# Edge-disjoint properly colored cycles in edge-colored complete graphs 

Xiaozheng Chen, Luyi Li, Xueliang Li<br>Center for Combinatorics and LPMC<br>Nankai University, Tianjin 300071, China<br>Email: cxz@mail.nankai.edu.cn, liluyi@mail.nankai.edu.cn, lxl@nankai.edu.cn


#### Abstract

In an edge-colored graph $G$, let $d^{m o n}(v)$ denote the maximum number of edges with the same color incident with a vertex $v$ in $G$, called the monochromatic-degree of $v$. The maximum value of $d^{m o n}(v)$ over all vertices $v \in V(G)$ is called the maximum monochromaticdegree of $G$, denoted by $\Delta^{\text {mon }}(G)$. Li et al. in 2019 conjectured that every edge-colored complete graph $G$ of order $n$ with $\Delta^{\text {mon }}(G) \leq n-3 k+1$ contains $k$ vertex-disjoint properly colored (PC for short) cycles of length at most 4, and they showed that the conjecture holds for $k=2$. Han et al. showed that every edge-colored complete graph $G$ of order $n$ with $\Delta^{\text {mon }}(G) \leq n-2 k$ contains $k$ PC cycles of different lengths. They further got the condition $\Delta^{m o n}(G) \leq n-6$ for the existence of two vertex-disjoint PC cycles of different lengths. In this paper, we consider the problems of the existence of edge-disjoint PC cycles of length at most 4 (different lengths) in an edge-colored complete graph $G$ of order $n$.


Keywords: edge-colored complete graph; (maximum) monochromatic-degree; properly colored ( $P C$ ) cycle; edge-disjoint.

AMS subject classification 2010: $05 \mathrm{C} 15,05 \mathrm{C} 38,05 \mathrm{C} 07$.

## 1 Introduction

Let $G$ be a simple graph consisting of a vertex-set $V(G)$ and an edge-set $E(G)$. An edgecoloring of $G$ is a mapping $c: E(G) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is called the color-set. An edge-colored graph is a graph equipped with an edge-coloring. In an edge-colored graph $G$, we use $c(e)$ to denote the color of an edge $e$ of $G$ and $c(G)$ to denote the set of colors assigned to the edges
of $G$. A subgraph is properly colored in an edge-colored graph $G$, or $P C$ for short, if no two adjacent edges of the subgraph have the same color. Similarly, a subgraph is rainbow in an edge-colored graph $G$ if no two edges of the subgraph have the same color. In an edge-colored graph $G$, for a color $\alpha \in c(G)$ and a vertex $v \in V$, we define the $\alpha$-neighbor of $v$ in $G$ by $N_{G}^{\alpha}(v)=\left\{u \in N_{G}(v): c(u v)=\alpha\right\}$, where $N_{G}(v)$ denotes the neighbors of $v$ in $G$, and we denote by $d_{G}^{\alpha}(v)$ the number of vertices in $N_{G}^{\alpha}(v)$. Let $d^{c}(v)$ denote the number of different colors on the edges incident with a vertex $v$ in $G$, called the color-degree of $v$. Denote by $\delta^{c}(G)$ the minimum value of $d^{c}(v)$ over all vertices $v$ in $G$, called the minimum color-degree of an edge-colored graph $G$. Similarly, denote by $\Delta^{c}(G)$ the maximum value of $d^{c}(v)$ over all vertices $v$ in $G$, called the maximum color-degree of an edge-colored graph $G$. Let $d^{\text {mon }}(v)$ denote the maximum number of edges with the same color incident with a vertex $v$ in $G$, called the monochromatic-degree of $v$. The maximum value of $d^{\text {mon }}(v)$ over all vertices $v \in V(G)$ is called the maximum monochromatic-degree of $G$, denoted by $\Delta^{\text {mon }}(G)$.

The length of a path or a cycle is the number of edges on the path or cycle. A cycle of length $\ell$ is denoted by $C_{\ell}$. For a vertex subset $X$ of $V(G), G[X]$ denotes the subgraph induced by $X$. For any two distinct vertex subsets $X$ and $Y$ in $G$, we use $E[X, Y]$ to denote the edge subset of $G$ such that one end of each edge of $E[X, Y]$ is in $X$ and the other end is in $Y$. For convenience, let $c(X, Y)=\{c(e), e \in E[X, Y]\}$. If $X=\{v\}$, then we write $c(v, Y)$ for $c(\{v\}, Y)$. For an edge subset $E^{\prime} \subseteq E(G)$, set $\psi\left(E^{\prime}\right)=V\left(E^{\prime}\right)$. In particular, if $E^{\prime}=\{e\}$, then $\psi(e)$ denotes the set of the two endvertices of $e$. For terminology and notation not defined here, we refer the reader to [4].

In recent years, the problems on the existence of PC cycles and rainbow cycles in an edgecolored complete graph attract much attention, and thus a lot of work has been done extensively. For more details, we refer the reader to literatures [14, 16, 17] (for rainbow cycles) and Chapter 16 in [2] (for PC cycles). In fact, there are a lot of researches on the existence of long PC cycles in edge-colored complete graphs; see [1, 19, 20]. It is also worthwhile to study the numbers of vertex-disjoint and/or edge-disjoint cycles in edge-colored graphs. The reader can find some results about vertex-disjoint cycles in [5-7], edge-disjoint cycles in [10, 15], and arc-disjoint subgraphs in [3].

The following result gives a sufficient condition for the existence of PC cycles in edge-colored graphs.

Theorem 1 ([11, 21]). Suppose $G$ is an edge-colored graph with no PC cycle. Then $G$ contains a vertex $v$ such that the edges from $v$ to every component of $G-v$ are assigned the same color.

In paper [18], Li et al. observed that in an edge-colored complete graph $G$, for any PC cycle $C$, each vertex $v$ in $V(C)$ is contained in a PC cycle $C^{\prime}$ of length at most 4 such that $V\left(C^{\prime}\right) \subseteq V(C)$. Combining this observation and Theorem1, they got the following result.

Theorem 2 ([18]). Suppose $G$ is an edge-colored graph of order $n \geq 3$ with $\Delta^{\text {mon }}(G) \leq n-2$. Then $G$ has a PC cycle of length at most 4.

Furthermore, the authors conjectured that there exists a positive function $h(k) \geq 3 k-1$ such that for every edge-colored complete graph $G$, if $\Delta^{\text {mon }}(G) \leq n-h(k)$, then $G$ has $k$ vertex-disjoint PC cycles, and they confirmed it for $k=2$.

Theorem 3 ([18]). Suppose $G$ is an edge-colored graph of order $n \geq 6$ with $\Delta^{\text {mon }}(G) \leq n-5$. Then $G$ has two vertex-disjoint PC cycles of length at most 4.

Motivated by the vertex-disjoint PC cycles, we pose the following problem for edge-disjoint PC cycles.

Problem 4. For every positive integer $k$, does there exist an integer $f(k)$ such that every edgecolored complete graph $G$ with $n$ vertices and $\Delta^{m o n}(G) \leq n-f(k)$ contains $k$ edge-disjoint PC cycles of length at most 4 .

We also consider this problem for $k=2$ and get the following result.
Theorem 5. Suppose $G$ is an edge-colored graph of order $n \geq 6$ with $\Delta^{\text {mon }}(G) \leq n-3$. Then $G$ contains two edge-disjoint PC cycles of length at most 4.

We continue here with two examples showing that the bounds on $\Delta^{m o n}(G)$ and $n$ in Theorem 5 are sharp, respectively.

Example 6. Let $G$ be an edge-colored complete graph of order $n \geq 6$ with $V(G)=\left\{x, v_{1}, v_{2}, \cdots\right.$, $\left.v_{n-1}\right\}$. Color all the edges in $G\left[\left\{v_{1}, v_{2}, \cdots, v_{n-1}\right\}\right]$ by $c_{1}$ and color the edges in $E\left[x,\left\{v_{1}, v_{2}, \cdots, v_{n-2}\right\}\right]$ by $c_{2}$ and $x v_{n-1}$ by $c_{3}$. Then it is easy to check that $\Delta^{m o n}(G)=n-2$, but $G$ does not contain two edge-disjoint PC cycles.

Example 7. Let $G$ be an edge-colored complete graph with $V(G)=\left\{v_{1}, v_{2}, \cdots, v_{5}\right\}$. Color the Hamilton cycle $v_{1} v_{2} \cdots v_{5} v_{1}$ by red and color the rest edges by blue. Then it is easy to check that $\Delta^{m o n}(G)=5-3=2$, but $G$ does not contain two edge-disjoint PC cycles.

As for the existence of (vertex-disjoint) PC cycles of different lengths under the maximum monochromatic-degree condition, Han et al. in [13] got the following results and the bounds are sharp.

Theorem 8 ([13]). Suppose $G$ is an edge-colored complete graph of order $n \geq 2 k+1$ with $\Delta^{\text {mon }}(G) \leq n-2 k$. Then $G$ contains $k P C$ cycles of different lengths.

Theorem 9 ([13]). Suppose $G$ is an edge-colored complete graph of order $n \geq 7$ with $\Delta^{\text {mon }}(G) \leq$ $n-6$. Then $G$ contains two vertex-disjoint $P C$ cycles of different lengths.

In this paper, we study the problem on the existence of edge-disjoint PC cycles of different lengths, and prove it for the case $k=2$.

Problem 10. For every positive integer $k$, does there exist an integer $g(k)$ such that every edge-colored complete graph $G$ with $n$ vertices and $\Delta^{\text {mon }}(G) \leq n-g(k)$ contains $k$ edge-disjoint PC cycles of different lengths.

Theorem 11. Suppose $G$ is an edge-colored complete graph of order $n \geq 5$ with $\Delta^{\text {mon }}(G) \leq$ $n-4$. Then $G$ contains two edge-disjoint PC cycles of different lengths.

Example 12. Let $G$ be an edge-colored complete graph of order $n \geq 5$ and $V(G)=\left\{v_{1}, v_{2}, \cdots\right.$, $\left.v_{n}\right\}$. Suppose $G$ has a vertex partition $\left(V_{1}, V_{2}\right)$ with $V_{1}=\left\{v_{1}, v_{2}, \cdots, v_{n-3}\right\}$ and $V_{2}=\{x, y, z\}$. Color the edge-cut $E\left(V_{1}, V_{2}\right)$ by $c_{1}$, and color the edges in $E\left(V_{1}\right)$ and $E\left(V_{2}\right)$ by $c_{2}$. Then it is easy to check that $\Delta^{\text {mon }}(G)=n-3$, but $G$ does not contain two edge-disjoint PC cycles of different lengths. Hence, this shows that the bound in Theorem 11 is sharp.

The paper is organized as follows. In the next section, we set up some basic terminology and useful lemmas for the proofs of our results. In Section 3, we are devoted to studying the existence of edge-disjoint PC short cycles in edge-colored complete graphs and proving Theorem 5 and some additional results. In Section 4, we consider the problem on the existence of edgedisjoint PC cycles of different lengths in edge-colored complete graphs and prove Theorem 11.

## 2 Terminology and lemmas

Let $G$ be an edge-colored complete graph. A partition of $G$ is a family of pairwise disjoint subsets $U_{1}, U_{2}, \cdots, U_{q}$ of $V(G)$ such that $\bigcup_{1 \leq i \leq q} U_{i}=V(G)$. Now we introduce some types of partitions which are useful in the following proofs of our theorems.

Definition 13. (Gallai partition). A partition $U_{1}, U_{2}, \cdots, U_{q}$ of an edge-colored complete graph $G$ is called a Gallai partition, if $\left|\bigcup_{1 \leq i<j \leq q} c\left(U_{i}, U_{j}\right)\right| \leq 2$ and $\left|c\left(U_{i}, U_{j}\right)\right|=1$ for $1 \leq i<j \leq q$ and $q \geq 2$.

Lemma 14 ([9]). If an edge-colored complete graph $G$ contains no rainbow triangle, then $G$ has a Gallai partition.

A vertex in an edge-colored graph is bad if there is no PC cycle passing through the vertex. In [18, Li et al. introduced a new partition for an edge-colored complete graph $G$ when there is a bad vertex in $G$.

Definition 15 ([18]). ( $v$-partition) Let $G$ be an edge-colored complete graph and $v \in V(G)$. Then $U_{0}, U_{1}, \cdots, U_{q}$ is a $v$-partition of $G$ if it is a partition of $G$ and satisfies the following conditions for some distinct colors $c_{1}, c_{2}, \cdots, c_{q} \in c(G)$ :
(a) $2 \leq q \leq d^{c}(v), v \in U_{0}$ and $\left|U_{i}\right| \geq 1$ for $0 \leq i \leq q$;
(b) $c\left(U_{0}, U_{i}\right)=\left\{c_{i}\right\}$ for $1 \leq i \leq q$;
(c) $c\left(U_{i}, U_{j}\right) \subseteq\left\{c_{i}, c_{j}\right\}$ for $1 \leq i<j \leq q$;
(d) $c\left(G\left[U_{i}\right]\right) \subseteq\left\{c_{i}\right\}$ for $1 \leq i \leq q$.

Lemma 16 ([18]). Let $G$ be an edge-colored complete graph with $\delta^{c}(G) \geq 2$. If $G$ contains a bad vertex $v_{0}$, then $G$ has a $v_{0}$-partition $U_{0}, U_{1}, \cdots, U_{q}$.

Lemma 17 ([18]). Suppose $G$ is an edge-colored complete graph with $\delta^{c}(G) \geq 2$. If $G$ has a Gallai partition and contains a bad vertex $v_{0}$, then $G$ has a $v_{0}$-partition $V_{0}, V_{1}, V_{2}$ with $v_{0} \in V_{0}$.

In [12], Han et al. gave a useful partition for an edge-colored complete graph $G$ with the minimum color-degree $\delta^{c}(G) \geq 2$ and the maximum color-degree $\Delta^{c}(G) \geq 3$ when $G$ contains no PC odd cycles, where a cycle is odd if its length is odd. We also use the following partition in our proofs.

Lemma 18 ([12]). Let $G$ be an edge-colored complete graph with $\delta^{c}(G) \geq 2$ and $\Delta^{c}(G) \geq 3$. If $G$ contains no PC odd cycles, then $G$ has a partition $\{X, Y, Z\}$ such that $c(X) \subseteq c(X, Z)=\left\{c_{1}\right\}$, $c(Y) \subseteq c(Y, Z)=\left\{c_{2}\right\}$ and $c(X, Y) \subseteq\left\{c_{1}, c_{2}\right\}$.

In the same paper, Han et al. gave an equivalent condition for an edge-colored complete graph $G$ not to contain PC odd cycle.

Lemma 19 ([12]). An edge-colored complete graph $G$ contains no PC odd cycle if and only if $G$ contains no $P C$ triangle or $C_{5}$.

## 3 Edge-disjoint PC cycles of length at most 4

We note that the existence of a monochromatic edge-cut acts a pivotal role when considering the number of edge-disjoint PC cycles of length at most 4 in an edge-colored complete graph. Thus, we begin this section with the following lemma.

Lemma 20. Suppose $G$ is an edge-colored complete graph of order $n \geq k+2$ with $\Delta^{\text {mon }}(G) \leq$ $n-k-1$. If $G$ contains a monochromatic edge-cut, then $G$ has $k$ edge-disjoint PC $C_{4}$ 's.

Proof. Since $G$ has a monochromatic edge-cut, $G$ has a partition ( $V_{1}, V_{2}$ ) such that (w.l.o.g.) $c\left(V_{1}, V_{2}\right)=\left\{c_{1}\right\}$. Since $\Delta^{\text {mon }}(G) \leq n-k-1$, there are $k$ distinct edges $x x_{1}, x x_{2}, \cdots, x x_{k}$
with colors distinct from $c_{1}$ for all $x \in V_{1}$. By symmetry, there are also $k$ distinct edges $y y_{1}, y y_{2}, \cdots, y y_{k}$ with colors distinct from $c_{1}$ for all $y \in V_{2}$. Thus, $x y_{i} y x_{i} x$ for $1 \leq i \leq k$ are $k$ edge-disjoint PC $C_{4}$ 's of $G$.

### 3.1 Proof of Theorem 5

By our observation, Theorem 5 can be divided into two lemmas by considering whether $G$ contains PC triangles or not. Therefore, the following results are stated.

Lemma 21. Suppose $G$ is an edge-colored complete graph of order $n \geq 5$ with $\Delta^{\text {mon }}(G) \leq n-3$. If $G$ contains a PC triangle, then $G$ has two edge-disjoint $P C$ cycles of length at most 4.

Lemma 22. Suppose $G$ is an edge-colored complete graph of order $n \geq 6$ with $\Delta^{\text {mon }}(G) \leq n-3$. If $G$ contains no $P C$ triangle, then $G$ has two edge-disjoint $P C C_{4}$ 's.

From Lemmas 21 and 22, one can get Theorem 5 easily. Now we give the proofs of the two lemmas.

Proof of Lemma 21: Suppose to the contrary that $G$ contains no two edge-disjoint PC cycles of length at most 4. W.l.o.g., label a PC triangle in $G$ by $C^{*}=v_{1} v_{2} v_{3} v_{1}$ and suppose that $c\left(v_{1} v_{2}\right)=c_{1}, c\left(v_{2} v_{3}\right)=c_{2}$ and $c\left(v_{3} v_{1}\right)=c_{3}$. In this proof, the index $i$ is taken by $\bmod 3$. Let $G_{i}=G-\left\{v_{i-1}, v_{i+1}\right\}, i=1,2,3$. Then $G_{i}$ contains no PC triangle. Clearly, $\Delta^{\text {mon }}\left(G_{i}\right)=\left|V\left(G_{i}\right)\right|-1=n-3$; otherwise from Theorem 2, $G_{i}$ contains a PC cycle, a contradiction. Thus, there exists a vertex $u_{i} \in V\left(G_{i}\right)$ such that $d_{G_{i}}^{c}\left(u_{i}\right)=1$, for $i=1,2,3$. Since $d_{G}^{\text {mon }}\left(u_{i}\right) \leq n-3$, we have

$$
\begin{equation*}
c\left(u_{i}, G_{i}\right) \cap\left\{c\left(u_{i} v_{i-1}\right), c\left(u_{i} v_{i+1}\right)\right\}=\emptyset . \tag{1}
\end{equation*}
$$

Note that $u_{i} \notin\left\{v_{i-1}, v_{i+1}\right\}$. In the following, we assert that $\left|\left\{u_{1}, u_{2}, u_{3}\right\} \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right| \geq 2$. If not, w.l.o.g., we may assume that $u_{1} \neq v_{1}$ and $u_{2} \neq v_{2}$. Since $n \geq 5$, there exists a vertex $w$ in $G-\left\{v_{1}, v_{2}, v_{3}, u_{i}\right\}$ such that $c\left(w u_{i}\right)=c\left(v_{i} u_{i}\right)$ for $i=1,2$. Apparently, $u_{1} \neq u_{2}$; otherwise $c\left(u_{1} v_{1}\right)=c\left(u_{1} w\right)=c\left(u_{2} w\right)=c\left(u_{2} v_{2}\right)=c\left(u_{1} v_{2}\right)$. While by (1), $c\left(u_{1} v_{1}\right) \neq c\left(u_{1} v_{2}\right)$, a contradiction. Hence, $c\left(v_{1} u_{1}\right)=c\left(u_{1} u_{2}\right)=c\left(u_{2} v_{2}\right)$. Together with (1), we get that $v_{1} u_{2} v_{2} u_{1} v_{1}$ is a PC $C_{4}$. Together with $C^{*}$, we obtain two edge-disjoint PC cycles of length at most 4 , a contradiction.

Now we distinguish two cases to proceed our discussion.
Case 1. $\left|\left\{u_{1}, u_{2}, u_{3}\right\} \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right|=2$.
By symmetry, we can assume $\left\{u_{1}, u_{2}, u_{3}\right\} \cap\left\{v_{1}, v_{2}, v_{3}\right\}=\left\{u_{1}, u_{2}\right\}$. Since $u_{1}$ is chosen from $G_{1}=G-\left\{v_{2}, v_{3}\right\}$, one has $u_{1}=v_{1}$. Since $u_{2}$ is chosen from $G_{2}=G-\left\{v_{1}, v_{3}\right\}$, we have $u_{2}=v_{2}$.

Hence, $v_{3} \neq u_{3}$ and $v_{i}=u_{i}$ for all $i=1,2$. Since $n \geq 5$, let $x$ be a vertex in $G-\left\{v_{1}, v_{2}, v_{3}, u_{3}\right\}$. By (1), we have

$$
\begin{align*}
& c\left(v_{1} x\right)=c\left(v_{1} u_{3}\right) \notin\left\{c_{1}, c_{3}\right\},  \tag{2}\\
& c\left(v_{2} x\right)=c\left(v_{2} u_{3}\right) \notin\left\{c_{1}, c_{2}\right\}, \tag{3}
\end{align*}
$$

and

$$
\begin{equation*}
c\left(x u_{3}\right)=c\left(v_{3} u_{3}\right) \notin\left\{c\left(u_{3} v_{1}\right), c\left(u_{3} v_{2}\right)\right\} . \tag{4}
\end{equation*}
$$

Combining these three equalities, we can get that $C=v_{1} v_{2} x u_{3} v_{1}$ is a PC $C_{4}$. From Equalities 3 and 4. we know that $c\left(v_{2} u_{3}\right) \neq c_{2}=c\left(v_{2} v_{3}\right)$ and $c\left(v_{2} u_{3}\right) \neq c\left(v_{3} u_{3}\right)$, respectively. If $c\left(u_{3} v_{3}\right) \neq c_{2}$, then $v_{2} v_{3} u_{3} v_{2}$ is a PC triangle edge-disjoint from $C$, a contradiction. Then $c\left(u_{3} v_{3}\right)=c_{2}$, which implies that $c\left(x u_{3}\right)=c_{2}$ and $c\left(v_{1} u_{3}\right) \neq c_{2}$. We can see that $v_{1} v_{3} u_{3} v_{1}$ is a PC triangle. To avoid that $x u_{3} v_{2} v_{3} x$ is a PC $C_{4}$, we have $c\left(x v_{3}\right)=c_{2}$. Thus, $c_{2}$ appears $n-2$ times at $v_{3}$, a contradiction.

Case 2. $v_{i}=u_{i}$ for all $i=1,2,3$.
If $c\left(v_{1}, G_{1}\right) \neq c\left(v_{2}, G_{2}\right)$, then $c\left(v_{1} w\right) \neq c\left(v_{2} w\right)$ for all $w \in V(G)-V\left(C^{*}\right)$. Since $n \geq 5$, there are two distinct vertices $x, y$ in $G-C^{*}$. Then $v_{1} v_{2} x v_{1}$ and $v_{1} v_{3} v_{2} y v_{1}$ are two edge-disjoint PC cycles of length at most 4, a contradiction. Hence by symmetry, we have that $c\left(v_{1}, G_{1}\right)=$ $c\left(v_{2}, G_{2}\right)=c\left(v_{3}, G_{3}\right)$, which means that $c\left(v_{1} w\right)=c\left(v_{2} w\right)=c\left(v_{3} w\right)$ for all $w \in V(G)-V\left(C^{*}\right)$. Then $E\left[V(G)-V\left(C^{*}\right), V\left(C^{*}\right)\right]$ is a monochromatic edge-cut. Since $\Delta^{\text {mon }}(G) \leq n-3=$ $n-2-1$, by Lemma 20, $G$ contains two edge-disjoint PC $C_{4}$ 's, a contradiction. The proof is thus complete.

Let $G$ be a graph and $X$ be a proper subset of $V(G)$. To shrink $X$ is to delete all the edges whose both ends are in $X$ and then identify all the vertices of $X$ into a single vertex. The resulting graph is denoted by $G / X$.

Proof of Lemma 22; Suppose to the contrary that $G$ does not contain two disjoint PC cycles of length at most 4. Since $G$ has no PC triangle, $G$ admits a Gallai partition $U_{1}, U_{2}, \cdots, U_{q}$. Assume that $c\left(U_{i}, U_{j}\right) \subset\left\{c_{1}, c_{2}\right\}$ for $(1 \leq i<j \leq q)$. We proceed by proving the following claims.

Claim 1. $G$ contains a PC $C_{4}$ with vertices from distinct sets of $U_{1}, U_{2}, \cdots, U_{q}$.
Proof. For this purpose, we first need the operation of shrinking. We obtain an edge-colored complete graph $H$ by shrinking $U_{i}$ to a single vertex $x_{i}$ for all $1 \leq i \leq q$ and deleting the parallel edges. The color of the edge $x_{i} x_{j}$ is just the one appearing on the edges between $U_{i}$ and $U_{j}$. If $G$ contains a monochromatic edge-cut, from Lemma 20 and set $k=2$, then there are two edge-disjoint PC $C_{4}$ 's in $G$, a contradiction. Hence, $G$ has no monochromatic edge-cut. This implies that $q \geq 4$. Then $c(x, H-x)=\left\{c_{1}, c_{2}\right\}$ for each $x \in V(H)$, which implies that
$\Delta^{\text {mon }}(H) \leq|V(H)|-2$. Thus by Theorem 2, $H$ contains a PC $C_{4}$. So, $G$ has a PC $C_{4}$ with vertices from different parts of the Gallai partition.
W.l.o.g., we label the PC $C_{4}$ in Claim 1 by $C^{*}=v_{1} v_{2} v_{3} v_{4} v_{1}$ with $c\left(v_{1} v_{2}\right)=c\left(v_{3} v_{4}\right)=$ $c_{1}$ and $c\left(v_{2} v_{3}\right)=c\left(v_{1} v_{4}\right)=c_{2}$. In this proof, the index $i$ is taken by mod 4 . Let $G_{i}=$ $G-\left\{v_{i+1}, v_{i+2}, v_{i+3}\right\}, i=1,2,3,4$. Since $\left|V\left(G_{i}\right)\right| \geq n-3 \geq 3, G_{i}$ is nonempty. Clearly, $\Delta^{m o n}\left(G_{i}\right)=\left|V\left(G_{i}\right)\right|-1=n-4$ for $i=1,2,3,4$; otherwise from Theorem 2, $G_{i}$ contains a PC cycle, a contradiction. Therefore, there exists a vertex $u_{i} \in V\left(G_{i}\right)$ such that $d_{G_{i}}^{c}\left(u_{i}\right)=1$ for $i=1,2,3,4$. Define $S_{i}=\left\{v \in V\left(G_{i}\right), d_{G_{i}}^{c}(v)=1\right\}$ for $i=1,2,3,4$. Clearly, there is only one color in $c\left(S_{i}, G_{i}-S_{i}\right) \cup c\left(S_{i}\right)$.

Claim 2. The color in $c\left(S_{i}, G_{i}-S_{i}\right) \cup c\left(S_{i}\right)$ must be $c_{1}$ or $c_{2}$ for all $1 \leq i \leq 4$.
Proof. Suppose not. Then we assume $c\left(S_{1}, G_{1}-S_{1}\right) \cup c\left(S_{1}\right)=\left\{c_{3}\right\}$. According to the definition of a Gallai partition, $V\left(G_{1}\right) \subseteq U_{i}$ for some $i$ with $1 \leq i \leq q$. Noticing that $v_{1} \in V\left(G_{1}\right)$, we have $c\left(v_{2}, G_{1}\right)=\left\{c_{1}\right\}$ and $c\left(v_{4}, G_{1}\right)=\left\{c_{2}\right\}$. According to Claim 1, $c\left(v_{2} v_{4}\right) \in\left\{c_{1}, c_{2}\right\}$. W.l.o.g., assume $c\left(v_{2} v_{4}\right)=c_{2}$. Then $d_{G}^{m o n}\left(v_{4}\right)=\left|V\left(G_{i}\right)\right|+1=n-2$, a contradiction. By symmetry, $c\left(S_{i}, G_{i}-S_{i}\right) \cup c\left(S_{i}\right)$ must be $c_{1}$ or $c_{2}$ for all $1 \leq i \leq 4$.

Claim 3. $S_{i}-v_{i} \neq \emptyset$ for all $1 \leq i \leq 4$.
Proof. Suppose not. Then we may assume $S_{1}=\left\{v_{1}\right\}$. By Claim 2, w.l.o.g., assume $c\left(v_{1}, G_{1}\right)=$ $\left\{c_{1}\right\}$. Since $d^{\text {mon }}\left(v_{1}\right) \leq n-3$, we have $c\left(v_{1} v_{3}\right)=c_{2}$. Noticing that $d^{\text {mon }}\left(v_{3}\right) \leq n-3$, there exists a vertex $x \in V(G)-V\left(C^{*}\right)$ such that $c\left(v_{3} x\right) \neq c_{2}$. If $c\left(v_{3} x\right) \neq c_{1}$, then $x v_{1} v_{3} x$ is a PC triangle edge-disjoint from $C^{*}$, a contradiction. Hence, $c\left(v_{3} x\right)=c_{1}$. Since $S_{1}-v_{1}=\emptyset$ and $n \geq 6$, there exists a vertex $y$ in $V(G)-V\left(C^{*}\right)$ distinct from $x$ such that $c(x y) \neq c_{1}$. Then $v_{1} v_{3} x y v_{1}$ is a PC $C_{4}$ edge-disjoint from $v_{1} v_{2} v_{3} v_{4} v_{1}$, a contradiction. By symmetry, we have $S_{i}-v_{i} \neq \emptyset$ for all $1 \leq i \leq 4$.

By Claim 2, we know that the color in $c\left(S_{1}, G_{1}-S_{1}\right) \cup c\left(S_{1}\right)$ must be $c_{1}$ or $c_{2}$. W.l.o.g., assume $c\left(S_{1}, G_{1}-S_{1}\right) \cup c\left(S_{1}\right)=\left\{c_{1}\right\}$. By Claim 3, we have $S_{2}-v_{2} \neq \emptyset$, which means that $S_{2} \cap G_{1} \neq \emptyset$. The condition $c\left(S_{1}, G_{1}-S_{1}\right) \cup c\left(S_{1}\right)=\left\{c_{1}\right\}$ implies $c\left(S_{1}, S_{2}\right)=\left\{c_{1}\right\}$. Using Claim 2 again, $c\left(S_{2}, G_{2}-S_{2}\right) \cup c\left(S_{2}\right)$ must be $c_{1}$. Hence, we have $c\left(S_{1} \cup S_{2}\right)=\left\{c_{1}\right\}$. Repeating the above discussion, we can get $c\left(\cup_{1 \leq i \leq 4} S_{i}\right)=\left\{c_{1}\right\}$.

Claim 4. $S_{i} \cap S_{i+1}=\emptyset$ for all $1 \leq i \leq 4$.
Proof. Suppose to the contrary that $S_{1} \cap S_{4} \neq \emptyset$. Then there exists a vertex $x \in S_{1} \cap S_{4}$ such that $c\left(x v_{1}\right)=c\left(x v_{4}\right)=c_{1}$. Since $d^{m o n}(x) \leq n-3$, neither $c\left(x v_{3}\right)$ nor $c\left(x v_{2}\right)$ is $c_{1}$. Therefore, $x v_{1} v_{4} v_{3} x$ is a PC $C_{4}$. Since by Claim 3, $S_{3}-v_{3} \neq \emptyset$, there exists another vertex $y$ distinct from $v_{3}$ in $S_{3}$ such that $c\left(y v_{3}\right)=c_{1}$. Then $c\left(y v_{1}\right)=c_{1}$; otherwise $y v_{3} v_{2} v_{1} y$ is a PC $C_{4}$ edge-disjoint
from $x v_{1} v_{4} v_{3} x$, a contradiction. Since $d^{m o n}(y) \leq n-3$, we have $c\left(y v_{4}\right) \neq c_{1}$. Therefore, $v_{4} y v_{3} x v_{4}$ is a PC $C_{4}$ edge-disjoint from $C^{*}$, a contradiction. By symmetry, we have $S_{i} \cap S_{i+1}=\emptyset$ for all $1 \leq i \leq 4$.

According to Claims 3 and $4, S_{1} \cap S_{2}=\emptyset$ and $S_{i}-v_{i} \neq \emptyset$ for $i=1,2$. There are four distinct vertices $u_{i} \in S_{i}$ such that $c\left(u_{i} v_{i}\right)=c_{1}$ for $1 \leq i \leq 4$. To avoid that $v_{1} u_{1} v_{2} u_{2} v_{1}$ is a PC $C_{4}$, we have $c\left(u_{1} v_{2}\right)$ or $c\left(u_{2} v_{1}\right)$ is $c_{1}$. W.l.o.g., assume $c\left(u_{1} v_{2}\right)=c_{1}$. Recall that $u_{1} \in S_{1}$ and $c\left(S_{1}, G_{1}-S_{1}\right) \cup c\left(S_{1}\right)=\left\{c_{1}\right\}$. The condition $\Delta^{m o n}(G) \leq n-3$ implies that $c\left(u_{1} v_{3}\right) \neq c_{1}$. If $c\left(u_{3} v_{1}\right) \neq c_{1}$, then $u_{1} v_{3} u_{3} v_{1} u_{1}$ is a PC $C_{4}$ edge-disjoint from $C^{*}$, a contradiction. Then $c\left(u_{3} v_{1}\right)=c_{1}$. Since $c\left(v_{3} u_{3}\right)=c_{1}$, if $c\left(v_{2} u_{3}\right) \neq c_{1}$, then $u_{1} v_{2} u_{3} v_{3} u_{1}$ is a PC $C_{4}$ edgedisjoint from $C^{*}$, a contradiction. Then $c\left(u_{3} v_{2}\right)=c_{1}$. Recall that $c\left(\cup_{1 \leq i \leq 4} S_{i}\right)=\left\{c_{1}\right\}$ and $c\left(S_{3}, G_{3}-S_{3}\right) \cup c\left(S_{3}\right)=\left\{c_{1}\right\}$. Then $d^{\text {mon }}\left(u_{3}\right) \geq n-2$, which contradicts the condition $\Delta^{\text {mon }}(G) \leq$ $n-3$. The proof is thus complete.

### 3.2 Some additional results

While for the case $k \geq 3$, we get some additional results related to Problem 4. The following results imply that if there exists a vertex in $G$ that is not contained in any PC cycle, then $G$ contains $k$ edge-disjoint PC cycles. For convenience, we call this type of vertex a bad vertex,

Theorem 23. Suppose $G$ is an edge-colored complete graph of order $n \geq 2 k+3$ with $\Delta^{\text {mon }}(G) \leq$ $n-k-1$. If $G$ has a Gallai partition, then at least one of the following statements holds.
(1) $G$ contains $k$ edge-disjoint PC cycles of length at most 4.
(2) $G$ contains no bad vertex.

Proof of Theorem 23: From Theorems 2 and 5, we can deduce that Theorem 23 holds for $k \in\{1,2\}$. Therefore, we set $k \geq 3$. Suppose to the contrary that $G$ does not contain $k$ edge-disjoint PC cycles of length at most 4 and $G$ has a bad vertex $v_{0}$. By induction on $k$, assume that the result holds for $k-1$. Since $G$ has a Gallai partition, by Lemma $17, G$ contains a $v_{0}$-partition $V_{0}, V_{1}$ and $V_{2}$ with $c\left(V_{0}, V_{i}\right)=\left\{c_{i}\right\}, c\left(V_{1}, V_{2}\right) \subseteq\left\{c_{1}, c_{2}\right\}$ and $c\left(G\left[V_{i}\right]\right)=\left\{c_{i}\right\}$, for $i=1,2$. Since $\Delta^{\text {mon }}(G) \leq n-k-1$, by Lemma 20, $G$ contains no monochromatic edge-cut. Hence, $\left|V_{0}\right|+\left(\left|V_{i}\right|-1\right)+1 \leq n-k-1$, for all $i=1,2$. Thus,

$$
\begin{gathered}
k+1 \leq\left|V_{i}\right| \leq n-k-2, \quad i=1,2, \\
1 \leq\left|V_{0}\right| \leq n-2 k-2 .
\end{gathered}
$$

Let $H_{1}=G-V_{0}$. Then for each $v \in V_{1}$, we have $\left|N_{H_{1}}^{c_{2}}(v)\right|=\left|N_{G}^{c_{2}}(v)\right|=n-\left|N_{G}^{c_{1}}(v)\right| \geq$ $n-(n-k-1)=k+1 \geq 4$, and $\left|N_{H_{1}}^{c_{1}}(v)\right| \geq\left|V_{1}\right|-1 \geq k \geq 3$. For each $v \in V_{2}$, we have $\left|N_{H_{1}}^{c_{2}}(v)\right|=\left|N_{G}^{c_{2}}(v)\right|=n-\left|N_{G}^{c_{1}}(v)\right| \geq n-(n-k-1)=k+1 \geq 4$, and $\left|N_{H_{1}}^{c_{1}}(v)\right| \geq\left|V_{1}\right|-1 \geq k \geq 3$.

This implies that $\Delta^{\text {mon }}\left(H_{1}\right) \leq\left|V\left(H_{1}\right)\right|-3$. Hence, by Theorem 5, $H_{1}$ contains two edge-disjoint PC cycles of length at most 4. Since $\left|c\left(H_{1}\right)\right|=2$, these two PC cycles must be $C_{4}$. It is easy to deduce that if $H_{1}$ has a PC $C_{4}$, then $H_{1}$ must contain two vertices, say $x$ and $z$, in $V_{1}$ and two vertices, say $y$ and $w$, in $V_{2}$. Since $c(x z) \neq c(y w)$, the $C_{4}$ must be $x y z w x$. Therefore, we label the two edge-disjoint PC $C_{4}^{\prime} s$ in $H_{1}$ by $x_{1} y_{1} x_{2} y_{2} x_{1}$ and $x_{3} y_{3} x_{4} y_{4} x_{3}$. We assume that $x_{i} \in V_{1}$ and $y_{i} \in V_{2}$ for $1 \leq i \leq 4$ (w.l.o.g., $x_{i}$ may be equal to $x_{i+2}$ for $i=1$, 2; see Figure 1).


Figure 1: Two edge-disjoint PC $C_{4}{ }^{\prime} s$ in $H_{1}$
Let $H_{2}=G-\left\{x_{1}, x_{2}, y_{3}, y_{4}\right\}$. For each vertex $v \in V_{1} \backslash\left\{x_{1}, x_{2}\right\}$, we have $c(v, G-v) \subseteq\left\{c_{1}, c_{2}\right\}$. Then $\left|N_{H_{2}}^{c_{1}}(v)\right| \leq\left|N_{G}^{c_{1}}(v)\right|-2 \leq n-k-3$, and $\left|N_{H_{2}}^{c_{2}}(v)\right| \leq\left|V_{2}\right|-2 \leq n-k-4$. For each $v \in V_{2} \backslash\left\{y_{3}, y_{4}\right\}$, we have $c(v, G-v) \subseteq\left\{c_{1}, c_{2}\right\}$. Then $\left|N_{H_{2}}^{c_{2}}(v)\right| \leq\left|N_{G}^{c_{2}}(v)\right|-2 \leq n-k-3$, and $\left|N_{H_{2}}^{c_{1}}(v)\right| \leq\left|V_{1}\right|-2 \leq n-k-4$. Furthermore, for each $v \in V_{0}$ we have $\left|N_{H_{2}}^{c_{i}}(v)\right| \leq\left|N_{G}^{c_{i}}(v)\right|-2 \leq$ $n-k-3$ for $i=1,2$, and $\left|N_{H_{2}}^{\alpha}(v)\right| \leq\left|N_{G}^{\alpha}(v)\right| \leq\left|V_{0}\right|-1 \leq n-2 k-3 \leq n-k-3$ for each color $\alpha \in c(G) \backslash\left\{c_{1}, c_{2}\right\}$. This implies that $\Delta^{m o n}\left(H_{2}\right) \leq n-k-3=\left|V\left(H_{2}\right)\right|-(k-2)-1$. From the choice of $G$ and the fact that $v_{0}$ is also a bad vertex in $H_{2}$, we know that $H_{2}$ contains $k-2$ edge-disjoint PC cycles of length at most 4 . Together with the PC cycles $x_{1} y_{1} x_{2} y_{2} x_{1}$ and $x_{3} y_{3} x_{4} y_{4} x_{3}$, there exist $k$ edge-disjoint PC cycles in $G$, a contradiction.

Theorem 24. Suppose $G$ is an edge-colored complete graph of order $n \geq 2 k+3$. If $\Delta^{\text {mon }}(G) \leq$ $n-2 k+1$, then at least one of the following statements holds.
(1) $G$ contains $k$ edge-disjoint PC cycles of length at most 4.
(2) $G$ contains no bad vertex.

Proof of Theorem 24; From Theorems 2 and 5, we can deduce that Theorem 24 holds for $k \in\{1,2\}$. Suppose to the contrary that $G$ does not contain $k$ edge-disjoint PC cycles of length at most 4 and $G$ has a bad vertex $v_{0} \in V(G)$. Let $G$ be such a graph that $k \geq 3$ is as small as possible. If $G$ contains a PC triangle $x y z x$, then let $H=G-\{y, z\}$. So, $\Delta^{m o n}(H) \leq \Delta^{m o n}(G) \leq n-2-2(k-1)+1$. Since $k$ is as small as possible, either $H$ has $k-1$ edge-disjoint PC cycles of length at most 4 or $H$ contains no bad vertex. This yields that either $G$ has $k$ edge-disjoint PC cycles of length at most 4 or $G$ contains no bad vertex, a contradiction. Thus, $G$ has no PC triangle. Then $G$ admits a Gallai partition. Since $\Delta^{\text {mon }}(G) \leq n-2 k+1 \leq n-k-1$, by Theorem 23 , either $G$ has $k$ edge-disjoint PC cycles
of length at most 4 or $G$ contains no bad vertex, a contradiction. The proof of Theorem 24 is thus complete.

## 4 Edge-disjoint PC cycles of different lengths

If $\Delta^{m o n}(G) \leq n-3$, one can get a lemma that $G$ contains a PC cycle of length at least 4, which was also proved by Han et al. in [13]. Moreover, from their proof for the case $\Delta^{\text {mon }}(G) \leq n-3$, we can see the following property.

Lemma 25. Suppose $G$ is an edge-colored complete graph of order $n \geq 4$ with $\Delta^{\text {mon }}(G) \leq n-3$. Then $G$ contains a PC cycle of length $k$ for any $k$ with $4 \leq k \leq 6$.

Next, we give some lemmas which will be used in the proof of Theorem 11 .
Lemma 26. Suppose $G$ is an edge-colored complete graph of order $n$ with $\Delta^{\text {mon }}(G) \leq n-4$. If $G$ contains a PC $C_{5}$ or $C_{6}$, then $G$ contains two edge-disjoint PC cycles of different lengths.

Proof. Suppose to the contrary that $G$ does not contain edge-disjoint PC cycles of different lengths. Assume that there exists a $\mathrm{PC} C_{\ell}$ in $G$ where $\ell=5$ or 6 . If $n=\ell$, then $G$ is properly colored when $n=5$, and $\delta^{c}(G) \geq 3$ when $n=6$. It is easy to find two edge-disjoint PC cycles of different lengths. Hence we assume $n>\ell$. If $\ell=6$, then $C_{\ell}=x_{1} y_{1} \cdots x_{3} y_{3} x_{1}$ and set $X=\left\{x_{1}, x_{2}, x_{3}\right\}, Y=\left\{y_{1}, y_{2}, y_{3}\right\}$. If $\ell=5$, then $C_{\ell}=a_{0} x_{1} y_{1} x_{2} y_{2} a_{0}$ and set $X=$ $\left\{a_{0}, x_{1}, x_{2}\right\}, Y=\left\{a_{0}, y_{1}, y_{2}\right\}$. Let $G_{1}=G-X$ and $G_{2}=G-Y$. Clearly, $G_{i}$ contains no PC cycle of length at most 4. From Theorem 2, there exist two vertices $v$ and $u$ in $G_{1}$ and $G_{2}$ respectively with $d_{G_{1}}^{m o n}(v)=\left|V\left(G_{1}\right)\right|-1$ and $d_{G_{2}}^{m o n}(u)=\left|V\left(G_{2}\right)\right|-1$. Suppose that $c\left(v, G_{1}\right)=\left\{c_{1}\right\}$ and $c\left(u, G_{2}\right)=\left\{c_{2}\right\}$.

Now, we distinguish four cases to proceed our proof.
Case 1. Neither $u$ nor $v$ is in $C_{\ell}$.
If $u \neq v$, then $c_{1}=c(u v)=c_{2}=c$. This implies from $\Delta^{m o n}(G) \leq n-4$ that there exist two distinct vertices $x \in X \backslash\left\{a_{0}\right\}$ and $y \in Y \backslash\left\{a_{0}\right\}$ such that $c(v x) \neq c$ and $c(u y) \neq c$. Since $c(v y)=c(u x)=c$, uxvyu is a PC $C_{4}$, a contradiction. Hence, it follows that $u=v$, and then $c_{1} \neq c_{2}$. Thus, $V(G) \backslash V\left(C_{\ell}\right)=\{u\}$.

Note that $c\left(u, X \backslash\left\{a_{0}\right\}\right)=\left\{c_{2}\right\}$ and $c\left(u, Y \backslash\left\{a_{0}\right\}\right)=\left\{c_{1}\right\}$. For any two vertices $x \in X \backslash\left\{a_{0}\right\}$ and $y \in Y \backslash\left\{a_{0}\right\}$, if $x y \notin C_{\ell}$, then $c(x y) \in\left\{c_{1}, c_{2}\right\}$; otherwise uxyu is a PC triangle. W.l.o.g., suppose $c(x y)=c_{1}$. Since uxyy'u is not a PC $C_{4}$ for $y^{\prime} \in Y \backslash\left\{a_{0}\right\}$, we have $c\left(y y^{\prime}\right)=c_{1}$. Then $d^{m o n}(y) \geq 3$ if $\ell=5$ and $d^{m o n}(y) \geq 4$ if $\ell=6$, a contradiction to the condition $\Delta^{m o n}(G) \leq n-4$.

Case 2. Both $u$ and $v$ are in $C_{\ell}$ and $u v \notin C_{\ell}$.

Note that $c(u v) \notin\left\{c_{1}, c_{2}\right\}$ as $\Delta^{\text {mon }}(G) \leq n-4$. Then $c_{1}=c_{2}=c$; otherwise wuvw is a PC triangle for $w \in V(G) \backslash C_{\ell}$. Since neither wuvyw nor wvuxw is PC, we have $c(w x)=c(w y)=c$ for $x \in X \backslash\left\{a_{0}\right\}, y \in Y \backslash\left\{a_{0}\right\}$ and $w \in V(G) \backslash V\left(C_{\ell}\right)$. Hence, there exists a distinct vertex $w^{\prime} \in V(G) \backslash C_{\ell}$ such that $c\left(w w^{\prime}\right) \neq c$. Then $w w^{\prime} u v w$ is a PC $C_{4}$, a contradiction.

When discussing the last two cases, we always divide them into two subcases by considering $\ell=5$ and $\ell=6$. We will prove them in detail for $\ell=6$ and always assume that $G$ contains no $\mathrm{PC} C_{6}$ for $\ell=5$.

Case 3. $u \in C_{\ell}$ while $v \notin C_{\ell}$.
Subcase 3.1. $\ell=6$. W.l.o.g., set $u=x_{1}$. The condition $\Delta^{\text {mon }}(G) \leq n-4$ implies that $c_{1} \neq c\left(v x_{i}\right)$ for all $i=1,2,3$. Recalling that $c\left(v x_{1}\right)=c_{2}$, we have $c_{1} \neq c_{2}$. Note that $c\left(v x_{1}\right)=c_{2}, c\left(v y_{2}\right)=c_{1}$ and $c\left(x_{1} y_{2}\right) \neq c_{2}$. To avoid that $v x_{1} y_{2} v$ is a PC triangle, we have $c\left(x_{1} y_{2}\right)=c_{1}$. Since neither $x_{1} y_{2} y_{1} v x_{1}$ nor $x_{1} y_{2} y_{3} v x_{1}$ is a PC $C_{4}$, we know $c\left(y_{2} y_{1}\right)=c\left(y_{2} y_{3}\right)=c_{1}$. Then, $c\left(y_{2},\left\{v, y_{1}, y_{3}, x_{1}\right\}\right)=\left\{c_{1}\right\}$. It follows from $\Delta^{\text {mon }}(G) \leq n-4$ that there is a vertex $w \in V \backslash V\left(C_{\ell}\right)$ such that $c\left(y_{2} w\right) \neq c_{1}$. The condition $c\left(v, G_{1}\right)=\left\{c_{1}\right\}$ implies that $c(v w)=c_{1}$. Then $y_{2} w v x_{1} y_{2}$ is a PC $C_{4}$ edge-disjoint from the $\mathrm{PC} C_{\ell}$, a contradiction.

Subcase 3.2. $\ell=5$. In this subcase, $c(u v)=c_{2}$ and $c_{1} \neq c_{2}$; otherwise $d^{m o n}(v) \geq n-3$. First suppose that $u=x_{1}$. Since $y_{1}, y_{2} \in G_{1}$ and $x_{2} \in G_{2}$, we have $c\left(v,\left\{y_{1}, y_{2}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(x_{1},\left\{x_{2}, v\right\}\right)=\left\{c_{2}\right\}$, respectively. The condition $\Delta^{\text {mon }}(G) \leq n-4$ implies that $c_{1} \notin$ $c\left(v,\left\{a_{0}, x_{1}, x_{2}\right\}\right)$ and $c_{2} \notin c\left(x_{1},\left\{a_{0}, y_{1}, y_{2}\right\}\right)$. Since neither $x_{1} v y_{1} x_{2} y_{2} a_{0} x_{1}$ nor $x_{2} v y_{2} a_{0} x_{1} y_{1} x_{2}$ is a PC $C_{6}$, we have $c\left(y_{1} x_{2}\right)=c\left(v y_{1}\right)=c_{1}$ and $c\left(y_{2} a_{0}\right)=c\left(v y_{2}\right)=c_{1}$. Then $a_{0} v x_{1} y_{1} x_{2} y_{2} a_{0}$ is a PC $C_{6}$, a contradiction.

Next, we set $u=x_{2}$. Since $y_{1}, y_{2} \in G_{1}$ and $x_{2} \in G_{2}$, we have $c\left(v,\left\{y_{1}, y_{2}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(x_{2},\left\{x_{1}, v\right\}\right)=\left\{c_{2}\right\}$, respectively. The condition $\Delta^{m o n}(G) \leq n-4$ implies that $c_{1} \notin$ $c\left(v,\left\{a_{0}, x_{1}, x_{2}\right\}\right)$ and $c_{2} \notin c\left(x_{2},\left\{a_{0}, y_{1}, y_{2}\right\}\right)$. Since neither $x_{2} v y_{1} x_{1} a_{0} y_{2} x_{2}$ nor $v y_{2} a_{0} x_{1} y_{1} x_{2} y_{2}$ is a PC $C_{6}$, we have $c\left(y_{1} x_{1}\right)=c\left(v y_{1}\right)=c_{1}$ and $c\left(y_{2} a_{0}\right)=c\left(v y_{2}\right)=c_{1}$. Recall that $C_{\ell}$ is a PC $C_{5}$. Then $c_{1} \notin\left\{c\left(a_{0} x_{1}\right), c\left(y_{1} x_{2}\right), c\left(x_{2} y_{2}\right)\right\}$. Thus, $T_{1}=x_{2} y_{2} v x_{2}$ is a PC triangle. Since $n \geq 7$, fixing a vertex $w \in V(G) \backslash\left(V\left(C_{\ell}\right) \cup\{v\}\right)$, the definitions of $v$ and $x_{2}$ implies that $c(w v)=c_{1}$ and $c\left(w x_{2}\right)=c_{2}$. Then $v x_{1} y_{1} x_{2} w v$ is a PC $C_{5}$ edge-disjoint from $T_{1}$, a contradiction.

Case 4. Both $u$ and $v$ are in $C_{\ell}$ and $u v \in C_{\ell}$.
Subcase 4.1. $\ell=6$. W.l.o.g., set $x_{1}=u$ and $y_{1}=v$. It is clear that $c\left(y_{1} x_{3}\right) \neq c_{1}$ and $c\left(x_{1} y_{2}\right) \neq c_{2}$. Note that $c\left(y_{1} x_{3}\right)=c_{2}$ or $c\left(x_{1} y_{2}\right)=c_{1}$ since $y_{1} y_{2} x_{1} x_{3} y_{1}$ is not a PC $C_{4}$. By symmetry, set $c\left(y_{1} x_{3}\right)=c_{2}$, which also implies $c_{1} \neq c_{2}$. Note that for each vertex $w \in$ $V(G) \backslash V\left(C_{\ell}\right), w x_{1} y_{1} w$ is a PC triangle. Then, $c\left(x_{2} y_{2}\right)=c\left(x_{1} x_{2}\right)=c_{2}$; otherwise $x_{1} x_{2} y_{2} x_{3} y_{3} x_{1}$ is a PC cycle of length 5 edge-disjoint from $w x_{1} y_{1} w$, a contradiction. Since $C_{\ell}=x_{1} y_{1} x_{2} y_{2} x_{3} y_{3} x_{1}$ is a PC cycle and $c\left(x_{2} y_{2}\right)=c_{2}$, we have $c\left(y_{1} x_{2}\right) \neq c_{2}$ and $c\left(y_{2} x_{3}\right) \neq c_{2}$. The hypothesis $c\left(y_{1} x_{3}\right)=c_{2}$ implies that $y_{1} x_{2} y_{2} x_{3} y_{1}$ is a PC $C_{4}$ edge-disjoint from $w x_{1} y_{1} w$, a contradiction.

Subcase 4.2. $\ell=5$. First, note that $a_{0} \notin\{u, v\}$. By symmetry, we only need to consider $u v=x_{2} y_{2}$ or $u v=y_{1} x_{2}$.

We assert that $c_{1}=c_{2}=c$. Suppose it is not the case. If $u v=x_{2} y_{2}\left(u=x_{2}\right.$ and $\left.v=y_{2}\right)$, then the condition $\Delta^{\text {mon }}(G) \leq n-4$ implies that $c\left(y_{2} a_{0}\right) \neq c_{1}$ and $c\left(x_{2} y_{1}\right) \neq c_{2}$. Fixing a vertex $w \in V(G) \backslash V\left(C_{\ell}\right)(n \geq 7)$, the definitions of $u$ and $v$ imply that $c\left(w y_{2}\right)=c_{1}$ and $c\left(w x_{2}\right)=c_{2}$. Then $x_{2} w y_{2} a_{0} x_{1} y_{1} x_{2}$ is a PC $C_{6}$, a contradiction. If $u v=y_{1} x_{2}\left(u=x_{2}\right.$ and $\left.v=y_{1}\right)$, then the condition $\Delta^{\text {mon }}(G) \leq n-4$ implies that $c\left(x_{2} y_{2}\right) \neq c_{2}$ and $c\left(y_{1} x_{1}\right) \neq c_{1}$. Fixing a vertex $w \in V(G) \backslash V\left(C_{\ell}\right)(n \geq 7)$, the definitions of $u$ and $v$ imply that $c\left(w x_{2}\right)=c_{2}$ and $c\left(w y_{1}\right)=c_{1}$. Then $w x_{2} y_{2} a_{0} x_{1} y_{1} w$ is a PC $C_{6}$, a contradiction.

First, suppose that $u v=x_{2} y_{2}\left(u=x_{2}\right.$ and $\left.v=y_{2}\right)$. Then $c\left(y_{2} y_{1}\right)=c\left(x_{2} x_{1}\right)=c$ and $c \notin\left\{c\left(y_{2} x_{1}\right), c\left(x_{2} y_{2}\right), c\left(x_{2} y_{1}\right)\right\}$. Choosing any two vertices $w_{1}, w_{2} \in V(G) \backslash V\left(C_{\ell}\right)$, if $c\left(w_{1} w_{2}\right) \neq c$, then $w_{1} w_{2} x_{2} y_{2} w_{1}$ is a PC $C_{4}$. This implies that $x_{1} x_{2} y_{1} x_{1}$ is not a PC triangle. Then $c\left(x_{1} y_{1}\right)=$ $c\left(x_{1} x_{2}\right)=c$. Hence, $w_{1} w_{2} x_{2} y_{1} x_{1} y_{2} w_{1}$ is a PC $C_{6}$, a contradiction. Then $c\left(V(G) \backslash V\left(C_{\ell}\right)=\{c\}\right.$. Fixing two vertices $w_{1}, w_{2} \in V(G) \backslash V\left(C_{\ell}\right)$, we have $c\left(w_{1} w_{2}\right)=c$. The condition $\Delta^{\text {mon }}(G) \leq$ $n-4$ implies that $c \notin c\left(w_{i},\left\{a_{0}, x_{1}, y_{1}\right\}\right)$ for $i=1,2$. Then $w_{1} w_{2} y_{1} y_{2} x_{2} x_{1} w_{1}$ is a PC $C_{6}$, a contradiction.

Now suppose that $u v=y_{1} x_{2}\left(u=x_{2}\right.$ and $\left.v=y_{1}\right)$. Then $c\left(x_{1} x_{2}\right)=c\left(y_{1} y_{2}\right)=c$ and $c \notin\left\{c\left(x_{1} y_{1}\right), c\left(y_{1} x_{2}\right), c\left(x_{2} y_{2}\right)\right\}$. Choosing any two vertices $w_{1}, w_{2} \in V(G) \backslash V\left(C_{\ell}\right)$, if $c\left(w_{1} w_{2}\right) \neq$ $c$, then $w_{1} w_{2} y_{1} x_{2} w_{1}$ is a PC $C_{4}$. This implies that $w_{1} y_{1} x_{1} w_{1}$ is not a PC triangle. Then $c\left(x_{1} w_{1}\right) \in\left\{c, c\left(x_{1} y_{1}\right)\right\}$. Note that $a_{0} x_{1} w_{1} y_{1} x_{2} y_{2} a_{0}$ is a PC $C_{6}$ when $c\left(x_{1} w_{1}\right)=c\left(x_{1} y_{1}\right)$, a contradiction. Hence, $c\left(w_{1} x_{1}\right)=c$. Since $w_{2} x_{2} y_{2} w_{2}$ is not a PC triangle, it follows that $c\left(y_{2} w_{2}\right) \in\left\{c\left(w_{2} x_{2}\right), c\left(x_{2} y_{2}\right)\right\}=\left\{c, c\left(x_{2} y_{2}\right)\right\}$. Since $a_{0} y_{2} w_{2} x_{2} y_{1} x_{1} a_{0}$ is a PC $C_{6}$ when $c\left(y_{2} w_{2}\right)=$ $c\left(x_{2} y_{2}\right)$, we have $c\left(y_{2} w_{2}\right)=c$. Thus, $x_{1} w_{1} w_{2} y_{2} x_{2} y_{1} x_{1}$ is a PC $C_{6}$, a contradiction. Then $c\left(V(G) \backslash V\left(C_{\ell}\right)\right)=\{c\}$. Fixing two vertices $w_{1}, w_{2} \in V(G) \backslash V\left(C_{\ell}\right)$, we have $c\left(w_{1} w_{2}\right)=c$. The condition $\Delta^{m o n}(G) \leq n-4$ implies that $c \notin c\left(w_{i},\left\{a_{0}, x_{1}, y_{2}\right\}\right)$ for $i=1,2$. Then $w_{1} w_{2} x_{1} x_{2} y_{1} y_{2} w_{1}$ is a PC $C_{6}$, a contradiction. The proof of Lemma 26 is thus complete.

Next, we prove that the existence of PC $C_{8}$ implies the existence of two edge-disjoint PC cycles of different lengths in a 2-edge-colored (using only two colors) complete graph of order $n$ with $\Delta^{m o n}(G) \leq n-4$.

Lemma 27. Suppose $G$ is an edge-colored complete graph of order $n$ with $c(G)=\left\{c_{1}, c_{2}\right\}$ and $\Delta^{\text {mon }}(G) \leq n-4$. If $G$ contains a PC $C_{8}$, then $G$ contains two edge-disjoint PC cycles of different lengths.

Proof. Suppose to the contrary that $G$ does not contain two edge-disjoint PC cycles of different lengths. By Lemma 26, $G$ contains no PC $C_{6}$. Assume that $C^{*}=x_{1} y_{1} x_{2} y_{2} x_{3} y_{3} x_{4} y_{4} x_{1}$ is a $\mathrm{PC} C_{8}$ with $c\left(x_{i} y_{i}\right)=c_{1}$ and $c\left(y_{i} x_{i+1}\right)=c_{2}$ for all $i=1,2,3,4$, where $x_{5}=x_{1}$. To avoid
that $y_{1} x_{2} y_{2} x_{3} y_{3} x_{4} y_{1}$ is a PC $C_{6}$, we have $c\left(y_{1} x_{4}\right)=c_{2}$. Then we have $c\left(x_{2} y_{3}\right)=c\left(x_{1} y_{2}\right)=$ $c\left(x_{3} y_{4}\right)=c_{2}$ and $c\left(y_{1} x_{3}\right)=c\left(x_{1} y_{3}\right)=c\left(x_{2} y_{4}\right)=c\left(y_{2} x_{4}\right)=c_{1}$. It is easily seen that $C^{\prime}=$ $x_{1} y_{2} x_{4} y_{1} x_{3} y_{4} x_{2} y_{3} x_{1}$ also is a PC $C_{8}$.

Let $G_{1}=G-\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$ and $G_{2}=G-\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$. From Theorem 2 and the hypothesis that $G$ does not contain two edge-disjoint PC cycles of different lengths, we have $\Delta^{\text {mon }}\left(G_{i}\right)=\left|V\left(G_{i}\right)\right|-1=n-5$, which implies that there are two vertices $x \in G_{1}$ and $y \in G_{2}$ such that $\left|c\left(x, G_{1}\right)\right|=\left|c\left(y, G_{2}\right)\right|=1$, respectively.

We assert that $x, y \notin C^{*}$. If $x \in C^{*}$, w.l.o.g., set $x=x_{1}$ and $c\left(x, G_{1}\right)=\left\{c_{1}\right\}$. Since $c\left(x_{1} y_{1}\right)=c\left(x_{1} y_{3}\right)=c_{1}, d_{G}^{c_{1}}\left(x_{1}\right)=n-3$, a contradiction. If $y \in C^{*}$, w.l.o.g., set $y=y_{1}$ and $c\left(y, G_{2}\right)=\left\{c_{1}\right\}$. Since $c\left(y_{1} x_{1}\right)=c\left(y_{1} x_{3}\right)=c_{1}$, we have $d_{G}^{c_{1}}\left(y_{1}\right)=n-3$, a contradiction.

If $x \neq y$, then $c\left(x, G_{1}\right)=c\left(y, G_{2}\right)=\{c(x y)\}$. W.l.o.g., set $c(x y)=c_{1}$. The condition $\Delta^{\text {mon }}(G) \leq n-4$ implies that $d_{C^{*}}^{c_{2}}(x) \geq 3$ and $d_{C^{*}}^{c_{2}}(y) \geq 3$. Choose an edge $x_{i} y_{i}$. Note that $c\left(x x_{i}\right)=c\left(y y_{i}\right)=c(x y)=c_{1}$. If $c\left(x y_{i}\right)=c\left(y x_{i}\right)=c_{2}$, then $x x_{i} y_{i} y x$ is a PC $C_{4}$ edge-disjoint from $C^{\prime}$, a contradiction. Hence, there is at most one edge colored by $c_{2}$ between $\{x, y\}$ and $\left\{x_{i}, y_{i}\right\}$. This means that there are at most four edges colored by $c_{2}$ between $\{x, y\}$ and $C^{*}$, which contradicts the fact that $d_{C^{*}}^{c_{2}}(x) \geq 3$ and $d_{C^{*}}^{c_{2}}(y) \geq 3$.

If $x=y$, then $c\left(x, G_{1}\right) \neq c\left(x, G_{2}\right)$; otherwise $|c(x, G)|=1$, a contradiction. Then, $n=9$ and $V(G)=V\left(C^{*}\right) \cup\{x\}$. W.l.o.g., set $c\left(x,\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}\right)=\left\{c_{2}\right\}$ and $c\left(x,\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}\right)=$ $\left\{c_{1}\right\}$. If $c\left(x_{1} x_{2}\right)=c_{1}$, then $x_{1} x_{2} x y_{4} x_{1}$ is a PC $C_{4}$ edge-disjoint from $C^{\prime}$, a contradiction. If $c\left(x_{1} x_{2}\right)=c_{2}$, then $\left\{x_{1}, x, y_{1}, y_{4}\right\} \subseteq N_{G}^{c_{2}}\left(x_{2}\right)$. The condition $\Delta^{m o n}(G) \leq n-4=5$ implies that $c\left(x_{2} x_{3}\right)=c_{1}$ or $c\left(x_{2} x_{4}\right)=c_{1}$. If $c\left(x_{2} x_{3}\right)=c_{1}$, then $x x_{2} x_{3} y_{4} x$ is a PC $C_{4}$ edge-disjoint from $C^{*}$, a contradiction. If $c\left(x_{2} x_{4}\right)=c_{1}$, then $x x_{2} x_{4} y_{1} x$ is a PC $C_{4}$ edge-disjoint from $C^{*}$, a contradiction. The result thus follows.

Lemma 28 ([13]). Suppose that $G$ is an edge-colored complete graph and $C=x y z w x$ is a $P C$ $C_{4}$ in $G$, where $c(x y)=c(z w)=c_{1}$ and $c(y z)=c(x w)=c_{2}$. If uv is an edge with no end on $C$ such that $c(u v)=c_{1}$ and $d_{C}^{c_{2}}(u)+d_{C}^{c_{2}}(v) \geq 5$, then $G[V(C) \cup\{u, v\}]$ contains a PC cycle of length 6.

Lemma 29 ([13]). Suppose that $G$ is an edge-colored complete graph with $c(G)=\left\{c_{1}, c_{2}\right\}$ and $G$ does not contain two vertex-disjoint PC cycles of different lengths. If $G$ contains $m \geq 3$ pairwise vertex-disjoint $P C C_{4}$ 's, say $Q_{1}, Q_{2}, \cdots, Q_{m}$, let $Q_{i}=x_{i} y_{i} z_{i} w_{i} x_{i}$ for each $i \in\{1,2, \ldots, m\}$. Then up to renaming the vertices, for every pair $i, j$ of integers with $1 \leq i<j \leq m$,

$$
\begin{gathered}
\left\{c\left(x_{i} z_{i}\right)\right\}=\left\{c\left(x_{i} y_{i}\right)\right\}=\left\{c\left(z_{i} w_{i}\right)\right\}=c\left(\left\{x_{i}, z_{i}\right\}, V\left(Q_{j}\right)\right)=\left\{c_{1}\right\}, \text { and } \\
\left\{c\left(y_{i} w_{i}\right)\right\}=\left\{c\left(x_{i} w_{i}\right)\right\}=\left\{c\left(z_{i} y_{i}\right)\right\}=c\left(\left\{y_{i}, w_{i}\right\}, V\left(Q_{j}\right)\right)=\left\{c_{2}\right\} .
\end{gathered}
$$

Lemma 30. Suppose that $G$ is an edge-colored complete graph with $c(G)=\left\{c_{1}, c_{2}\right\}$ and $\Delta^{\text {mon }}(G) \leq n-4$, and suppose $G$ does not contain two edge-disjoint PC cycles of differen$t$ lengths. If $G$ contains $m \geq 2$ pairwise vertex-disjoint $P C C_{4}$ 's, say $Q_{1}, Q_{2}, \cdots, Q_{m}$, let $Q_{i}=x_{i} y_{i} z_{i} w_{i} x_{i}$ for each $i \in\{1,2, \ldots, m\}$. Then up to renaming the vertices, for every pair $i, j$ of integers with $1 \leq i<j \leq m$,

$$
\begin{gathered}
\left\{c\left(x_{i} z_{i}\right)\right\}=\left\{c\left(x_{i} y_{i}\right)\right\}=\left\{c\left(z_{i} w_{i}\right)\right\}=c\left(\left\{x_{i}, z_{i}\right\}, V\left(Q_{j}\right)\right)=\left\{c_{1}\right\}, \text { and } \\
\left\{c\left(y_{i} w_{i}\right)\right\}=\left\{c\left(x_{i} w_{i}\right)\right\}=\left\{c\left(z_{i} y_{i}\right)\right\}=c\left(\left\{y_{i}, w_{i}\right\}, V\left(Q_{j}\right)\right)=\left\{c_{2}\right\}
\end{gathered}
$$

Proof. If $m \geq 3$, then the result follows from Lemma 29. Hence we suppose $m=2$. W.l.o.g., we assume that $c\left(x_{i} y_{i}\right)=c\left(z_{i} w_{i}\right)=c_{1}$ and $c\left(y_{i} z_{i}\right)=c\left(w_{i} x_{i}\right)=c_{2}$ for $i=1,2$. From Lemmas 26 and 27, $G\left[V\left(Q_{1}\right) \cup V\left(Q_{2}\right)\right]$ contains neither PC $C_{6}$ nor PC $C_{8}$. By symmetry of $c_{1}$ and $c_{2}$, we assume that $c\left(x_{1} x_{2}\right)=c_{1}$. Since $y_{1} z_{1} w_{1} x_{1} x_{2} w_{2} y_{1}, w_{1} x_{1} x_{2} w_{2} z_{2} y_{2} w_{1}$ and $z_{1} w_{1} x_{1} x_{2} w_{2} z_{2} z_{1}$ are not PC $C_{6}$, we have $c\left(y_{1} w_{2}\right)=c\left(w_{1} y_{2}\right)=c_{2}$ and $c\left(z_{1} z_{2}\right)=c_{1}$.

If $c\left(x_{1} z_{2}\right)=c_{2}$, then since $y_{1} x_{1} z_{2} w_{2} x_{2} y_{2} y_{1}, w_{1} z_{1} y_{1} x_{1} z_{2} w_{2} w_{1}$ and $z_{1} y_{1} x_{1} z_{2} w_{2} x_{2} z_{1}$ are not PC $C_{6}$, we have $c\left(y_{1} y_{2}\right)=c\left(w_{1} w_{2}\right)=c_{1}$ and $c\left(z_{1} x_{2}\right)=c_{2}$. This implies that $x_{1} y_{1} z_{1} w_{1} y_{2} x_{2} w_{2} z_{2} x_{1}$ is a PC $C_{8}$, a contradiction. So we have $c\left(x_{1} z_{2}\right)=c_{1}$. Since $y_{1} z_{1} w_{1} x_{1} z_{2} y_{2} y_{1}, w_{1} x_{1} z_{2} y_{2} x_{2} w_{2} w_{1}$ and $z_{1} w_{1} x_{1} z_{2} y_{2} x_{2} z_{1}$ are not PC $C_{6}$, we have $c\left(y_{1} y_{2}\right)=c\left(w_{1} w_{2}\right)=c_{2}$ and $c\left(z_{1} x_{2}\right)=c_{1}$. Then $c\left(\left\{x_{1}, z_{1}\right\},\left\{x_{2}, z_{2}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(\left\{y_{1}, w_{1}\right\},\left\{y_{2}, w_{2}\right\}\right)=\left\{c_{2}\right\}$. By symmetry of $c_{1}$ and $c_{2}$, we assume that $c\left(x_{1} y_{2}\right)=c_{1}$. Then we can also have $c\left(\left\{x_{1}, z_{1}\right\},\left\{y_{2}, w_{2}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(\left\{y_{1}, w_{1}\right\},\left\{x_{2}, z_{2}\right\}\right)=\left\{c_{2}\right\}$. Then $c\left(\left\{x_{1}, z_{1}\right\}, V\left(Q_{2}\right)\right)=\left\{c_{1}\right\}$ and $c\left(\left\{y_{1}, w_{1}\right\}, V\left(Q_{2}\right)\right)=\left\{c_{2}\right\}$. Since $x_{1} z_{1} x_{2} w_{2} z_{2} y_{2} x_{1}$ and $y_{1} w_{1} x_{2} y_{2} z_{2} w_{2} y_{1}$ are not PC $C_{6}$, we have $c\left(x_{1} z_{1}\right)=c_{1}$ and $c\left(y_{1} w_{1}\right)=c_{2}$. The result thus follows.

Lemma 31. Suppose $G$ is an edge-colored complete graph of order $n \geq 5$ with $\Delta^{\text {mon }}(G) \leq$ $n-4$. If $G$ contains a monochromatic edge-cut, then $G$ contains two edge-disjoint PC cycles of different lengths.

Proof. Suppose not. Since $G$ contains a monochromatic edge-cut, $G$ has a bipartition $\left\{V_{1}, V_{2}\right\}$ with (w.l.o.g.,) $c\left(V_{1}, V_{2}\right)=\left\{c_{1}\right\}$. Since $\Delta^{\text {mon }}(G) \leq n-4$, each vertex $v \in V_{i}$ is connected with at least three vertices in $V_{i}$ by edges with colors distinct from $c_{1}$ for $i=1,2$. Clearly, there are two independent edges $x_{1} x_{2}$ and $x_{3} x_{4}$ in $V_{1}$ and two independent edges $y_{1} y_{2}$ and $y_{3} y_{4}$ in $V_{2}$ such that $c_{1} \notin\left\{c\left(x_{1} x_{2}\right), c\left(x_{3} x_{4}\right), c\left(y_{1} y_{2}\right), c\left(y_{3} y_{4}\right)\right\}$. Then we can find a PC cycle $x_{1} x_{2} y_{1} y_{2} x_{3} x_{4} y_{3} y_{4} x_{1}$ of length 8. Let $G_{1}=G-\left\{x_{1}, x_{3}, y_{1}, y_{3}\right\}$. Then $\Delta^{\text {mon }}\left(G_{1}\right)=\left|V\left(G_{1}\right)\right|-1$; otherwise $G_{1}$ contains a PC cycle of length at most 4 by Theorem 2. W.l.o.g., assume that the vertex $v$ with $d_{G_{1}}^{c}(v)=1$ is in $V_{1}$. It is obvious that $c\left(v, G_{1}\right)=\left\{c_{1}\right\}$. Then $c\left(v y_{1}\right)=c\left(v y_{3}\right)=c_{1}$ follows from $c\left(V_{1}, V_{2}\right)=\left\{c_{1}\right\}$. Thus $d_{G}^{c_{1}}(v) \geq n-3$, which contradicts the condition $\Delta^{\text {mon }}(G) \leq n-4$. The lemma thus follows.

In 2018, Fujita et al. in [8] gave a sufficient condition for the existence of PC triangles in edge-colored complete graphs.

Lemma 32 ([8]). Let $G$ be an edge-colored complete graph order $n \geq 3$ with $\delta^{c}(G)>\log _{2} n$. Then $G$ contains a $P C$ triangle.

### 4.1 Proof of Theorem 11

Note that $n=5$ implies that $\Delta^{m o n}(G)=1$ and $\delta^{c}(G)=4$. i.e., $G$ is a properly edge-colored complete graph. We can easily find two edge-disjoint PC cycles of different lengths. Thus, we always assume $n \geq 6$ in the proof. Theorem 11 can be divided into two lemmas by considering whether $G$ contains PC triangles or not. Therefore, the following results are stated.

Lemma 33. Let $G$ be an edge-colored complete graph of order $n$ with $\Delta^{\text {mon }}(G) \leq n-4$. If $G$ contains a PC triangle, then $G$ contains two edge-disjoint PC cycles of different lengths.

Lemma 34. Let $G$ is an edge-colored complete graph of order $n$ with $\Delta^{\text {mon }}(G) \leq n-4$. If $G$ contains no PC triangle, then $G$ still contains two edge-disjoint PC cycles of different lengths.

From Lemmas 33 and 34, we can get Theorem 11 easily.

igure 2: The edge-colored graph $F^{*}$

First, for convenience we construct a graph $F^{*}$, which is an edge-colored graph with 6 vertices $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ as shown in Figure 2. In $F^{*}, v_{1} v_{2} v_{3} v_{1}, v_{4} v_{5} v_{6} v_{4}$ and $v_{2} v_{3} v_{4} v_{5} v_{2}$ are PC cycles. Now we are ready to prove Lemmas 33 and 34 .

Proof of Lemma 33: Suppose to the contrary that $G$ does not contain two edge-disjoint PC cycles of different lengths. Assume that the PC $C_{3}$ in $G$ is $C^{*}=v_{1} v_{2} v_{3} v_{1}$.

Next, we prove the following claims.
Claim 1. Any two PCs $C_{3}$ and $C_{4}$ in $G$ share exactly one edge.
Proof. Suppose to the contrary that there exists a PC $C_{4}$ that shares two edges $v_{1} v_{2}$ and $v_{2} v_{3}$ with $C^{*}$. We label the PC $C_{4}$ by $v_{1} v_{2} v_{3} v_{4} v_{1}$. Let $G_{1}=G-\left\{v_{1}, v_{3}\right\}$. Then $\Delta^{\text {mon }}\left(G_{1}\right) \leq$
$\Delta^{\text {mon }}(G) \leq\left|V\left(G_{1}\right)\right|-2$. Theorem 2 guarantees that $G_{1}$ contains a PC cycle of length at most 4. Apparently, this cycle is edge-disjoint from both $v_{1} v_{2} v_{3} v_{4} v_{1}$ and $v_{1} v_{2} v_{3} v_{1}$, a contradiction.

Let $G^{\prime}=G-v_{1}$. Then $\Delta^{\text {mon }}\left(G^{\prime}\right) \leq \Delta^{\text {mon }}(G) \leq\left|V\left(G^{\prime}\right)\right|-3$. Hence, Lemma 25 guarantees that $G^{\prime}$ contains a PC cycle of length $k$ for $4 \leq k \leq 6$. While from Lemma 26, $G$ contains no PC $C_{5}$ or $C_{6}$. Thus, $G$ has a PC $C_{4}$ which is connected with $C^{*}$ by the edge $v_{2} v_{3}$, and we label the PC $C_{4}$ by $v_{2} v_{3} v_{4} v_{5} v_{2}$.

Claim 2. $G$ contains a copy of $F^{*}$.
Proof. Let $G_{1}=G-\left\{v_{1}, v_{2}\right\}$. Then the condition $\Delta^{\text {mon }}\left(G_{1}\right) \leq n-4=\left|V\left(G_{1}\right)\right|-2$ guarantees that $G_{1}$ contains a PC cycle of length at most 4 by Theorem 2. If $G_{1}$ contains a PC $C_{4}$, then this cycle is edge-disjoint from $C^{*}$, a contradiction. Thus, $G_{1}$ has one PC triangle containing $v_{3} v_{4}$ or $v_{4} v_{5}$. According to Claim 1, $v_{3} v_{4} v_{5} v_{3}$ is not a PC triangle. If there exists a vertex $u \in V(G) \backslash\left\{v_{1}, \cdots, v_{5}\right\}$ such that $u v_{4} v_{5} u$ is a PC triangle, then the claim follows. Hence, we assume that there exists no such vertex. Then there exists a vertex $u_{1} \in V(G) \backslash\left\{v_{1}, \cdots, v_{5}\right\}$ such that $u_{1} v_{3} v_{4} u_{1}$ is a PC triangle. By symmetry, let $G_{2}=G-\left\{v_{1}, v_{3}\right\}$. We can also get a PC triangle $u_{2} v_{2} v_{5} u_{2}$ with $u_{2} \in V(G) \backslash\left\{v_{1}, \cdots, v_{5}\right\}$. If $u_{1} \neq u_{2}$, the claim follows from $G\left[\left\{v_{2}, \cdots, v_{5}, u_{1}, u_{2}\right\}\right]$. If $u_{1}=u_{2}$, then set $G_{3}=G-\left\{v_{3}, u_{1}\right\}$. Repeating the analysis above, we know that $v_{1} v_{4} v_{5} v_{1}$ is a PC triangle. Then we get a copy of graph shown as Figure 3. Then

$\mathrm{PC} C_{4}: v_{2} v_{3} v_{4} v_{5} v_{2}$
$\mathrm{PC} C_{3}^{\prime} \mathrm{s}: v_{1} v_{2} v_{3} v_{1}$
$v_{4} v_{5} v_{2} v_{4}$
$v_{4} v_{5} v_{3} v_{4}$

Figure 3: The edge-colored graph in Claim 2
let $G_{4}=G-\left\{v_{2}, v_{4}\right\}$ and $G_{5}=G-\left\{v_{3}, v_{5}\right\}$. The condition $\Delta^{\text {mon }}\left(G_{4}\right) \leq n-4=\left|V\left(G_{4}\right)\right|-2$ guarantees that $G_{4}$ contains a PC cycle of length at most 4 by Theorem 2. If $G_{4}$ contains a PC triangle, then this PC triangle is edge-disjoint from PC cycle $v_{2} v_{3} v_{4} v_{5} v_{2}$. Hence, $G_{4}$ contains a PC $C_{4}$. Note that such a PC $C_{4}$ must contains at least one edge from each PC triangles of $\left\{u_{1} v_{2} v_{5} u_{1}, u_{1} v_{3} v_{4} u_{1}, C^{*}, v_{1} v_{4} v_{5} v_{1}\right\}$. Hence, $v_{1} v_{5} u_{1} v_{3} v_{1}$ is the desired PC $C_{4}$. Since $G_{4}$ and $G_{5}$ are isomorphic, we know that $v_{2} v_{1} v_{4} u_{1} v_{2}$ is a PC $C_{4}$.

Then, to avoid $u_{1} v_{3} v_{5} u_{1}$ and $v_{1} v_{3} v_{5} v_{1}$ being PC triangles, w.l.o.g. we have $c\left(u_{1} v_{3}\right)=$ $c\left(v_{3} v_{5}\right)=c\left(v_{1} v_{5}\right)=c_{1}$. Recalling that both $v_{1} v_{4} v_{5} v_{1}$ and $u_{1} v_{4} v_{3} u_{1}$ are PC triangles, we have $c_{1} \notin\left\{c\left(v_{3} v_{4}\right), c\left(v_{4} v_{5}\right)\right\}$. Therefore, $v_{3} v_{4} v_{5} v_{3}$ is a PC $C_{3}$ edge-disjoint from $v_{2} v_{1} v_{4} u_{1} v_{2}$, a contradiction.

Continuing the proof of Lemma 33, we label the vertices of the copy $F$ of $F^{*}$ in $G$ by $\left\{v_{1}, \cdots, v_{6}\right\}$ such that $v_{1} v_{2} v_{3} v_{1}, v_{4} v_{5} v_{6} v_{4}$ and $v_{2} v_{3} v_{4} v_{5} v_{2}$ are PC cycles. Set $G_{i}=G-\left\{v_{i}, v_{i+2}, v_{i+4}\right\}$ for $i=1,2$. Clearly, neither $G_{1}$ nor $G_{2}$ contains a PC cycle. Hence, $G_{i}$ contains a vertex $u_{i}$ such that $d_{G_{i}}^{c}\left(u_{i}\right)=1$ for $i=1,2$. W.l.o.g., set $c\left(u_{i}, G_{i}\right)=\left\{c_{i}\right\}$ for $i=1,2$. Especially, $c\left(u_{1},\left\{v_{2}, v_{4}, v_{6}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(u_{2},\left\{v_{1}, v_{3}, v_{5}\right\}\right)=\left\{c_{2}\right\}$. Since $\Delta^{\text {mon }}(G) \leq n-4$, we have $c_{1} \notin c\left(u_{1},\left\{v_{1}, v_{3}, v_{5}\right\}\right)$ and $c_{2} \notin c\left(u_{2},\left\{v_{2}, v_{4}, v_{6}\right\}\right)$.

Claim 3. $u_{i} \notin V(F)$ for $i=1,2$.
Proof. We prove the claim by contradiction. Note that $u_{1} \notin\left\{v_{1}, v_{3}, v_{5}\right\}$ and $u_{2} \notin\left\{v_{2}, v_{4}, v_{6}\right\}$. If $u_{1}=v_{2}$, then we have $c\left(v_{2} v_{4}\right)=c_{1}$ and $c_{1} \notin\left\{c\left(v_{2} v_{3}\right), c\left(v_{2} v_{5}\right)\right\}$. From Claim 1, we know that neither $v_{2} v_{3} v_{4} v_{2}$ nor $v_{2} v_{5} v_{4} v_{2}$ is a PC triangle. Since $c\left(v_{2} v_{3}\right) \neq c\left(v_{3} v_{4}\right)$ and $c\left(v_{2} v_{5}\right) \neq c\left(v_{4} v_{5}\right)$, we have $c\left(v_{3} v_{4}\right)=c\left(v_{4} v_{5}\right)=c_{1}$, which contradicts the fact that $v_{2} v_{3} v_{4} v_{5} v_{2}$ is a PC $C_{4}$. Hence, $u_{1} \neq v_{2}$. Since $G_{1}$ and $G_{2}$ are isomorphic, we can deduce that $u_{1} \neq v_{4}$ and $u_{2} \notin\left\{v_{3}, v_{5}\right\}$. If $u_{1}=v_{6}$ and $u_{2} \notin V(F)$, it is clear that $c_{1} \neq c_{2}$. To avoid that $u_{2} v_{6} v_{5} v_{4} v_{3} u_{2}$ is a PC cycle of length 5 , we have $c\left(v_{4} v_{3}\right)=c\left(v_{3} u_{2}\right)=c_{2}$. Since $u_{2} v_{4} v_{3} v_{1} u_{2}$ and $v_{4} v_{5} v_{6} v_{4}$ are not edge-disjoint, we have $c\left(v_{1} v_{3}\right)=c_{2}$. Recall that $v_{1} v_{2} v_{3} v_{1}$ is a PC triangle. Then $c_{2} \notin\left\{c\left(v_{1} v_{2}\right), c\left(v_{2} v_{3}\right)\right\}$, which implies that $u_{2} v_{4} v_{3} v_{2} v_{1} u_{2}$ is a PC cycle of length 5 , a contradiction. Since $G_{1}$ and $G_{2}$ are isomorphic, we can get a contradiction when $u_{2}=v_{1}$ and $u_{1} \notin V(F)$. Consequently, we only have $u_{1}=v_{6}$ and $u_{2}=v_{1}$. Then $c\left(v_{1} v_{6}\right) \notin\left\{c_{1}, c_{2}\right\}$ and $v_{1} v_{3} v_{2} v_{5} v_{4} v_{6} v_{1}$ is a PC cycle of length 6 , a contradiction. The claim thus follows.

While if $u_{i} \notin V(F)$ for $i=1,2$, then $c\left(u_{1},\left\{v_{2}, v_{4}, v_{6}\right\}\right)=\left\{c_{1}\right\}$ and $c\left(u_{2},\left\{v_{1}, v_{3}, v_{5}\right\}\right)=$ $\left\{c_{2}\right\}$. If $u_{1} \neq u_{2}$, then $c_{1}=c\left(u_{1} u_{2}\right)=c_{2}$. This implies from $\Delta^{\text {mon }}\left(u_{i}\right)=n-4$ that $c_{1} \notin$ $c\left(u_{i},\left\{v_{i}, v_{i+2}, v_{i+4}\right\}\right)$ for $i=1,2$. Hence, $u_{1} v_{1} u_{2} v_{2} u_{1}$ is a PC $C_{4}$ edge-disjoint from the PC triangle $v_{4} v_{5} v_{6} v_{4}$, a contradiction.

If $u_{1}=u_{2}$, then $c_{1} \neq c_{2}$, and so $V(G) \backslash V\left(F^{*}\right)=\left\{u_{1}\right\}$ and $n=7$. Since $v_{6} u_{1} v_{3} v_{6}$ is not a PC triangle, we have $c\left(v_{6} v_{3}\right) \in\left\{c\left(u_{1} v_{6}\right)=c_{1}, c\left(u_{1} v_{3}\right)=c_{2}\right\}$. W.l.o.g., we assume $c\left(v_{6} v_{3}\right)=c_{1}$. As neither $v_{6} v_{3} u_{1} v_{2} v_{6}$ nor $v_{6} v_{3} u_{1} v_{4} v_{6}$ is a PC $C_{4}$, we get $c\left(v_{2} v_{6}\right)=c_{1}$ and $c\left(v_{4} v_{6}\right)=c_{1}$. Then $\left|N_{G}^{c_{1}}\left(v_{6}\right)\right| \geq 4=n-3$, a contradiction. The proof of Lemma 33 is thus complete.
Proof of Lemma 34: Suppose to the contrary that $G$ does not contain two edge-disjoint PC cycles of different lengths. If $n=6$, from Lemma 32, $G$ contains a PC triangle, a contradiction. Hence, $n \geq 7$. Since $G$ contains no PC triangle, there is a Gallai partition $\left\{U_{1}, U_{2}, \cdots, U_{q}\right\}$ in $G$. We assume $c\left(U_{i}, U_{j}\right) \subset\left\{c_{1}, c_{2}\right\}$ for $1 \leq i<j \leq q$. Furthermore, by Lemma $31 G$ contains no monochromatic edge-cut, which implies $q \geq 4$.

Let $H$ be an edge-colored complete graph obtained from $G$ by shrinking each $U_{i}$ to a vertex $u_{i}$ and deleting all parallel edges. Since $G$ contains no monochromatic edge-cut, one can see that $d^{c}\left(u_{i}\right)=2$ for each $i \in\{1, \ldots, q\}$ and $\Delta^{\text {mon }}(H) \leq|H|-2$. One can also see that $H$
contains only two colors $c_{1}$ and $c_{2}$. From Theorem 2, $H$ contains a PC cycle of length at most 4. If $H$ has a PC triangle, then $G$ has a PC triangle, a contradiction. Hence, $H$ contains a PC $C_{4}$ with only two colors. Thus, there is a PC $C_{4}: C^{*}=x_{0} y_{0} z_{0} w_{0} x_{0}$ with only two colors in $G$ and each vertex is from different parts in the Gallai partition. W.l.o.g., we suppose that $c\left(x_{0} y_{0}\right)=c\left(z_{0} w_{0}\right)=c_{1}$ and $c\left(y_{0} z_{0}\right)=c\left(w_{0} x_{0}\right)=c_{2}$. Let $G_{1}=G-\left\{y_{0}, w_{0}, z_{0}\right\}$.

The subgraph of an edge-colored graph $F$ induced by all the edges with color $\alpha$ is called the $i$-color subgraph, denoted by $F^{\alpha}$.

Claim 1. (1) If there is a color $\alpha$ such that $G_{1}^{\alpha}$ is spanning and connected, then $\alpha \in\left\{c_{1}, c_{2}\right\}$.
(2) $c\left(\left\{y_{0}, z_{0}, w_{0}\right\}, V\left(G_{1}\right)\right) \subseteq\left\{c_{1}, c_{2}\right\}$.
(3) $\Delta^{\text {mon }}\left(G_{1}\right) \leq\left|V\left(G_{1}\right)\right|-2$.

Proof. (1) Suppose to the contrary that $\alpha \notin\left\{c_{1}, c_{2}\right\}$. Since $c\left(U_{i}, U_{j}\right) \subset\left\{c_{1}, c_{2}\right\}$ for $1 \leq i, j \leq q$, $V\left(G_{1}\right) \subseteq U_{i_{0}}$ for some $i_{0} \in\{1,2, \cdots, q\}$. Since $x_{0} \in V\left(G_{1}\right)$, one has $y_{0} \notin U_{i_{0}}$. Thus, $d_{G}^{m o n}\left(y_{0}\right) \geq$ $\left|V\left(G_{1}\right)\right|=n-3$, a contradiction.
(2) Suppose to the contrary that there is a vertex $v \in V\left(G_{1}\right) \backslash\left\{x_{0}\right\}$ such that $c\left(y_{0} v\right)=c_{3} \notin$ $\left\{c_{1}, c_{2}\right\}$. Then $y_{0}, v$ are in the same part of the Gallai partition. Hence, $c\left(x_{0} v\right)=c\left(x_{0} y_{0}\right)=c_{1}$. Then, $v y_{0} z_{0} w_{0} x_{0} v$ is a PC $C_{5}$. Lemma 26 implies that $G$ contains two edge-disjoint PC cycles of different lengths, a contradiction.
(3) If not, there is one vertex $v \in G_{1}$ such that $d_{G_{1}}^{c}(v)=1$, say $c\left(v, G_{1}\right)=\{f\}$. Since $G_{1}$ is a complete graph, $G_{1}^{f}$ is spanning and connected. By (1), we know $f \in\left\{c_{1}, c_{2}\right\}$. W.l.o.g., we assume $c\left(v, G_{1}\right)=\{f\}=\left\{c_{1}\right\}$. If $c_{1} \in\left(v,\left\{y_{0}, z_{0}, w_{0}\right\}\right)$, then $\Delta^{m o n}(v) \geq n-3$, a contradiction. Hence $c\left(v,\left\{y_{0}, z_{0}, w_{0}\right\}\right)=\left\{c_{2}\right\}$ and $v \neq x_{0}$.

We assert that for each $u \in V\left(G_{1}\right) \backslash\left\{v, x_{0}\right\}, c\left(u,\left\{x_{0}, y_{0}, z_{0}\right\}\right)=\left\{c_{1}\right\}$. If not, choose an arbitrary vertex $u \in V\left(G_{1}\right) \backslash\left\{v, x_{0}\right\}$. If $c\left(u y_{0}\right) \neq c_{1}$, then $u y_{0} x_{0} w_{0} z_{0} v u$ is a PC $C_{6}$. If $c\left(u z_{0}\right) \neq c_{1}$, then $u z_{0} w_{0} y_{0} z_{0} v u$ is a PC $C_{6}$. If $c\left(u w_{0}\right) \neq c_{1}$, then $u w_{0} z_{0} y_{0} x_{0} v u$ is a PC $C_{6}$. From Lemma 26, there are two edge-disjoint PC cycles of different lengths, a contradiction.

If there exists a vertex $u \in V\left(G_{1}\right) \backslash\left\{v, x_{0}\right\}$ such that $c\left(x_{0} u\right) \neq c_{1}$, then $v u x_{0} y_{0} z_{0} w_{0} v$ is a PC $C_{6}$. Hence, $c\left(x_{0}, G_{1}\right)=\left\{c_{1}\right\}$. Then $\Delta^{\text {mon }}\left(x_{0}\right) \geq n-3$, a contradiction. The claim thus follows.

Next, we distinguish two cases by considering the maximum color-degree of $G$.
Case 1: $\Delta^{c}(G)=2$.
Recall that $G$ has a Gallai partition $U_{1}, U_{2}, \ldots, U_{p}$ and $\cup_{1 \leq i<j \leq p} c\left(U_{i}, U_{j}\right)=\left\{c_{1}, c_{2}\right\}$. Then $\left\{c_{1}, c_{2}\right\} \subseteq c(G)$. Since $G$ contains no monochromatic-cut, we have $\left\{c_{1}, c_{2}\right\} \subseteq c(v, G)$ for each vertex $v \in G$. So, $c(G)=\left\{c_{1}, c_{2}\right\}$.

Claim 1.1 $G$ contains at least two vertex-disjoint PC $C_{4}$ 's.

Proof. Since $\Delta^{\text {mon }}\left(G_{1}\right) \leq\left|V\left(G_{1}\right)\right|-2$, there is a PC cycle $Q_{1}$ of length 4 in $G_{1}$ by Theorem 2. Suppose to the contrary that all PC $C_{4}$ 's in $G_{1}$ intersect $C^{*}$ at $x_{0}$. We can assume that $Q_{1}=x_{1} y_{1} z_{1} w_{1} x_{1}$ is a PC $C_{4}$ in $G_{1}$ and $x_{1}=x_{0}$. W.l.o.g., we suppose $c\left(x_{1} y_{1}\right)=c\left(z_{1} w_{1}\right)=c_{1}$ and $c\left(y_{1} z_{1}\right)=c\left(w_{1} x_{1}\right)=c_{2}$. Let $W=V\left(C^{*}\right) \cup V\left(Q_{1}\right)$ and $G_{2}=G-W$.

First, we assert that for all $v \in G_{2}, c\left(v y_{0}\right)=c\left(v y_{1}\right), c\left(v z_{0}\right)=c\left(v z_{1}\right)$ and $c\left(v w_{0}\right)=c\left(v w_{1}\right)$. In fact, if $c\left(v y_{0}\right) \neq c\left(v y_{1}\right)$, we suppose that $c\left(v y_{0}\right)=c_{1}$ and $c\left(v y_{1}\right)=c_{2}$. Then $v y_{0} z_{0} w_{0} x_{0} y_{1} v$ is a PC cycle of length 6, which contradicts Lemma 26. By a similar discussion, we can show the other cases.

Moreover, since $x_{0} y_{0} z_{0} w_{0} z_{1} w_{1} x_{0}$ is not a PC cycle of length 6 , we have $c\left(w_{0} z_{1}\right)=c_{1}$. By symmetry, we can conclude that $c\left(z_{0} w_{1}\right)=c_{1}$ and $c\left(y_{0} z_{1}\right)=c\left(y_{1} z_{0}\right)=c_{2}$. Since neither $y_{0} w_{0} x_{0} y_{1} w_{1} z_{0} y_{0}$ nor $y_{1} w_{1} x_{0} y_{0} w_{0} z_{1} y_{1}$ is a PC cycle of length 6 , we have $c\left(y_{0} w_{0}\right)=c\left(y_{1} w_{1}\right)=c_{i}$. W.l.o.g., assume that $c\left(y_{0} w_{0}\right)=c\left(y_{1} w_{1}\right)=c_{1}$. Then we get the subgraph $G[W]$ as shown in Figure 4.

Now we define two vertex sets $S=\left\{v \in G_{2}: c\left(v w_{0}\right)=c\left(v w_{1}\right)=c_{1}\right\}$ and $T=\{v \in$ $\left.G_{2}: c\left(v w_{0}\right)=c\left(v w_{1}\right)=c_{2}\right\}$. From Claim 2.1, we know that $c\left(v w_{0}\right)=c\left(v w_{1}\right)$ for all $v \in G_{2}$. Recall that $c(G)=\left\{c_{1}, c_{2}\right\}$. Then for each vertex $v \in G_{2}$, we have $c\left(v w_{0}\right)=c\left(v w_{1}\right)=c_{1}$ or $c\left(v w_{0}\right)=c\left(v w_{1}\right)=c_{2}$. So, $(S, T)$ is a vertex partition of $G_{2}$. Note that $T \neq \emptyset$; otherwise $\Delta^{m o n}(G) \leq n-4$ implies that $c\left(w_{1} y_{0}\right)=c\left(w_{1} w_{0}\right)=c_{2}$, which means that $w_{1} y_{0} w_{0} x_{0} y_{1} z_{1} w_{1}$ is a PC $C_{6}$, a contradiction.


Figure 4: The edge-colored graph $G[W]$ and the vertex partition $(S, T)$ of $G_{2}$

Next, we assert that $c\left(T, T \cup W \backslash\left\{x_{0}\right\}\right)=\left\{c_{2}\right\}$. For any $t \in T$, to avoid both $t z_{1} y_{1} w_{1} t$ and $t y_{1} z_{1} w_{1} t$ being properly colored, we have $c\left(t,\left\{z_{1}, y_{1}\right\}\right)=\left\{c_{2}\right\}$. By symmetry, $c\left(t,\left\{z_{0}, y_{0}\right\}\right)=$ $\left\{c_{2}\right\}$. If there is an edge in $G[T]$ such that $c\left(t_{1} t_{2}\right) \neq c_{2}$, then $t_{1} t_{2} z_{1} w_{1} t_{1}$ is a PC $C_{4}$ vertex-disjoint from $C^{*}$, a contradiction. Then $c(T) \subseteq\left\{c_{2}\right\}$.

Hence, referring to $G[T, S]$, we assert the following statements.
(1) For each vertex $t \in T$, there is a vertex $s_{t} \in S$ such that $c\left(t s_{t}\right)=c_{1}$. Note that $d_{G[W \cup T]}^{c_{2}}(t) \geq|W \cup T|-2$ for each vertex $t \in T$. Thus, there must be a vertex $s_{t} \in S$ such that $c\left(s_{t} t\right)=c_{1}$.
(2) If $c_{1} \in c(s, T)$ for $s \in S$, there is a vertex $t_{s} \in T$ such that $c\left(s t_{s}\right)=c_{2}$. Suppose there is
a vertex $t \in T$ such that $c(s t)=c_{1}$. Since we want to avoid $t s z_{1} w_{1} t$ and $t s y_{1} w_{1} t$ being properly colored, we have $c\left(s,\left\{y_{1}, z_{1}\right\}\right)=\left\{c_{1}\right\}$. Since $c\left(s, W \backslash\left\{x_{0}\right\}\right)=\left\{c_{1}\right\}$, for any $s^{\prime} \in S \backslash\{s\}$, as $t s s^{\prime} w_{1} t$ is not a PC $C_{4}$, we have $c\left(s s^{\prime}\right)=c_{1}$. This implies that $c\left(s, S \cup W \backslash\left\{x_{0}\right\}\right)=\left\{c_{1}\right\}$. Then there must exist a vertex $t_{s} \in T$ such that $c\left(s t_{s}\right)=c_{2}$.

Next, we show that there is a PC $C_{4}$ in $G_{2}$. According to the above statements, there exists an alternatively colored path in $G[S, T]$. Let $P=v_{1} v_{2} \cdots v_{l}$ be a longest one which begins with a vertex in $T\left(v_{1} \in T\right)$ and $c\left(v_{1} v_{2}\right)=c_{1}$. Recall that $(S, T)$ is a vertex partition of $G_{2}$. Then $v_{l} \in S$ or $v_{l} \in T$. In fact, the proofs of the two cases are similar. Hence, we assume $v_{l} \in S$. Then, the length of $P$ is odd and $c\left(v_{l-1} v_{l}\right)=c_{1}$. Thus by (2), there is a vertex $v_{l+1} \in T$ such that $c\left(v_{l} v_{l+1}\right)=c_{2}$. Since $P$ is longest, $v_{l+1}=v_{k} \in P$. Then, $C=v_{k} v_{k+1} \cdots v_{l} v_{k}$ is a PC cycle vertex-disjoint from $C^{*}$. If $|C| \geq 6$, we can get the lemma. Therefore, $|C|=4$.

Then by the claim above, let $C^{*}=Q_{0}, Q_{1}, Q_{2}, \ldots, Q_{m}$ be vertex-disjoint PC $C_{4}$ 's in $G$ such that $m$ is as large as possible. Apparently, $m \geq 1$. Define $Q_{i}=x_{i} y_{i} z_{i} w_{i} x_{i}$ for each $i \in$ $\{1,2, \ldots, m\}$, where $c\left(x_{i} y_{i}\right)=c\left(z_{i} w_{i}\right)=c_{1}$ and $c\left(x_{i} w_{i}\right)=c\left(y_{i} z_{i}\right)=c_{2}$. Let $W=\bigcup_{0 \leq i \leq m} V\left(Q_{i}\right)$ and $G_{2}=G-W$. Since $G_{2}$ contains no PC cycle, we know that $G_{2}$ contains a vertex $v$ such that $\left|c\left(v, G_{2}\right)\right|=1$. W.l.o.g., we set $c\left(v, G_{2}\right)=\left\{c_{1}\right\}$. From Lemma 30, there is an integer $i_{0} \in\{0,1, \ldots, m\}$ such that $d_{G[W]}^{c_{2}}\left(x_{i_{0}}\right)=d_{G[W]}^{c_{2}}\left(z_{i_{0}}\right)=1$ and $d_{G[W]}^{c_{1}}\left(y_{i_{0}}\right)=d_{G[W]}^{c_{1}}\left(w_{i_{0}}\right)=1$. W.l.o.g., we suppose $i_{0}=0$.

Firstly, we assert that $c\left(v, Q_{0}\right)=\left\{c_{1}\right\}$. Suppose not. If $c\left(v x_{0}\right)=c_{2}$ for any $u \in G_{2}-\{v\}$, since $v x_{0} y_{0} w_{0} z_{0} u v$ is not a PC cycle of length 6 , we have $c\left(u z_{0}\right)=c_{1}$. Then, $d_{G}^{c_{1}}\left(z_{0}\right) \geq n-3$, a contradiction. Thus, $c\left(v x_{0}\right)=c_{1}$. By a similar argument, we have $c\left(v z_{0}\right)=c\left(v y_{0}\right)=c\left(v w_{0}\right)=$ $c_{1}$.

Since $d_{G[W]}^{c_{2}}\left(x_{0}\right)=d_{G[W]}^{c_{2}}\left(z_{0}\right)=1$, we can find two distinct vertices $u_{1}, u_{2} \in G_{2}$ such that $c\left(u_{1} x_{0}\right)=c\left(u_{2} z_{0}\right)=c_{2}$. If $c\left(u_{1} u_{2}\right)=c_{1}$, then $u_{1} u_{2} z_{0} w_{0} y_{0} x_{0} u_{1}$ is a PC cycle of length 6 . So, we have $c\left(u_{1} u_{2}\right)=c_{2}$, which implies that $v \neq u_{i}$ for $i=1,2$. Recall that $c\left(v, G_{2}\right)=c\left(v, Q_{0}\right)=$ $\left\{c_{1}\right\}$. Since $\Delta^{\text {mon }}(G) \leq n-4$, there exists an integer $i \in\{1,2, . ., m\}$ such that $c_{2} \in c\left(v, Q_{i}\right)$. W.l.o.g., we assume $i=1$. If $c\left(v x_{1}\right)=c_{2}$, then $v u_{1} x_{0} y_{0} y_{1} x_{1} v$ is a PC $C_{6}$, a contradiction. Since $v u_{2} z_{0} w_{0} x_{1} y_{1} v, v u_{2} z_{0} w_{0} w_{1} z_{1} v$ and $v u_{2} z_{0} w_{0} z_{1} w_{1} v$ are not PC cycles of length six, we have $c\left(v y_{1}\right) \neq c_{2}, c\left(v z_{1}\right) \neq c_{2}$ and $c\left(v w_{1}\right) \neq c_{2}$. This implies $c_{2} \notin c\left(v, Q_{1}\right)$, a contradiction.

Case 2. $\Delta^{c}(G) \geq 3$.
Recall that $G$ contains no PC triangle and Lemma 26 implies that $G$ contains no PC $C_{5}$. Then by Lemma 19, $G$ contains no PC odd cycles. It is easily seen that $G$ satisfies the conditions of Lemma 18. Then we can get that $G$ admits a partition $\{X, Y, Z\}$ such that $c(X) \subseteq c(X, Z)=\left\{c_{1}\right\}, c(Y) \subseteq c(Y, Z)=\left\{c_{2}\right\}$ and $c(X, Y) \subseteq\left\{c_{1}, c_{2}\right\}$. Since $\Delta^{c}(G) \geq 3$, there exist at least three colors in $G$. However, $c(X \cup Y) \cup c(X \cup Y, Z) \subseteq\left\{c_{1}, c_{2}\right\}$, which means $Z \neq \emptyset$.

We assert that $|X| \geq 4$ and $|Y| \geq 4$. By symmetry, we only need to prove $|X| \geq 4$.

Otherwise, for each vertex $y \in Y$, we have $d_{G}^{c_{2}}(y) \geq n-3$, which contradicts the condition $\Delta^{\text {mon }}(G) \leq n-4$.

If $G[X \cup Y]$ contains no PC cycle, then $G[X \cup Y]$ contains a vertex $v$ such that $\mid c(v, G[X \cup$ $Y]) \mid=1$. W.l.o.g., we set $v \in X$. Since $|X| \geq 4$ and $c(X) \subseteq\left\{c_{1}\right\}$, we have $c(v, G[X \cup Y])=\left\{c_{1}\right\}$. Recall that $c(X, Z)=\left\{c_{1}\right\}$. This implies that $d_{G}^{c_{1}}(x)=n-1$, a contradiction. Then by Theorem 2. there is a PC $C_{4}$ in $G[X \cup Y]$. Thus we assume that the $C_{4}$ in $G[X \cup Y]$ is $Q_{0}=x_{0} y_{0} x_{1} y_{1} x_{0}$, where $x_{i} \in X$ and $y_{i} \in Y$ for $i=0,1$. Thus, $G[X \cup Y]$ contains at least two vertex-disjoint PC $C_{4}$ 's.

Let $X^{\prime}=X \backslash\left\{x_{0}, x_{1}\right\}$ and $Y^{\prime}=Y \backslash\left\{y_{0}, y_{1}\right\}$. Next, we only need to show that $G\left[X^{\prime} \cup Y^{\prime}\right]$ contains a PC $C_{4}$. Suppose not. Then $G\left[X^{\prime} \cup Y^{\prime}\right]$ contains a vertex $v$ such that $\mid c\left(v, G\left[X^{\prime} \cup\right.\right.$ $\left.\left.Y^{\prime}\right]\right) \mid=1$. W.l.o.g., we set $v \in X^{\prime}$. Note that $c\left(v,\left\{x_{0}, x_{1}\right\}\right)=c(v, Z)=\left\{c_{1}\right\}$, which implies that $d_{G}^{c_{1}}(v) \geq n-3$, a contradiction. Thus, we can conclude that $G\left[X^{\prime} \cup Y^{\prime}\right]$ contains a PC $C_{4}$.

Since $G[X \cup Y]$ contains at least two vertex-disjoint PC $C_{4}$ 's and $c(G[X \cup Y])=\left\{c_{1}, c_{2}\right\}$, noticing that $\Delta^{m o n}(G[X \cup Y]) \leq|X \cup Y|-4$. Let $C^{*}=Q_{0}, Q_{1}, Q_{2}, \ldots, Q_{m}$ be vertex-disjoint $\mathrm{PC} C_{4}$ 's in $G[X \cup Y]$ such that $m$ is as large as possible. Recall that $G[X \cup Y]$ contains at least two vertex-disjoint PC $C_{4}$ 's. Then $m \geq 1$. Define $Q_{i}=x_{i} y_{i} z_{i} w_{i} x_{i}$ for each $i \in\{1,2, \ldots, m\}$, where $c\left(x_{i} y_{i}\right)=c\left(z_{i} w_{i}\right)=c_{1}$ and $c\left(x_{i} w_{i}\right)=c\left(y_{i} z_{i}\right)=c_{2}$. Let $W=\bigcup_{0 \leq i \leq m} V\left(Q_{i}\right)$ and $G_{2}=$ $G[X \cup Y]-W$. Since $G_{2}$ contains no PC cycle, we know that $G_{2}$ contains a vertex $v$ such that $\left|c\left(v, G_{2}\right)\right|=1$. W.l.o.g., we set $c\left(v, G_{2}\right)=\left\{c_{1}\right\}$. From Lemma 30, there is an integer $i_{0} \in\{0,1, \ldots, m\}$ such that $d_{G[W]}^{c_{2}}\left(x_{i_{0}}\right)=d_{G[W]}^{c_{2}}\left(z_{i_{0}}\right)=1$ and $d_{G[W]}^{c_{1}}\left(y_{i_{0}}\right)=d_{G[W]}^{c_{1}}\left(w_{i_{0}}\right)=1$. W.l.o.g., we suppose $i_{0}=0$.

Since $d_{G[W]}^{c_{2}}\left(x_{0}\right)=d_{G[W]}^{c_{2}}\left(z_{0}\right)=c_{1}, c\left(\left\{x_{0}, z_{0}\right\}, Z\right)=\left\{c_{1}\right\}$ and $\Delta^{m o n}(G) \leq n-4$, there are two vertices $u_{1}, u_{2} \in Y \cap V\left(G_{2}\right)$ such that $c\left(x_{0} u_{1}\right)=c\left(z_{0} u_{2}\right)=c_{2}$, where $u_{1}$ and $u_{2}$ may be identical. From Lemma 30, we have $c\left(\left\{y_{0}, w_{0}\right\}, Q_{1}\right)=c_{2}$. By the definitions of $G[X]$ and $G[Y]$, assume that $x_{0}, z_{0} \in X$ and $y_{0}, w_{0} \in Y$. Since $v \in X$, we have $c\left(v,\left\{x_{0}, z_{0}\right\}\right)=\left\{c_{1}\right\}$. As $c\left(v, G_{2}\right)=c\left(v,\left\{x_{0}, z_{0}\right\}\right)=\left\{c_{1}\right\}$ and $\Delta^{m o n}(G) \leq n-4$, there is an integer $i \in\{1,2, \ldots, m\}$ such that $c_{2} \in c\left(v, Q_{i}\right)$, say $i=1$.

If $c\left(v x_{1}\right)=c_{2}$, then $v u_{1} x_{0} y_{0} y_{1} x_{1} v$ is a PC cycle of length six, a contradiction. If $c\left(v w_{1}\right)=c_{2}$, then $v u_{1} x_{0} y_{0} z_{1} w_{1} v$ is a PC cycle of length six, a contradiction. If $c\left(v y_{1}\right)=c_{2}$, then $v u_{2} z_{0} w_{0} x_{1} y_{1} v$ is a PC cycle of length six, a contradiction. If $c\left(v z_{1}\right)=c_{2}$, then $v u_{2} z_{0} w_{0} w_{1} z_{1} v$ is a PC cycle of length six, a contradiction.

Combining Cases 1 and 2, the proof of Lemma 34 is now complete.
Acknowledgement. The authors are very grateful to the reviewers and the editor for their very useful suggestions and comments, which helped to improving the presentation of the paper greatly. The work was supported by the National Natural Science Foundation of China (Nos. 12131013, 12161141006 and 12301457.) and the Tianjin Research Innovation Project for

Postgraduate Students (No.2022BKY039).

## References

[1] N. Alon, G. Gutin, Properly colored Hamilton cycles in edge-colored complete graphs, Random Struct. Algorithms 11(1997), 179-186.
[2] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Second edition, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2009.
[3] J. Bang-Jensen, G. Gutin, A. Yeo, Arc-disjoint strong spanning subdigraphs of semicomplete compositions, J. Graph Theory 95(2020), 267-289.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, Berlin, 2008.
[5] Y. Bai, Y. Manoussakis, On the number of vertex-disjoint cycles in digraphs, SIAM J. Discrete Math. 33(2019), 2444-2451.
[6] J.C. Bermond, C. Thomassen, Cycles in digraphs - A survey, J. Graph Theory 5(1981), 1-43.
[7] S. Chiba, T. Yamashita, Degree conditions for the existence of vertex-disjoint cycles and paths: a survey, Graphs Combin. 34(2018), 1-83.
[8] S. Fujita, R. Li, S. Zhang, Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs, J. Graph Theory 87(2018), 362-373.
[9] T. Gallai, Transitiv orientierbare graphen, Acta Math. Hungar. 18(1967), 25-66.
[10] R.J. Gould, Recent advances on the Hamiltonian problem: survey III, Graphs Combin. 30(2014), 1-46.
[11] J.W. Grossman, R. Häggkvist, Alternating cycles in edge-partitioned graphs, J. Combin. Theory Ser.B 34(1983), 77-81.
[12] T. Han, Y. Bai, S. Zhang, Edge-colored complete graphs containing no properly colored odd cycles, Graphs Combin. 37(2021), 1129-1138.
[13] T. Han, Y. Bai, S. Zhang, Properly colored cycles of different lengths in edge-colored complete graph, preprint 2020.
[14] M. Kano, X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs a survey, Graphs Combin. 24(2008), 237-263.
[15] D. Kühn, D. Osthus, A survey on Hamilton cycles in directed graphs, Eurpean J. Combin. 33(2012), 750-766.
[16] B. Li, B. Ning, C. Xu, S. Zhang, Rainbow triangles in edge-colored graphs, Eurpean J. Combin. 36(2014), 453-459.
[17] H. Li, G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, Eurpean J. Combin. 33 (2012), 1958-1964.
[18] R. Li, H. Broersma, S. Zhang, Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs, J. Graph Theory 94(2020), 476-493.
[19] A. Lo, Properly coloured Hamiltonian cycles in edge-coloured complete graphs, Combinatorica 36(2016), 471-492.
[20] G. Wang, T. Wang, G. Liu, Long properly colored cycles in edge colored complete graphs, Discrete Math. 324(2014), 56-61.
[21] A. Yeo, A note on alternating cycles in edge-colored graphs, J. Combin. Theory Ser.B 69(1997), 222-225.

