we know that $\widetilde{G}$ is nonempty. Then, $\widetilde{G}$ contains no two vertex-disjoint cycles as $\widetilde{G} \subseteq G$. By the definition of $\widetilde{G}$, we have $\delta(\widetilde{G}) \geq 3$. By Lemma 2.3, we get that $\widetilde{G} \in \{2K_1 + C_3, K_1 + C_{|\widetilde{G}|-1}\}$. If $\widetilde{G} \cong K_1 + C_{|\widetilde{G}|-1}$, then

$$e(G) \leq e(\widetilde{G}) + 2(n - |\widetilde{G}|) = 2(|\widetilde{G}| - 1) + 2(n - |\widetilde{G}|) = 2n - 2,$$

a contradiction. Thus, $\widetilde{G} \cong 2K_1 + C_3$. Now, $e(G) \leq e(\widetilde{G}) + 2(n-5) = 2n-1$. Therefore, $e(G) = 2n-1$, which implies that $\mathrm{ex}_{\mathscr{P}}(n, 2\mathscr{C}) = 2n-1$ and $G \in \mathscr{G}_n$. By Lemma 2.4, we have $G \in \mathscr{G}_n^*$. This completes the proof of Theorem 1.5. □

3 Proof of Theorem 1.6

We shall further introduce some notations on a plane graph G. A vertex or an edge of G is said to be *incident* with a face F, if it lies on the boundary of F. Clearly, every edge of G is incident with at most two faces. A face of size i is called an i-face. The numbers of i-faces and total faces are denoted by $f_i(G)$ and $f(G)$, respectively. Let $E_3(G)$ be the set of edges incident with at least one 3-face, and particularly, let $E_{3,3}(G)$ be the set of edges incident with two 3-faces. Moreover, let $e_3(G)$ and $e_{3,3}(G)$ denote the cardinalities of $E_3(G)$ and $E_{3,3}(G)$, respectively. We can easily see that $3f_3(G) = e_3(G) + e_{3,3}(G)$.

Lan, Shi and Song proved that $ex_{\mathscr{P}}(n, K_1 + P_3) \leq \frac{12(n-2)}{5}$, with equality when $n \equiv 12 \pmod{20}$ (see [13]), and $ex_{\mathscr{P}}(n, K_1 + P_{k+1}) \leq \frac{13kn}{4k+2} - \frac{12k}{2k+1}$ for $k \in \{3,4,5\}$ (see [14]). For $k \geq 6$, one can easily see that $ex_{\mathscr{P}}(n, K_1 + P_{k+1}) = 3n - 6$. In [8], the authors obtained the following sharp result.

Lemma 3.1. [8]) *Let n, k be two integers with $k \in \{2,3,4,5\}$ and $n \geq \frac{12}{6-k} + 1$. Then $ex_{\mathscr{P}}(n, K_1 + P_{k+1}) \leq \frac{24k}{7k+6}(n-2)$, with equality if $n \equiv \frac{12(k+2)}{6-k} \pmod{\frac{28k+24}{6-k}}$.*

To prove Theorem 1.6, we also need an edge-extremal result on outerplanar graphs. Let $\mathrm{ex}_{\mathscr{OP}}(n, C_k)$ denote the maximum number of edges in an n-vertex C_k-free outerplanar graph.

Lemma 3.2. [9]) *Let n, k, λ be three integers with $n \geq k \geq 3$ and $\lambda = \lfloor \frac{kn-2k-1}{k^2-2k-1} \rfloor + 1$. Then*

$$\mathrm{ex}_{\mathscr{OP}}(n, C_k) = \begin{cases} 2n - \lambda + 2\lfloor \frac{\lambda}{k} \rfloor - 3 & \text{if } k \mid \lambda, \\ 2n - \lambda + 2\lfloor \frac{\lambda}{k} \rfloor - 2 & \text{otherwise.} \end{cases}$$

In particular, we can obtain the following corollary.

Corollary 3.1.

$$\mathrm{ex}_{\mathscr{OP}}(n-1, C_4) = \begin{cases} \frac{12}{7}n - 5 & \text{if } 7 \mid n, \\ \lfloor \frac{12n-27}{7} \rfloor & \text{otherwise.} \end{cases}$$

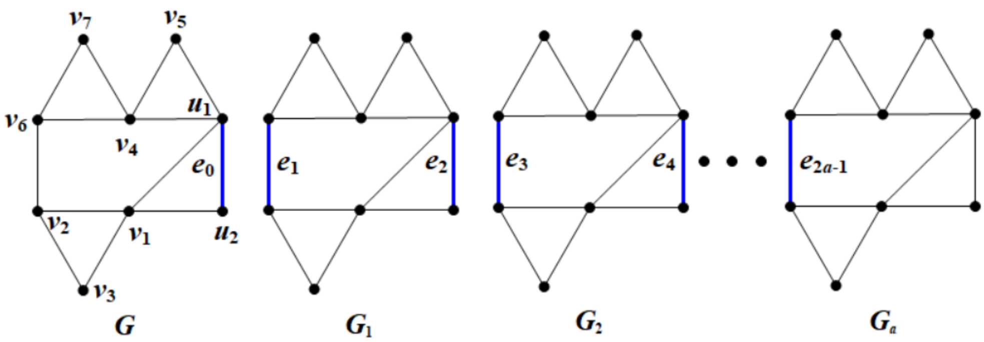

Figure 8: The constructions of $G, G_1, G_2, \ldots, G_a$.

For arbitrary integer $n \geq 4$, we can find a unique (a,b) such that $a \geq 0$, $1 \leq b \leq 7$ and $n - 1 = 7a + b + 2$. Let G be a 9-vertex outerplanar graph and $G_1, \ldots, G_a$ be a copies of G (see Fig. 8). Then, we define G_0 as the subgraph of G induced by $\{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_b\}$. One can check that $|G_0| = b + 2$ and

$$e(G_0) = \begin{cases} \frac{12(b+2)-23}{7} & \text{if } 7 \mid (b+2-6), \\ \lfloor \frac{12(b+2)-15}{7} \rfloor & \text{otherwise.} \end{cases}$$

We now construct a new graph G^* from $G_0, G_1, \ldots, G_a$ by identifying the edges e_{2i} and e_{2i+1} for each $i \in \{0, \ldots, a-1\}$. Clearly, G^* is a connected C_4-free outerplanar graph with

$$|G^*| = \sum_{i=0}^{a} |G_i| - 2a = (2+b) + 9a - 2a = n - 1.$$

Moreover, since $n \equiv b+2-6 (\mod 7)$, we have

$$e(G^*) = \sum_{i=0}^{a} e(G_i) - a = e(G_0) + 12a = \begin{cases} \frac{12}{7}n - 5 & \text{if } 7 \mid n, \\ \lfloor \frac{12n-27}{7} \rfloor & \text{otherwise.} \end{cases}$$

Combining Corollary 3.1, G^* is an extremal graph corresponding to $\mathrm{ex}_{\mathscr{OP}}(n-1, C_4)$.

Lemma 3.3. *Let $n \geq 2661$ and G^{**} be an extremal plane graph corresponding to $\mathrm{ex}_{\mathscr{P}}(n, 2C_4)$. Then G^{**} contains at least fourteen quadrilaterals and all of them share exactly one vertex.*

Proof. Note that $e(G^*) = \mathrm{ex}_{\mathscr{OP}}(n-1, C_4) \geq \frac{12}{7}n - 5$ and G^* is a C_4-free outerplanar graph of order $n-1$. Then $K_1 + G^*$ is an n-vertex $2C_4$-free planar graph, and thus

$$e(G^{**}) \geq e(K_1 + G^*) = e(G^*) + n - 1 \geq \frac{19}{7}n - 6.$$

On the other hand, by Lemma 3.1, we have

$$ex_{\mathscr{P}}(n, K_1 + P_4) \leq \frac{8}{3}(n-2).$$

Note that $\frac{19}{7}n - 6 > \frac{8}{3}(n-2)$ for $n \geq 2661$. Then G^{**} contains a copy, say H_1, of $K_1 + P_4$. Let G_1 be the graph obtained from G^{**} by deleting all edges within $V(H_1)$. Since $|H_1| = 5$, we have $e(G_1) \geq e(G^{**}) - (3|H_1| - 6) = e(G^{**}) - 9 > \frac{8}{3}(n-2)$. Thus, G_1 contains a copy, say H_2, of $K_1 + P_4$. Now we can obtain a new graph G_2 from G_1 by deleting all edges within $V(H_2)$. Note that $e(G^{**}) - 14 \times 9 > \frac{8}{3}(n-2)$. Repeating above steps, we can obtain a graph sequence $G_1, G_2, \ldots, G_{14}$ and fourteen copies $H_1, H_2, \cdots, H_{14}$ of $K_1 + P_4$ such that $H_i \subseteq G_{i-1}$ and G_i is obtained from G_{i-1} by deleting all edges within $V(H_i)$. This also implies that G^{**} contains at least fourteen quadrilaterals. We next give four claims on those copies of $K_1 + P_4$.

Claim 3.1. *Let i, j be two integers with $1 \leq i < j \leq 14$ and $v \in V(H_i) \cap V(H_j)$. Then, $V(H_i) \cap N_{H_j}(v) = \varnothing$.*

Proof. Suppose to the contrary that there exists a vertex $w \in V(H_i) \cap N_{H_j}(v)$. Note that $v, w \in V(H_i)$. By the definition of G_i, whether $vw \in E(H_i)$ or not, we can see that $vw \notin E(G_i)$. On the other hand, note that $H_j \subseteq G_{j-1} \subseteq G_i$, then $vw \in E(H_j) \subseteq E(G_i)$, contradicting $vw \notin E(G_i)$. Hence, the claim holds. □

Claim 3.2. $|V(H_i) \cap V(H_j)| \in \{1, 2\}$ *for any two integers i, j with* $1 \leq i < j \leq 14$.

Proof. If H_i and H_j are vertex-disjoint, then G^{**} contains $2C_4$, a contradiction. Now suppose that there exist three vertices $v_1, v_2, v_3 \in V(H_i) \cap V(H_j)$. Observe that $K_1 + P_4$ contains no an independent set of size 3. Then $H_j[\{v_1, v_2, v_3\}]$ is nonempty. Assume without loss of generality that $v_1 v_2 \in E(H_j)$. Then $v_2 \in V(H_i) \cap N_{H_j}(v_1)$, which contradicts Claim 3.1. Therefore, $1 \leq |V(H_i) \cap V(H_j)| \leq 2$. □

Now for convenience, a vertex v in a graph G is called a 2-*vertex* if $d_G(v) = 2$, and a 2^+-*vertex* if $d_G(v) > 2$. Clearly, every copy of $K_1 + P_4$ contains two 2-vertices and three 2^+-vertices.

Claim 3.3. *Let $\mathscr{H}$ be the family of graphs H_i $(1 \leq i \leq 14)$ such that every 2-vertex in H_i is a 2^+-vertex in H_1. Then $|\mathscr{H}| \leq 3$.*

Proof. Note that H_1 contains only three 2^+-vertices, say v_1, v_2 and v_3. Then every graph $H_i \in \mathscr{H}$ must contain two of v_1, v_2 and v_3 as 2-vertices. Suppose to the contrary that $|\mathscr{H}| \geq 4$. By pigeonhole principle, there exist two graphs $H_{i_1}, H_{i_2} \in \mathscr{H}$ such that they contain the same two 2-vertices, say v_1, v_2. It follows that $H_{i_j} - \{v_j\}$ contains a 4-cycle for $j \in \{1,2\}$. By Claim 3.2, we have $V(H_{i_1}) \cap V(H_{i_2}) = \{v_1, v_2\}$, which implies that $H_{i_1} - \{v_1\}$ and $H_{i_2} - \{v_2\}$ are vertex-disjoint. Hence, G^{**} contains two vertex-disjoint 4-cycles, a contradiction. □

Claim 3.4. *Let j be an integer with $2 \leq j \leq 14$ and $H_j \notin \mathscr{H}$. Then, there exists a vertex $v \in V(H_1) \cap V(H_j)$ such that $d_{H_1}(v) \geq 3$ and $d_{H_j}(v) \geq 3$.*

Proof. By Claim 3.2, we have $1 \leq |V(H_1) \cap V(H_j)| \leq 2$. We first assume that $V(H_1) \cap V(H_j) = \{u\}$. If $d_{H_1}(u) \geq 3$ and $d_{H_j}(u) \geq 3$, then there is nothing to prove. If $d_{H_1}(u) = 2$, then G^{**} contains two vertex-disjoint subgraphs $H_1 - \{u\}$ and H_j, and thus $2C_4$, a contradiction. If $d_{H_j}(u) = 2$, then we can similarly get a contradiction. Therefore, $|V(H_1) \cap V(H_j)| = 2$.

Now, assume that $V(H_1) \cap V(H_j) = \{u_1, u_2\}$. We first deal with the case $d_{H_j}(u_1) = d_{H_j}(u_2) = 2$. Since $H_j \notin \mathscr{H}$, one of $\{u_1, u_2\}$, say u_1, is a 2-vertex in H_1. Hence, G^{**} contains two vertex-disjoint subgraphs $H_1 - \{u_1\}$ and $H_j - \{u_2\}$, and so $2C_4$, a contradiction. Thus, there exists some $i \in \{1,2\}$ with $d_{H_j}(u_i) \geq 3$. If $d_{H_1}(u_i) \geq 3$, then we are done. If $d_{H_1}(u_i) = 2$, then we define H'_j as the subgraph of H_j induced by $N_{H_j}(u_i) \cup \{u_i\}$. Since $d_{H_j}(u_i) \geq 3$, we can check that H'_j always contains a C_4. Moreover, since $d_{H_1}(u_i) = 2$, we can see that $H_1 - \{u_i\}$ also contains a C_4. On the other hand, by Claim 3.1, we have $N_{H_j}(u_i) \cap V(H_1) = \varnothing$, which implies that H'_j and $H_1 - \{u_i\}$ are vertex-disjoint. Therefore, G^{**} contains $2C_4$, a contradiction. □

By Claim 3.3, $|\mathscr{H}| \leq 3$, thus there are at least ten graphs in $\{H_j \mid 2 \leq j \leq 14\} \setminus \mathscr{H}$. However, H_1 has only three 2^+-vertices. By Claim 3.4 and pigeonhole principle, there exists a 2^+-vertex w in H_1 and four graphs, say H_2, H_3, H_4, H_5, of $\{H_j \mid 2 \leq j \leq 14\} \setminus \mathscr{H}$. By Claim 3.1, we get that $N_{H_j}(w) \cap V(H_i) = \varnothing$, and so $N_{H_j}(w) \cap N_{H_i}(w) = \varnothing$, for any i, j with $1 \leq i < j \leq 5$. If $G^{**} - \{w\}$ contains a quadrilateral C', then there exists some $j' \leq 5$ such that $N_{H_{j'}}(w) \cap V(C') = \varnothing$ as $|C'| = 4$. Since w is a 2^+-vertex in $H_{j'}$, we can observe that the subgraph of $H_{j'}$ induced by $N_{H_{j'}}(w) \cup \{w\}$ must contain a C_4. Consequently, G^{**} is not $2C_4$-free, a contradiction. Thus, $G^{**} - \{w\}$ is C_4-free, which implies that all quadrilaterals of G^{**} share exactly one vertex. This completes the proof of Lemma 3.3. □

Now we are ready to give the proof of Theorem 1.6.

Proof. Recall that G^* is an extremal graph corresponding to $\mathrm{ex}_{\mathscr{OP}}(n-1, C_4)$. Then $K_1 + G^*$ is planar and $2C_4$-free. By Corollary 3.1, we have

$$e(K_1 + G^*) = \mathrm{ex}_{\mathscr{OP}}(n-1, C_4) + n - 1 = \begin{cases} \frac{19}{7}n - 6 & \text{if } 7 \mid n, \\ \left\lfloor \frac{19n-34}{7} \right\rfloor & \text{otherwise.} \end{cases} \tag{1}$$

To prove Theorem 1.6, it suffices to show $\mathrm{ex}_{\mathscr{P}}(n, 2C_4) = e(K_1 + G^*)$. Since G^{**} is an extremal plane graph corresponding to $\mathrm{ex}_{\mathscr{P}}(n, 2C_4)$, we have $e(G^{**}) \geq e(K_1 + G^*)$. In the following, we show that $e(G^{**}) \leq e(K_1 + G^*)$.

By Lemma 3.3, all quadrilaterals of G^{**} share a vertex w. Thus, $G^{**} - \{w\}$ is C_4-free. Assume that $d_{G^{**}}(w) = s$ and $w_1, \ldots, w_s$ are around w in clockwise order, with subscripts interpreted modulo s. Let e be an arbitrary edge in $E_{3,3}(G^{**})$, that is, e is incident with two 3-faces, say F and

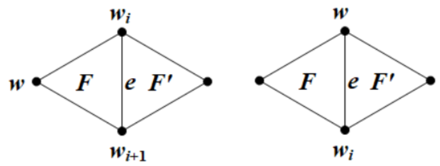

Figure 9: Two possible structures of $H(e)$.

F'. We define $H(e)$ as the plane subgraph induced by all edges incident with F and F'. Clearly, $H(e) \cong K_1 + P_3$ and so it contains a C_4. Recall that all quadrilaterals of G^{**} share exactly one vertex w. Then, $w \in V(H_e)$ and w is incident with at least one face of F and F' (see Fig. 9). Note that e is incident with F. Then, either $e = ww_i$ or $e = w_iw_{i+1}$ for some $i \in \{1, 2, \ldots, s\}$. By the choice of e, we have

$$E_{3,3}(G^{**}) \subseteq \{ww_i, w_iw_{i+1} \mid 1 \le i \le s\}. \tag{2}$$

Assume first that $f_4(G^{**}) = t \ge 1$ and $F_1, \ldots, F_t$ are 4-faces in G^{**}. Since every 4-face is a quadrilateral, w is incident with each 4-face. Consequently, there exists $j_i \in \{1, \ldots, s\}$ such that ww_{j_i}, ww_{j_i+1} are incident with F_i for each $i \in \{1, \ldots, t\}$. Thus, $ww_{j_i} \notin E_{3,3}(G^{**})$ for $1 \le i \le t$. On the other hand, if $w_{j_i}w_{j_i+1} \in E_{3,3}(G^{**})$, then $H(w_{j_i}w_{j_i+1})$ contains a C_4, and so $w \in V(H(w_{j_i}w_{j_i+1}))$. This implies that $ww_{j_i}w_{j_i+1}w$ is a 3-face in G^{**}, contradicting the fact that ww_{j_i}, ww_{j_i+1} are incident with the 4-face F_i. Thus, we also have $w_{j_i}w_{j_i+1} \notin E_{3,3}(G^{**})$ for $1 \le i \le t$.

By the argument above, we can see that

$$E_{3,3}(G^{**}) \cap \{ww_{j_i}, w_{j_i}w_{j_i+1} \mid 1 \le i \le t\} = \varnothing. \tag{3}$$

Using (2) and (3) gives $e_{3,3}(G^{**}) \le 2s - 2t = 2s - 2f_4(G^{**})$. Hence,

$$3f_3(G^{**}) = e_3(G^{**}) + e_{3,3}(G^{**}) \le e(G^{**}) + 2s - 2f_4(G^{**}). \tag{4}$$

On the other hand,

$$2e(G^{**}) = \sum_{i \ge 3} if_i(G^{**}) \ge 3f_3(G^{**}) + 4f_4(G^{**}) + 5(f(G^{**}) - f_3(G^{**}) - f_4(G^{**})),$$

which yields $f(G^{**}) \le \frac{1}{5}\left(2e(G^{**}) + 2f_3(G^{**}) + f_4(G^{**})\right)$. Combining this with Euler's formula $f(G^{**}) = e(G^{**}) - (n-2)$, we obtain

$$e(G^{**}) \le \frac{5}{3}(n-2) + \frac{2}{3}f_3(G^{**}) + \frac{1}{3}f_4(G^{**}). \tag{5}$$

If $f_4(G^{**}) = t = 0$, then (4) and (5) hold directly. Combining (4) and (5), we have $e(G^{**}) \le \frac{15}{7}(n-2) + \frac{4}{7}s - \frac{1}{7}f_4(G^{**})$. Recall that $d_{G^{**}}(w) = s \le n-1$. If $s \le n-2$, then $e(G^{**}) \le \lfloor \frac{19}{7}(n-2) \rfloor \le e(K_1 + G^*)$ by (1), as desired. If $s = n-1$, then w is a dominating vertex of the planar graph G^{**}, which implies that $G^{**} - \{w\}$ is outerplanar. Recall that $G^{**} - \{w\}$ is C_4-free. Thus, $e(G^{**} - \{w\}) \le \mathrm{ex}_{\mathcal{OP}}(n-1, C_4)$, and so $e(G^{**}) \le \mathrm{ex}_{\mathcal{OP}}(n-1, C_4) + n - 1$. Combining (1), we get $e(G^{**}) \le e(K_1 + G^*)$, as required. This completes the proof of Theorem 1.6. $\square$

4 Proof of Theorem 1.1

Let $A(G)$ be the adjacency matrix of a planar graph G, and $\rho(G)$ be its spectral radius, i.e., the maximum modulus of eigenvalues of $A(G)$. Throughout this section, let G be an extremal graph to $spex_{\mathscr{P}}(n,\mathscr{F})$, and ρ denote this spectral radius. By Perron-Frobenius theorem, there exists a Perron eigenvector $X=(x_1,\ldots,x_n)^T$ corresponding to ρ. Choose $u'\in V(G)$ with $x_{u'}=\max\{x_i\mid i=1,2,\ldots,n\}=1$. For a vertex u and a positive integer i, let $N_i(u)$ denote the set of vertices at distance i from u in G. For two disjoint subset $S,T\subset V(G)$, denote by $G[S,T]$ the bipartite subgraph of G with vertex set $S\cup T$ that consist of all edges with one endpoint in S and the other endpoint in T. Set $e(S)=|E(G[S])|$ and $e(S,T)=|E(G[S,T])|$. Since G is a planar graph, we have

$$e(S)\le 3|S|-6 \quad\text{and}\quad e(S,T)\le 2(|S|+|T|)-4. \tag{6}$$

In this section we will often assume that $n\ge 2.16\times 10^{17}$. We first give the lower bound of ρ.

Lemma 4.1. $\rho\ge\sqrt{2n-4}$.

Proof. Note that $K_{2,n-2}$ is planar and $\mathscr{F}$-free. Then, $\rho\ge\rho(K_{2,n-2})=\sqrt{2n-4}$, since G is an extremal graph to $spex_{\mathscr{P}}(n,\mathscr{F})$. □

Set $L^\lambda=\{u\in V(G)\mid x_u\ge\frac{1}{10^3\lambda}\}$ for some constant $\lambda\ge\frac{1}{10^3}$. The following lemmas is used to give an upper bound for $|L^1|$ and a lower bound for degrees of vertices in L^1.

Lemma 4.2. $|L^\lambda|\le\frac{\lambda n}{10^5}$.

Proof. By Lemma 4.1, $\rho\ge\sqrt{2n-4}$. Hence,

$$\frac{\sqrt{2n-4}}{10^3\lambda}\le\rho x_u=\sum_{v\in N_G(u)}x_v\le d_G(u)$$

for each $u\in L^\lambda$. Summing this inequality over all vertices $u\in L^\lambda$, we obtain

$$|L^\lambda|\frac{\sqrt{2n-4}}{10^3\lambda}\le\sum_{u\in L^\lambda}d_G(u)\le\sum_{u\in V(G)}d_G(u)\le 2(3n-6).$$

It follows that $|L^\lambda|\le 3\times 10^3\lambda\sqrt{2n-4}\le\frac{\lambda n}{10^5}$ as $n\ge 2.16\times 10^{17}$. □

Lemma 4.3. $|L^1|\le 6\times 10^4$.

Proof. Let $u\in V(G)$ be an arbitrary vertex. For convenience, we use N_i, L_i^λ and $\overline{L_i^\lambda}$ instead of $N_i(u)$, $N_i(u)\cap L^\lambda$ and $N_i(u)\setminus L^\lambda$, respectively. By Lemma 4.1, $\rho\ge\sqrt{2n-4}$. Then

$$(2n-4)x_u\le\rho^2x_u=d_G(u)x_u+\sum_{v\in N_1}\sum_{w\in N_1(v)\setminus\{u\}}x_w. \tag{7}$$

Note that $N_1(v)\setminus\{u\}\subseteq N_1\cup N_2=L_1^\lambda\cup L_2^\lambda\cup\overline{L_1^\lambda}\cup\overline{L_2^\lambda}$. We can calculate $\sum_{v\in N_1}\sum_{w\in N_1(v)\setminus\{u\}}x_w$ according to two cases $w\in L_1^\lambda\cup L_2^\lambda$ or $w\in\overline{L_1^\lambda}\cup\overline{L_2^\lambda}$. We first consider the case $w\in L_1^\lambda\cup L_2^\lambda$. Clearly, $N_1=L_1^\lambda\cup\overline{L_1^\lambda}$ and $x_w\le 1$ for $w\in L_1^\lambda\cup L_2^\lambda$. We can see that

$$\sum_{v\in N_1}\sum_{w\in(L_1^\lambda\cup L_2^\lambda)}x_w\le\left(2e(L_1^\lambda)+e(L_1^\lambda,L_2^\lambda)\right)+\sum_{v\in\overline{L_1^\lambda}}\sum_{w\in(L_1^\lambda\cup L_2^\lambda)}x_w. \tag{8}$$

By Lemma 4.2, we have $|L^{\lambda}| \leq \frac{\lambda n}{10^5}$. Moreover, $L_1^{\lambda} \cup L_2^{\lambda} \subseteq L^{\lambda}$. Then, by (6), we have

$$2e(L_1^{\lambda}) + e(L_1^{\lambda}, L_2^{\lambda}) \leq 2(3|L_1^{\lambda}| - 6) + (2(|L_1^{\lambda}| + |L_2^{\lambda}|) - 4) < 8|L^{\lambda}| \leq \frac{8\lambda n}{10^5}. \tag{9}$$

Next, we consider the remain case $w \in \overline{L_1^{\lambda}} \cup \overline{L_2^{\lambda}}$. Clearly, $x_w \leq \frac{1}{10^3\lambda}$ for $w \in \overline{L_1^{\lambda}} \cup \overline{L_2^{\lambda}}$. Then

$$\sum_{v\in N_1} \sum_{w\in \overline{L_1^{\lambda}} \cup \overline{L_2^{\lambda}}} x_w \leq \left(e(L_1^{\lambda}, \overline{L_1^{\lambda}} \cup \overline{L_2^{\lambda}}) + 2e(\overline{L_1^{\lambda}}) + e(\overline{L_1^{\lambda}}, \overline{L_2^{\lambda}}) \right) \frac{1}{10^3\lambda} < \frac{6n}{10^3\lambda}, \tag{10}$$

where $e(L_1^{\lambda}, \overline{L_1^{\lambda}} \cup \overline{L_2^{\lambda}}) + 2e(\overline{L_1^{\lambda}}) + e(\overline{L_1^{\lambda}}, \overline{L_2^{\lambda}}) \leq 2e(G) < 6n$.

Combining (7-10), we obtain

$$(2n-4)x_u < d_G(u)x_u + \sum_{v\in \overline{L_1^{\lambda}}} \sum_{w\in (L_1^{\lambda} \cup L_2^{\lambda})} x_w + \left(\frac{8\lambda}{10} + \frac{60}{\lambda} \right) \frac{n}{10^4}. \tag{11}$$

Now we prove that $d_G(u) \geq \frac{n}{10^4}$ for each $u \in L^1$. Suppose to the contrary that there exists a vertex $\widetilde{u} \in L^1$ with $d_G(\widetilde{u}) < \frac{n}{10^4}$. Note that $x_{\widetilde{u}} \geq \frac{1}{10^3}$ as $\widetilde{u} \in L^1$. Setting $u = \widetilde{u}$, $\lambda = 10$ and combining (11), we have

$$\frac{2n-4}{10^3} < d_G(\widetilde{u})x_{\widetilde{u}} + \sum_{v\in \overline{L_1^{10}}} \sum_{w\in (L_1^{10} \cup L_2^{10})} x_w + \frac{14n}{10^4}. \tag{12}$$

By (6), we have

$$e\left(\overline{L_1^{10}}, L_1^{10} \cup L_2^{10}\right) < 2\left(|\overline{L_1^{10}}| + |L_1^{10} \cup L_2^{10}|\right) \leq 2\left(|N_1(\widetilde{u})| + |L^{10}|\right) \leq \frac{4n}{10^4},$$

where $|N_1(\widetilde{u})| \leq \frac{n}{10^4}$ and $|L^{10}| \leq \frac{n}{10^4}$ by Lemma 4.2. Combining this with $d_G(\widetilde{u}) \leq \frac{n}{10^4}$ gives

$$d_G(\widetilde{u})x_u + \sum_{v\in \overline{L_1^{10}}} \sum_{w\in (L_1^{10} \cup L_2^{10})} x_w + \frac{14n}{10^4} \leq d_G(\widetilde{u}) + e\left(\overline{L_1^{10}}, L_1^{10} \cup L_2^{10}\right) + \frac{14n}{10^4} \leq \frac{19n}{10^4},$$

which contradicts (12). Therefore, $d_G(u) \geq \frac{n}{10^4}$ for each $u \in L^1$. Summing this inequality over all vertices $u \in L^1$, we obtain

$$|L^1| \frac{n}{10^4} \leq \sum_{u\in L^1} d_G(u) \leq 2e(G) \leq 6n,$$

which yields that $|L^1| \leq 6 \times 10^4$. □

For convenience, we use L, L_i and $\overline{L_i}$ instead of L^1, $N_i(u) \cap L^1$ and $N_i(u) \setminus L^1$, respectively.

Lemma 4.4. *For every $u \in L$, we have $d_G(u) \geq (x_u - \frac{4}{1000})n$.*

Proof. Let $\overline{L_1}'$ be the subset of $\overline{L_1}$ in which each vertex has at least 2 neighbors in L. We first claim that $|\overline{L_1}'| \leq |L|^2$. If $|L| = 1$, then $\overline{L_1}'$ is empty, as desired. It remains the case $|L| \geq 2$. Suppose to the contrary that $|\overline{L_1}'| > |L|^2$. Since there are only $\binom{|L|}{2}$ options for vertices in $\overline{L_1}'$ to choose a set of 2 neighbors from L, we can find a set of 2 vertices in L with at least $\left\lceil |\overline{L_1}'| / \binom{|L|}{2} \right\rceil \geq 3$ common neighbors in $\overline{L_1}'$. Moreover, note that $u \notin L$ and $\overline{L_1}' \subseteq \overline{L_1} \subseteq N_1(u)$. Hence, G contains a copy of

$K_{3,3}$, contradicting that G is planar. Therefore, $|\overline{L_1}'| \le |L|^2$. Thus, $|L||\overline{L_1}'| \le (6\times 10^4)^3 \le \frac{n}{10^3}$ as $n \ge 2.16\times 10^{17}$. Hence

$$e(\overline{L_1}, L) \le |\overline{L_1} \setminus \overline{L_1}'| + |L||\overline{L_1}'| \le d_G(u) + \frac{n}{1000}.$$

Since $L_1 \cup L_2 \subseteq L$, we have

$$\sum_{v\in \overline{L_1}} \sum_{w\in (L_1\cup L_2)} x_w \quad \le \quad e\big(\overline{L_1}, L_1\cup L_2\big) \le d_G(u) + \frac{n}{1000}.$$

Setting $\lambda = 1$ and combining the above inequality with (11), we have

$$(2n-4)x_u \le d_G(u) + \left(d_G(u) + \frac{n}{10^3}\right) + \frac{61n}{10^4}.$$

which yields $d_G(u) \ge (n-2)x_u - \frac{71n}{2\times 10^4} \ge (x_u - \frac{4}{1000})n$. $\square$

Lemma 4.5. *There exists a vertex $u'' \in L_1 \cup L_2$ such that $x_{u''} \ge \frac{988}{1000}$.*

Proof. Setting $u = u'$, $\lambda = 1$ and combining (11), we have

$$2n-4 < d_G(u') + \sum_{v\in \overline{L_1}} \sum_{w\in L_1\cup L_2} x_w + \frac{60.8n}{10^4},$$

which yields that

$$\sum_{v\in \overline{L_1}} \sum_{w\in L_1\cup L_2} x_w \ge 2n-4 - \frac{60.8n}{10^4} - d_G(u') \ge \frac{993n}{1000}.$$

From Lemma 4.4 we have $d_G(u') \ge \frac{996n}{1000}$ as $u' \in L$. It infers that

$$d_{\overline{L_1}}(u') = d_G(u') - d_{L_1}(u') \ge d_G(u') - |L| \ge \frac{995n}{1000}$$

as $|L| \le 6\times 10^4$ by Lemma 4.3. By (6), we further get

$$e(\overline{L_1}, L_1\cup L_2) \le e(\overline{L_1}, L) - d_{\overline{L_1}}(u') \le (2n-4) - \frac{995n}{1000} \le \frac{1005n}{1000}.$$

By averaging, there is a vertex u'' such that

$$x_{u''} \ge \frac{\sum_{v\in \overline{L_1}} \sum_{w\in (L_1\cup L_2)} x_w}{e(\overline{L_1}, L_1\cup L_2)} \ge \frac{\frac{993n}{1000}}{\frac{1005n}{1000}} \ge \frac{988}{1000},$$

as desired. $\square$

Notice that $x_{u'} = 1$ and $x_{u''} \ge \frac{988}{1000}$. By Lemma 4.4, we have $d_G(u') \ge \frac{996}{1000}n$ and $d_G(u'') \ge \frac{984}{1000}n$. Now, let $D = \{u', u''\}$, R be the subset of $V(G)\setminus\{u', u''\}$ in which every vertex is a non-neighbor of some vertex in $\{u', u''\}$ and $R' = V(G)\setminus(\{u', u''\}\cup R)$. Thus, $|R| \le (n - d_G(u')) + (n - d_G(u'')) \le \frac{2n}{100}$.

Lemma 4.6. *Let $u \in V(G)\setminus\{u', u''\}$. Then $x_u \le \frac{1}{10}$.*

Proof. For any vertex $u \in R$, we have $d_D(u) \leq 1$ by the definition of R. Moreover, $d_{R'}(u) \leq 2$ (otherwise G contains a copy of $K_{3,3}$, contradicting that G is planar). Then, $d_G(u) = d_D(u) + d_{R'}(u) + d_R(u) \leq 3 + d_R(u)$. Note that $e(R) \leq 3|R|$ and $|R| \leq \frac{2n}{100}$. Thus

$$\rho \sum_{u\in R} x_u \leq \sum_{u\in R} d_G(u) \leq \sum_{u\in R}(3+d_R(u)) \leq 3|R| + 2e(R) \leq 9|R| \leq \frac{18n}{100},$$

which yields $\sum_{u\in U} x_u \leq \frac{18n}{100\sqrt{2n-4}}$ as $\rho \geq \sqrt{2n-4}$. Since G is $K_{3,3}$-free, $d_{D\cup R'}(u) \leq 4$ for any vertex $u \in R\cup R'$. It follows that

$$\rho x_u = \sum_{v\in N_G(u)} x_v \leq 4 + \sum_{v\in R} x_v \leq 4 + \frac{18n}{100\sqrt{2n-4}}.$$

Dividing both sides by ρ, we get $x_u \leq \frac{1}{10}$. □

Now, we are ready to give the proof of Theorem 1.1.

Proof. We first claim that R is empty. Suppose to the contrary that $|R| \geq 1$. Then, $G[R]$ is planar, and so there exists a vertex $v \in R$ with $d_R(v) \leq 5$. Furthermore, v has at most 2 neighbours in R' and at most one neighbour in $\{u', u''\}$. Then,

$$\sum_{w\in N_G(v)} x_w \leq x_{u'} + \sum_{v\in R'} x_w + \sum_{v\in R} x_w \leq \frac{17}{10}, \tag{13}$$

where the last inequality holds as $x_w < \frac{1}{10}$ for any $w \in R\cup R'$ by Lemma 4.6. We modify the graph G by deleting all edges incident to v and adding edges vu' and vu'', to obtain the graph G'. We first claim that G' is $\mathscr{F}$-free. Otherwise, G' contains a subgraph $F \in \mathscr{F}$. From the modification, we can see that $v \in V(F)$ and $d_F(v) \subseteq \{u', u''\}$. Note that

$$|R'| = |N_G(u') \cap N_G(u'')| \geq |N_G(u')| + |N_G(u'')| - n \geq \frac{980}{1000}n > |V(F)|.$$

Thus, there exists a vertex $v' \in R' \setminus V(F)$ such that $d_G(v') = \{u', u''\}$. This indicates that an isomorphic copy of F is already present in G, a contradiction. On the other hand, $x_{u'} + x_{u''} \geq \frac{1988}{1000}$. Combining this with (13), we have

$$\rho(G') - \rho(G) \geq \frac{2x_v}{X^TX}\left((x_{u'} + x_{u''}) - \sum_{w\in N_G(v)} x_w\right) > 0.$$

Thus, $\rho(G') > \rho(G)$, contradicting that G is extremal to $spex_{\mathscr{P}}(n, \mathscr{F})$. Hence, the claim holds. So, $R' = V(G) \setminus \{u', u''\}$. Therefore, G contains a copy of $K_{2,n-2}$. □

5 Spectral extremal problems on planar graphs

In case $\mathscr{F} = \{C_3\}$, by Theorem 1.1, G contains a copy of $K_{2,n-2}$. we further obtain that $G \cong K_{2,n-2}$ as G is triangle-free (otherwise, adding any edge increases triangles, a contradiction). In case $\mathscr{F} = \{C_4\}$, clearly, J_n is planar. Nikiforov [20] and Zhai and Wang [23] determined $spex(n, C_4)$ for odd and even n, respectively. This implies that J_n is the extremal graph to $spex_{\mathscr{P}}(n, C_4)$. In this section, we always assume that G is an extremal graph to $spex_{\mathscr{P}}(n, F)$, where $F \in \{C_\ell \mid \ell \geq 5\} \cup \{2C_\ell \mid \ell \geq 3\}$. Clearly, $K_{2,n-2}$ is F-free. By Theorem 1.1, G contains a copy of $K_{2,n-2}$, where $V(K_2) = \{u', u''\}$. We first prove that u' is adjacent to u'' in G.

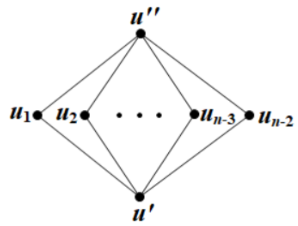

Figure 10: A local structure of G^*.

Lemma 5.1. *We have $u'u'' \in E(G)$.*

Proof. Suppose to the contrary that $u'u'' \notin E(G)$. Assume that G^* is a planar embedding of G, and $u_1,\dots,u_{n-2}$ are around u' in clockwise order in G^*, with subscripts interpreted modulo $n-2$ (see G^* in Fig. 10). If R' induces an cycle $u_1u_2\dots u_{n-2}u_1$, then we can easily check that G contains a copy of F, a contradiction. Thus, there exists an integer $i \le n-2$ such that $u_iu_{i+1} \notin E(G[R'])$. Then, $u'u_iu''u_{i+1}u'$ is a 4-face of the plane graph G^*.

We modify the graph G^* by adding the edge $u'u''$ crossing the above 4-face, to obtain the graph G'. Clearly, G' is a plane graph. We claim that G' is F-free. Suppose to the contrary that G' contains a copy of F. If $F = C_\ell$ for some $\ell \ge 5$, then G' contains an ℓ-cycle containing $u'u''$, say $u'u''u'_1u'_2\dots u'_{\ell-2}u'$. It follows that G already contains a copy of ℓ-cycle $u'u'_1u''u'_2\dots u'_{\ell-2}u'$, a contradiction. If $F = 2C_\ell$ for some $\ell \ge 3$, then F contains two disjoint ℓ-cycles C^1 and C^2. From the modification, we can see that one of C^1 and C^2, say C^1, contains the edge $u'u''$. This implies that C^2 is a subgraph of $G[R']$. However, $G[V(C^2)\cup\{u',u''\}]$ contains a K_5-minor, contradicting the fact that G is planar. Hence, the claim holds. However, $\rho(G') > \rho$, contradicting that G is extremal to $spex_{\mathscr{P}}(n,F)$. Therefore, $u'u'' \in E(G)$. $\square$

Lemma 5.2. *$G[R']$ is a disjoint union of paths.*

Proof. Theorem 1.1 and Lemma 5.1 imply that u' and u'' are dominating vertices. Furthermore, since G is K_5-minor-free and $K_{3,3}$-minor-free, we can find that $G[R']$ is K_3-minor-free and $K_{1,3}$-minor-free. Furthermore, this implies that $G[R']$ is an acyclic graph with maximum degree at most 2. Thus, $G[R']$ is a disjoint union of paths. $\square$

We shall give characterizations of eigenvector entries of vertices in R' in the following lemmas.

Lemma 5.3. *For any vertex $u \in R'$, we have $x_u \in [\frac{2}{\rho}, \frac{2}{\rho} + \frac{6}{\rho^2}]$.*

Proof. Recall that u' and u'' are dominating vertices of G. So, $x_{u'} = x_{u''} = 1$. Then

$$\rho x_u = x_{u'} + x_{u''} + \sum_{v \in N_G(u)\cap R'} x_v = 2 + \sum_{v \in N_G(u)\cap R'} x_v. \tag{14}$$

Moreover, $d_{R'}(v) \le 2$ for any vertex $v \in R'$. Combining this with Lemma 4.6 and (14), we have $x_u \in \left[\frac{2}{\rho}, \frac{3}{\rho}\right]$. Furthermore, by (14), $\rho x_u \in \left[2, 2+\frac{6}{\rho}\right]$, which yields that $x_u \in \left[\frac{2}{\rho}, \frac{2}{\rho}+\frac{6}{\rho^2}\right]$. $\square$

Now we give a transformation that we will use in subsequent proof.

Definition 5.1. *Let s_1, s_2 and s_3 be three integers with $s_1 \geq s_2 \geq s_3+1 \geq 1$, and let $H = P_{s_1} \cup P_{s_2} \cup H_0$, where H_0 is a disjoint union of paths. We say that H^* is a (s_1,s_2,s_3)-transformation of H if*

$$H^* = \begin{cases} P_{s_3} \cup P_{s_1+s_2-s_3} \cup H_0 & \text{if } 1 \leq s_3 \leq s_2-1, \\ P_{s_1+s_2} \cup H_0 & \text{if } s_3 = 0. \end{cases}$$

Lemma 5.4. *Let H and H^* be defined as in Definition 5.1. Assume that $G[R'] \cong H$. Then, $\rho(K_2+H^*) > \rho$ for $n \geq \max\{2.16\times 10^{17}, 9\times 2^{s_3+2}+3\}$.*

Proof. Assume that $P^1 := v_1v_2\ldots v_{s_1}$ and $P^2 := w_1w_2\ldots w_{s_2}$ are two components of H. Clearly, $G \cong K_2+H$ as $G[R'] \cong H$. If $s_3 = 0$, then $H \subset H^*$, and so $G \subset K_2+H^*$. It follows that $\rho(P_2+H^*) > \rho(G)$, the result holds. Next, we deal with the case $s_3 = 1$. If $x_{v_1} \leq x_{w_1}$, then let H' be obtained from H by deleting the edge $x_{v_1}x_{v_2}$ and adding the edge $x_{v_2}x_{w_1}$. Clearly, $H' \cong H^*$. Moreover,

$$\rho(K_2+H^*) - \rho(G) \geq \frac{X^T\left(A(K_2+H^*)-A(G)\right)X}{X^TX} \geq \frac{2}{X^TX}(x_{w_1}-x_{v_1})x_{v_2} \geq 0.$$

Since $K_2+H^* \ncong G$, we have $\rho(K_2+H^*) > \rho(G)$, the result holds. The case $x_{v_1} > x_{w_1}$ is similar and hence omitted here.

It remains the case $s_3 \geq 2$. So, $s_1 \geq s_2 \geq 3$.

Claim 5.1. *(i) For any $i \in \{1,\ldots,\lfloor\frac{s_1-1}{2}\rfloor\}$, $x_{v_{i+1}} - x_{v_i} \in \left[\frac{2}{\rho^{i+1}} - \frac{6\times 2^i}{\rho^{i+2}}, \frac{2}{\rho^{i+1}} + \frac{6\times 2^i}{\rho^{i+2}}\right]$.*
(ii) For any $i \in \{1,\ldots,\lfloor\frac{s_2-1}{2}\rfloor\}$, $x_{w_{i+1}} - x_{w_i} \in \left[\frac{2}{\rho^{i+1}} - \frac{6\times 2^i}{\rho^{i+2}}, \frac{2}{\rho^{i+1}} + \frac{6\times 2^i}{\rho^{i+2}}\right]$.
(iii) For any $i \in \{1,\ldots,\lfloor\frac{s_2-1}{2}\rfloor\}$, $x_{v_i} - x_{w_i} \in \left[-\frac{6\times 2^i}{\rho^{i+2}}, \frac{6\times 2^i}{\rho^{i+2}}\right]$.

Proof. (i) It suffices to prove that for any $i \in \{1,\ldots,\lfloor\frac{s_1-1}{2}\rfloor\}$,

$$\rho^i(x_{v_{j+1}} - x_{v_j}) \in A_i = \begin{cases} \left[\frac{2}{\rho} - \frac{6\times 2^i}{\rho^2}, \frac{2}{\rho} + \frac{6\times 2^i}{\rho^2}\right] & \text{if } j = i, \\ \left[-\frac{6\times 2^i}{\rho^2}, \frac{6\times 2^i}{\rho^2}\right] & \text{if } i+1 \leq j \leq s_1-i-1. \end{cases}$$

We shall proceed the proof by induction on i. Clearly,

$$\rho x_{v_j} = \sum_{v\in N_G(v_j)} x_v = \begin{cases} 2+x_{v_2} & \text{if } j = 1, \\ 2+x_{v_{j-1}}+x_{v_{j+1}} & \text{if } 2 \leq j \leq s_1-1. \end{cases} \tag{15}$$

Then,

$$\rho(x_{v_{j+1}} - x_{v_j}) = \begin{cases} x_{v_1} + x_{v_3} - x_{v_2} \in A_1 & \text{if } j = 1, \\ (x_{v_j} - x_{v_{j-1}}) + (x_{v_{j+2}} - x_{j+1}) \in A_1 & \text{if } 2 \leq j \leq s_1-2. \end{cases}$$

So the result is true when $i = 1$. Assume then that $2 \leq i \leq \lfloor\frac{s_1-1}{2}\rfloor$, which implies that $s_1 \geq 2i+1$. For $i \leq j \leq s_1-i$, $\rho(x_{v_{j+1}} - x_{v_j}) = (x_{v_j} - x_{v_{j-1}}) + (x_{v_{j+2}} - x_{v_{j+1}})$. By the induction hypothesis,

$$\rho^i(x_{v_{j+1}} - x_{v_j}) = \rho^{i-1}(x_{v_j} - x_{v_{j-1}}) + \rho^{i-1}(x_{v_{j+2}} - x_{v_{j+1}}) \in A_i.$$

So the result holds.

The proof of (ii) is similar to that of (i) and hence omitted here.

(iii) It suffices to prove that for any $i \in \{1,\ldots,\lfloor\frac{s_2-1}{2}\rfloor\}$ and $j \in \{i,\ldots,s_2-i\}$,

$$\rho^i(x_{v_j} - x_{w_j}) \in B_i = \left[-\frac{6\times 2^i}{\rho^2}, \frac{6\times 2^i}{\rho^2}\right].$$

We shall proceed the proof by induction on i. Clearly,

$$\rho x_{w_j} = \sum_{w\in N_G(w_j)} x_w = \begin{cases} 2+x_{w_2} & \text{if } j=1, \\ 2+x_{w_{j-1}}+x_{w_{j+1}} & \text{if } 2\le j\le s_2-1. \end{cases}$$

Combining this with (15) and Lemma 5.3 gives

$$\rho(x_{v_j}-x_{w_j}) = \begin{cases} x_{v_2}-x_{w_2}\in\left[-\frac{6}{\rho^2},\frac{6}{\rho^2}\right]\subset B_1 & \text{if } j=1, \\ (x_{v_{j+1}}-x_{w_{j+1}})+(x_{v_{j-1}}-x_{w_{j-1}})\in B_1 & \text{if } 2\le j\le s_2-1. \end{cases}$$

So the claim is true when $i=1$. Assume then that $2\le i\le\lfloor\frac{s_2-2}{2}\rfloor$, which implies that $s_2\ge 2i+2$. For $i\le j\le s_2-i$, $\rho(x_{v_{j+1}}-x_{v_j})=(x_{v_j}-x_{v_{j-1}})+(x_{v_{j+2}}-x_{v_{j+1}})$. By the induction hypothesis, we have

$$\rho^i(x_{v_j}-x_{w_j}) \quad = \quad \rho^{i-1}(x_{v_{j-1}}-x_{w_{j-1}})+\rho^{i-1}(x_{v_{j+1}}-x_{w_{j+1}})\in B_i$$

if $i\le j\le s_2-i$. The result holds. This completes the proof of Claim 5.1. □

Let t_1,t_2 be integers with $1\le t_i\le s_i-1$ for each $i\in\{1,2\}$, and H' be obtained from H by deleting edges $x_{v_{t_1}}x_{v_{t_1+1}}, x_{w_{t_2}}x_{w_{t_2+1}}$ and adding edges $x_{v_{t_1}}x_{w_{t_2}}, x_{v_{t_1+1}}x_{w_{t_2+1}}$. Then

$$\rho(K_2+H^*)-\rho(G)\ge\frac{X^T\big(A(K_2+H^*)-A(G)\big)X}{X^TX}\ge\frac{2}{X^TX}(x_{v_{t_1+1}}-x_{w_{t_2}})(x_{w_{t_2+1}}-x_{v_{t_1}}). \tag{16}$$

Since $n\ge 9\times 2^{s_3+2}+3$, we have

$$\rho\ge\sqrt{2n-4}>6\times 2^{(s_3+1)/2}. \tag{17}$$

Now, we consider the following three cases:

Case 1. s_3 is even.

Set $t_1=\frac{s_3}{2}$ and $t_2=\frac{s_3}{2}$. Clearly, $t_1+t_2=s_3$, and so $H'\cong H^*$. By Claim 5.1 and (17), we have

$$x_{v_{s_3/2+1}}-x_{v_{s_3/2}}\ge\frac{2}{\rho^{s_3/2+1}}-\frac{6\times 2^{s_3/2}}{\rho^{s_3/2+2}}>0.$$

Then, by (16), we have $\rho(K_2+H^*)>\rho(G)$, as desired.

Case 2. $s_1=s_2$ and s_3 is odd.

Note that $s_1=s_2$. Then, by symmetry, we have $x_{v_{(s_3+1)/2}}=x_{w_{(s_3+1)/2}}$. Set $t_1=\frac{s_3+1}{2}$ and $t_2=\frac{s_3-1}{2}$. Clearly, $t_1+t_2=s_3$, and so $H'\cong H^*$. By (16), we have $\rho(K_2+H^*)\ge\rho(G)$. Furthermore, since $K_2+H^*\ncong G$, we have $\rho(K_2+H^*)>\rho(G)$, as desired.

Case 3. $s_1\ge s_2+1$ and s_3 is odd.

We first consider the subcase $x_{v_{(s_3+1)/2}}\le x_{w_{(s_3+1)/2}}$. Set $t_1=\frac{s_3+1}{2}$ and $t_2=\frac{s_3-1}{2}$. Then, $x_{w_{t_2+1}}\ge x_{v_{t_1}}$. Clearly, $t_1+t_2=s_3$, and so $H'\cong H^*$. Obviously, $s_1\ge s_3+2$, and so $\left\lfloor\frac{s_3+1}{2}\right\rfloor\le\left\lfloor\frac{s_1-1}{2}\right\rfloor$. By Claim 5.1 and (17), we have

$$x_{v_{i+1}}-x_{w_i}=(x_{v_{i+1}}-x_{v_i})+(x_{v_i}-x_{w_i})\ge\left(\frac{2}{\rho^{i+1}}-\frac{6\times 2^i}{\rho^{i+2}}\right)-\frac{6\times 2^i}{\rho^{i+2}}>0\ \text{ for } i\le\left\lfloor\frac{s_3+1}{2}\right\rfloor,$$

which implies that $x_{v_{t_1+1}} > x_{w_{t_2+1}} \geq x_{v_{t_1}} > x_{w_{t_2}}$. By (16), we have $\rho(K_2+H^*) \geq \rho(G)$. Furthermore, $\rho(K_2+H^*) > \rho(G)$ as $K_2+H^* \ncong G$, as desired.

We then consider the subcase $x_{v_{(s_3+1)/2}} > x_{w_{(s_3+1)/2}}$. Let $t_1 = \frac{s_3-1}{2}$ and $t_2 = \frac{s_3+1}{2}$. Clearly, $t_1+t_2 = s_3$, and so $H' \cong H^*$. By Claim 5.1 and (17), we have

$$x_{w_{i+1}} - x_{v_i} = (x_{w_{i+1}} - x_{w_i}) + (x_{w_i} - x_{v_i}) \geq \frac{2}{\rho^{i+1}} - \frac{12 \times 2^i}{\rho^{i+2}} > 0 \tag{18}$$

for $i \leq \min\left\{\left\lfloor \frac{s_3+1}{2} \right\rfloor, \left\lfloor \frac{s_2-1}{2} \right\rfloor\right\}$. Then, $x_{v_{t_1+1}} > x_{w_{t_2}}$.

If $s_2 = s_3+1$, then s_2 is even. By symmetry, $x_{w_{(s_3+1)/2}} = x_{w_{(s_3+3)/2}}$, that is, $x_{w_{t_2}} = x_{w_{t_2+1}}$. Moreover, by (18), $x_{w_{t_2}} > x_{v_{t_1}}$. So, $x_{v_{t_1+1}} > x_{w_{t_2+1}} = x_{w_{t_2}} > x_{v_{t_1}}$. By (16), $\rho(K_2+H^*) \geq \rho(G)$, as desired. If $s_2 \geq s_3+2$, then by (18), $x_{w_{t_2+1}} > x_{v_{t_1+1}} > x_{w_{t_2}} > x_{v_{t_1}}$. By (16), we have $\rho(K_2+H^*) > \rho(G)$, as desired.

This completes the proof of Lemma 5.4. □

According to Lemma 5.2, assume that $G[R'] \cong \cup_{i=1}^{t} P_{n_i}$, where $n_1 \geq \cdots \geq n_t \geq 1$. Having Lemmas 5.3 and 5.4, we are ready to give the proof of Theorems 1.2-1.4.

Proof of Theorem 1.2. We first give the following claim.

Claim 5.2. *Assume that $H = \cup_{i=1}^{t} P_{n_i(H)}$, where $n_1(H) \geq \cdots \geq n_t(H) \geq 1$. Then, K_2+H is $2C_\ell$-free if and only if $n_1(H) \leq 2\ell-3$ and $n_2(H) \leq \ell-2$.*

Proof. We first claim that K_2+H is $2C_\ell$-free if and only if H is $2P_{\ell-1}$-free. Equivalently, K_2+H contains a copy of $2C_\ell$ if and only if H contains a copy of $2P_{\ell-1}$. Assume that K_2+H contains two vertex-disjoint ℓ-cycles C^1 and C^2, and $V(K_2) = \{u', u''\}$. Since H is acyclic, we can see that C^i must contain at least one vertex of u' and u'' for any $i \in \{1,2\}$. Without loss of generality, assume that $u' \in V(C^1)$ and $u'' \in V(C^2)$. Then, $C^1 - \{u'\} \cong C^2 - \{u''\} \cong P_{\ell-1}$, and so H contains a copy of $2P_{\ell-1}$. Conversely, assume that H contains two vertex-disjoint paths P^1 and P^2 such that $P^1 \cong P^2 \cong P_{\ell-1}$. Thus, the subgraph induced by $V(P^1) \cup \{u'\}$ contains a copy of C_ℓ. Similarly, the subgraph induced by $V(P^2) \cup \{u''\}$ contains a copy of C_ℓ. This implies that K_2+H contains a copy of $2C_\ell$.

We can easily check that H is $2P_{\ell-1}$-free if and only if $P_{n_1(H)} \cup P_{n_2(H)}$ is $2P_{\ell-1}$-free. Moreover, we claim that $P_{n_1(H)} \cup P_{n_2(H)}$ is $2P_{\ell-1}$-free if and only if $n_1(H) \leq 2\ell-3$ and $n_2(H) \leq \ell-2$. If $P_{n_1(H)} \cup P_{n_2(H)}$ is $2P_{\ell-1}$-free, then $n_1(H) \leq 2\ell-3$ (otherwise, $P_{n_1(H)}$ contains a copy of $2P_{\ell-1}$, a contradiction); $n_2(H) \leq \ell-2$ (otherwise, $P_{n_1(H)} \cup P_{n_2(H)}$ contains a copy of $2P_{\ell-1}$, a contradiction). Conversely, if $n_1(H) \leq 2\ell-3$ and $n_2(H) \leq \ell-2$, then $P_{n_1(H)} \cup P_{n_2(H)}$ is $2P_{\ell-1}$-free.

This completes the proof of Claim 5.2. □

By Claim 5.2, $n_1 \leq 2\ell-3$ and $n_2 \leq \ell-2$. We first claim that $n_1 = 2\ell-3$. Suppose to the contrary that $n_1 \leq 2\ell-4$. If $n_1+n_t > 2\ell-3$, then let H' be a $(n_1, n_t, n_1+n_t-2\ell+3)$-transformation of $G[R']$. Clearly, $n_1(H') = 2\ell-3$ and $n_2(H') = n_2$. By Claim 5.2, K_2+H' is $2C_\ell$-free. Moreover, $n_1+n_t-2\ell+3 \leq n_t-1 \leq n_2-1 \leq \ell-3$. By Lemma 5.4 and $n \geq \max\{2.16 \times 10^{17}, 9 \times 2^{\ell-1}+3\}$, we have $\rho(K_2+H') > \rho$, a contradiction. If $n_1+n_t \leq 2\ell-3$, then let H' be a $(n_1, n_t, 0)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1+n_t < 2\ell-3$ and $n_2(H') = n_2$. By Claim 5.2, K_2+H' is $2C_\ell$-free. Moreover, by Lemma 5.4, we have $\rho(K_2+H') > \rho$, a contradiction.

In the following, we shall show that $n_i = \ell - 2$ for $i \in \{2, \ldots, t-1\}$. Suppose to the contrary, then set

$$i_0 = \min\{i \mid 2 \le i \le t-1, n_i < \ell - 2\}.$$

If $n_{i_0} + n_t \le \ell - 2$, then let H' be a $(n_{i_0}, n_t, 0)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1 = 2\ell - 3$ and $n_2(H') = \max\{n_2, n_{i_0} + n_t\} \le \ell - 2$. By Claim 5.2, $K_2 + H'$ is $2C_\ell$-free. Moreover, by Lemma 5.4, we have $\rho(K_2 + H') > \rho$, a contradiction. If $n_{i_0} + n_t > \ell - 2$, then let H' be a $(n_{i_0}, n_t, n_{i_0} + n_t - \ell + 2)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1 = 2\ell - 3$ and $n_2(H') = \ell - 2$. By Claim 5.2, $K_2 + H'$ is $2C_\ell$-free. Moreover, by Lemma 5.4, we have $\rho(K_2 + H') > \rho$, a contradiction.

Since $n_1 = 2\ell - 3$ and $n_i = \ell - 2$ for $i \in \{2, \ldots, t-1\}$, we have $G[R'] \cong H(2\ell - 3, \ell - 2)$. This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3. By Theorem 1.2, $K_2 + H(3,1)$ is the extremal graph to $spex_{\mathscr{P}}(n, 2C_3)$ for $n \ge 2.16 \times 10^{17}$. One can observe that if a planar graph G' is $2\mathscr{C}$-free then G' is also $2C_3$-free. Moreover, by Theorem 1.5, $K_2 + H(3,1)$ is $2\mathscr{C}$-free for $n \ge 5$. Thus, $K_2 + H(3,1)$ is the extremal graph to $spex_{\mathscr{P}}(n, 2\mathscr{C})$ for $n \ge 2.16 \times 10^{17}$. $\square$

Proof of Theorem 1.4. It remains the case $\ell \ge 5$. We first give the following claim.

Claim 5.3. *Assume that $H = \cup_{i=1}^{t} P_{n_i(H)}$, where $n_1(H) \ge \cdots \ge n_t(H) \ge 1$. Then, $P_2 + H$ is C_ℓ-free if and only if $n_1(H) + n_2(H) \le \ell - 3$.*

Proof. It is not hard to verify that $K_2 + (P_{n_1(H)} \cup P_{n_2(H)})$ contains all cycles of order at most $n_1(H) + n_2(H) + 2$. Since $K_2 + H$ is C_ℓ-free, $n_1(H) + n_2(H) + 2 \le \ell - 1$, yielding $n_1(H) + n_2(H) \le \ell - 3$. Conversely, if $n_1(H) + n_2(H) \le \ell - 3$, then every cycle in $K_2 + H$ contains vertices in at most two paths of H. It implies that the longest cycle in $K_2 + H$ is $n_1(H) + n_2(H) + 2$. Clearly, $n_1(H) + n_2(H) + 2 \le \ell - 1$. Thus, $K_2 + H$ is C_ℓ-free. $\square$

Since $n_1 + n_2 \le \ell - 3$ and $n_1 \ge n_2$, we have $n_2 \le \lfloor \frac{\ell-3}{2} \rfloor$. We then prove that $n_i = \ell - 3 - n_1$ for each $i \in \{2, \ldots, t-1\}$. Suppose to the contrary, then set

$$i_0 = \min\{i \mid 2 \le i \le t-1, n_i < \ell - 3 - n_1\}.$$

If $n_{i_0} + n_t \le \ell - 3 - n_1$, then let H' be a $(n_{i_0}, n_t, 0)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1$ and $n_2(H') \le \max\{n_2, n_{i_0} + n_t\} \le \ell - 3 - n_1$, and so $n_1(H') + n_2(H') \le \ell - 3$. By Claim 5.3, $K_2 + H'$ is C_ℓ-free. Moreover, by Lemma 5.4, we have $\rho(K_2 + H') > \rho$, a contradiction. If $n_{i_0} + n_t > \ell - 3 - n_1$, then let H' be a $(n_{i_0}, n_t, n_{i_0} + n_t - \ell + 3 + n_1)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1$ and $n_2(H') = \ell - 3 - n_1$, and so $n_1(H') + n_2(H') = \ell - 3$. By Claim 5.3, $K_2 + H'$ is C_ℓ-free. Note that $n_t \le n_{i_0} \le \ell - 4 - n_1$. Then, $n_{i_0} + n_t - \ell + 3 + n_1 \le n_{i_0} - 1 \le \lfloor \frac{\ell-5}{2} \rfloor$. By Lemma 5.4 and $n \ge 9 \times 2^{\lfloor \frac{\ell-1}{2} \rfloor} + 3$, we have $\rho(K_2 + H') > \rho$, a contradiction. It follows that $G = K_2 + (P_{n_1} \cup (t-2)P_{n_2} \cup P_{n_t})$.

By claim 5.3, $n_1 + n_2 \le \ell - 3$. Moreover, since $n_1 \ge n_2$, we have $n_2 \le \lfloor \frac{\ell-3}{2} \rfloor$. Now we prove that $n_i = \lfloor \frac{\ell-3}{2} \rfloor$ for any $i \in \{2, \ldots, t-1\}$. Otherwise, let $G' := K_2 + (P_{n_1 - 1} \cup (n_2 + 1)P_{n_2+1} \cup (t - n_2 - 4)P_{n_2} \cup P_{n_t})$. More precisely, we can see that G' can be obtained from G by

(i) deleting an edge incident to one endpoint u of P_{n_1} and connecting u to one endpoint of P_{n_2};

(ii) deleting all edges in one path P of order n_2 and recursively connecting every vertex in $V(P)$ to

an endpoint of a new P_{n_2}.
So, G' is obtained from G by deleting n_2 edges and adding n_2+1 edges. By Lemma 5.3, we have

$$\frac{4}{\rho^2} < x_{u_i}x_{u_j} < \frac{4}{\rho^2} + \frac{24}{\rho^3} + \frac{144}{\rho^4} < \frac{4}{\rho^2} + \frac{25}{\rho^3}$$

for any vertices $u_i, u_j \in R'$. Then

$$\rho(G') - \rho \geq \frac{X^T(A(G') - A(G))X}{X^TX} = \frac{2}{X^TX}\left(\frac{4}{\rho^2} - \frac{25n_2}{\rho^3}\right) > 0,$$

where $n_2 < \lfloor\frac{\ell-3}{2}\rfloor \leq \frac{4}{25}\sqrt{2n-4}$ for $n \geq \frac{625}{32}\lfloor\frac{\ell-3}{2}\rfloor^2 + 2$. So, $\rho(G') > \rho$, a contradiction.

Since $n_1 + n_2 \leq \ell - 3$ and $n_2 = \lfloor\frac{\ell-3}{2}\rfloor$, we have $n_1 \leq \lceil\frac{\ell-3}{2}\rceil$. Finally, we prove that $n_1 = \lceil\frac{\ell-3}{2}\rceil$. Suppose to the contrary, then $n_1 < \lceil\frac{\ell-3}{2}\rceil$. Note that $n_1 \geq n_2 = \lfloor\frac{\ell-3}{2}\rfloor$. It implies that ℓ is odd and $n_1 = \lfloor\frac{\ell-3}{2}\rfloor$. Let H' be a $(n_1, n_t, n_t - 1)$-transformation of $G[R']$. Clearly, $n_1(H') = n_1 + 1 = \lceil\frac{\ell-3}{2}\rceil$ and $n_2(H') = n_2$. By Claim 5.3, $K_2 + H'$ is C_ℓ-free. Moreover, setting $s_3 = n_t - 1$, we have $s_3 \leq n_t - 1 \leq n_2 - 1 \leq \lfloor\frac{\ell-5}{2}\rfloor$. By Lemma 5.4 and $n \geq 9 \times 2^{\lfloor\frac{\ell-1}{2}\rfloor} + 3$, we have $\rho(K_2 + H') > \rho$, a contradiction. Thus, $n_1 = \lceil\frac{\ell-3}{2}\rceil$.

Since $n_1 = \lceil\frac{\ell-3}{2}\rceil$ and $n_i = \lfloor\frac{\ell-3}{2}\rfloor$ for $i \in \{2, \ldots, t-1\}$, we have $G[R'] \cong H(\lceil\frac{\ell-3}{2}\rceil, \lfloor\frac{\ell-3}{2}\rfloor)$. This completes the proof of Theorem 1.4. $\square$

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