Vertex-bipancyclicity in a bipartite graph collection

Jie Hu, Luyi Li, Xueliang Li, Ningyan Xu

Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

Email: hujie@nankai.edu.cn, liluyi@mail.nankai.edu.cn

lxl@nankai.edu.cn, xny@mail.nankai.edu.cn

Abstract

Let $\mathbf{G} = \{G_1, \ldots, G_{2n}\}$ be a bipartite graph collection on the common vertex bipartition (X, Y) with $|X| = |Y| = n$. We say that $\mathbf{G}$ is *bipancyclic* if there exists a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for each even integer $\ell \in [4, 2n]$, while $\mathbf{G}$ is *vertex-bipancyclic* if any vertex $v \in X \cup Y$ is contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for each even integer $\ell \in [4, 2n]$. Bradshaw in [Transversals and bipancyclicity in bipartite graph families, Electron. J. Comb., 2021] showed that for each $i \in [2n]$, if $d_{G_i}(x) > \frac{n}{2}$ for each $x \in X$ and $d_{G_i}(y) \geq \frac{n}{2}$ for each $y \in Y$, then $\mathbf{G}$ is bipancyclic, which generalizes a classical result of Schmeichel and Mitchem in [Bipartite graphs with cycles of all even lengths, J. Graph Theory, 1982] on the bipancyclicity of bipartite graphs to the setting of graph transversals. Motivated by their work, we study vertex-bipancyclicity in bipartite graph collections and prove that if $\delta(G_i) \geq \frac{n+1}{2}$ for any $i \in [2n]$, then $\mathbf{G}$ is vertex-bipancyclic unless $n = 3$ and $\mathbf{G}$ consists of 6 identical copies of a 6-cycle. Moreover, we also show the Hamiltonian connectivity of $\mathbf{G}$.

Keywords: bipartite graph collection; transversal; vertex-bipancyclicity; Hamiltonian connectivity; minimum degree.

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1 Introduction

Over the last decades, there has been much research on Hamiltonicity of graphs, which is one of the most fundamental topics in graph theory. The classical Dirac's theorem [12] in 1952 states that every n-vertex graph with minimum degree at least $\frac{n}{2}$ is Hamiltonian. In 1971, Bondy [4] proved that every n-vertex graph is pancyclic under the same degree condition expect for $K_{\frac{n}{2},\frac{n}{2}}$. Later, Bondy in [5] posed the following meta-conjecture: Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that it is pancyclic (except for possibly a simple family of exceptional graphs). This has been verified for many sufficient conditions for Hamiltonicity. Actually, some sufficient conditions forcing Hamiltonicity can even guarantee vertex-pancyclicity with minor adjustments. For example, in 1990, Hendry in [14] proved that every n-vertex graph with minimum degree at least $\frac{n+1}{2}$ is vertex-pancyclic. Note that when the graph is a balanced bipartite graph, it is natural to study whether conditions forcing Hamiltonicity can guarantee bipancyclicity, or even vertex-bipancyclicity. Indeed, in 1963, Moon and Moser in [17] established a minimum degree condition for the existence of Hamiltonian cycle in a balanced bipartite graph, which can be seen as a bipartite analogue of Dirac's theorem. Later in 1982, Schmeichel and Mitchem in [21] generalized the above result to bipancyclicity under the same degree condition.

Very recently, the study of transversals over graph collections has received much attention, and some classical results in extremal graph theory have been extended to the setting of graph transversals. The concept of a graph transversal was first raised by Joos and Kim [15] in 2020. Let $\mathbf{G} = \{G_1, \dots, G_s\}$ be a graph collection with common vertex set V and H be a graph with $V(H) \subseteq V$. We say that (H, ϕ) is a *partial* $\mathbf{G}$*-transversal* if there exists an injection $\phi : E(H) \to [s]$ such that $e \in E(G_{\phi(e)})$ for each $e \in E(H)$. In particular, if $|E(H)| = s$, then we call (H, ϕ) a $\mathbf{G}$*-transversal*. Aharoni, DeVos, González Hermosillo de la Maza, Montejano and Šámal [2] considered Mantel's theorem in the setting of graph transversals and proposed the following conjecture motivated by Dirac's theorem.

Conjecture 1 ([2]). Let $\mathbf{G} = \{G_1, \cdots, G_n\}$ be a graph collection on the common vertex set V of size n. If the minimum degree of G_i is at least $\frac{n}{2}$ for each $i \in [n]$, then there exists a $\mathbf{G}$-transversal isomorphic to a Hamiltonian cycle on V.

This conjecture was verified asymptotically by Cheng, Wang and Zhao in [11], and completely by Joos and Kim in [15]. Besides Hamiltonian cycles, results on other structures in extremal graph theory have also been generalized to the setting of graph transversals, including cycles [7, 9, 20], matchings [1, 3, 13], trees [8, 16] and factors [10, 18]. Bradshaw in [7] initiated the study of (partial) transversals in bipartite graph collections and obtained the following theorem analogous to Moon and Moser's result in [17].

Theorem 1.1 ([7]). *Let $\mathbf{G} = \{G_1, \cdots, G_{2n}\}$ be a bipartite graph collection on the common vertex bipartition (X, Y) with $|X| = |Y| = n$. If for each $i \in [2n]$, $d_{G_i}(x) \geq \frac{n}{2}$ and $d_{G_i}(y) > \frac{n}{2}$ for any $x \in X$ and $y \in Y$, then there exists a $\mathbf{G}$-transversal isomorphic to a Hamiltonian cycle on $X \cup Y$.*

For convenience, in the following context we always use $\mathbf{G}$ to denote a bipartite graph collection $\{G_1, \cdots, G_{2n}\}$ on the common vertex bipartition (X, Y) with $|X| = |Y| = n$, unless otherwise stated. We define $\delta(\mathbf{G}) = \min\{\delta(G_i) : i \in [2n]\}$, where $\delta(G)$ denotes the minimum degree of a graph G. We say that $\mathbf{G}$ is *bipancyclic* if there exists a partial

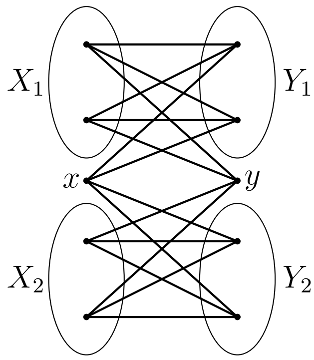

Figure 1: $F = (\{x\} \cup X_1 \cup X_2, \{y\} \cup Y_1 \cup Y_2)$

$\mathbf{G}$-transversal isomorphic to an ℓ-cycle for each even integer $\ell \in [4, 2n]$, while $\mathbf{G}$ is *vertex-bipancyclic* if each vertex $v \in X \cup Y$ is contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for each even integer $\ell \in [4, 2n]$. Bradshaw in [7] proved a stronger result which states that $\mathbf{G}$ is bipancyclic under the same degree condition as Theorem 1.1, generalizing the result of Schmeichel and Mitchem in [21]. Hence, the degree condition in Theorem 1.1 fits Bondy's meta-conjecture in the setting of graph transversals. This motivates us to verify if this degree condition (with a minor adjustment) can guarantee the vertex-bipancyclicity of $\mathbf{G}$, since vertex-bipancyclicity implies bipancyclicity. It is noteworthy that $K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$ contains no Hamiltonian cycles. Thus, the condition $\delta(\mathbf{G}) = \frac{n}{2}$ can not guarantee the vertex-bipancyclicity of $\mathbf{G}$. As a result, we get a positive answer as follows.

Theorem 1.2. *If $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, then $\mathbf{G}$ is vertex-bipancyclic, unless $n = 3$ and $\mathbf{G}$ consists of six identical copies of a 6-cycle.*

Hamiltonian connectivity is closely related with Hamiltonicity and it is a significant property in graph theory (see e.g. [19]). Inspired by this, we study the Hamiltonian connectivity of bipartite graph collections. We say that $\mathbf{G}$ is *Hamiltonian connected* if for any two vertices $x \in X$ and $y \in Y$, there is a partial $\mathbf{G}$-transversal isomorphic to a Hamiltonian path from x to y. To state our result, we define the following graph collection: Let n be an odd integer and $F = (\{x\} \cup X_1 \cup X_2, \{y\} \cup Y_1 \cup Y_2)$ be a balanced bipartite graph with $|X_i| = |Y_i| = \frac{n-1}{2}$ for $i = 1, 2$ such that $E(F)$ consists of edges of 4 complete bipartite graphs $F[\{x\}, Y_1 \cup Y_2]$, $F[\{y\}, X_1 \cup X_2]$ and $F[X_i, Y_i]$ for $i = 1, 2$. Let $\mathbf{F} = \{F_1, \cdots, F_{2n}\}$ be a bipartite graph collection on the common vertex bipartition such that $F_i = F$ or $F_i = F \cup \{xy\}$ for each $i \in [2n]$. Note that $\delta(\mathbf{F}) = \frac{n+1}{2}$ and there is no partial $\mathbf{F}$-transversal isomorphic to a Hamiltonian path from x to y, see Figure 1. Hence, we give the following result on Hamiltonian connectivity of bipartite graph collections.

Theorem 1.3. *If $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, then $\mathbf{G}$ is Hamiltonian connected or $\mathbf{G} = \mathbf{F}$.*

In Section 3, we will give the proofs of Theorems 1.2 and 1.3.

2 Preliminaries

We first give some necessary notation and lemmas in this section, which will be used in next sections.

2.1 Notation

In this paper, we only consider finite, undirected, connected and simple graphs. For terminology and notation used but not defined here, we refer the reader to [6]. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. For a vertex subset $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U. We use $G_1 \cup G_2$ to denote the union of two vertex-disjoint graphs G_1 and G_2. A path or cycle of order k is called a *k-path* or *k-cycle*, respectively. For two distinct vertices v_i and v_j in a cycle $C = v_1v_2 \cdots v_\ell v_1$, the segment $v_iv_{i+1} \cdots v_{j-1}v_j$ and $v_iv_{i-1} \cdots v_{j+1}v_j$ are denoted by v_iCv_j and $v_iC^-v_j$, respectively, where the subscripts are taken modulo $|V(C)|$. Meanwhile, we use u_iPu_j to denote the subpath of $P = u_1u_2 \cdots u_\ell$ between u_i and u_j. For two positive integers $a < b$, let $[a] = \{1, 2, \ldots, a\}$ and $[a, b] = \{a, a+1, \ldots, b\}$.

Recall that when (H, ϕ) is a partial $\mathbf{G}$-transversal, ϕ is an injection from $E(H)$ to $[2n]$. Let $im(\phi)$ be the image of ϕ. For an integer $i \in [2n]$, if $i \notin im(\phi)$, then we say that i is *missed* by (H, ϕ). If there exists a partial $\mathbf{G}$-transversal (H, ϕ), then we also say that $\mathbf{G}$ contains a partial transversal (H, ϕ). When there is no possible confusion, we replace (H, ϕ) by H in the following context.

2.2 Lemmas

Now we will give some lemmas which will be used in the sequel.

Lemma 2.1. *Let $\mathbf{G} = \{G_i : i \in [2n]\}$ be a bipartite graph collection on the same bipartition $V = (X, Y)$ with $|X| = |Y| = n$. If $\delta(\mathbf{G}) \geq \frac{n}{2}$, then one of the following statements holds:*

(1) $\mathbf{G}$ contains a partial transversal isomorphic to a Hamiltonian path;

(2) n is even and $G_1 = \cdots = G_{2n} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. Suppose that neither statements (1) nor (2) holds. Let P be a partial $\mathbf{G}$-transversal isomorphic to a path with $|V(P)|$ maximum. We divide into two cases to discuss depending on the parity of $|V(P)|$.

Case 1. P is a partial $\mathbf{G}$-transversal isomorphic to a 2ℓ-path.

Evidently, $\ell \leq n - 1$. We first prove an easy but crucial claim.

Claim 2.2. There is no partial $\mathbf{G}$-transversal isomorphic to a 2ℓ-cycle.

Proof. Suppose $\mathbf{G}$ has a partial transversal isomorphic to a cycle $C = x_1y_1x_2y_2\cdots x_\ell y_\ell x_1$. Without loss of generality, assume that C has an associated injection $\phi : E(C) \to [2n]$ and $[2n]\backslash im(\phi) = [2\ell+1, 2n]$.

We assert that there is no edge between $V(C)$ and $V\backslash V(C)$ in G_i for each $i \in [2\ell+1, 2n]$. Otherwise, if there exists some $i \in [2\ell+1, 2n]$, $j \in [\ell]$ and a vertex $x \in V\backslash V(C)$ (let $x \in X$ by symmetry) such that $xy_j \in E(G_i)$, then $P_1 = xy_jCx_j$ is a partial $\mathbf{G}$-transversal isomorphic to a $(2\ell+1)$-path, contradicting with the maximality of P. Since $\delta(G_i) \geq \frac{n}{2}$ for each $i \in [2\ell+1, 2n]$, it follows that $|V(C)| \geq n$ and $|V\backslash V(C)| \geq n$. Then n is even and $G_{2\ell+1} = \cdots = G_{2n} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$.

We also assert that there is no edge between $V(C)$ and $V\backslash V(C)$ in G_i for each $i \in [2\ell]$. Otherwise, there exists some $i \in [2\ell]$, $j \in [\ell]$ and a vertex $x \in V\backslash V(C)$ (let $x \in X$ by symmetry) such that $xy_j \in E(G_i)$. Let uv be the edge of $E(C)$ with $\phi(uv) = i$. As $G_{2n} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$, we have $uv \in E(G_{2n})$. Then (P_2, ϕ_1) with $P_2 = xy_jCx_j$ is a partial $\mathbf{G}$-transversal isomorphic to a $(2\ell+1)$-path where ϕ_1 arises from ϕ by setting $\phi_1(uv) := 2n$ (if uv exists in P_2) and $\phi_1(xy_j) := i$, a contradiction. Similarly, we can deduce that $G_1 = \cdots = G_{2\ell} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$.

Hence, Lemma 2.1 (2) holds and the assumption is wrong. Thus Claim 2.2 follows. □

Without loss of generality, let $P = x_1y_1x_2y_2\cdots x_\ell y_\ell$ have an associated injection $\phi : E(P) \to [2n]$ with $[2n]\backslash im(\phi) = [2\ell, 2n]$. Choose two arbitrary integers $c_1, c_2 \in [2\ell, 2n]$. Let I_{c_1} and I_{c_2} be the following sets:

$$I_{c_1} := \{i \in [\ell-1] : x_1y_i \in E(G_{c_1})\},$$

$$I_{c_2} := \{i \in [2,\ell] : x_iy_\ell \in E(G_{c_2})\}.$$

By the maximality of P and Claim 2.2, we have $N_{G_{c_1}}(x_1) \subseteq V(P)\backslash\{y_\ell\}$ and $N_{G_{c_2}}(y_\ell) \subseteq V(P)\backslash\{x_1\}$. We also have $I_{c_1} \cap I_{c_2} = \emptyset$, otherwise there exists a partial $\mathbf{G}$-transversal isomorphic to a 2ℓ-cycle. Therefore, $n \leq |I_{c_1}| + |I_{c_2}| = |I_{c_1} \cup I_{c_2}| \leq \ell \leq n-1$, a contradiction.

Case 2. P is a partial $\mathbf{G}$-transversal isomorphic to a $(2\ell-1)$-path.

Let $P = x_1y_1x_2y_2\cdots x_{\ell-1}y_{\ell-1}x_\ell$ with its associated injection $\phi^* : E(P) \to [2n]$. Without loss of generality, set $[2n]\backslash im(\phi^*) = [2\ell-1, 2n]$. Obviously, $\ell \leq n$. We first prove the following result.

Claim 2.3. There is a partial $\mathbf{G}$-transversal isomorphic to a $(2\ell-2)$-cycle.

Proof. Suppose $\mathbf{G}$ has no partial transversal isomorphic to a $(2\ell-2)$-cycle. Then $x_1y_{\ell-1} \notin E(G_{2n})$ and $x_\ell y_1 \notin E(G_{2n-1})$. We consider the following two sets:

$$I_{2n} := \{i \in [\ell-2] : x_1y_i \in E(G_{2n})\},$$

$$I_{2n-1} := \{i \in [3,\ell] : x_\ell y_{i-1} \in E(G_{2n-1})\}.$$

By the maximality of P, $N_{G_{2n}}(x_1) \subseteq V(C)\backslash\{y_{\ell-1}\}$ and $N_{G_{2n-1}}(x_\ell) \subseteq V(C)\backslash\{y_1\}$. No partial **G**-transversal isomorphic to a $(2\ell-2)$-cycle guarantees that $I_{2n} \cap I_{2n-1} = \emptyset$, which implies that $n \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq \ell \leq n$. Hence $\ell = n$. Now we define the following sets:

$$I'_{2n} := \{i \in [2, n-1] : y_n x_i \in E(G_{2n})\},$$

$$I'_{2n-1} := \{i \in [n-2] : x_1 y_i \in E(G_{2n-1})\}.$$

We can deduce $I'_{2n} \cap I'_{2n-1} = \emptyset$, otherwise, there is a partial **G**-transversal isomorphic to a Hamiltonian path. It follows that $n \leq |I'_{2n}| + |I'_{2n-1}| = |I'_{2n} \cup I'_{2n-1}| \leq n-1$, a contradiction. Thus the claim follows. $\square$

Without loss of generality, let (C, ϕ) be a partial **G**-transversal isomorphic to a $(2\ell-2)$-cycle with $C = x_1y_1x_2y_2 \cdots x_{\ell-1}y_{\ell-1}x_1$ and ϕ missing $[2\ell-1, 2n]$. We assert that $G_i[V\backslash V(C)]$ contains at least one edge for some integer $i \in [2\ell-1, 2n]$. Otherwise, $G_i[V\backslash V(C)]$ is an empty graph for each $i \in [2\ell-1, 2n]$. We define

$$I_{2n} := \{i \in [\ell-1] : x_i y \in E(G_{2n})\},$$

$$I_{2n-1} := \{i \in [\ell-1] : x y_i \in E(G_{2n-1})\},$$

where $x \in X\backslash V(C), y \in Y\backslash V(C)$. We have $I_{2n} \cap I_{2n-1} = \emptyset$, otherwise there exists some $i \in I_{2n} \cap I_{2n-1}$ such that $P_1 = yx_iC^-y_ix$ is a partial **G**-transversal isomorphic to a 2ℓ-path, contradicting with the maximality of P. Hence, $n \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq \ell - 1 \leq n-1$, a contradiction.

Without loss of generality, assume that $G_{2n}[V\backslash V(C)]$ contains at least one edge. Choose an arbitrary edge xy in $G_{2n}[V\backslash V(C)]$. We assert that $N_{G_j}(x) \cap V(C) = \emptyset$ and $N_{G_j}(y) \cap V(C) = \emptyset$ for each $j \in [2\ell-1, 2n-1]$. Otherwise, there exists some $j \in [2\ell-1, 2n-1]$ and $k \in [\ell-1]$ such that $xy_k \in E(G_j)$. Then $P_2 = yxy_kCx_k$ is a partial **G**-transversal isomorphic to a 2ℓ-path, a contradiction.

Fix an integer $j \in [2\ell-1, 2n-1]$, since $d_{G_j}(x) \geq \frac{n}{2}$ and $d_{G_j}(y) \geq \frac{n}{2}$, we have that $|V\backslash V(C)| \geq n$ and $G_j[V\backslash V(C)]$ is not empty. Choosing an edge in $G_j[V\backslash V(C)]$ and repeating the above discussion, we have $N_{G_{2n}}(x) \cap V(C) = \emptyset$ and $N_{G_{2n}}(y) \cap V(C) = \emptyset$. Thus, for each $v \in V\backslash V(C)$ and $i \in [2\ell-1, 2n]$, we have $N_{G_i}(v) \cap V(C) = \emptyset$. It follows that there is no edge between $V(C)$ and $V\backslash V(C)$ in G_i for each $i \in [2\ell-1, 2n]$. Hence, n is even and $G_{2\ell-1} = \cdots = G_{2n} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$.

Next we consider G_i for each $i \in [2\ell-2]$. Suppose that there exists a vertex $x \in X\backslash V(C)$ such that $xy_k \in E(G_i)$ for some $k \in [\ell-1]$. Let uv be the edge of C with $uv \in E(G_i)$. As $G_j = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$ for each $j \in [2\ell-1, 2n]$, we have $uv \in E(G_{2n-1})$ and $xy \in E(G_{2n})$ for each $y \in Y\backslash V(C)$. Then (P_4, ϕ'') is a partial **G**-transversal isomorphic to a 2ℓ-path such that $P_4 = yxy_kCx_k$ and ϕ'' arises from ϕ by letting $\phi''(uv) := 2n-1$ (if uv exists in P_4), $\phi''(xy) := 2n$ and $\phi''(xy_k) := i$, a contradiction. Hence for each $v \in V\backslash V(C)$ and $i \in [2\ell-2]$, we have $N_{G_i}(v) \cap V(C) = \emptyset$, which implies that $G_1 = \cdots = G_{2\ell-2} = K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}}$.

Thus Lemma 2.1 (2) holds. This completes the proof of Lemma 2.1. □

Lemma 2.4. *Let $\mathbf{G} = \{G_i : i \in [2n]\}$ be a bipartite graph collection on the same bipartition (X, Y) with $|X| = |Y| = n-1$ such that $\delta(\mathbf{G}) \geq \frac{n-1}{2}$. Let $P = x_1y_1 \cdots x_{n-1}y_{n-1}$ be a partial $\mathbf{G}$-transversal isomorphic to a $(2n-2)$-path. Then one of the following statements holds:*

(1) there is a partial $\mathbf{G}$-transversal isomorphic to a $(2n-2)$-cycle;

(2) there is a partial $\mathbf{G}$-transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2;

(3) $n-1$ is even and for each i missed by P, we have $N_{G_i}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}\}$.

Proof. Without loss of generality, let $\phi : E(P) \to [2n]$ be the associated injection of P with $\phi(x_iy_i) = 2i-1$ for $i \in [n-1]$ and $\phi(y_ix_{i+1}) = 2i$ for $i \in [n-2]$. Then $[2n] \backslash im(\phi) = [2n-2, 2n]$. Define the following sets:

$$I_{2n} := \{i \in [n-1] : x_1y_i \in E(G_{2n})\},$$

$$I_{2n-1} := \{i \in [n-1] : x_iy_{n-1} \in E(G_{2n-1})\}.$$

If $I_{2n} \cap I_{2n-1} \neq \emptyset$, then there exists some $i \in I_{2n} \cap I_{2n-1}$. Thus $C_1 = x_1Px_iy_{n-1}Py_ix_1$ is a partial $\mathbf{G}$-transversal isomorphic to a $(2n-2)$-cycle and so statement (1) holds. If $I_{2n} \cap I_{2n-1} = \emptyset$, then $n-1 \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq n-1$. It follows that $n-1$ is even, $I_{2n} \cup I_{2n-1} = [n-1]$ and $|I_{2n}| = |I_{2n-1}| = \frac{n-1}{2}$. If there exists some $i \in I_{2n-1}$ such that $i+1 \in I_{2n}$, then $C_2 = x_1Px_iy_{n-1}Py_{i+1}x_1$ and y_ix_{i+1} is a partial $\mathbf{G}$-transversal isomorphic to the disjoint of a $(2n-4)$-cycle and a copy of K_2 and statement (2) thus holds. Otherwise, $I_{2n} = [\frac{n-1}{2}]$ and $I_{2n-1} = [\frac{n+1}{2}, n-1]$, which implies that $N_{G_{2n}}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$ and $N_{G_{2n-1}}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}\}$. By symmetry, we have $N_{G_i}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}\}$ for each $i \in [2n-2, 2n]$. Hence, statement (3) holds. □

3 Proofs of Theorems 1.2 and 1.3

After the preparations of the above section, we are ready to give the proofs of our main results, Theorems 1.2 and 1.3.

3.1 Proof of Theorem 1.2

Given a positive integer d and a set $A \subseteq \mathbb{Z}$, let $A + d = \{i + d : i \in A\}$ and $A - d = \{i - d : i \in A\}$. We first introduce a useful tool obtained in [7].

Lemma 3.1 ([7]). *Let n be an integer and $(\mathbb{Z}_{2n},+)$ be the cyclic group of $2n$ elements. Let $A \subseteq \mathbb{Z}_{2n}$ and $B=(A+d)\cup(A-d)$ with $d\in[2n-1]$. If $|A|=|B|$, then $A=A+2d$.*

Now we are ready to prove Theorem 1.2.

Proof. By Theorem 1.1, each vertex of $X\cup Y$ is contained in a $\mathbf{G}$-transversal isomorphic to a Hamiltonian cycle. It remains to show that each vertex of $X\cup Y$ is contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for every even integer $\ell\in[4,2n-2]$.

First, we consider the case $n=3$. Let (C,ϕ) be a $\mathbf{G}$-transversal isomorphic to a 6-cycle C with $C=v_1v_2\cdots v_6v_1$ and its associated injection ϕ satisfying $\phi(v_iv_{i+1}):=i$ for each $i\in[6]$ (identify v_7 with v_1). Without loss of generality, assume that v_1 is not contained in any partial $\mathbf{G}$-transversal isomorphic to a 4-cycle. Now we show that $v_1v_4, v_2v_5, v_3v_6\notin E(G_i)$ for all $i\in[6]$. If $v_1v_4\in E(G_i)$ for some $i\in[6]$, then $v_1v_2v_3v_4v_1$ or $v_1v_4v_5v_6v_1$ is a partial $\mathbf{G}$-transversal isomorphic to a 4-cycle, a contradiction. Since $d_{G_i}(v_1)\geq 2$ and v_1,v_3,v_5 belong to the same part, we have $v_1v_2, v_1v_6\in E(G_i)$ for each $i\in[6]$. If $v_2v_5\in E(G_i)$ for some $i\in[6]\backslash\{5\}$, then $v_1v_2v_5v_6v_1$ is a partial $\mathbf{G}$-transversal isomorphic to a 4-cycle, a contradiction. So, $v_2v_5\notin E(G_i)$ for each $i\in[6]\backslash\{5\}$. It follows from $d_{G_i}(v_5)\geq 2$ that $v_5v_6\in E(G_i)$ for each $i\in[6]\backslash\{5\}$. Therefore, $v_2v_5\notin E(G_5)$, otherwise, $v_1v_2v_5v_6v_1$ is a partial $\mathbf{G}$-transversal isomorphic to a 4-cycle. By symmetry, we have $v_3v_6\notin E(G_i)$ for each $i\in[6]$. Thus, $G_i=C$ for all $i\in[6]$.

Next we consider the case $n\geq 4$. Fix an arbitrary vertex $x\in X$ and suppose that x is not contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ'-cycle for some even integer $\ell'\in[4,2n-2]$. Since $\delta(\mathbf{G})\geq\frac{n+1}{2}$, we have $\sum_{i=1}^{2n} d_{G_i}(x)\geq n(n+1)$. By an averaging argument, we can find a vertex $y\in Y$ such that the edge xy appears on at least $n+1$ graphs of $\mathbf{G}$. Set $M_i=G_i-\{x,y\}$ for each $i\in[2n]$ and $\mathbf{M}'=\{M_1,\cdots,M_{2n-2}\}$. Then $|M_i|=2n-2$ and $\delta(M_i)\geq\frac{n-1}{2}$ for each $i\in[2n]$. By Lemma 2.1, $\mathbf{M}'$ contains a partial transversal isomorphic to a $(2n-2)$-path or $n-1$ is even and $M_1=\cdots=M_{2n-2}=K_{\frac{n-1}{2},\frac{n-1}{2}}\cup K_{\frac{n-1}{2},\frac{n-1}{2}}$.

We first assume that $n-1$ is even and $M_1=\cdots=M_{2n-2}=K_{\frac{n-1}{2},\frac{n-1}{2}}\cup K_{\frac{n-1}{2},\frac{n-1}{2}}$. Set $X-\{x\}=X_1\cup X_2$ and $Y-\{y\}=Y_1\cup Y_2$ with $X_1=\{x_1,x_2,\cdots,x_{\frac{n-1}{2}}\}$, $X_2=\{x_{\frac{n+1}{2}},x_{\frac{n+3}{2}},\cdots,x_{n-1}\}$, $Y_1=\{y_1,y_2,\cdots,y_{\frac{n-1}{2}}\}$ and $Y_2=\{y_{\frac{n+1}{2}},y_{\frac{n+3}{2}},\cdots,y_{n-1}\}$. Furthermore, $M_i[X_j\cup Y_j]=K_{\frac{n-1}{2},\frac{n-1}{2}}$ for every $i\in[2n-2]$ and $j\in\{1,2\}$. As $\delta(\mathbf{G})\geq\frac{n+1}{2}$, we have $xy_j, yx_j\in E(G_i)$ for each $i\in[2n-2]$ and $j\in[n-1]$. It is not difficult to find that x is contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for every even integer $\ell\in[4,2n-2]$, a contradiction.

Now we assume that $\mathbf{M}'$ contains a partial transversal, denoted by P, isomorphic to a $(2n-2)$-path. It is clear that P is also a partial transversal isomorphic to a $(2n-2)$-path in $\mathbf{M}=\{M_1,\cdots,M_{2n}\}$. Thus applying Lemma 2.4 on $\mathbf{M}$, we know that one of the three statements in Lemma 2.4 holds. Next, we proceed with our proof by distinguishing three cases according to the three statements.

Case 1. There is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle.

Assume that $\mathbf{M}$ has a partial transversal isomorphic to a cycle $C = v_1v_2\cdots v_{2n-3}v_{2n-2}v_1$. Without loss of generality, let ϕ be its associated injection with $\phi(v_iv_{i+1}) := i$ for each $i \in [2n-2]$ (identify v_{2n-1} with v_1). Set $X = \{v_1, v_3, \cdots, v_{2n-3}, x\}$ and $Y = \{v_2, v_4, \cdots, v_{2n-2}, y\}$. Let ℓ be an even integer with $\ell \in [4, 2n-2]$. We define

$$I_{2n} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_i \in E(G_{2n})\}$$

and

$$I_{2n-1} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_{i+\ell-2} \in E(G_{2n-1})\}.$$

If $I_{2n} \cap I_{2n-1} \neq \emptyset$, then choose an integer $i \in I_{2n} \cap I_{2n-1}$, and so $v_iCv_{i+\ell-2}xv_i$ is a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle, a contradiction. Hence, $I_{2n} \cap I_{2n-1} = \emptyset$. Then, $\frac{n-1}{2} + \frac{n-1}{2} \leq |I_{2n}| + |I_{2n-1}| \leq n-1$, which implies that $I_{2n} \cup I_{2n-1} = [2n-2] \cap 2\mathbb{Z}$ and $|I_{2n}| = |I_{2n-1}| = \frac{n-1}{2}$. Since $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have $xy \in E(G_{2n}) \cap E(G_{2n-1})$. In fact, for any partial $\mathbf{M}$-transversal (C', ϕ') isomorphic to a $(2n-2)$-cycle with $V(C') = V(C)$, we have $xy \in E(G_i)$ for each $i \in [2n]$ missed by (C', ϕ').

Claim 3.2. For any two consecutive edges e and f on C, we have $xy \in E(G_{\phi(e)}) \cup E(G_{\phi(f)})$.

Proof. Without loss of generality, we only need to prove $xy \in E(G_1) \cup E(G_2)$. Suppose that $xy \notin E(G_1)$ and $xy \notin E(G_2)$. If $v_1v_2 \in E(G_{2n})$, then (C, ϕ_1) is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle where ϕ_1 arises from ϕ by setting $\phi_1(v_1v_2) := 2n$. Observe that (C, ϕ_1) misses 1, and then $xy \in E(G_1)$, a contradiction. Thus $v_1v_2 \notin E(G_{2n})$. By symmetry, $v_1v_2 \notin E(G_{2n-1})$.

For each even integer $j \in [2n-2]$, we pair $\{v_1, v_j\}$ with $\{v_2, v_{j+1}\}$. Note that if $v_1v_j \in E(G_{2n})$, then $v_2v_{j+1} \notin E(G_{2n-1})$. Otherwise, (C', ϕ_2) with $C' = C - v_1v_2 - v_jv_{j+1} + v_1v_j + v_2v_{j+1}$ is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle where ϕ_2 arises from ϕ by setting $\phi_2(v_1v_j) := 2n$ and $\phi_2(v_2v_{j+1}) := 2n-1$. Observe that (C', ϕ_2) misses 1, which implies $xy \in E(G_1)$, a contradiction. Therefore, $|N_{G_{2n}}(v_1) \cap V(C)| + |N_{G_{2n-1}}(v_2) \cap V(C)| \leq n-1$. On the other hand, since $\delta(G_{2n}) \geq \frac{n+1}{2}$ and $\delta(G_{2n-1}) \geq \frac{n+1}{2}$, we have $|N_{G_{2n}}(v_1) \cap V(C)| + |N_{G_{2n-1}}(v_2) \cap V(C)| \geq n-1$. Then, $|N_{G_{2n}}(v_1) \cap V(C)| = |N_{G_{2n-1}}(v_2) \cap V(C)| = \frac{n-1}{2}$. Hence for each even integer $j \in [2n-2]$, either $v_1v_j \in E(G_{2n})$ and $v_2v_{j+1} \notin E(G_{2n-1})$ or $v_1v_j \notin E(G_{2n})$ and $v_2v_{j+1} \in E(G_{2n-1})$. When $j = 2$, we have $v_2v_3 \in E(G_{2n-1})$ since $v_1v_2 \notin E(G_{2n})$. Therefore, (C, ϕ_3) is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle where ϕ_3 arises from ϕ by setting $\phi_3(v_2v_3) := 2n-1$. So, (C, ϕ_3) misses 2 which implies $xy \in E(G_2)$, a contradiction. The claim thus follows. $\square$

Recall that $I_{2n} := \{i \in [2n-2] \cap 2\mathbb{Z} : xv_i \in E(G_{2n})\}$ and $|I_{2n}| = \frac{n-1}{2}$. In fact, we will see that I_{2n} can be seen as a subgroup of $\mathbb{Z}_{2n-2}$. We consider the following sets

$$B := (I_{2n} + (\ell-2)) \cup (I_{2n} - (\ell-2)),$$

$$B' := (I_{2n} + (\ell-3)) \cup (I_{2n} - (\ell-3)),$$

where $\ell \in [4, 2n-2] \cap 2\mathbb{Z}$. So, $|B| \geq |I_{2n}|$ and $|B'| \geq |I_{2n}|$.

Set $A := \{j \in [2n-2] \cap 2\mathbb{Z} : xv_j \in E(G_{2n-1})\} \setminus B$. If $|B| \geq |I_{2n}|+1$, then $|A| \leq n-1-(|I_{2n}|+1) \leq \frac{n-3}{2} < \frac{n-1}{2}$. So, there exists some $i \in I_{2n}$ satisfying $j = i+(\ell-2)$ or $j = i-(\ell-2)$ such that $xv_j \in E(G_{2n-1})$. Then for every even integer $\ell \in [4, 2n-2]$, (C_1, ϕ_1) with $C_1 = xv_iCv_{i+\ell-2}x$ (or $C_1 = xv_iC^-v_{i-(\ell-2)}x$) is a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle, where ϕ_1 arises from ϕ by setting $\phi_1(xv_i) := 2n$ and $\phi_1(xv_{i+\ell-2}) := 2n-1$ (or $\phi_1(xv_{i-(\ell-2)}) := 2n-1$), a contradiction. Then $|I_{2n}| = |B|$, and by Lemma 3.1, $I_{2n} = I_{2n} + (2\ell-4)$.

Set $A' := \{k \in [2n-2] \backslash (2\mathbb{Z} \cup B') : yv_k \in G_{2n-1}\}$. If $|B'| \geq |I_{2n}|+1$, then $|A'| \leq \frac{n-3}{2} < \frac{n-1}{2}$. Thus, there exists some $i \in I_{2n}$ such that $k = i+(\ell-3)$ or $k = i-(\ell-3)$ such that $yv_k \in E(G_{2n-1})$. By Claim 3.2, we know that $xy \in E(G_{i-1})$ or $xy \in E(G_{i-2})$, and $xy \in E(G_i)$ or $xy \in E(G_{i+1})$. Without loss of generality, assume that $xy \in E(G_{i-1})$ and $xy \in E(G_i)$. Then we obtain that (C_2, ϕ_2) with $C_2 = xv_iCv_{i+\ell-3}yx$ (or $C_2 = xv_iC^-v_{i-(\ell-3)}yx$) is a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle, where ϕ_2 arises from ϕ by setting $\phi_2(xv_i) := 2n$, $\phi_2(yv_{i+\ell-3}) := 2n-1$ (or $\phi_2(yv_{i-(\ell-3)}) := 2n-1$) and $\phi_2(xy) := i-1$ (or $\phi_2(xy) := i$), a contradiction. Then $|I_{2n}| = |B|$, and by Lemma 3.1, $I_{2n} = I_{2n} + (2\ell-6)$.

Since $I_{2n} = I_{2n} + (2\ell-4)$ and $I_{2n} = I_{2n} + (2\ell-6)$, it follows that $I_{2n} = I_{2n} + 2$. Then $|I_{2n}| = n-1$, which contradicts with $|I_{2n}| = \frac{n-1}{2}$. Therefore for each even integer $\ell \in [4, 2n-2]$, there exists a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle containing x, a contradiction.

Case 2. There is a partial $\mathbf{M}$-transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2.

Let $C = v_1v_2\cdots v_{2n-4}v_1$ and $C \cup \{wz\}$ be the partial $\mathbf{M}$-transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2. Let ϕ be its associated injection with $\phi(wz) = 2n-3$ and $\phi(v_iv_{i+1}) = i$ for each $i \in [2n-4]$ (identify v_{2n-3} with v_1). Set $X = \{v_2, v_4, \cdots, v_{2n-4}, x, z\}$ and $Y = \{v_1, v_3, \cdots, v_{2n-5}, y, w\}$. We define

$$I_{2n} = \{i \in [2n-4] \cap 2\mathbb{Z} : zv_{i+1} \in E(G_{2n})\},$$

$$I_{2n-1} = \{i \in [2n-4] \cap 2\mathbb{Z} : wv_i \in E(G_{2n-1})\}.$$

Since $\mathbf{M}$ contains no partial transversal isomorphic to a $(2n-2)$-cycle, we have $I_{2n} \cap I_{2n-1} = \emptyset$. Note that $d_{G_{2n}[V(C)]}(z) \geq \frac{n-3}{2}$ and $d_{G_{2n-1}[V(C)]}(w) \geq \frac{n-3}{2}$. So, $n-3 \leq |I_{2n}| + |I_{2n-1}| = |I_{2n} \cup I_{2n-1}| \leq n-2$.

If $|I_{2n} \cup I_{2n-1}| = n-2$, then $I_{2n} \cup I_{2n-1} = [2n-4] \cap 2\mathbb{Z}$. So, there exists some $i \in [2n-4] \cap 2\mathbb{Z}$ satisfying $i \in I_{2n}$ and $i+2 \in I_{2n-1}$. Therefore, (C', ϕ') is partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle with $C' = zwv_{i+2}Cv_{i+1}z$ and ϕ' originating from ϕ by adding $\phi'(v_{i+1}z) := 2n$ and $\phi'(v_{i+2}w) := 2n-1$, a contradiction. Hence $|I_{2n} \cup I_{2n-1}| = n-3$. It follows that n is odd and $|I_{2n}| = |I_{2n-1}| = \frac{n-3}{2}$. There exists some $i \in [2n-4] \cap 2\mathbb{Z}$ such that $i \notin I_{2n} \cup I_{2n-1}$. Without loss of generality, we assume

$2n-4 \notin I_{2n} \cup I_{2n-1}$, implying that $I_{2n} \cup I_{2n-1} = [2n-6] \cap 2\mathbb{Z}$. As $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have $zw, zy \in E(G_{2n})$ and $wx, wz \in E(G_{2n-1})$.

If there exists an $i \in [2n-6] \cap 2\mathbb{Z}$ such that $i \in I_{2n}$ and $i+2 \in I_{2n-1}$, then (C_1, ϕ_1) is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle with $C_1 = zwv_{i+2}Cv_{i+1}z$ and ϕ_1 obtained from ϕ by setting $\phi_1(zv_{i+1}) := 2n$ and $\phi_1(wv_{i+2}) := 2n-1$, a contradiction. Therefore, $I_{2n} = \{n-1, n+1, \cdots, 2n-6\}$ and $I_{2n-1} = \{2, 4, \cdots, n-3\}$, which means that $N_{G_{2n}}(z) = \{v_n, v_{n+2}, \cdots, v_{2n-5}, y, w\}$ and $N_{G_{2n-1}}(w) = \{v_2, v_4, \cdots, v_{n-3}, x, z\}$. By symmetry, for each $i \in [2n-3, 2n]$, we have $N_{G_i}(z) = \{v_n, v_{n+2}, \cdots, v_{2n-5}, y, w\}$ and $N_{G_i}(w) = \{v_2, v_4, \cdots, v_{n-3}, x, z\}$.

Note that $\mathbf{G} - \{x, w\}$ contains no partial transversal isomorphic to a $(2n-2)$-cycle, since otherwise, the proof of Case 1 ensures that x is contained in a partial $\mathbf{G}$-transversal isomorphic to an ℓ-cycle for each even integer $\ell \in [4, 2n-2]$, a contradiction. Since $yz \in E(G_{2n-3})$, we have that $C \cup \{yz\}$ is also a partial transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2 in $\mathbf{G}$. Moreover, so far we have not used the property that xy appears on at least $n+1$ graphs of $\mathbf{G}$. Hence, we can exchange the roles of y and w in the above proof of Case 2, and obtain that $N_{G_i}(y) = \{v_2, v_4, \cdots, v_{n-3}, x, z\}$ for each $i \in [2n-3, 2n]$.

Actually, for any partial $\mathbf{M}$-transversal (C', ϕ') with $V(C') = V(C)$, we have $xy \in E(G_i)$ for each $i \in [2n]$ missed by (C', ϕ'). Next, we prove one more property for the edge xy.

Claim 3.3. For any three consecutive edges e, f, g on C, we have $xy \in E(G_{\phi(e)}) \cup E(G_{\phi(f)}) \cup E(G_{\phi(g)})$.

Proof. Without loss of generality, we only need to consider the case that $e = v_1v_2$, $f = v_2v_3$ and $g = v_3v_4$, and prove that $xy \in E(G_1) \cup E(G_2) \cup E(G_3)$. Suppose to the contrary that $xy \notin E(G_1) \cup E(G_2) \cup E(G_3)$.

If $v_2v_3 \in E(G_{2n})$, then (C, ϕ') is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-4)$-cycle where ϕ' arises from ϕ by setting $\phi'(v_2v_3) := 2n$. So, (C, ϕ') misses 2. Hence $xy \in E(G_2)$, a contradiction. Thus $v_2v_3 \notin E(G_{2n})$. By symmetry, we have $v_2v_3 \notin E(G_i)$ for each $i \in [2n-3, 2n]$. Similarly, we have $v_iv_{i+1} \notin E(G_j)$ for each $i \in \{1, 2, 3\}$ and each $j \in [2n-3, 2n]$.

For each odd integer $a \in [2n-4]$, we pair $\{v_2, v_a\}$ with $\{v_3, v_{a+1}\}$. The total number of such pairs is $n-2$. For any $\{j_1, j_2\} \subseteq [2n-3, 2n]$, if $v_2v_a \in E(G_{j_1})$, then $v_3v_{a+1} \notin E(G_{j_2})$. Otherwise, (C', ϕ'') is a partial transversal isomorphic to a $(2n-4)$-cycle with $C' = v_2v_aC^-v_3v_{a+1}Cv_2$, where ϕ'' arises from ϕ by setting $\phi''(v_2v_a) = j_1$ and $\phi''(v_3v_{a+1}) = j_2$. Observing that $V(C') = V(C)$ and (C', ϕ'') misses 2, we have $xy \in G_2$, a contradiction.

Notice that $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are a pair, while $\{v_2, v_3\}$ and $\{v_3, v_4\}$ are a pair as well. Recall that $v_iv_{i+1} \notin E(G_j)$ for each $i \in \{1, 2, 3\}$ and $j \in [2n-3, 2n]$. Then $v_1v_2 \notin E(G_{j_1})$, $v_2v_3 \notin E(G_{j_2})$, $v_2v_3 \notin E(G_{j_1})$ and $v_3v_4 \notin E(G_{j_2})$, which indicates that $|N_{G_{j_1}}(v_2) \cap V(C)| + |N_{G_{j_2}}(v_3) \cap V(C)| \leq n-4$. Since $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have $|N_{G_{j_1}}(v_2) \cap$

$V(C)| + |N_{G_{j_2}}(v_3) \cap V(C)| \geq n-3$, a contradiction. $\square$

Claim 3.4. For each $i \in [2n-3, 2n]$, we have $N_{G_i}(x) = \{v_n, v_{n+2}, \cdots, v_{2n-5}, y, w\}$.

Proof. Suppose to the contrary that there exists some $i \in [2n-3, 2n]$ and an odd integer $a \in [n-1]$ such that $xv_a \in E(G_i)$. Without loss of generality, we assume that $xv_a \in E(G_{2n})$. Recalling that n is odd, we have $a \neq n-1$.

If $a = n-2$, then we can find a partial **G**-transversal isomorphic to an ℓ-cycle containing x for each even integer $\ell \in [4, 2n-2]$. In fact, for each even integer $\ell \in [4, n-1]$, (C_2, ϕ_2) is a partial **G**-transversal isomorphic to an ℓ-cycle containing x, where $C_2 = yv_{n-\ell+1}Cv_{n-2}xy$ and ϕ_2 arises from ϕ by setting $\phi_2(xv_{n-2}) := 2n, \phi(yv_{n-\ell+1}) := 2n-1, \phi(xy) := 2n-2$. For each even integer $\ell \in [n+3, 2n-2]$, (C_2, ϕ_2) with $C_2 = yv_{n-\ell+1}C^{-}v_{n-2}xy$ is a partial **G**-transversal isomorphic to an ℓ-cycle containing x. For $\ell = n+1$, (C_2, ϕ_2) is a partial **G**-transversal isomorphic to an ℓ-cycle containing x, where $C_2 = xyzwv_2Cv_{n-2}x$ and ϕ_2 arises from ϕ by setting $\phi_2(xv_{n-2}) := 2n, \phi(yz) := 2n-1, \phi(zw) := 2n-2, \phi(wv_2) := 2n-3, \phi(xy) \in \{n-2, n-1, n\}$ (by Claim 3.3). In conclusion, for each even integer $\ell \in [4, 2n-2]$, x is contained in a partial **G**-transversal isomorphic to an ℓ-cycle, a contradiction.

Now we assume $a \neq n-2$. Then the odd integer a is in $[n-4]$. For each even integer $\ell \in [4, n-a]$, (C_3, ϕ_3) is a partial **G**-transversal isomorphic to an ℓ-cycle containing x, where $C_3 = xv_aCv_{a+\ell-3}yx$ and ϕ_3 arises from ϕ by setting $\phi_3(xv_a) := 2n, \phi_3(yv_{a+\ell-3}) := 2n-1, \phi_3(xy) := 2n-2$. By symmetry, for each even integer $\ell \in [n+a+2, 2n-2]$, set $\ell^* + \ell = 2n+2$, then $\ell^* \in [4, n-a]$. Thus, (C_3, ϕ_3) with $C_3 = xv_aC^{-}v_{a+\ell^*-3}yx$ is a partial **G**-transversal isomorphic to an ℓ-cycle containing x.

For $\ell = n-a+2$, (C_3, ϕ_3) is a partial **G**-transversal isomorphic to an $(n+2-a)$-cycle with $C_3 = v_aCv_{n-3}wzyxv_a$ and ϕ_3 obtained from ϕ by setting $\phi_3(xv_a) := 2n$, $\phi_3(wv_{n-3}) := 2n-1$, $\phi_3(yz) := 2n-2$, $\phi_3(zw) := 2n-3$, $\phi_3(xy) \in \{a-3, a-2, a-1\}$ (by Claim 3.3).

For each even integer $\ell \in [n-a+4, 2n-a-1]$, (C_3, ϕ_3) with $C_3 = xv_aCv_{a+\ell-4}zyx$ is a partial **G**-transversal isomorphic to an ℓ-cycle, where ϕ_3 arises from ϕ by setting $\phi_3(xv_a) := 2n$, $\phi_3(xy) := 2n-1$, $\phi_3(yz) := 2n-2$, $\phi_3(zv_{a+\ell-4}) := 2n-3$. Likewise, for each even integer $\ell \in [a+5, n+a]$, set $\ell^* + \ell = 2n+4$, then $\ell^* \in [n-a+4, 2n-a-1]$. Thus, (C_3, ϕ_3) with $C_3 = xv_aC^{-}v_{a+\ell^*-4}zyx$ is a partial **G**-transversal isomorphic to an ℓ-cycle, where ϕ_3 arises from ϕ by setting $\phi_3(xv_a) := 2n$, $\phi_3(xy) := 2n-1$, $\phi_3(yz) := 2n-2$, $\phi_3(zv_{a+\ell^*-4}) := 2n-3$.

Since $a \leq n-4$, we have $2n-a-1 > a+5$. Consequently, x is contained in a partial **G**-transversal isomorphic to an ℓ-cycle for every even integer $\ell \in [2n-2]$, a contradiction. Therefore, we get that $N_{G_{2n}}(x) = \{v_n, v_{n+2}, \cdots, v_{2n-5}, y, w\}$. By symmetry, $N_{G_i}(x) = \{v_n, v_{n+2}, \cdots, v_{2n-5}, y, w\}$ for each $i \in [2n-3, 2n]$. The claim thus follows. $\square$

Therefore, for each even integer $\ell \in [6, n+1]$, (C', ϕ') is a partial **G**-transversal isomorphic to an ℓ-cycle with $C' = yv_{n-3}Cv_{n+\ell-6}xy$ and ϕ' arising from ϕ by setting

$\phi'(xy) := 2n$, $\phi'(xv_{n+\ell-6}) := 2n-1$, $\phi'(yv_2) := 2n-2$. For each even integer $\ell \in [n+1, 2n-4]$, (C'', ϕ'') is a partial **G**-transversal isomorphic to an ℓ-cycle with $C'' = yv_2Cv_{\ell-1}xy$ and ϕ'' arising from ϕ by setting $\phi''(xy) := 2n$, $\phi''(xv_{\ell-1}) := 2n-1$, $\phi''(yv_2) := 2n-2$. Apparently, $xyzwz$ and xyv_4Cv_2wx are partial **G**-transversals isomorphic to a 4-cycle and a $(2n-2)$-cycle, respectively. Hence, x is contained in a partial **G**-transversal isomorphic to an ℓ-cycle with any even integer $\ell \in [2n-2]$, a contradiction.

Case 3. Lemma 2.4 (3) holds in **M**.

Without loss of generality, we assume that $P = x_1y_1 \cdots x_{n-1}y_{n-1}$ with $\phi(x_iy_i) = 2i-1$ for each $i \in [n-1]$ and $\phi(y_jx_{j+1}) = 2j$ for each $j \in [n-2]$.

Claim 3.5. For each $i \in \{\phi(x_1y_1), \phi(x_{n-1}y_{n-1})\}$ or i missed by P, we have $N_{G_i}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}, y\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}, x\}$.

Proof. Note that P misses $2n-2$, $2n-1$ and $2n$. By Lemma 2.4 (3), we know that $N_{M_i}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$ and $N_{M_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}\}$ for each $i \in [2n-2, 2n]$. Since $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have $N_{G_i}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}, y\}$ and $N_{G_i}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}, x\}$ for each $i \in [2n-2, 2n]$.

Let (P, ϕ_1) be a partial **M**-transversal isomorphic to a $(2n-2)$-path with ϕ_1 arising from ϕ by setting $\phi'(x_1y_1) := 2n$. Note that (P, ϕ_1) misses 1. Then by Lemma 2.4 (3) and $\delta(G_1) \geq \frac{n+1}{2}$, we get that $N_{G_1}(x_1) = N_{G_{2n}}(x_1)$ and $N_{G_1}(y_{n-1}) = N_{G_{2n}}(y_{n-1})$. Since $x_{n-1}y_{n-1} \in E(G_{2n-1})$, by the similar analysis, we can deduce that $N_{G_{2n-3}}(y_{n-1}) = N_{G_{2n-1}}(y_{n-1})$ and $N_{G_{2n-3}}(x_1) = N_{G_{2n-1}}(x_1)$. The claim thus follows. □

Denote $C(uv) := \{i \in [2n] : uv \in E(G_i)\}$. Let (P_i, ϕ_i) be a partial **M**-transversal isomorphic to a $(2n-2)$-path with $P_i = x_iPx_1y_iPy_{n-1}$ and ϕ_i arising from ϕ by setting $\phi_i(x_1y_i) := 2n$ for each $i \in [2, \frac{n-1}{2}]$. Observe that (P_i, ϕ_i) misses $2i-1$ and $\phi_i(x_iy_{i-1}) = 2i-2$. By Claim 3.5, we know that $[1, n-2] \cup [2n-3, 2n] \subseteq C(xy_{n-1})$. Let (P'_i, ϕ'_i) be a partial **M**-transversal isomorphic to a $(2n-2)$-path with $P'_i = y_iPy_{n-1}x_iPx_1$ and ϕ'_i arising from ϕ by setting $\phi'_i(x_iy_{n-1}) := 2n$ for each $i \in [\frac{n+1}{2}, n-2]$. By the similar analysis, we have $\{1\} \cup [n, 2n] \subseteq C(x_1y)$.

Since $|C(xy)| \geq n+1 > 1$, there exists some $a \in C(xy)$ satisfying $a \neq n-1$. If $a \in [2n-2, 2n]$, say $a = 2n$, then $yx_1Py_{n-1}xy$ is a partial **G**-transversal isomorphic to a Hamiltonian cycle and $xy_{n-1}x_{n-1}y_{n-2}x$ is a partial **G**-transversal isomorphic to a 4-cycle. Moreover, for each $k \in [3, n-1]$, (C_{2k}, ϕ^*_{2k}) is a partial transversal isomorphic to a $2k$-cycle with $C_{2k} = yx_1y_{\frac{n-1}{2}-\lfloor\frac{k-3}{2}\rfloor}Px_{\frac{n+1}{2}+\lfloor\frac{k-3}{2}\rfloor}y_{n-1}xy$ and ϕ^*_{2k} arising from ϕ by adding $\phi^*_{2k}(x_1y_{\frac{n-1}{2}-\lfloor\frac{k-3}{2}\rfloor}) := 1$, $\phi^*_{2k}(x_{\frac{n+1}{2}+\lfloor\frac{k-3}{2}\rfloor}y_{n-1}) := 2n-3$, $\phi^*_{2k}(xy) := 2n$, $\phi^*_{2k}(yx_1) := 2n-1$, $\phi^*_{2k}(xy_{n-1}) := 2n-2$, a contradiction. Furthermore, for any Hamiltonian **G**-transversal C satisfying $xy \in E(C)$, there exists a partial **G**-transversal isomorphic to a $2k$-cycle containing x for each $k \in [2, n]$.

Next, we consider the case $a < n-1$. If $\phi(x_jy_j) = a$, then (P', ϕ') is a partial **M**-transversal isomorphic to a $(2n-2)$-path with $P' = P_j$ and $\phi' = \phi_j$. Observe that (P', ϕ')

misses a. By Claim 3.5, (C, ϕ^*) is a $\mathbf{G}$-transversal isomorphic to a Hamiltonian cycle with $C = yx_jP'y_{n-1}xy$, where ϕ^* arises from ϕ' by setting $\phi^*(xy) := a$, $\phi^*(x_jy) := 2n-1$ and $\phi^*(xy_{n-1}) := 2n-2$, a contradiction.

If $\phi(y_jx_{j+1}) = a$, then (P_{j+1}, ϕ_{j+1}) is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-path. Applying Claim 3.5 yields that $y_jx_{j+1} \in E(G_{2n-1})$. Therefore (P', ϕ') is a partial transversal isomorphic to a $(2n-2)$-path as well, with $P' = P_{j+1}$ and ϕ' arising from ϕ_{j+1} by setting $\phi'(y_jx_{j+1}) := 2n-1$. So, (P', ϕ') misses a and $2j+1$. Hence, (C, ϕ^*) is a Hamiltonian $\mathbf{G}$-transversal with $C = yx_{j+1}P'y_{n-1}xy$, where ϕ^* arises from ϕ' by setting $\phi^*(xy) := a$, $\phi^*(x_{j+1}y) := 2n-2$ and $\phi^*(xy_{n-1}) := 2j+1$, a contradiction. By symmetry, we deduce that if $a \in [n, 2n-3]$, then x is contained in a partial $\mathbf{G}$-transversal isomorphic to a $2k$-cycle for each $k \in [2, n]$, a contradiction.

This complete the proof for the case $n \geq 4$, and Theorem 2.1 thus follows. $\square$

3.2 Proof of Theorem 1.3

Proof. Take arbitrary vertices $x \in X$ and $y \in Y$. Set $M_i = G_i - \{x, y\}$ for each $i \in [2n]$ with bipartition $X_M = X \setminus \{x\}$ and $Y_M = Y \setminus \{y\}$. Let $\mathbf{M} = \{M_1, M_2, \cdots, M_{2n}\}$. Then $|M_i| = 2n-2$ and $\delta(M_i) \geq \frac{n-1}{2}$ for each $i \in [2n]$. By Lemma 2.1, $\mathbf{M}$ contains a partial transversal isomorphic to a $(2n-2)$-path or $M_1 = \cdots = M_{2n} = K_{\frac{n-1}{2},\frac{n-1}{2}} \cup K_{\frac{n-1}{2},\frac{n-1}{2}}$ (n is odd).

If $M_1 = \cdots = M_{2n} = K_{\frac{n-1}{2},\frac{n-1}{2}} \cup K_{\frac{n-1}{2},\frac{n-1}{2}}$, then without loss of generality, let $X_M = X_1 \cup X_2$, $Y_M = Y_1 \cup Y_2$ with $X_1 = \{x_1, x_2, \cdots, x_{\frac{n-1}{2}}\}$, $X_2 = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}\}$, $Y_1 = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$, $Y_2 = \{y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, \cdots, y_{n-1}\}$. Moreover, $M_i[X_j \cup Y_j] = K_{\frac{n-1}{2},\frac{n-1}{2}}$ for every $i \in [2n]$ and $j \in \{1, 2\}$. As $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have xy_i, $yx_i \in E(G_j)$ for each $i \in [n-1]$ and $j \in [2n]$. Therefore $\mathbf{G} = \mathbf{F}$.

Note that each partial $\mathbf{M}$-transversal is also a partial $\mathbf{G}$-transversal. If $\mathbf{M}$ contains a partial transversal isomorphic to a $(2n-2)$-path, then $\mathbf{M}$ satisfies one of the three statements by Lemma 2.4. Next, we distinguish the following three cases to proceed the proof.

Case 1. There is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle.

Assume that $C = x_1y_1 \cdots x_{n-1}y_{n-1}x_1$ is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle. Let ϕ be an associated injection of C (identify x_n with x_1) with $\phi(x_iy_i) = 2i-1$ and $\phi(y_ix_{i+1}) = 2i$ for each $i \in [n-1]$. Then $[2n] \backslash im(\phi) = \{2n-1, 2n\}$. We define

$$T_{2n} := \{i \in [n-1] : xy_i \in E(G_{2n})\},$$

$$T_{2n-1} := \{i \in [n-1] : x_iy \in E(G_{2n-1})\}.$$

If $T_{2n} \cap T_{2n-1} \neq \emptyset$, then xy_iCx_iy is a partial transversal isomorphic to a Hamiltonian path from x to y. If $T_{2n} \cap T_{2n-1} = \emptyset$, then $\frac{n-1}{2} + \frac{n-1}{2} \leq |T_{2n}| + |T_{2n-1}| \leq n-1$, which implies that $T_{2n} \cup T_{2n-1} = [n-1]$ and $|T_{2n}| = |T_{2n-1}| = \frac{n-1}{2}$. Therefore, there exists some

$i \in [n-1]$ such that $i \in T_{2n}$ and $i+1 \in T_{2n-1}$. Then $yx_{i+1}Cy_ix$ is a partial transversal isomorphic to a Hamiltonian path from y and x.

Case 2. There is a partial $\mathbf{M}$-transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2.

Let $C = x_1y_1 \cdots x_{n-2}y_{n-2}x_1$ (identify x_{n-1} with x_1) and $C \cup \{wz\}$ ($z \in X_M$ and $w \in Y_M$) be the partial $\mathbf{M}$-transversal isomorphic to the disjoint union of a $(2n-4)$-cycle and a copy of K_2 with associated injection ϕ, where $\phi(wz) = 2n-3$ and $\phi(x_iy_i) = 2i-1$ and $\phi(y_ix_{i+1}) = 2i$ for each $i \in [n-2]$. We define the following sets

$$T_{2n} := \{i \in [n-2] : zy_i \in E(G_{2n})\},$$

$$T_{2n-1} := \{i \in [n-2] : wx_i \in E(G_{2n-1})\}.$$

Note that if $\mathbf{M}$ contains a partial transversal isomorphic to a $(2n-2)$-cycle, then we are done by Case 1. So, we have $T_{2n} \cap T_{2n-1} = \emptyset$. Then, $n-3 \leq |T_{2n}| + |T_{2n-1}| = |T_{2n} \cup T_{2n-1}| \leq n-2$.

If $|T_{2n} \cup T_{2n-1}| = n-2$, then we have $T_{2n} \cup T_{2n-1} = [n-2]$. Observe that there exists some $i \in [n-2]$ satisfying $i \in T_{2n}$, $i+1 \in T_{2n-1}$. Thus, (C, ϕ') is a partial $\mathbf{M}$-transversal isomorphic to a $(2n-2)$-cycle such that $C' = y_izwx_{i+1}Cy_i$ and ϕ' arises from ϕ by setting $\phi'(y_iz) := 2n$, $\phi'(wx_{i+1}) := 2n-1$, a contradiction.

If $|T_{2n} \cup T_{2n-1}| = n-3$, then we have $|T_{2n}| = |T_{2n-1}| = \frac{n-3}{2}$, which indicates that n is odd. Hence, there exists some $i \in [n-2]$ such that $i \notin T_{2n} \cup T_{2n-1}$. Without loss of generality, say $i = 1$, implying that $T_{2n} \cup T_{2n-1} = [2, n-2]$. As $\delta(\mathbf{G}) \geq \frac{n+1}{2}$, we have zw, $zy \in E(G_{2n})$, wx, $wz \in E(G_{2n-1})$.

If there exists some $i \in [n-3]$ such that $i \in T_{2n}$ and $i+1 \in T_{2n-1}$, then $y_izwx_{i+1}Cy_i$ is a partial transversal isomorphic to a $(2n-2)$-cycle, a contradiction. Hence, $T_{2n-1} = [2, \frac{n-1}{2}]$ and $T_{2n} = [\frac{n+1}{2}, n-2]$. It follows that $N_{G_{2n-1}}(w) = \{x_2, x_3, \cdots, x_{\frac{n-1}{2}}, x, z\}$ and $N_{G_{2n}}(z) = \{y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, \cdots, y_{n-2}, y, w\}$. By symmetry, we deduce that for any $i \in [2n-3, 2n]$, $N_{G_i}(w) = \{x_2, x_3, \cdots, x_{\frac{n-1}{2}}, x, z\}$ and $N_{G_i}(z) = \{y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, \cdots, y_{n-2}, y, w\}$.

Observe that $(C' \cup \{x_1y_1\}, \phi')$ is a partial $\mathbf{M}$-transversal with $C' = wx_2Cy_{n-2}zw$ (identify x_{n-1} with z), where ϕ' arises from ϕ by setting $\phi'(wx_2) := 2n$, $\phi'(zy_{n-2}) := 2n-1$, $\phi'(zw) := 2n-2$. Let

$$T_{2n-3} := \{i \in [2, n-2] : x_1y_i \in E(G_{2n-3})\},$$

$$T_{2n-4} := \{i \in [n-2] : x_{i+1}y_1 \in E(G_{2n-4})\}.$$

Since $x_1 \notin N_{G_{2n-3}}(w)$, by the similar analysis we obtain that $|T_{2n-3}| = |T_{2n-4}| = \frac{n-3}{2}$, $T_{2n-3} = [2, \frac{n-1}{2}]$ and $T_{2n-4} = [\frac{n+1}{2}, n-2]$. Recall that $x_{n-1} = z$. Hence, $N_{G_{2n-3}}(x_1) = \{y, y_1, y_2, \cdots, y_{\frac{n-1}{2}}\}$ and $N_{G_{2n-4}}(y_1) = \{x, x_1, x_{\frac{n+3}{2}}, \cdots, x_{n-2}, z\}$. By symmetry, $N_{G_{2n-3}}(y_1) = N_{G_{2n-4}}(y_1)$, contradicting with $y_1 \notin N_{G_{2n-3}}(z)$.

Case 3. Lemma 2.4 (3) holds in $\mathbf{M}$.

Let $P = x_1y_1 \cdots x_{n-1}y_{n-1}$ be a partial **M**-transversal isomorphic to a $(2n-2)$-path with $\phi(x_iy_i) = 2i-1$ for each $i \in [n-1]$ and $\phi(y_ix_{i+1}) = 2i$ for each $i \in [n-2]$. Then $[2n]\backslash im(\phi) = [2n-2, 2n]$. By Lemma 2.4 (3) and $\delta(G_j) \geq \frac{n+1}{2}$ for each $j \in \{2n-1, 2n\}$, we have $N_{G_{2n}}(x_1) = \{y_1, y_2, \cdots, y_{\frac{n-1}{2}}, y\}$ and $N_{G_{2n-1}}(y_{n-1}) = \{x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, \cdots, x_{n-1}, x\}$. Hence, (P', ϕ') is a partial **G**-transversal isomorphic to a Hamiltonian path from x to y such that $P' = xy_{n-1}Px_1y$ and ϕ' arises from ϕ by setting $\phi'(xy_{n-1}) := 2n-1$ and $\phi'(yx_1) := 2n$.

This completes the proof of Theorem 1.3. □

Note that Theorem 1.3 is about the rainbow Hamiltonian connectivity of a collection of $2n$ bipartite graphs rather than $2n-1$ bipartite graphs. The main reason is that in the above proof, we need to use two colors not appearing on the rainbow cycle C of length $2n-2$ to find a rainbow Hamiltonian path for for any two vertices $x \in X$ and $y \in Y$. It would be interesting to study the rainbow Hamiltonian connectivity of a collection $2n-1$ bipartite graphs under the same degree condition.

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