3-component domination numbers in graphs

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Abstract

Let k be a positive integer and let $G = (V(G), E(G))$ be a graph. A vertex set D is a k-component dominating set of G if every vertex outside D in G has a neighbor in D and every component of the subgraph $G[D]$ of G induced by D contains at least k vertices. The minimum cardinality of a k-component dominating set of G is the k-component domination number $\gamma_k(G)$ of G. It was conjectured that if G is a connected graph of order $n \geq k+1$, and minimum degree at least 2, then $\gamma_k(G) \leq \frac{2kn}{2k+3}$ except for a finite set of graphs. In this paper, we focus on the parameter $\gamma_3(G)$ of G. We first determine the exact values of 3-component domination numbers of paths and cycles. We then proceed to show that if G is a connected graph of order n with minimum degree at least 2 and maximum degree at most 3, then $\gamma_3(G) \leq \frac{2n}{3}$, unless G is one of seven special graphs. This result provides positive support for the conjecture and also generalizes a result by Alvarado et al. [Discrete Math., 2016].

AMS Math. Subj. Class. (2010): 05C69, 05C07

Keywords: Domination, 3-component domination.

1 Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph. The order of G is $n(G) := |V(G)|$. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of v in G is $d_G(v) = |N_G(v)|$, and the minimum and maximum degree in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If G is clear from the context, we omit writing it in the above expressions. Let $v \in V(G)$. We denote the graph obtained by deleting v from G by $G - v$. For a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$, $G[V \backslash S]$ is denoted by $G - S$, and the edge set of $G[S]$ is denoted by $E[S]$. The *girth* $g(G)$ of G is the length of the shortest cycle in G. We use $d(u, v)$ to denote the distance between u and v in G. We denote the *path*, *cycle*, and *complete graph* on n vertices by P_n, C_n and K_n, respectively. Remark that for positive integers t, let $[t] = \{1, 2, \ldots, t\}$. We follow [3] for notation and terminology not defined here.

Domination in graphs, together with its many variants, is a widely studied problem in graph theory [5,6,10,21]. The two most prominent domination parameters, the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph G, have been extensively studied and many rich results have been obtained. For detailed surveys on domination and total domination, we refer the reader to [11–13, 15]. In order to unify results and proofs that generalize statements obtained separately for $\gamma(G)$ and $\gamma_t(G)$, Alvarado et al. [2] introduced the important domination parameter $\gamma_k(G)$, which is defined as follows. A set D of vertices in G is *dominating* if every vertex not in D is adjacent to a vertex in D. Given a positive integer k, the set D is a *k-component dominating set* of G if it is dominating and every component of the subgraph $G[D]$ of G has order at least k. The minimum cardinality of a k-component dominating set of G is the *k-component domination number* $\gamma_k(G)$ of G. Clearly, $\gamma_1(G)$ coincides with the domination number of G, and $\gamma_2(G)$ coincides with the total domination number of G. A k-component dominating set with cardinality $\gamma_k(G)$ will be referred to as a $\gamma_k(G)$-set. If a graph G has a k-component dominating set, then we say G can be k-component dominated, and in this paper, when there is no ambiguity, we simplify say "G can be dominated" instead of "G can be k-component dominated". Similarly, we say a graph G is c-dominated if $\gamma_k(G) \leq c \cdot n(G)$, where $c \leq 1$ is a positive real number, and a set D is called c-dominating if it is a k-component dominating set and $|D| \leq c \cdot n(G)$.

Recall that an outerplanar graph is a planar graph that can be drawn in the plane with all vertices on the outer face. A maximal outerplanar graph is an outerplanar graph where no edge can be added and the graph remains outerplanar. For maximal outerplanar graphs G, Matheson and Tarjan [16] derived the bound $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$, and Dorfling et al. [8] showed that $\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$ unless G is isomorphic to one of two exceptional graphs of order 12. In view of the results mentioned in the previous sentence, Alvarado et al. [2] were able to obtain a common generalization using a unified proof. The result is stated in what follows.

$$\gamma_k(G) = \begin{cases} \lceil \frac{kn}{2k+1} \rceil, & \text{if } G \in \mathcal{H}_k, \\ \lfloor \frac{kn}{2k+1} \rfloor, & \text{otherwise.} \end{cases}$$

where $\mathcal{H}_k$ is a set of well-defined graphs of order $4k+4 \leq n(\mathcal{H}_k) \leq 4k^2 - 2k$.

Given a connected graph G of order n, recall that in [4,7,9,18,19], the bounds $\gamma(G) \leq \frac{n}{2}$ and $\gamma_t(G) \leq \frac{2n}{3}$ are derived, and graphs achieving these bounds are characterized. In order to unify the above results, Alvarado et al. [1] provided another graceful result. To state it we define the following construction. Let F be the graph with vertex set $\{u_1, \ldots, u_n\}$ and let k be an integer. The graph $F \circ P_k$ is formed by the disjoint union of F and n copies P_k, adding an edge between u_i and one end-vertex of i^{th} copy of P_k for every $i \in [n]$.

Theorem 1.1. *[1] If G is a connected graph of order n at least $k+1$, where k is a positive integer, then $\gamma_k(G) \leq \frac{kn}{k+1}$ with equality if and only if G either has order $k+1$, or is $C_{2(k+1)}$, or is $F \circ P_k$ for some connected graph F of order at least 2.*

Recall that if G is a connected graph with minimum degree at least 2, then $\gamma(G) \leq \frac{2n}{5}$, unless G is one of the seven exceptional graphs given in [17], and $\gamma_t(G) \leq \frac{4n}{7}$, unless G is one of the six exceptional cases given in [14, 20]. Based on these results, Alvarado et al. gave the following conjecture.

Conjecture 1.2. *[1] If G is a connected graph of order n at least $k+1$, and minimum degree at least 2, then $\gamma_k(G) \leq \frac{2kn}{2k+3}$ unless G belongs to a finite set of exceptional graphs.*

In order to support their conjecture, they also get the following result.

Theorem 1.3. *[1] If G is a graph of order n, minimum degree at least 2, maximum degree at most 3, and girth at least 29, then $\gamma_3(G) \leq \frac{8n}{11}$.*

Part of the proof of Theorem 1.3 relies on bounds for the 3-component domination number of paths and cycles. In this paper, we will follow their steps and focus on the 3-component domination numbers of graphs with maximum degree at most 3. We will first derive the exact values for the 3-component domination numbers for paths and cycles and then prove the following main result of this paper.

Theorem 1.4. *If G is a connected graph of order n, minimum degree at least 2, maximum degree at most 3, then $\gamma_3(G) \leq \frac{2n}{3}$ unless $G \in \mathcal{B} := \{C_3, C_4, K_4 - e, K_4, C_7, C_8, C_{13}\}$.*

This result improves the upper bound given in Theorem 1.3, and also verifies Conjecture 1.2 when $k = 3$ and $\Delta(G) \leq 3$.

We proceed as follows. In Section 2, we determine the exact values of $\gamma_3(P_n)$ and $\gamma_3(C_n)$, and derive upper bounds for graphs of small order. This will assist in establishing the base cases of the induction hypothesis. In Section 3, we present the proof of our main result by employing induction and providing a detailed case analysis. In addition, we close Section 3 by demonstrating that our derived bound is sharp.

2 The 3-component domination number for special graphs

In this section, we first give the exact values of 3-component domination numbers of paths and cycles, which are stated as follows.

Theorem 2.1. *Let P_n and C_n ($n \geq 3$) be a path and cycle, respectively. Then*

$$\gamma_3(P_n) = \gamma_3(C_n) = \begin{cases} \frac{3n}{5}, & n \equiv 0 (\text{mod } 5), \\ \frac{3n+2}{5}, & n \equiv 1 (\text{mod } 5), \\ \frac{3n+4}{5}, & n \equiv 2 (\text{mod } 5), \\ \frac{3n+6}{5}, & n \equiv 3 (\text{mod } 5), \\ \frac{3n+3}{5}, & n \equiv 4 (\text{mod } 5). \end{cases}$$

Proof. We will first prove that the result holds for P_n. Without loss of generality, assume that $P_n = v_1 v_2 \dots v_n$. Let D be a 3-component dominating set of P_n. Define the function $f : V(P_n) \to \{0, 1\}$ as $f(v_i) = 1$ if $v_i \in D$ and $f(v_i) = 0$ otherwise. The cardinality of D is $w(f) := \sum_{i \in [n]} f(v_i)$.

In order to give the upper bounds, we consider the following cases to construct a 3-component dominating set of P_n. Whenever $n \equiv 4 (\text{mod } 5)$, we define f by $f(v_{5i+2}) = f(v_{5i+3}) = f(v_{5i+4}) = 1$ for $0 \leq i \leq \lfloor \frac{n}{5} \rfloor$ and $f(v) = 0$ otherwise. It is easy to check that the set of vertices v with $f(v) = 1$ forms a 3-component dominating set of P_n, and that $\gamma_3(P_n) \leq \frac{3(n-4)}{5} + 3 = \frac{3n+3}{5}$. We now consider the other cases. Define f as $f(v_{5i+2}) = f(v_{5i+3}) = f(v_{5i+4}) = 1$ for $0 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$. If $n \equiv 0 (\text{mod } 5)$, then assign 0 to the remaining vertices; if $n \equiv 1 (\text{mod } 5)$, then let $f(v_{n-1}) = 1$; if $n \equiv 2 (\text{mod } 5)$, then define $f(v_{n-2}) = f(v_{n-1}) = 1$; lastly, if $n \equiv 3 (\text{mod } 5)$, then let $f(v_{n-2}) = f(v_{n-1}) = f(v_n) = 1$. All the unassigned vertices in cases of $n \equiv 1, 2, 3 (\text{mod } 5)$ are assigned 0. It is clear that the set of vertices assigned 1 forms a 3-component dominating set of P_n. Hence,

$$\gamma_3(P_n) \leq \begin{cases} \frac{3n}{5}, & n \equiv 0 (\bmod\ 5), \\ \frac{3n+2}{5}, & n \equiv 1 (\bmod\ 5), \\ \frac{3n+4}{5}, & n \equiv 2 (\bmod\ 5), \\ \frac{3n+6}{5}, & n \equiv 3 (\bmod\ 5), \\ \frac{3n+3}{5}, & n \equiv 4 (\bmod\ 5). \end{cases}$$

Now we demonstrate the inverse inequality. One can check that $\gamma_3(P_3) = \gamma_3(P_4) = 3$. For $n \geq 5$, let g be the function corresponding to a minimum 3-component dominating set. It is easy to see that $\sum_{t=i}^{i+4} g(v_t) \geq 3$ for any $i \in [n-4]$. Furthermore, we also note that $\sum_{i=1}^{3} g(v_i) \geq 2$ (similarly $\sum_{i=n-2}^{n} g(v_i) \geq 2$), and $\sum_{i=1}^{4} g(v_i) \geq 3$ (similarly $\sum_{i=n-3}^{n} g(v_i) \geq 3$) hold. Thus we get that $w(g) = \sum_{i \in [n]} g(v_i) \geq \frac{3n}{5}$ for $n \equiv 0 (\bmod\ 5)$; for $n \equiv 1 (\bmod\ 5)$, $w(g) = \sum_{i=1}^{3} g(v_i) + \sum_{i \in [4,n-3]} g(v_i) + \sum_{i=n-2}^{n} g(v_i) \geq 4 + \frac{3(n-6)}{5} = \frac{3n+2}{5}$; for $n \equiv 2 (\bmod\ 5)$, $w(g) = \sum_{i=1}^{3} g(v_i) + \sum_{i \in [4,n-4]} g(v_i) + \sum_{i=n-3}^{n} g(v_i) \geq 5 + \frac{3(n-7)}{5} = \frac{3n+4}{5}$; for $n \equiv 3 (\bmod\ 5)$, $w(g) = \sum_{i=1}^{4} g(v_i) + \sum_{i \in [5,n-4]} g(v_i) + \sum_{i=n-3}^{n} g(v_i) \geq 6 + \frac{3(n-8)}{5} = \frac{3n+6}{5}$; for $n \equiv 4 (\bmod\ 5)$, $w(g) = \sum_{i=1}^{4} g(v_i) + \sum_{i \in [5,n]} g(v_i) \geq 3 + \frac{3(n-4)}{5} = \frac{3n+3}{5}$. Hence, the result holds for P_n.

In order to complete the proof of this theorem, we will show that $\gamma_3(C_n) = \gamma_3(P_n)$, where P_n and C_n have the common vertex set V. Without loss of generality, we assume that $V = v_1 v_2 \ldots v_n$. Clearly $\gamma_3(C_3) = \gamma_3(P_3) = 3$ and $\gamma_3(C_n) \leq \gamma_3(P_n)$ always hold. Now we claim that $\gamma_3(C_n) \geq \gamma_3(P_n)$ for $n \geq 4$. Let D be a $\gamma_3(C_n)$-set of C_n, and let g be the corresponding function $g : V \to \{0,1\}$, $g(v_i) = 1$ if $v_i \in D$ and $g(v_i) = 0$ otherwise. Then the cardinality of D is $\gamma_3(C_n) = w(g) = \sum_{i \in [n]} g(v_i)$. Note that there must exist a vertex assigned 0 under g. Now without loss of generality, we assume first that $g(v_i) = 0$, and one of its neighbors is also assigned 0 under g, say $g(v_{i+1}) = 0$ and $g(v_{i-1}) = 1$, where the subscript is modulo n. We form P_n by removing from C_n the edge $v_i v_{i+1}$. For this P_n define a new function $h : V \to \{0,1\}$ as $h(v_i) = g(v_i)$ for $i \in [n]$. Clearly, the vertex set D is a 3-component dominating set of P_n. Thus $\gamma_3(C_n) = w(g) = w(h) \geq \gamma_3(P_n)$. We now may assume that $g(v_i) = 0$ and $g(v_{i-1}) = g(v_{i+1}) = 1$. Removing the edge $v_i v_{i+1}$, the resulted graph is also a P_n. For this P_n define $h : V \to \{0,1\}$ as $h(v_i) = g(v_i)$ for $i \in [n]$, and then we also have $\gamma_3(C_n) = w(g) = w(h) \geq \gamma_3(P_n)$. Therefore, the proof of the result is complete. $\square$

Lemma 2.2. $\left\lceil \frac{kn}{k+2} \right\rceil \leq \gamma_k(P_n) = \gamma_k(C_n) \leq \left\lceil \frac{kn}{k+2} \right\rceil + 1.$

Proof. $\gamma_3(P_n) = \gamma_3(C_n)$ is already shown in Theorem 2.1, one can follow the same idea to verify that $\gamma_k(P_n) = \gamma_k(C_n)$. Let D be a $\gamma_k(P_n)$-set and K_1 be a component of $G[D]$ with order at least k, then the total number of vertices dominated by K_1, including $V(K_1)$, is at most $k+2$. Assume that there are m components in $G[D]$, and each component is of order s_i over $1 \leq i \leq m$, then the maximum number of vertices that D can dominate is $\sum_{i=1}^{m}(s_i + 2) = |D| + 2m$. From the definition of k-component domination, we know that $s_i \geq k$ for all $i \in [m]$. Thus $m \leq \lfloor \frac{|D|}{k} \rfloor$. Since D is also a dominating set, we have $|D| + 2\lfloor \frac{|D|}{k} \rfloor \geq |D| + 2m \geq n$, that is $\gamma_k(P_n) = |D| \geq \lceil \frac{kn}{k+2} \rceil$.

The upper bound is proved by construction. Let f be a function defined similarly to the one in the above theorem. Now we define f as follows: If $n \equiv i \bmod (k+2)$, where $0 \leq i \leq k+1$, then $f\left(v_{(k+2)j+2}\right) = f\left(v_{(k+2)j+3}\right) = \cdots = f\left(v_{(k+2)j+k+1}\right) = 1$ for

$0 \leq j \leq \left\lfloor \frac{n}{k+2} \right\rfloor - 1$, and $f(v_{n-i}) = f(v_{n-i+1}) = \cdots = f(v_{n-1}) = 1$. Thus, $\sum_{v \in V} f(v) = (\left\lfloor \frac{n}{k+2} \right\rfloor - 1)k + k + i = \frac{nk+2i}{k+2} < \frac{kn}{k+2} + 2 \leq \lceil \frac{kn}{k+2} \rceil + 2$. Since $\frac{nk+2i}{k+2}$ is an integer, and $\frac{nk+2i}{k+2} < \lceil \frac{kn}{k+2} \rceil + 2$, we have that the desired result is obtained. $\square$

We give the following result by utilizing some simple calculations from Theorem 2.1.

Corollary 2.3. *(1) Let G be a path of order n. Then $\gamma_3(G) \leq \frac{2n}{3}$ unless $G \in \{P_3, P_4, P_7, P_8, P_{13}\}$.*

(2) Let G be a cycle of order n. Then $\gamma_3(G) \leq \frac{2n}{3}$ unless $G \in \{C_3, C_4, C_7, C_8, C_{13}\}$.

Now we examine the 3-component domination number of the graph G with a small number of vertices. Before that, let us make some observations.

Observation 2.4. *(1) Let G be a connected graph of order n with $\delta(G) \geq 2$ and $\Delta(G) \leq 3$. If G has exactly two vertices of degree 3, then G has a spanning path P_n.*

(2) Let C_n be a cycle. Then for any vertex $v \in V(C_n)$, there exists a $\gamma_3(C_n)$-set of C_n which contains v.

(3) Let G be a connected graph of order $n(G) \leq 6$ with $\Delta(G) \leq 3$ and $\delta(G) \geq 2$ (or there is exactly one vertex with degree 1). Then for any vertex $v \in V(G)$, there exists a $\frac{2}{3}$-dominating set D which contains v.

(4) Let G be a 7-vertex connected graph with $\delta(G) = 1$ and $\Delta(G) \leq 3$. Then $\gamma_3(G) \leq 4$ except when $G = P_7$ with $\gamma_3(P_7) = 5$.

Proof. (1) and (2) are clearly obtained.

(3) The statement clearly holds when G is a cycle. Now we assume that G is a connected graph with $\delta(G) \geq 2$ (or there is at most one vertex with degree 1) which contains a vertex of degree 3, say v_1. For $n(G) = 5$, suppose that $N(v_1) = \{v_2, v_3, v_4\}$ and the remaining vertex v_5. Since G is connected, without loss of generality, we assume $v_2 \in N(v_5)$. Thus v_1, v_2 and any other vertex can be added to get a 3-component dominating set of size 3, the result holds. For $n(G) = 6$, let $N(v_1) = \{v_2, v_3, v_4\}$ and the remaining vertices v_5, v_6. If v_5 is not adjacent to v_6, then v_5 and v_6 are at least adjacent to a vertex in $N(v_1)$ by connectivity of G. If there is a vertex in $N(v_1)$ such that $N(v_5) \cap N(v_6) \neq \emptyset$, say $v_3 \in N(v_5) \cap N(v_6)$, then v_1, v_3 and any other vertex can be added to get a 3-component dominating set of size 3. The result holds. Now we assume that v_5 and v_6 are adjacent to distinct vertices in $N(v_1)$, say v_2 and v_3. Then v_1, v_2, v_3 forms a 3-component dominating set of size 3. If v_5 is adjacent to v_6, then assume that there exists an edge between v_5 and $N(v_1)$, say v_2v_5. If follows v_1, v_2, v_5 forms a 3-component dominating set of size 3. Combining the above cases, we get the result.

(4) From Theorem 2.1, we know that $\gamma_3(P_7) = 5$. Without loss of generality, we assume that G contains a vertex v of degree 3. Let T be a BFS-spanning tree of G with v as the root vertex. Let $vx_1 \ldots x_r$, $vy_1 \ldots y_s$ and $vz_1 \ldots z_t$ be the three paths that start in v in T, where $r, s, t \geq 1$. Furthermore, if $r, s, t \geq 2$, then we choose $x_1, \ldots, x_{r-1}, y_1, \ldots, y_{s-1}, z_1, \ldots, z_{t-1}$ and v to form a 3-component dominating set in G; if there is a value of 1 among r, s and t, then we do not choose any vertex on that branch except for vertex v. Thus the number of the chosen vertices is $n(G) - 3$, whence $\gamma_3(G) \leq 4$. $\square$

Next we examine the 3-component domination number for graphs of small order. This will establish the base cases for the inductive hypothesis.

Lemma 2.5. *Let G be a connected graph of order $n(G) \leq 13$ with minimum degree at least 2 and maximum degree at most 3. Then $\gamma_3(G) \leq \frac{2n}{3}$ unless $G \in \{C_3, C_4, K_4 - e, K_4, C_7, C_8, C_{13}\}$.*

Proof. From Corollary 2.3, we know that the statement is true when G is a cycle, and it is easy to verify that the result also holds when $n(G) = 4$. Thus we only consider the graphs with order $5 \leq n(G) \leq 13$ containing at least one vertex of degree 3. Furthermore, since $\gamma_3(G)$ is an integer, we get that $\gamma_3(G) \leq \frac{2n}{3}$ holds for $n(G) = 5, 6, 8, 9, 12$ by Theorem 1.1 (Note that when $n(G) = 8$ or 12, the bound in Theorem 1.1 is only achieved if G is C_8). Thus we only consider the remaining cases $n(G) = 7, 10, 11$ or 13.

Case 1. $n(G) = 7$.

In this case, we want to show $\gamma_3(G) \leq 4$. Suppose that v_1 is a vertex of degree 3 and $N(v_1) = \{v_2, v_3, v_4\}$. Let $H = \{u_1, u_2, u_3\}$ be the remaining vertices of G. If $|E(G[H])| \geq 2$, then, without loss of generality, we assume that u_2 is adjacent to u_1 and u_3. Moreover, there exists an edge $e = uv$ where $u \in H$ and $v \in N(v_1)$ by the connectivity of G. Now we choose $\{v_1, v, u, u_2\}$ (note that u may be u_2) as a 3-component dominating set of G. Hence $\gamma_3(G) \leq 4$. If $|E(G[H])| \leq 1$, then every vertex in H is adjacent to some vertex in $N(v_1)$ because $\delta(G) \geq 2$. And thus we choose $\{v_1, v_2, v_3, v_4\}$ as a 3-component dominating set of G, so $\gamma_3(G) \leq 4$.

Up to now, the known upper bounds on the 3-component domination number of a connected graph G ($n(G) \leq 9$) with $\Delta(G) \leq 3$ and $\delta(G) = 1$ or $\delta(G) \geq 2$ are summarized in the following table.

Table 1: $\gamma_3(G)$ of a connected graph G with $\Delta(G) \leq 3$ when $n(G) \leq 9$.

$\delta(G)$ \ $n(G)$	≤ 5	6	7	8	9
1	$\gamma_3(G) = 3$	$\gamma_3(G) \leq 4$	$\gamma_3(G) \leq 4$ expect $\gamma_3(P_7) = 5$	$\gamma_3(G) \leq 5$ except $\gamma_3(P_8) = 6$	$\gamma_3(G) \leq 6$
≥ 2	$\gamma_3(G) = 3$	$\gamma_3(G) \leq 4$	$\gamma_3(G) \leq 4$ except $\gamma_3(C_7) = 5$	$\gamma_3(G) \leq 5$ except $\gamma_3(C_8) = 6$	$\gamma_3(G) \leq 6$

We next show that when $n(G) = 10, 11$ or 13, the 3-component domination number of G is at most $6, 7$ or 8, respectively. First, assume that G is a graph with minimum number of edges satisfying $\delta(G) \geq 2$, $\Delta(G) \leq 3$ and G is connected. This is because removing edges does not decrease the 3-component domination number of G. Moreover, according to the edge minimality of G, we will not consider the occurrence of the configuration $K_4 - e$ later, and only consider C_4 instead.

Case 2. $n(G) \in \{10, 11\}$.

We say a cycle is a *pendant cycle* if the cycle is attached by a vertex of degree 3 in G (see Figure 1). In particular, if the cycle is C_3, then we call it a *pendant triangle.* Now we give the following claims regarding the graph with $n(G) \in \{10, 11\}$.

Claim 2.1. G contains no pendant triangles.

Proof of Claim 2.1. Suppose, to the contrary, that G contains a pendant triangle C_3. And let u be the vertex of degree 3 on C_3 and v be the neighbor of u that is not on C_3. Let $G' = G - C_3$. Note $n(G') = 7$ for $n(G) = 10$ (or $n(G') = 8$ for $n(G) = 11$). By Table 1, we know that $\gamma_3(G') \leq 4$, except when G' is P_7 or C_7, in which case $\gamma_3(G') = 5$ (or

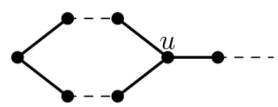

Figure 1: Pendant cycle attached by the vertex u.

$\gamma_3(G') \leq 5$, except when G' is P_8 or C_8, in which case $\gamma_3(G') = 6$). First, it is clear that G' is not P_7 (or P_8), since the only possible vertex of degree 1 in G' is v. Furthermore, if G' is C_7 (or C_8), then there is a spanning path in G by Observation 2.4 (1). Since G has a spanning path P_n, we have that $\gamma_3(G) \leq \gamma_3(P_n) \leq \frac{2n}{3}$, where $n = 10$ or $n = 11$. Thus $\gamma_3(G') \leq 4$ when $n(G) = 10$ (or $\gamma_3(G') \leq 5$ when $n(G) = 11$). We then only need to add at most two vertices u and v in order to dominate G, regardless of whether v belongs to the $\gamma_3(G')$-set. Thus, $\gamma_3(G) \leq 6$ for $n(G) = 10$ (or $\gamma_3(G) \leq 7$ for $n(G) = 11$). □

Claim 2.2. G contains no pendant 4-cycles.

Proof of Claim 2.2. Suppose, to the contrary, that G contains a pendant 4-cycle $C_4 := uu_1u_2u_3$. Let u be the vertex of degree 3 in C_4, and v be the neighbor of u that is not on C_4. Let $G' = G - C_4$. Note that $n(G') = 6$ for $n(G) = 10$ (or $n(G') = 7$ for $n(G) = 11$). We know from Table 1 that $\gamma_3(G') \leq 4$ (or $\gamma_3(G') \leq 4$, except when G' is C_7 or P_7, in which case $\gamma_3(G') = 5$). Further for $n(G) = 10$, we have, by Observation 2.4 (3), that G' has a 3-component dominating set of size at most 4 containing v, and we can add two more vertices u, u_1 from C_4 to form a 3-component dominating set of G, thus $\gamma_3(G) \leq 6$, and so we are done. For $n(G) = 11$, firstly, G' is clearly not a P_7 because the only possible vertex of degree 1 in G' is v. If G' is a C_7, then Observation 2.4 (1) implies that there exists a spanning path P_{11} in G, and so $\gamma_3(G) \leq \gamma_3(P_{11}) \leq 7$, whence we are done. Thus, recall that Table 1 gives $\gamma_3(G') \leq 4$. We can add at most three vertices u, v, u_1 to the $\gamma_3(G')$-set to form a 3-component dominating set of G. It follows that $\gamma_3(G) \leq 7$, the result holds true. □

Note that according to the assumption of G having a minimal number of edges, if there is an edge between two vertices of degree 3, the edge must be a bridge. Therefore, we give the following claim.

Claim 2.3. No two degree 3 vertices of G are joined by a bridge.

Proof of Claim 2.3. Let $e = uv$ be the bridge which joins two vertices u, v of degree 3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components formed by removing e from G. Considering Claim 2.1 and Claim 2.2, we know that $(n(G_1), n(G_2))$ is $(5, 5)$ and $(5, 6)$ for $n(G) = 10$ and $n(G) = 11$, respectively. Thus, we get $\gamma_3(G) \leq \frac{2n}{3}$ from Table 1. □

Claim 2.3 and the edge minimality of G imply that G contains no two adjacent vertices of degree 3. If we let V_i be the set of vertices of degree i in G, where $i \in \{1, 2, 3\}$, then we can make the following assumption.

$$V_3 \text{ is an independent set in } G. \tag{\textbf{I}}$$

Set $n_i(G) = |V_i(G)|$, where $i \in \{1, 2, 3\}$. For any $H \subset G$, we get the following result.

Claim 2.4. $n_3(H) \leq \frac{2n(H) - n_1(H)}{5}$.

Proof of Claim 2.4. Considering the Handshaking lemma $\sum_{v \in H} d(v) = 2|E(H)|$, its left side is equal to $n_1(H) + 2n_2(H) + 3n_3(H)$, and noting that $n(H) = n_1(H) + n_2(H) + n_3(H)$, we have the left side as $2n(H) - n_1(H) + n_3(H)$. Furthermore, considering the right side, it must be at least $6n_3(H)$ due to the assumption (**I**). Hence, we have $n_3(H) \leq \frac{2n(H)-n_1(H)}{5}$. $\square$

In particular, we have $n_3(G) \leq \frac{2n(G)}{5}$ since $n_1(G) = 0$. Remark that for $n(G) \leq 11$ (which is the current position we are in), we have $n_3(G) \leq 4$ and in fact, $n_3(G)$ can only take the values of 2 or 4, since the number of vertices of odd degree is even.

Let $C_\ell(t)$ be a cycle of length ℓ containing t vertices of degree 3. Let u_i be a vertex of degree 3 on the cycle $C_\ell(t)$, and let v_i be the neighbor of u_i in $G - C_\ell(t)$, where $1 \leq i \leq t$. Note that the vertices v_i may be the same. The example $C_6(2)$ is illustrated in the Figure 2. We know that $\{u_i | i \in [t]\}$ is an independent set in G and $d(v_i) = 2$ for each $i \in [t]$.

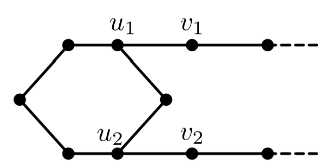

Figure 2: The example $C_6(2)$.

We will discuss the cases based on the occurrence of $C_\ell(t)$ in G, where $\ell \geq 3$ and $1 \leq t \leq 4$ by considering the assumption (**I**).

If $t = 1$, then $\ell \geq 5$ for $C_\ell(1)$ by Claim 2.1 and Claim 2.2. Let $G' = G - C_\ell(1)$, then we have $n(G') \leq 5$ for $n(G) = 10$ and $n(G') \leq 6$ for $n(G) = 11$. Moreover, since there is only one vertex v_i of degree 1 in G', applying Claim 2.4 to graph G' yields $n_3(G - C_\ell(1)) < 3$, and hence $n_3(G) \leq 3$. Since G has at least one vertex of degree 3 and an even number of vertices of odd degree, we have that $n_3(G) = 2$. Thus, we know from Observation (1) that there is a spanning path in G, further, $\gamma_3(G) \leq \gamma_3(P_{10}) \leq 6$ for $n(G) = 10$ and $\gamma_3(G) \leq \gamma_3(P_{11}) \leq 7$ for $n(G) = 11$, and so we are done.

If $t = 2$, then $\ell \geq 4$ for $C_\ell(2)$ by (**I**). If $\ell = 4$, we first observe that v_1 and v_2 are distinct vertices, and v_1 is not adjacent to v_2 because (**I**) and $n(G) \in \{10, 11\}$. Delete $C_4(2)$ and two vertices v_1, v_2, and let $G' = G - C_4(2) - \{v_1, v_2\}$. It follows that $n(G') = 4$ when $n(G) = 10$ and $n(G') = 5$ for $n(G) = 11$. Moreover, G' is a connected graph, otherwise, it would contradict assumption (**I**), or a pendant triangle would appear in G, which contradicts Claim 2.1. Now $\gamma_3(G') \leq 3$ and we add 3 vertices u_1, u_2 and any other vertex on the cycle $C_4(2)$ to dominate G. Thus we have that $\gamma_3(G) \leq 6$ for $n(G)$ is 10 or 11, and so we are done. If $\ell \geq 5$, then we claim that $n_3(G) = 2$ always holds in this case. When $v_1 = v_2$, then $\ell = 9$ for $n(G) = 10$ and $\ell = 10$ for $n(G) = 11$, whence $n_3(G) = 2$. When $v_1 \neq v_2$, let $G' = G - C_\ell(2) - \{v_1, v_2\}$. Note that $n(G') \leq 4$. Thus by Claim 2.4, $n_3(G') < 2$. Furthermore $n_3(G) \leq 3$, and so $n_3(G) = 2$. Thus there is a spanning path P_n by Observation 2.4 (1), further $\gamma_3(G) \leq \gamma_3(P_n) \leq \frac{2n}{3}$ holds true for $n(G) \in \{10, 11\}$.

If $t = 3$, then $6 \leq \ell \leq 7$ by considering (**I**). Specifically, $\ell = 6$ for $n(G) = 10$ and $\ell \in \{6, 7\}$ for $n(G) = 11$. Firstly, we consider the cases $\ell = 6$ for $n(G) = 10$ and $\ell = 7$ for $n(G) = 11$. Remark that vertices v_i $(i \in [3])$ are distinct since assumption (**I**). Let w be the single vertex of $G - C_\ell(3) - \{v_1, v_2, v_3\}$. The only possibility in this case is that $N(w) = \{v_1, v_2, v_3\}$. Now we can choose $\{u_1, u_2, u_3, v_1\}$ and any other two vertices on $C_\ell(3)$ to form a 3-component dominating set. Thus $\gamma_3(G) \leq 6$ for $n(G) \in \{10, 11\}$. Next, we consider the case $\ell = 6$ for $n(G) = 11$. Note that vertices v_i $(i \in [3])$ are distinct, which can be inferred by considering assumption (**I**) and the absence of pendant triangles in G. Let w_1 and w_2 be the two vertices of $G - C_6(3) - \{v_1, v_2, v_3\}$. Note that w_1 must be adjacent to w_2, and without loss of generality, we assume $N(w_1) = \{v_1, v_2, w_2\}$. Thus $\{u_1.u_2, u_3, v_1, w_1\}$ and any other two vertices on $C_6(3)$ form a 3-component dominating set of G. Thus $\gamma_3(G) \leq 7$, and so we are done.

If $t = 4$, then $\ell = 8$ for $n = 10$ and $\ell \in \{8, 9\}$ for $n = 11$, as illustrated in Figure 3. In the third subfigure, u_3 could be located at either of the two positions marked with P. We can easily verify that $\gamma_3(G) \leq 6$ holds true for all of these cases.

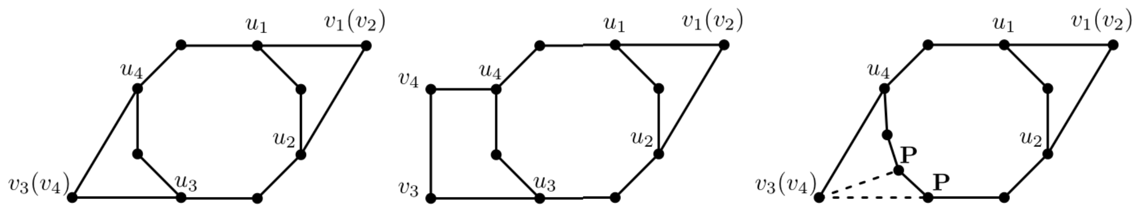

Figure 3: The cases when $t = 4$ for $n(G) \in \{10, 11\}$.

Case 3. $n(G) = 13$.

In this case, we want to show $\gamma_3(G) \leq 8$. Now we give the following claims regarding the graph with $n(G) = 13$.

Claim 2.5. G contains no pendant triangles.

Proof of Claim 2.5. Suppose that G contains a pendant triangle C_3. Let u be the vertex of degree 3 in C_3 and let v be the neighbor of u not in C_3. If $d(v) = 3$, then let $G' = G - C_3$. We know that G' is a 10-vertex graph with $\delta(G') \geq 2$, and that from Case 2, $\gamma_3(G') \leq 6$. We then only need to add at most two vertices u and v in order to dominate G, regardless of whether v belongs to the $\gamma_3(G')$-set. Thus, $\gamma_3(G) \leq 8$.

Now $d(v) = 2$. Let w be another neighbor of v different from u. If $d(w) = 2$, then further let $G' = G - C_3 - \{v, w\}$. Note that $n(G') = 8$, and G' is clearly not a P_8, since x, the neighbor of w different from v, is only one possible vertex of degree 1 in G'. Moreover, if G' is a C_8, then we can check that $\gamma_3(G) \leq 8$. Thus, by Table 1, $\gamma_3(G') \leq 5$ holds. Then we add three vertices u, v, w to dominate G, whence $\gamma_3(G) \leq 8$. We may assume that $d(w) = 3$. Let the other two neighbors of w be x_1 and x_2. By the edge minimality of G and the fact that $n(G) = 13$, we have that x_1 and x_2 are not adjacent. Delete $C_3 + \{v, w\}$ and join x_1, x_2, and call the resulting graph G'. Note $n(G') = 8$ and $\delta(G') \geq 2$. If G' is C_8, then we can check that $\gamma_3(G) \leq 7$. Thus, by Table 1, $\gamma_3(G') \leq 5$. We add three vertices u, v, w to dominate G, thus $\gamma_3(G) \leq 8$. □

Claim 2.6. G contains no pendant 4-cycles.

Proof of Claim 2.6. If not, suppose that G contains a pendant 4-cycle C_4. Let u be the vertex of degree 3 in C_4, and v be the neighbor of u not in C_4. If $d(v) = 2$, then further let $G' = G - C_4 - \{v\}$. Note that $n(G') = 8$, and we can easily infer that G' is not a P_8. Moreover, if G' is a C_8, then we can check that $\gamma_3(G) \leq 8$. Thus, by Table 1, $\gamma_3(G') \leq 5$. Then we add three vertices u, v and another neighbor of u to dominate G, whence $\gamma_3(G) \leq 8$. If $d(v) = 3$, then further let the other two neighbors of v be w_1

and w_2. Firstly, w_1 and w_2 are not adjacent, as this is due to considering the fact that $n(G) = 13$ and G is edge minimal. Delete $C_4 + \{v\}$ and join w_1, w_2. Call the resulting graph G'. Note $n(G') = 8$ and $\delta(G') \geq 2$. If G' is C_8, then we can check that $\gamma_3(G) \leq 8$. Thus, by Table 1, $\gamma_3(G') \leq 5$. Then we add three vertices u, v and another neighbor of u to dominate G. Thus, $\gamma_3(G) \leq 8$. □

Recall that by the edge minimality of G, if there is an edge between two vertices of degree 3, the edge must be a bridge. Therefore, we give the following claim.

Claim 2.7. If G contains two vertices of degree 3 which are joined by a bridge, then $\gamma_3(G) \leq 8$.

Proof of Claim 2.7. Let u, v be two adjacent vertices of degree 3 and uv be a bridge whose removal yields components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Also, assume that u is contained in G_1. Considering Claim 2.5 and Claim 2.6, the two possibilities for the pair $(n(G_1), n(G_2))$ are $(5, 8)$ and $(6, 7)$ by symmetry. Suppose $(n(G_1), n(G_2)) = (5, 8)$, and G_2 is a C_8. Firstly, for G_1, there exists a 3-component dominating set of size 3 containing u by Observation 2.4 (3). Additionally, we notice that $G_2 - v$ can be dominated by 5 vertices. Hence, $\gamma_3(G) \leq 8$ holds. Thus, we have $\gamma_3(G_2) \leq 5$ from Table 1, further $\gamma_3(G) \leq 5 + 3 = 8$, and so we are done. Assume now that $(n(G_1), n(G_2)) = (6, 7)$, and G_2 is a C_7. Firstly, for G_1, there exists a 3-component dominating set of size 4 containing u. Also, the graph $G_2 - v$ can be dominated by only 4 vertices. Hence, $\gamma_3(G) \leq 8$. Thus we have $\gamma_3(G_2) \leq 4$ from Table 1, further $\gamma_3(G) \leq 8$, and so we are done. □

Now we can also make assumption (**I**) by Claim 2.7.

Define $C_\ell(t)$ as in Case 2. Let u_i be the i'th vertex of degree 3 contained on the cycle $C_\ell(t)$, and let the neighbor of u_i in $G - C_\ell(t)$ be v_i, where $1 \leq i \leq t$. Note that the vertices v_i may be the same. Moreover, we know that the set of vertices consisting of all u_i is independent, and for each i, $d(v_i) = 2$. We will discuss the cases based on the occurrence of $C_\ell(t)$ in G, where $3 \leq \ell \leq 12$ and $1 \leq t \leq 4$ by considering the assumption (**I**).

Case 3.1. $\ell = 11$ or $\ell = 12$.

We can find that for $\ell \in \{11, 12\}$, there is a 3-component dominating set of size 8 that contains any two nonadjacent vertices on $C_\ell(t)$. This can be inferred by observing the structure of $\gamma_3(C_n)$-set in any cycle C_n, and the result that a cycle of order 11 or 12 has a $\frac{2}{3}$-dominating set.

Case 3.2. $\ell = 10$.

Firstly, we have $1 \leq t \leq 5$, and considering assumption (**I**), V_3 is an independent set. We further deduce that $t \neq 1, 3, 5$ in this case. If $t = 2$, then v_1 and v_2 are distinct. Let w be the single vertex of $G - C_{10}(2) - \{v_1, v_2\}$. Moreover, the only possibility is $N(w) = \{v_1, v_2\}$. Now, for $C_{10}(2)$, there is a 3-component dominating set of size 7 containing the nonadjacent two vertices of degree 3, u_1 and u_2. Furthermore, add one more vertex v_1 in order to dominate G. Thus $\gamma_3(G) \leq 8$, and so we are done. If $t = 4$, then the only possibility is that among three 2-degree vertices, say v_1, v_2, v_3, outside C_ℓ, two of them, say v_1 and v_2, are adjacent to the distinct 3-degree vertices u_1 and u_2 of C_{10}, respectively. The remaining 2-degree v_3 is adjacent to u_3 and u_4. Now we can easily check that $\gamma_3(G) \leq 6$, whence we are done.

Case 3.3. $\ell = 9$.

In this case, we have $1 \leq t \leq 4$. If $t = 1$, then $|G - (C_9(1) + v_1)| = 3$, and further a pendant triangle appears, which contradicts Claim 2.5.

If $t = 2$, we first note that $v_1 \neq v_2$ by considering $n(G) = 13$ and assumption (**I**). Let w_1 and w_2 be the vertices of $G - C_9(2) - \{v_1, v_2\}$. Note $d(w_1) = d(w_2) = 2$, and without loss of generality we assume that v_i is adjacent to w_i for each $i \in [2]$. Now, for $C_9(2)$, there is a 3-component dominating set of size 6 containing u_1 and u_2. Furthermore, we only need to add two more vertices v_1 and v_2 in order to dominate G. Thus $\gamma_3(G) \leq 8$.

If $t = 3$, we note, once again, that vertices v_i for $i \in [3]$ are distinct. Let w be the single vertex in $G - C_9(3) + \{v_i\}_{i=1}^3$. The only possibility is $N(w) = \{v_i\}_{i=1}^3$. Now, for $C_9(3)$, there is a 3-component dominating set of size 7 containing u_i for all $i \in [3]$. Furthermore, we add one more vertex v_1 in order to dominate G. Thus $\gamma_3(G) \leq 8$.

If $t = 4$, we can find a 3-component dominating set of size 6 containing each u_i. Furthermore, if all v_i adjacent to u_i are distinct, where $i \in [4]$, then $\gamma_3(G) \leq 6$, and so we are done. If there are some vertices v_i that are the same, say $v_1 = v_2$, then v_3 must be different from v_4. Moreover, there exists a single vertex w, and without loss generality, we say w and v_3 are adjacent. Now, we add one more vertex v_3 in order to dominate G. Thus $\gamma_3(G) \leq 7$.

Case 3.4. $\ell = 8$.

In this case, we have $1 \leq t \leq 4$. If $t = 1$, then $|G - (C_8(1) + v_1)| = 4$, and further a pendant triangle or a pendant 4-cycle appears, which contradicts Claim 2.5 or Claim 2.6.

Assume that $t = 2$, we first note that $v_1 \neq v_2$ by considering $n(G) = 13$ and assumption (**I**). Let w_1, w_2 and w_3 be the vertices of $G - C_8(2) - \{v_1, v_2\}$. The graph $G - C_8(2)$ has no isolated vertices. If $G - C_8(2)$ has no 3-degree vertex then there exists a $v_1 - v_2$ path in $G - C_8(2)$ on 5 vertices. Hence, G has a cycle on more than 8 vertices and these cases have been considered. We may assume that $G - C_8(2)$ has a 3-degree vertex, say w_1. The vertex w_1 needs to be adjacent to either v_1 or v_2, say v_1. To avoid a contradiction with assumption (**I**), G will have a vertex of degree 1, which is impossible.

Assume that $t = 3$. Consider the case where, without loss of generality, $v_1 = v_2$. Let w_1, w_2, w_3 be the vertices in $G - C_8(3) - \{v_1, v_3\}$. Then w_1, w_2, w_3 forms a pendant triangle, which contradicts Claim 2.5. Thus the vertices v_i are distinct, and let the remaining vertices be w_1 and w_2. Then we can assume $\{v_1, w_2\} \subset N(w_1)$. Now, for $C_8(3)$, there is a 3-component dominating set of size 6 containing u_i ($i \in [3]$), further we add two more vertices $\{v_1, w_1\}$ in order to dominate G. Thus, $\gamma_3(G) \leq 8$.

If $t = 4$ and $v_1 = v_2$, then there exists a $v_3 - v_4$ path in $G - C_8(4)$ on 4 vertices, as can be seen in the first subfigure in Figure 4. Hence, G has a cycle on more than 8 vertices and the case has been considered. Thus v_i are distinct from each other, as illustrated in the second subfigure in Figure 4. We can easily check that $\gamma_3(G) \leq 7$.

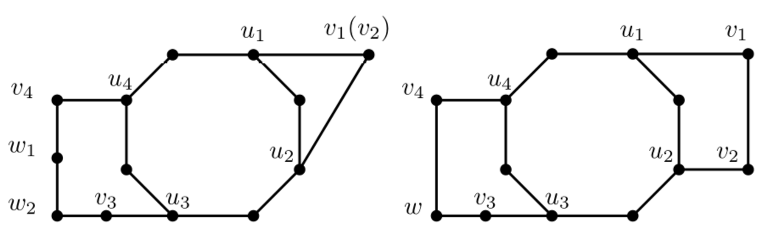

Figure 4: The cases when $t = 4$ for $n(G) = 13$.

Case 3.5. $\ell = 7$.

In this case, we have $1 \le t \le 3$. Let $G' = G - (C_7(t) + \{v_i\}_{i=1}^t)$. If $t = 2$, then v_i is clearly distinct, since $n(G) = 13$ and assumption (**I**). If $t = 3$, assume, without loss of generality, $v_1 = v_2$. To avoid contradiction with (**I**), a pendant triangle or a pendant 4-cycle appears, which also contradicts Claim 2.5 or Claim 2.6. Thus, v_i are distinct from each other. Now we know that $3 \le n(G') \le 5$. To avoid a pendant triangle and satisfy assumption (**I**), we need G' to be connected. Now, for G', only 3 vertices are needed to dominate G', further, $C_7(t)$ has a 3-component dominating set of size 5 containing all u_i ($i \in [t]$). Thus, $\gamma_3(G) \le 8$.

Case 3.6. $\ell = 6$.

In this case, we have $1 \le t \le 3$. Let $G' = G - (C_6(t) + \{v_i\}_{i=1}^t)$. Moreover, v_i ($i \in [t]$) are clearly distinct when $t = 2$, since $n(G) = 13$ and (**I**). Now, if $t \in [2]$, then we know that $n(G') \le 6$, and to avoid a pendant triangle and satisfy assumption (**I**), we need G' to be connected. Thus, G' can be dominated by at most 4 vertices. Also $C_6(t)$ has a 3-component dominating set of size 4 containing u_i ($i \in [t]$). Whence $\gamma_3(G) \le 8$. If $t = 3$, then $C_6(3)$ has a 3-component dominating set of size 5 containing u_i ($i \in [3]$). Furthermore, $n(G') = 4$ or 5, depending on whether v_i are distinct or not. We note, once again, that to avoid a pendant triangle and satisfy assumption (**I**) and the condition of degree, it can be deduced that G' must be connected. Thus $\gamma_3(G') \le 3$ always holds, further $\gamma_3(G) \le 8$.

Case 3.7. $\ell = 5$ or 4.

In this case, we have $1 \le t \le 2$. Moreover, we note that if $t = 1$, then ℓ can only be 5, since there are no pendant 4-cycles. Thus for $t = 1$, let $G' = G - C_5(1)$. We know $n(G') = 8$ and G' is not a P_8, since v_1 is only one vertex of degree 1 in G'. Thus $\gamma_3(G') \le 5$ from Table 1, and $\gamma_3(G) \le 8$ since we can choose 3 vertices on $C_5(1)$ to dominate G.

If $t = 2$ and $\ell = 4$ or 5, then v_i are distinct by considering $n(G) = 13$ and (**I**). Let $G' = G - C_\ell(2) - \{v_1, v_2\}$. We notice that $n(G') \le 7$, and further G' is connected, since otherwise a pendant 4-cycle or triangle appears in G, contradicting Claim 2.5 and 2.6. Consequently, $\gamma_3(G') \le 5$, and $\gamma_3(G) \le 8$ since we can choose 3 vertices containing $\{u_1, u_2\}$ on $C_\ell(2)$, in order to dominate G. Hence,we are done.

The result in Lemma 2.5 is true based on the analysis of all the above cases. $\square$

It now follows that if $n(G) \le 13$ with minimum degree at least 2 and maximum degree at most 3, then $\gamma_3(G) \le \frac{2n}{3}$ unless $G \in \{C_3, C_4, K_4 - e, K_4, C_7, C_8, C_{13}\}$. This establishes the base cases for the subsequent induction hypothesis.

3 Proof of the main result

In this section, we will prove Theorem 1.4, which is stated as follows. If G is a connected graph satisfying minimum degree at least 2, maximum degree at most 3, then $\gamma_3(G) \le \frac{2n}{3}$, unless $G \in \mathcal{B} := \{C_3, C_4, K_4 - e, K_4, C_7, C_8, C_{13}\}$.

Proof. Firstly, from Lemma 2.5, the result is true when $n \le 13$. Furthermore, suppose that the result holds for graphs satisfying conditions in Theorem 1.4 and of order less than n. Thus our aim is to show that for a connected graph G with order $n(G) \ge 14$, $\delta(G) \ge 2$

and $\Delta(G) \leq 3$, the result $\gamma_3(G) \leq \frac{2n}{3}$ always holds true. In the sequel, we will refer to "induction hypothesis" simply as "(**IH**)".

We now assume that G is a connected graph, with minimum size, such that $\delta(G) \geq 2$ and $\Delta(G) \leq 3$. This is because removing edges does not decrease the 3-component domination number of G. Moreover, according to the edge minimality of G and $\Delta(G) \leq 3$, we will not consider the occurrence of the 4-vertex configurations $K_4 - e$ and K_4 later, and only consider C_4 instead.

In order to complete the proof, we will need the following claims. Recall that V_i is the set of vertices of degree i in G, where $i \in \{2, 3\}$.

Claim 3.1. For any vertex $v \in V_3$, there is no path $vv_1v_2v_3v_4v_5$ starting from v, where $v_i \in V_2$ for $1 \leq i \leq 5$.

Proof of Claim 3.1. Suppose, to the contrary, that there is a path $vv_1v_2v_3v_4v_5$ starting with the vertex $v \in V_3$, and $v_i \in V_2$ for $1 \leq i \leq 5$. Let v' be the neighbor of v_5, other than v_4. We first notice that v and v' are not adjacent, since otherwise there is a contradiction with the edge minimality of G when $d(v') = 3$, or a contradiction with the requirement in Claim 3.1 when $d(v') = 2$. Thus, delete $\{v_i\}_{i=1}^5$ from G and join vv', specifically, let $G' = G - \{v_i\}_{i=1}^5 + vv'$ (This implies that seven vertices $vv_1v_2v_3v_4v_5$ and another 2-degree neighbor of v_5 do not form a cycle of length 7, otherwise, G' has a parallel edge). Note that G' is connected with $\delta(G') \geq 2$ and $\Delta(G') \leq 3$. Actually G' cannot be a cycle, since $d_G(v) = d_{G'}(v) = 3$. Hence, $\gamma_3(G') \leq \frac{2(n(G)-5)}{3}$ by (**IH**). Now we only need to add three more vertices to dominate G, thus $\gamma_3(G) < \frac{2n}{3}$. □

Recall that $C_\ell(t)$ denotes the cycle of length ℓ containing t vertices of degree 3. If the path between two end vertices of degree 3 only contains inter vertices of degree 2, we define such a path as a *special path*. Moreover, we define the *length* of a special path as the number of 2-degree vertices on this path. Notice that if two 3-degree vertices are adjacent, then the length we define is 0. If there is no ambiguity, we will simply refer to a special path as a path. Thus from the above claim, we know that the paths between two 3-degree vertices have length at most 4, and further $3 \leq \ell \leq 7$ for $C_\ell(1)$. Let H_1, H_2 be two disjoint cycles, and let H be a graph of order at least 14 constructed in the following way. Join a 2-degree vertex of H_1 to a 2-degree vertex of H_2 with an edge e. Subdivide the e edge 0 or more times.

Claim 3.2. $G \neq H$. In particular, no two graphs belong to $\mathcal{B}$ are joined by an edge or a path in G.

Proof of Claim 3.2. Suppose $G = H$. Then there is a spanning path of length at least 14 in G, and it implies, by Corollary 2.3, that $\gamma_3(G) \leq \frac{2n}{3}$, we are done. Thus, we can assume that no two graphs belong to $\mathcal{B}$ are joined by an edge or a path, since we recall that the configuration can be taken from $\mathcal{B}$ is $\{C_3, C_4, C_7, C_8, C_{13}\}$. □

Recall that the edge minimality of G, if there is an edge between two vertices of degree 3, the edge must be a bridge. Therefore, we give the following claim.

Claim 3.3. If G contains two adjacent vertices of degree 3 joined by a bridge, then $\gamma_3(G) \leq \frac{2n}{3}$.

Proof of Claim 3.3. Let u, v be two adjacent vertices of degree 3 where uv is a bridge whose removal yields two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Also, assume that u is contained in G_1. Note that G_1 and G_2 do not both belong to $\mathcal{B}$, otherwise it contradicts

Claim 3.2. If G_1 and G_2 both have a $\frac{2}{3}$-dominating set, we are done. Thus, by symmetry, we may assume that $G_1 \in \mathcal{B}$ and G_2 has a $\frac{2}{3}$-dominating set. Further, $G_1 \in \{C_3, C_4, C_7\}$ by Claim 3.1. Now we prove the claim by examining the following cases.

Case 1. G_1 is a C_3.

Since G_2 has a $\frac{2}{3}$-dominating set, we only need to add at most two vertices u and v in order to dominate G, regardless of whether v belongs to the $\gamma_3(G_2)$-set. Thus, $\gamma_3(G) \leq \frac{2(n-3)}{3} + 2 = \frac{2n}{3}$.

Case 2. G_1 is a C_4.

Let w_1 and w_2 be two neighbors of v in G_2 different from u. Assume w_1 and w_2 are adjacent. If w_1 and w_2 both have degree two, then we contradict the fact that $n(G) \geq 14$. If, without loss of generality, $deg(w_1) = 3$, then we contradict the edge minimality of G. We may therefore assume that w_1 and w_2 are not adjacent.

Thus, we delete $C_4 + \{v\}$ from G and join w_1w_2. Call the resulting graph G'. Note that G' is still connected with $n(G') \geq 9$ and $\delta(G') \geq 2$. Furthermore $G' \notin \mathcal{B}$, since otherwise Claim 3.2 is contradicted. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$ by (**IH**). Now we only need to add $\{u, v\}$ and another neighbor of u to dominate G. Specially, $\gamma_3(G) \leq \frac{2(n-5)}{3} + 3 < \frac{2n}{3}$.

Case 3. G_1 is a C_7.

Let w_1 and w_2 be two neighbors of v in G_2 different from u. Similar to the proof of Case 2, we assert that w_1 and w_2 are not adjacent. Thus, we let $G' = G - C_7 - \{v\} + \{w_1w_2\}$, and note that $n(G') \geq 6$ and $\delta(G') \geq 2$. Furthermore $G' \notin \mathcal{B}$, since otherwise Claim 3.2 is contradicted. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$ by (**IH**). Let D' be a $\gamma_3(G')$-set of G'. If $w_1, w_2 \notin D'$, then we add five consecutive vertices of C_7 containing u, in order to dominate G. For other cases, we add u, v and three more consecutive vertices of C_7. Thus, we have that $\gamma_3(G) \leq \frac{2(n-8)}{3} + 5 < \frac{2n}{3}$. □

By Claim 3.3, we may assume that V_3 is an independent set.

Claim 3.4. *G does not contain a pendant cycle $C_\ell(1)$.*

Proof of Claim 3.4. Suppose, to the contrary, that there is a pendant cycle $C_\ell(1)$ attached by the vertex u. Let P be a path on which vertex u is located. Furthermore, let v be the other end vertex of the path with degree 3, and let w be the last 2-degree vertex on this path. Removing the bridge wv, we get two components G_1 and G_2. Assume that G_1 contains u, w, and G_2 contains v. As shown in Figure 5, the configurations are all cases of G_1 by considering Claim 3.1.

If $G_2 \in \mathcal{B}$, then there is a spanning path of G, and further $\gamma_3(G) \leq \frac{2n}{3}$ by Corollary 2.3. We may assume that $G_2 \notin \mathcal{B}$. Now G_2 has a $\frac{2}{3}$-dominating set by (**IH**). Furthermore, it can be easily examined that the graphs in Figure 5 also has a $\frac{2}{3}$-dominating set, except for (A1). Thus, when G_1 is not configuration (A1), we have $\gamma_3(G) \leq \gamma_3(G_1) + \gamma_3(G_2) \leq \frac{2n}{3}$.

We now consider the case where G_1 resembles configuration (A1). Since $d(v) = 3$, let the two neighbors of v other than w be w_1 and w_2. Note that $d(w_1) = d(w_2) = 2$, and w_1 is not adjacent to w_2, since otherwise there is a contradiction with $n(G) \geq 14$. Thus, let $G' = G - C_3 - \{w, v\} + \{w_1, w_2\}$. Then $\delta(G') \geq 2$ and G' is connected. Also, $G' \notin \mathcal{B}$, since otherwise there is a contradiction with Claim 3.2. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$, and we only need to add three more vertices u, w, v to dominate G, regardless of whether w_i belong to the $\gamma_3(G')$-set. Specifically, $\gamma_3(G) \leq \gamma_3(G') + 3 \leq \frac{2(n-5)}{3} + 3 < \frac{2n}{3}$. □

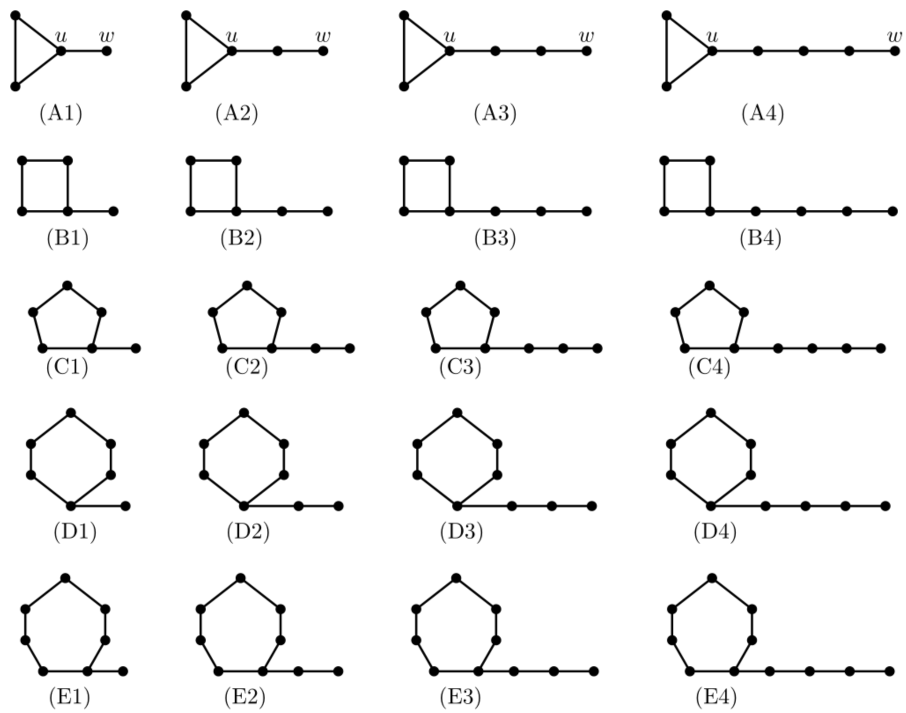

Figure 5: The configurations of G_1 in Claim 3.4.

We may assume that G contains no $C_\ell(1)$, and the length of paths between any two vertices of degree 3 in G is at least 1 and at most 4.

Claim 3.5. *G does not contain $C_\ell(2)$.*

Proof of Claim 3.5. Suppose, to the contrary, that G contains $C_\ell(2)$. From Claim 3.1, we know that $4 \leq \ell \leq 10$. Let u_i be the vertex of degree 3 contained on the cycle $C_\ell(2)$, and let v_i be the neighbor of u_i in $G - C_\ell(2)$, where $i = 1, 2$. Since V_3 is an independent set in G, u_1 is not adjacent to u_2 and $d(v_i) = 2$ for $i = 1, 2$. We first notice that v_1 and v_2 are distinct and are not adjacent, by considering $n(G) \geq 14$ and assumption (**I**).

When $\ell = 5, 6, 9, 10$, we let $G_1 = C_\ell(2)$ and $G' = G - G_1 + v_1v_2$. Then G' is a connected graph with $\delta(G') \geq 2$ and $\Delta(G') \leq 3$. Moreover, if $G' \in \mathcal{B}$, then a spanning path of G appears. By Observation 2.4 $\gamma_3(G) \leq \frac{2n}{3}$ and so we are done. Thus G' has a $\frac{2}{3}$-dominating set by (**IH**). Furthermore, we can examine graph G_1 and find that it can be dominated by three or four consecutive vertices containing both u_1 and u_2 when $\ell = 5$ or 6. When $\ell = 9$, G_1 can be dominated by two vertex disjoint P_3 paths that include both u_1 and u_2. In short, G_1 has a $\frac{2}{3}$-dominating set. Thus $\gamma_3(G) \leq \frac{2n}{3}$.

When $\ell = 4, 7, 8$, we discuss the cases in detail based on whether v_1 and v_2 have a common neighbor.

First we consider $N(v_1) \cap N(v_2) = \emptyset$. Let the neighbors of v_i, which are distinct from u_i, be w_i for $i \in [2]$. Delete $C_\ell(2) + \{v_1\}$ and join w_1v_2, the resulting graph we call G'. Note that G' is a connected graph with $\delta(G') \geq 2$ and $\Delta(G') \leq 3$. Moreover, if $G' \in \mathcal{B}$, then there exists a spanning path of G by Observation 2.4, whence $\gamma_3(G) \leq \frac{2n(G)}{3}$. Thus G' has a $\frac{2}{3}$-dominating set by (**IH**), namely D'. Furthermore, for $C_\ell + \{v_1\}$, we can find that it has a $\frac{2}{3}$-dominating set. Specifically, we can indeed choose 3 vertices when $\ell = 4$, choose 5 vertices when $\ell = 7$, and 6 vertices when $\ell = 8$. Note that if neither w_1 nor v_2 belongs to set D', all chosen vertices should contain u_1. If $w_1 \in D'$ and $v_2 \notin D'$, all chosen vertices should contain three consecutive vertices including u_2. If $w_1 \notin D'$ and $v_2 \in D'$, all chosen vertices should contain three consecutive vertices including v_1, u_1 and one of u_1's neighbors. If $w_1, v_2 \in D'$, all chosen vertices should contain two consecutive vertices including v_1 and u_1, or u_2 and one of its neighbors. Thus $\gamma_3(G) \leq \frac{2n(G)}{3}$.

Now $N(v_1) \cap N(v_2) \neq \emptyset$, and let their common neighbor be w. By Claim 3.1, we know that the third path P with w as a starting vertex has length s at most 4, in other words, P contains at most four vertices of degree 2, and let the vertices on the path be x_i ($i \in [s]$). Let $G_1 = C_\ell(2) + \{v_1, v_2, w\} + \{x_i\}_{i=1}^{s}$, and let $G' = G - G_1$. We notice that $n(G_1) = \ell + 3 + s$, and $\delta(G') \geq 2$. Claims 3.1 and 3.4 imply that $G' \notin \mathcal{B}$. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$, and if $\gamma_3(G_1) \leq \frac{2n(G_1)}{3}$, we are done.

For $\ell = 4$, we have that there exists a 7-vertex subgraph formed by $C_4(2) + \{v_1, v_2, w\}$, and it can be dominated by 4 vertices containing w by Table 1. Now we only need to choose $\{x_i\}_{i=1}^{s-1}$ to dominate G_1. Thus $\gamma_3(G_1) \leq s + 3$, and $\frac{s+3}{s+7} \leq \frac{2}{3}$ holds for $s \leq 5$.

For $\ell = 7$, first note $C_7(2)$ can be dominated by 5 vertices containing u_1 and u_2. Now if $s \geq 2$, then G_1 can be dominated by 8 vertices, since we can add $\{x_i\}_{i=1}^{3}$ or w, x_1, x_2. Moreover, $\frac{8}{10+s} \leq \frac{2}{3}$ holds for $s \geq 2$. If $s = 1$, we only add two vertices v_1 and w to dominate G_1. Note $\frac{7}{11} < \frac{2}{3}$.

For $\ell = 8$, first note $C_8(2)$ can be dominated by 6 vertices containing u_1 and u_2. Now if $s \geq 3$, then G_1 can be dominated by 9 vertices, since we can add $\{x_i\}_{i=1}^{3}$. Moreover, $\frac{9}{11+s} \leq \frac{2}{3}$ holds for $s \geq 3$. If $s \leq 2$, then only 5 vertices are needed to dominate all vertices but one vertex u_1 of $C_8(2)$. And further, we choose 3 vertices v_1, w and x_1 to dominate G_1. Note that $\frac{8}{11+s} \leq \frac{2}{3}$ holds for $s \geq 1$. Thus $\gamma_3(G_1) \leq \frac{2n(G_1)}{3}$. □

Claim 3.6. There are no paths that have length 1 between two 3-degree vertices u and v.

Proof of Claim 3.6. Suppose that w is a vertex of degree 2 between two 3-degree vertices u and v, and let the two longest paths starting from u be $ux_1 \ldots x_r$ and $uy_1 \ldots y_s$, with the endpoints of these paths being u_1 and u_2, respectively. Similarly, let the two longest paths starting from v be $ux'_1 \ldots x'_{r'}$ and $uy'_1 \ldots y'_{s'}$, with endpoints v_1 and v_2, respectively, as illustrated in Figure 6.

First recall that V_3 is an independent set, and by Claim 3.1, $r, s, r', s' \in \{1, 2, 3, 4\}$. By Claim 3.5, we may assume that $u_1 \neq u_2$ and $v_1 \neq v_2$. By symmetry, we assume $r \geq s$.

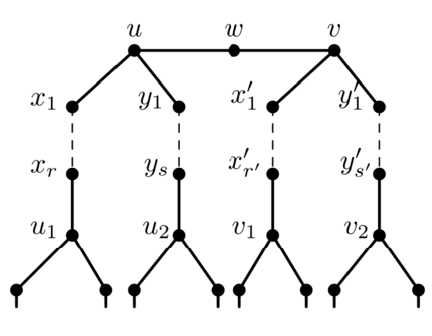

Figure 6: The pattern in Claim 3.6.

Case 1. $(r, s) \notin \{(1,1),(4,4)\}$.

By Claim 3.5, u_1, u_2 and v are distinct. Take the vertex u and all 2-degree vertices on the three paths attached to it. Specifically, let G_1 be the subgraph formed by $\{u, w, x_1, \ldots, x_r, y_1, \ldots, y_s\}$. Note $n(G_1) = r+s+2$ and G_1 has a $\frac{2}{3}$-dominating set, since we can choose $\{u, w, x_1, \ldots, x_{r-1}, y_1, \ldots, y_{s-1}\}$, a total of $n(G_1)-3$ vertices, or choose 3 vertices $\{u, x_1, y_1\}$ when $r=2, s=1$. Note that $\frac{n(G_1)-3}{n(G_1)} \leq \frac{2}{3}$ holds for $n(G_1) \leq 9$. Indeed, this can be achieved since $r+s \leq 7$ in this case, and we note that $\frac{3}{5} < \frac{2}{3}$ when $(r,s) = (2,1)$. Thus $\gamma_3(G_1) \leq \frac{2n(G_1)}{3}$. Now, let $G' = G - G_1$. If $\gamma_3(G') \leq \frac{2n(G')}{3}$, then we are done. Next, we discuss the cases to confirm that $\gamma_3(G') \leq \frac{2n(G')}{3}$ indeed holds.

If G' has three components, say u_1-component, u_2-component and v-component, respectively, then each component is not in $\mathcal{B}$. This is because we know from Claim 3.4 that no pendant cycle appears. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$ by (**IH**).

If G' has two two components K_1 and K_2, where K_1 contains u_1, and K_2 contains u_2 and v. By Claim 3.4, we know that $K_1 \notin \mathcal{B}$. Moreover, K_2 has at least one degree 3 vertex in $\{v_1, v_2\}$. If $K_2 \in B$, then K_2 can only be $K_4 - e$, and this contradicts the edge minimality of G. Thus, $K_2 \notin \mathcal{B}$, and so, by (**IH**), we are done.

If G' is a connected graph, then $G' \in \mathcal{B}$. Otherwise by (**IH**), $\gamma_3(G') \leq \frac{2n(G')}{3}$, and so we are done. Note that $n(G) \geq 14$, and $5 \leq n(G_1) \leq 9$. Hence, $n(G') \geq 5$ and so $G' \in \{C_7, C_8, C_{13}\}$. Note that if G' is C_7, C_8 or C_{13}, then there exists a $\gamma_3(G')$-set of size 5, 6, or 9 respectively, containing u_1 and u_2. In short, $\gamma_3(G') \leq \frac{2n(G')+2}{3}$ for $G' \in \{C_7, C_8, C_{13}\}$.

Now, in order to dominate G, we only need to add $n(G_1)-5$ vertices, namely $\{u, x_1, \ldots, x_{r-2}, y_1, \ldots, y_{s-2}\}$ when $n(G_1) \geq 8$. Then $\gamma_3(G) \leq n(G_1)-5+\frac{2n(G')+2}{3} = \frac{2n(G)+n(G_1)-13}{3} < \frac{2n(G)}{3}$. When $6 \leq n(G_1) \leq 7$, we add at most 3 vertices, thus $\gamma_3(G) \leq 3 + \frac{2n(G')+2}{3} = \frac{2n(G)+11-2n(G_1)}{3} < \frac{2n(G)}{3}$. Lastly, when $n(G_1) = 5$, we consider $n(G) \geq 14$ and $n(G') \geq 9$, thus G' can only be a C_{13} in $\mathcal{B}$. Then $\gamma_3(G) \leq 3 + 9 = 12 = \frac{2 \times 18}{3}$.

Case 2. $(r, s) = (4, 4)$.

Let $G_1 = \{u, \{x_i\}_{i=1}^4, \{y_i\}_{i=1}^4\}$, and set $G' = G - G_1 + wu_2$. Note that $\delta(G') \geq 2$, and $d_{G'}(u_2) = 3$. Then the graph G' is not in $\mathcal{B}$ if G' is connected, or the component of G' containing u_2 is not in $\mathcal{B}$ if G' is disconnected, and further, the u_1-component of G' is not in $\mathcal{B}$ by Claim 3.4. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$. Let D' be the $\gamma_3(G')$-set of G'.

Now, we only need to add 6 vertices $\{\{x_i\}_{i=1}^3, \{y_i\}_{i=2}^4\}\}$ if $u_2 \notin D'$, or add 6 vertices $\{\{x_i\}_{i=1}^3, u, y_3, y_4\}$ if $u_2 \in D'$, whence $\gamma_3(G) \leq \frac{2(n(G)-9)}{3} + 6 = \frac{2n(G)}{3}$, and so we are done.

Case 3. $(r, s) = (1, 1)$.

By Case 1 and 2 as well as symmetry, the paths starting from u and v have length of 1. Label the vertices as Figure 7 (a). By Claim 3.5, $u_1 \neq u_2$ and $v_1 \neq v_2$. By symmetry, next we claim that $u_2 \neq v_1$ ($u_1 \neq v_2$ is similar).

Suppose that $u_2 = v_1$ (see Figure 7 (b)). Note that $d(u_2) = 3$ and $d(w') = 2$. Let $G_1 = C_6(3) + x_1$. Also, we know from Claim 3.5 that $w' \neq y_1'$ and w' is not adjacent to y_1'. Let $G' = G - G_1 + w'y_1'$. Note that $\delta(G') \geq 2$, and $d_{G'}(v_2) = 3$. Then the graph G' is not in $\mathcal{B}$ if G' is connected, or the component of G' containing w', y_1' is not in $\mathcal{B}$ if G' is disconnected, and further, the u_1-component of G' is not in $\mathcal{B}$ by Claim 3.4. Thus

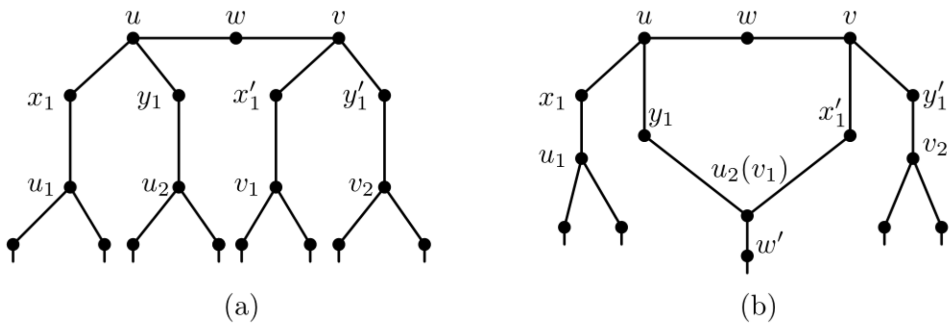

Figure 7: The configurations of $(r, s) = (1, 1)$ in Claim 3.6.

$\gamma_3(G') \leq \frac{2n(G')}{3}$.

Let D' be the $\gamma_3(G')$-set of G', we only need to add 4 vertices to dominate G. Specifically, if $w', y_1' \notin D'$, then we add $\{u, w, v, x_1'\}$. If $w' \notin D'$ and $y_1' \in D'$ or if $w', y_1' \in D'$ and apart from the vertex y_1', w' does not have any neighbors in G' that belong to D', then we add $\{u, y_1, u_2, v\}$. For other cases, we add $\{u, w, v, u_2\}$. Thus, $\gamma_3(G) \leq \frac{2(n(G)-7)}{3} + 4 < \frac{2n(G)}{3}$.

Thus, now u_1, u_2, v_1 and v_2 are all distinct. Redefine the new 7-vertex G_1 with vertex set $\{u, w, v, x_1, y_1, x_1', y_1'\}$ and $G' = G - G_1$. For G_1, we have that $\gamma_3(G_1) \leq \frac{2n(G_1)}{3}$, since $\{u, w, v\}$ can dominate G_1. Thus if $\gamma_3(G') \leq \frac{2n(G')}{3}$, then $\gamma_3(G) \leq 3 + \frac{2(n(G)-7)}{3} < \frac{2n(G)}{3}$, we are done. Next, we discuss the cases to confirm that $\gamma_3(G') \leq \frac{2n(G')}{3}$ indeed holds.

First we consider G' has four connected components K_i $(i \in [4])$, which contain u_1, u_2, v_1 and v_2, respectively, then $K_i \notin \mathcal{B}$ from Claim 3.4. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$.

If G' has three components K_1, K_2 and K_3, where K_1 contains u_1, K_2 contains u_2 and v_1, and K_3 contains v_2. By Claim 3.4, K_1 and K_3 are not in $\mathcal{B}$. Note that K_2 is not a C_3 since V_3 is an independent set in G, and not a C_4, C_7 or C_8 by Claim 3.5, and not a C_{13} by Claim 3.1. Hence, $K_2 \notin \mathcal{B}$. Thus using (**IH**), we get that $\gamma_3(K_i) \leq \frac{2n(K_i)}{3}$ for $i = 1, 2, 3$, and $\gamma_3(G') \leq \frac{2n(G')}{3}$.

If G' has two components K_1 and K_2, with K_1 containing u_1 and u_2 and K_2 containing v_1 and v_2, then using a similar analysis to that for the K_2 component in the previous paragraph, we can get that K_i $(i \in [2])$ is not in $\mathcal{B}$. Thus we are done. If G' has two components K_1 and K_2, with K_1 containing u_1 and K_2 containing u_2, v_1 and v_2, then $K_1 \notin \mathcal{B}$, and K_2 cannot be C_3, C_4 from assumption (**I**). Further if $K_2 \in \{C_7, C_8, C_{13}\}$, then we know that $\gamma_3(K_2) \leq \frac{2n(K_2)+2}{3}$. Thus $\gamma_3(G) \leq 3 + \frac{2n(K_1)}{3} + \frac{2n(K_2)+2}{3} = \frac{2n(G)-3}{3}$, and so we are done. Hence, $K_2 \notin \mathcal{B}$, and so, by (**IH**), $\gamma_3(G') \leq \frac{2n(G')}{3}$.

We may assume that G' is connected. Clearly $G_2 \notin \{C_3, C_4, C_7\}$. Moreover, if $G' \in \{C_8, C_{13}\}$, then $\gamma_3(G) \leq 3 + \frac{2n(G')+2}{3} = \frac{2n(G)-3}{3}$, and so we are done. Thus, $G' \notin \mathcal{B}$ and so $\gamma_3(G') \leq \frac{2n(G')}{3}$. By (**IH**), we are done. □

Claim 3.7. No paths between two 3-degree vertices u and v have length 2.

Proof of Claim 3.7. Let w_1 and w_2 be 2-degree vertices between u and v, and let the two longest paths starting from u be $ux_1 \dots x_r$ and $uy_1 \dots y_s$, with the other endpoints of these paths being u_1 and u_2. Recall that V_3 is an independent set. Claims 3.1 and 3.6 imply that $r, s \in \{2, 3, 4\}$. Furthermore, u_1, u_2 and v are distinct by Claim 3.5. By symmetry, we assume $r \geq s$.

Case 1. $(r, s) \notin \{(4, 3), (4, 4)\}$.

Let $G_1 = \{u, w_1, w_2, x_1, \ldots, x_r, y_1, \ldots, y_s\}$, and $G' = G - G_1$. Note that $n(G_1) = r + s + 3$,
and we can choose $\{u, w_1, x_1, \ldots, x_{r-1}, y_1, \ldots, y_{s-1}\}$, a total of $n(G_1) - 3$ vertices to dominate G_1. Moreover, $n(G_1) - 3 \leq \frac{2n(G_1)}{3}$ holds for $n(G_1) \leq 9$. Indeed, this can be achieved in this case. Thus, if $\gamma_3(G') \leq \frac{2n(G')}{3}$, then we are done. Next, we discuss the cases to confirm that $\gamma_3(G') \leq \frac{2n(G')}{3}$ indeed holds.

Now if G' has three connected components K_i ($i \in [3]$), which contain u_1, u_2 and v, respectively, then $K_i \notin \mathcal{B}$ from Claim 3.4. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$.

If G' has two components K_1 and K_2, where K_1 contains u_1, and K_2 contains u_2 and v. Clearly, $K_1 \notin \mathcal{B}$ from Claim 3.4. By assumption (**I**), K_2 cannot be C_3. Claim 3.5 implies that K_2 cannot be a C_4, C_7 or C_8. Lastly, by Claim 3.1, we can deduce that K_2 cannot be C_{13}. Thus using (**IH**), we get that $\gamma_3(K_i) \leq \frac{2n(K_i)}{3}$ for $i = 1, 2$, and so we are done.

Now G' is connected. Clearly $G' \notin \{C_3, C_4, C_7, C_8\}$. Moreover, if G' is C_{13}, then there is a 3-component dominating set containing v with a size of 9 to dominate C_{13}, thus we only need to choose $\{u, x_1, \ldots, x_{r-1}, y_1, \ldots, y_{s-1}\}$ in G_1, a total of $n(G_1) - 4$ vertices to dominate G_1. Whence, $\gamma_3(G) \leq n(G_1) - 4 + 9 = n(G) - 8$. Note that $n(G) - 8 \leq \frac{2n(G)}{3}$ holds when $n(G) \leq 24$. Indeed, this is true in this case. Thus, $G' \notin \mathcal{B}$ and $\gamma_3(G') \leq \frac{2n(G')}{3}$ by (**IH**).

Case 2. $(r, s) \in \{(4, 3), (4, 4)\}$.

Let $G_1 = \{u, w_1, w_2, x_1, x_2, x_3, x_4, y_1, y_2\}$ and set $G' = G - G_1 + u_1y_3$. Note that $d_{G'}(u_1) = d_{G'}(u_2) = 3$. The graph G' is not in $\mathcal{B}$ if G' is connected, or the component of G' containing u_1 and y_3 is not in $\mathcal{B}$ if G' is disconnected, and further, the v-component of G' is not in $\mathcal{B}$ by Claim 3.4. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$.

Let D' be the $\gamma_3(G')$-set of G'. We now only need to add 6 vertices $\{u, w_1, y_1, x_1, x_2, x_3\}$ if $u_1, y_3 \notin D'$, or add 6 vertices $\{u, w_1, y_1, x_2, x_3, x_4\}$ if $u_1 \notin D'$ and $y_3 \in D'$. For the other cases, we add $\{u, w_1, y_1, y_2, x_3, x_4\}$. Then $\gamma_3(G) \leq \frac{2(n(G)-9)}{3} + 6 = \frac{2n(G)}{3}$. □

Claim 3.8. There are no three vertices u, u_1 and u_2 of degree 3 such that the lengths between u and u_1 as well as u and u_2 are both 3.

Proof of Claim 3.8. Let $x_1x_2x_3$ and $y_1y_2y_3$ be the two paths between u and u_1 as well as u and u_2, respectively. Let $w_1 \ldots w_t$ be the third path linking u to a vertex v of degree 3, where all internal vertices in the path are of degree 2. Note that t is 3 or 4 by above claims. Also u_1, u_2 and v are distinct by Claim 3.5. Now we let $G_1 = \{u, w_1, w_2, x_1, x_2, x_3, y_1, y_2, y_3\}$, and $G' = G - G_1 + u_2w_3$.

Note $d_{G'}(u_2) = d_{G'}(v) = 3$. Then G' is not in $\mathcal{B}$ if G' is connected, or the component of G' containing u_2 and w_3 is not in $\mathcal{B}$ if G' is disconnected, and further, the u_1-component of G' is not in $\mathcal{B}$ by Claim 3.4. Thus $\gamma_3(G') \leq \frac{2n(G')}{3}$. Let D' be the $\gamma_3(G')$-set of G'.

Now, in order to dominate G, we only need to add 6 vertices. Indeed, we can choose u, x_1, x_2 and three additional vertices apart from those. Specifically, the three additional vertices can be chosen as w_1, y_1, y_2 when $u_2, w_3 \notin D'$, as w_1, w_2, y_1 when $u_2 \in D'$ and $w_3 \notin D'$, as y_1, y_2, y_3 when $u_2 \notin D'$ and $w_3 \in D'$. We may assume that $u_2, w_3 \in D'$. If apart from the vertex w_3, u_2 does not have any neighbors in G' that belong to D', we can choose y_2, y_3, w_2. Finally, for other cases, we can choose w_1, w_2, y_3. Thus, $\gamma_3(G) \leq \frac{2(n(G)-9)}{3} + 6 = \frac{2n(G)}{3}$. $\square$

Up to now, there is no $C_\ell(1)$ in G, and the length of paths between two vertices of degree 3 in G is 3 or 4, and there are no two paths of length 3 that link the same vertex of degree 3.

Note that all components of $G - V_3$ are paths of length 3 or 4, and let p_1 and p_2 be the number of the paths of $G - V_3$ of length 3 and 4, respectively. Thus $p_1 + p_2 = \frac{3n_3}{2}$ and $p_1 \leq \frac{n_3}{2}$ by Claim 3.8. This implies that $n \geq n_3 + 3\frac{n_3}{2} + 4n_3 = \frac{13n_3}{2}$. Choose all vertices in V_3, and one end-vertex on each path of $G - V_3$ of length 3, and two end vertices on each path of $G - V_3$ of length 4. This choice produces a 3-component dominating set D of G such that

$$\begin{aligned} |D| &= n_3 + p_1 + 2p_2 \\ &\leq 4n_3 \\ &\leq \frac{8n}{13}. \end{aligned}$$

Clearly $\frac{8n}{13} < \frac{2n}{3}$, this completes the proof. $\square$

To end this section, we remark that the bound in Theorem 1.4 is sharp. Let H be the graph formed by the disjoint union of two cycles C_9 and joining them with an edge. It is easy to check that if $G \in \{C_6, C_9, C_{12}, C_{18}, H\}$, then $\gamma_3(G) = \frac{2n}{3}$.

Acknowledgments

We would like to express our sincere gratitude to the referees for their very careful reading of the paper and for insightful comments and valuable suggestions, leading to an improvement of an earlier version. Gao and Xi were partially supported by the National Natural Science Foundation of China (No. 12161141006), the Natural Science Foundation of Tianjin (Nos. 20JCJQJC00090 and 20JCZDJC00840) and the Fundamental Research Funds for the Central Universities, Nankai University.

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