On the dominating (induced) cycles of iterated line graphs

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Abstract

In this paper, we give characterizations of graphs with line graphs or iterated line graphs that have dominating cycles. The characterization of graphs with dominating cycles in its line graphs and its i-iterated line graphs for $i \geq 2$ are different: we may not unify them.

As an application, we give characterizations of graphs with iterated line graphs that have dominating induced cycles. They are very different from the known results, although those characterizations for dominating cycles have some similarities with results on hamiltonian iterated line graphs of Harary and Nash-Williams (1965) and Xiong and Liu (2002).

Using these results, we also give some analysis on the complexity of determining the existence of dominating cycles. It is NP-complete to decide whether a given graph has a dominating induced cycle, even for a 2-iterated line graph.

Keyword: Iterated line graph; Dominating cycle; Dominating induced cycle.

1 Introduction

The graphs considered in this paper are finite undirected graphs without loops. For graph-theoretical notation and terminology not defined here we refer the reader to [2].

Let $G = (V(G), E(G))$ be a graph. The *line graph* $L(G)$ of G has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. The *i-iterated line graph* $L^i(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^i(G) = L(L^{i-1}(G))$, and $L^{i-1}(G)$ is assumed to be nonempty.

A graph G is *hamiltonian* if G has a *Hamilton cycle* containing all vertices of G. Harary and Nash-Williams[13] characterized those graphs G for which $L(G)$ is hamiltonian. Xiong and Liu [28] characterized the graphs whose i-iterated line graphs are hamiltonian for $i \geq 2$.

A cycle C of a graph G is *dominating* if $G - V(C)$ is edgeless. Inspired by the research on Hamilton cycles, we consider similar problems about dominating cycles. It is natural to ask that for any integer k, does there exist a characterization of those graphs G for which $L^k(G)$ has dominating cycles? We will answer this question and give a characterization: Theorems 2 and 3. More results on dominating cycles can be found in [7, 17, 25].

We need some more notation and terminology. For a graph G and a nonnegative integer k, we denote $V_k(G) = \{x \in V(G) : d_G(x) = k\}$, where $d_G(x)$ is the degree of x in G. The *distance* $d_G(H_1, H_2)$ between two subgraphs H_1 and H_2 of G is defined to be $\min\{d_G(v_1, v_2) : v_1 \in V(H_1),\ v_2 \in V(H_2)\}$, where $d_G(v_1, v_2)$ denotes the length of a shortest path between v_1 and v_2 in G. If $d_G(e, H) = 0$ for an edge e of G, we say that H dominates e. For a subgraph H of G, let $\overline{E}(H)$ denote the set of edges of G that are incident with some vertices of H, that is, dominated by H. A subgraph H of G is called *dominating* if it dominates all edges of G, that is, $\overline{E}(H) = E(G)$. For $X \subseteq V(G)$, let $G[X]$ be the *vertex-induced subgraph* of G, and let $G - X = G[V(G) \setminus X]$. For $S \subseteq E(G)$, let $G[S]$ be the *edge-induced subgraph* of G, and let $G - S = G[E(G) \setminus S]$.

A subgraph of G is called *eulerian* if it is connected and even. If an eulerian subgraph is nontrivial, then it contains at least one cycle. An eulerian subgraph D of a graph G is called a D_λ-*eulerian* subgraph if every component of $G - V(D)$ has order less than λ. Moreover, if D is a cycle such that every component of $G - V(D)$ has order less than λ, we call it a D_λ-*cycle* of G. Some results about D_λ-cycle refer to [26]. In particular, a D_1-eulerian subgraph is a spanning eulerian subgraph, and a D_1-cycle is a Hamilton cycle. In connected graphs, a D_2-eulerian subgraph is a dominating eulerian subgraph, and a D_2-cycle is a dominating cycle.

A graph is *trivial* if it has only one vertex, *nontrivial* otherwise. A *branch* in G is a nontrivial path with internal vertices, if any, of degree two in G and neither endvertex of degree two in G. We denote by $\mathcal{B}(G)$ the set of branches of G and $\mathcal{B}_1(G)$ the subset of $\mathcal{B}(G)$ in which at least one endvertex has degree one. For a subgraph H of G, $\mathcal{B}_H(G)$ denotes the set of branches of G with all edges in H.

In order to characterize the graphs G whose i-iterated line graphs are hamiltonian for $i \geq 2$, Xiong and Liu [28] defined $EU_k(G)$, where $EU_k(G)(k \geq 2)$ is the set of those subgraphs H of a graph G that satisfy the following conditions:

(I) H is an even graph,

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$,

(III) $d_G(H_1, H - H_1) \leq k - 1$ for every subgraph H_1 of H,

(IV) $|E(B)| \leq k + 1$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$,

(V) $|E(B)| \leq k$ for every branch $B \in \mathcal{B}_1(G)$.

Theorem 1. Let G be a connected graph with at least three edges. Then the following holds.

(1) (Harary and Nash-Williams [13]) $L(G)$ is hamiltonian if and only if G has a D_2-eulerian subgraph.

(2) (Xiong and Liu [28]) For $i \geq 2$, $L^i(G)$ is hamiltonian if and only if $EU_i(G) \neq \emptyset$.

We start with a characterization of those graphs with a line graph that has a dominating cycle.

Theorem 2. Let G be a connected simple graph that is not a path. Then $L(G)$ has a dominating cycle if and only if G has a D_3-eulerian subgraph.

By modifying conditions (IV) and (V) of $EU_k(G)$, we introduce $EDU_k(G)$, where $EDU_k(G)(k \geq 2)$ is the set of those subgraphs H of a graph G that satisfy (I),(II),(III) and the following conditions:

(IV*) $|E(B)| \leq k+2$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$,

(V*) $|E(B)| \leq k+1$ for every branch $B \in \mathcal{B}_1(G)$.

Theorem 3. Let G be a connected graph with at least three edges and $i \geq 2$. Then $L^i(G)$ has a dominating cycle if and only if $EDU_i(G) \neq \emptyset$.

Chartrand [3] was one of the first to study properties of iterated line graphs. And in [4] he introduced the *hamiltonian index* of a graph G, denoted $h(G)$, which is the least nonnegative integer k such that $L^k(G)$ is hamiltonian, see also [18]. More generally, we have the following definition[19].

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

By the definition of $\mathcal{P}(G)$, the *dominating cycle index* of a graph G, denoted $dc(G)$, is the least nonnegative integer k such that $L^k(G)$ has a dominating cycle. By the relation of dominating cycle and Hamilton cycle, for a connected graph G that is not a path, $dc(G)$ exists and it has a natural bound, $h(G) - 1 \leq dc(G) \leq h(G)$.

Besides hamiltonicity, many cycle properties of iterated line graphs have been studied, including k-orderability[16], k-ordered hamiltonicity[14], pancyclicity[22], Hamilton-connectivity[6], existence of 2-factors[10] and existence of even factors[27]. Some other properties on iterated line graphs were also considered. The connectivity of iterated line graphs was discussed in [5, 15, 23]. Planarity and outerplanarity refer to [12] and [20].

Proofs of Theorem 2 and 3 are presented in Section 2. Section 3 is a further research on dominating induced cycles of iterated line graphs. Section 4 is devoted to the analysis of the complexity of the problem to determine these subgraphs. The last section is devoted to the concluding remarks.

2 Characterization of graphs with iterated line graphs that have dominating cycles

Lemma 4. Let G be a connected graph and C be a cycle of $L(G)$. Then there exists an eulerian subgraph H of G such that $E(H) \subseteq V(C) \subseteq \overline{E}(H)$.

Proof. Let $C = e_1 e_2 \dots e_m e_1$ be a cycle of $L(G)$ with $m \geq 3$, where $e_i \in E(G)$. We construct a subgraph H of G induced by the set of the remaining edges of $\{e_1, e_2, \dots, e_m\}$ by deleting the $s_i + t_i - 1$ edges $e_{i-t_i+1}, \dots, e_{i+s_i-1}$ as many as possible such that those edges $e_{i-t_i}, \dots, e_i, \dots, e_{i+s_i}$ have the same common endvertex with $s_i + t_i - 1 \geq 1$ for any possible i, where the subscript takes modules by m. The above resulting graph H is possible an edgeless graph, then the edge set of $e_1, e_2, \dots, e_m$ induces a star of G and we let H be the center vertex of the star. By our construct, H is a connected graph in which every vertex has even degree. Hence H is an eulerian subgraph. Since H is a subgraph of G induced by subset of edges $e_1, \dots, e_m$, it holds that $E(H) \subseteq V(C) \subseteq \overline{E}(H)$. This completes the proof. □

Proof of Theorem 2. Suppose that C is a dominating cycle of $L(G)$. Then by Lemma 4, there exists an eulerian subgraph H of G such that $E(H) \subseteq V(C) \subseteq \overline{E}(H)$. We claim that H is a D_3-eulerian subgraph of G. Otherwise suppose H_1 is a connected component of $G - V(H)$ of order at least three, then $|E(H_1)| \geq 2$ and there exist two adjacent edges in H_1, say e_1 and e_2. Since $e_1, e_2 \notin \overline{E}(H)$, $d_{L(G)}(e_1e_2, V(C)) = min\{d_{L(G)}(e_1, V(C)), d_{L(G)}(e_2, V(C))\} \geq 1$, contradicting that C is dominating.

Conversely, suppose first that G is a cycle. Then $L(G)$ is a cycle and the assertion clearly holds. Suppose then that G is not a cycle. Then $\Delta(G) \geq 3$ since G is simple other than a path. We may assume that H is a D_3-eulerian subgraph of G. Then H contains a vertex of degree at least three in G, and there exists a cycle C in $L(G)$ such that $V(C) = \overline{E}(H)$. Since all connected components of $G - V(H)$ have at most one edge, for each edge $ef \in E(L(G))$, at least one of vertices e and f is in $\overline{E}(H)$, thus C is a dominating cycle of $L(G)$. $\square$

Our main result, Theorem 3, is a direct consequence of Theorems 5 and 6. One easily derives Theorem 3 by induction, which can be considered as an analogue of Theorem 1(2) on dominating cycles of iterated line graphs.

Theorem 5. Let G be a connected graph and $k \geq 1$ be an integer. Then $EDU_k(L(G)) \neq \emptyset$ if and only if $EDU_{k+1}(G) \neq \emptyset$.

Theorem 6. Let G be a connected graph with at least three edges. Then $L^2(G)$ has a dominating cycle if and only if $EDU_2(G) \neq \emptyset$.

We start our proof with some auxiliary results.

Lemma 7. (Xiong and Liu [28]) Let $B = u_1u_2 \dots u_s (s \geq 3)$ be a path of G and let $e_i = u_iu_{i+1}$. Then B is a branch of G if and only if $B' = e_1e_2 \dots e_{s-1}$ is a branch of $L(G)$.

Lemma 8. (Xiong and Liu [28]) Let G be a connected graph of order at least three. Then H is a nontrivial eulerian subgraph of the line graph $L(G)$ if and only if there exists a subgraph D of G such that

(1) D is an even graph,
(2) $V_0(D) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G)$,
(3) $G[V(D)]$ is connected, i.e., $d_G(D_1, D - D_1) \leq 1$ for any subgraph D_1 of D,
(4) $E(D) \subseteq V(H) \subseteq \overline{E}(D)$.

Proof of Theorem 5. Suppose that $EDU_{k+1}(G) \neq \emptyset$, and choose an $H \in EDU_{k+1}(G)$ with a minimum number of components $H_1, \dots, H_t$. For each H_i, we can find a cycle C_i of $L(G)$ that spans $\overline{E}(H_i)$. Let $H' = \bigcup_{i=1}^{t} C_i$. We will show that $H' \in EDU_k(L(G))$.

Since $d_G(H_i, H_j) \geq 1$, we claim that any C_i and C_j are edge-disjoint. Otherwise there would exist two components H_i, H_j and edges e_1, e_2 in $\overline{E}(H_i) \cap \overline{E}(H_j)$ with the same set of endvertices. And $H + \{e_1, e_2\}$ is a subgraph of G in $EDU_{k+1}(G)$ that contains fewer components than H, a contradiction. Hence H' is a union of edge-disjoint cycles, satisfying (I).

By the definition of H, $\bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$. As $H' = \bigcup_{i=1}^{t} \overline{E}(H_i)$, we have $\bigcup_{i=3}^{\Delta(L(G))} V_i(L(G)) \subseteq V(H')$ and there are no isolated vertices in H'. Hence H' satisfies (II).

Take an arbitrary $T \subseteq \{1, \ldots, t\}$. Let $P = xu_1 \ldots u_s y$ be a shortest path from $\bigcup_{i \in T} H_i$ to $H - \bigcup_{i \in T} H_i$ with $x \in \bigcup_{i \in T} H_i$ and $y \in H - \bigcup_{i \in T} H_i$. By the choice of H, it follows that $s = d_G(\bigcup_{i \in T} H_i, H - \bigcup_{i \in T} H_i) - 1 \leq (k+1) - 1 - 1 = k - 1$. Then $L(P)$ is a path from $\bigcup_{i \in T} C_i$ to $H - \bigcup_{i \in T} C_i$ with length $s \leq k-1$, which implies that (III) holds. By Lemma 7, we can immediately see that H' satisfies (IV*) and (V*) since H satisfies (IV*) and (V*).

Conversely, suppose that $EDU_k(L(G)) \neq \emptyset$. Let H be a subgraph of $L(G)$ in $EDU_k(L(G))$ with a minimum number of isolated vertices. We claim that H actually has no isolated vertices. By the definition of H, any isolated vertex e of H has degree at least three in $L(G)$. Since $L(G)$ is claw-free, the vertex e lies on some triangle in $L(G)$, say ee_1e_2.

Construct a subgraph H_0 as follows.

$$H_0 = \begin{cases} H + \{ee_1, ee_2, e_1e_2\} & \text{if } e_1e_2 \notin E(H), \\ H + \{ee_1, ee_2\} - \{e_1e_2\} & \text{if } e_1e_2 \in E(H). \end{cases}$$

Clearly H_0 is in $EDU_k(L(G))$ and it has fewer isolated vertices than H, verifying the claim.

Let $H_1, \ldots, H_t$ be the components of H, each of which is a nontrivial eulerian subgraph of $L(G)$. Thus, by Lemma 8, for each H_i there exists a subgraph D_i of G satisfying the four given conditions. Set $D = (\bigcup_{i=1}^{t} D_i) \cup (\bigcup_{i=3}^{\Delta(G)} V_i(G))$.

We now show that D is in $EDU_{k+1}(G)$. Since each H_i is vertex disjoint with other components of H and $E(D_i) \subseteq V(H_i)$ for all i, each D_i is edge-disjoint with other components of D. Hence D is an even subgraph, satisfying (I). Each D_i also satisfies (2), hence $d_G(x) \geq 3$ for every $x \in V(D)$ with $d_D(x) = 0$, and thus (II) holds.

Take an arbitrary $T \subseteq \{1, \ldots, t\}$. By the choice of H, it follows that $d_{L(G)}(\bigcup_{i \in T} H_i, H - \bigcup_{i \in T} H_i) \leq k - 1$. Let $P = e_1 e_2 \ldots e_s$ be a shortest path with $s \leq k$ from $\bigcup_{i \in T} H_i$ to $H - \bigcup_{i \in T} H_i$ with $e_1 \in V(\bigcup_{i \in T} H_i) \subset \overline{E}(\bigcup_{i \in T} D_i)$ and $e_2 \in V(H - \bigcup_{i \in T} H_i) \subset \overline{E}(D - \bigcup_{i \in T} D_i)$. Since e_j and e_{j+1} are two adjacent edges in G for each $j \in \{1, \ldots, s-1\}$, it follows that the subgraph of G induced by edge set $\{e_1, e_2, \ldots, e_s\}$ is a path between $\bigcup_{i \in T} D_i$ and $H - \bigcup_{i \in T} D_i$. Hence $d_G(\bigcup_{i \in T} D_i, H - \bigcup_{i \in T} D_i) = s \leq k$. Together with $d_G(x, V(D) - \{x\}) \leq k$ for every isolated vertex x in D, D satisfies (III).

As H satisfies (IV*) and (V*), Lemma 7 yields that D satisfies (IV*) and (V*). $\square$

Proof of Theorem 6. Suppose that $EDU_2(G) \neq \emptyset$. We choose an $H \in EDU_2(G)$ with a minimum number of components $H_1, H_2, \ldots, H_t$.

By the definition of H, $|\overline{E}(H_i)| \geq 3$ holds for each $H_i (1 \leq i \leq t)$. Hence we can find a cycle C_i of $L(G)$ with length at least 3 such that $V(C_i) = |\overline{E}(H_i)|$. Let $C = \bigcup_{i=1}^{t} C_i$.

We claim that $C_1, C_2, \ldots, C_t$ are edge-disjoint. Otherwise there would exist two components H_i, H_j and edges e_1, e_2 in $\overline{E}(H_i) \cap \overline{E}(H_j)$ with the same set of endvertices. And $H + \{e_1, e_2\}$ is a subgraph of G in $EDU_2(G)$ that contains fewer components than H, a contradiction. Hence C is an even subgraph of $L(G)$.

Furthermore, since $d_G(H_1, H - H_1) \leq 1$, C is connected. Hence C is an eulerian subgraph of $L(G)$. By Lemma 7, any branch in $\mathfrak{B}(L(G)) \setminus \mathfrak{B}_H(L(G))$ has length at most three and any branch in $\mathfrak{B}_1(L(G))$ has length at most two. Then any component of $G - V(C)$ is of order at most two.

Therefore C is a D_3-eulerian subgraph of $L(G)$, which implies that $L_2(G)$ has a dominating cycle by Theorem 2.

Conversely, suppose that $L_2(G)$ has a dominating cycle. By Theorem 2, $L(G)$ has a D_3-eulerian subgraph C since $L(G)$ is simple. Then by Lemma 8, G has an even subgraph H corresponding to C satisfying (1) to (4). Let $H' = H \cup (\bigcup_{i=3}^{\Delta(G)} V_i(G))$. We will prove that $H' \in EDU_2(G)$.

We claim that for any $x \in \bigcup_{i=3}^{\Delta(G)} V_i(G)$, $d_G(x, H) \leq 1$. Otherwise the edges adjacent to x form a clique of $L(G)$. Such clique is contained in a component of $L(G) - V(C)$ of order at least three, a contradiction. Together with property (3) of H, $d_G(H'_1, H' - H'_1) \leq 1$ holds for every subgraph $H'_1 \in H'$. Hence H' satisfies (III).

Since C is a D_3-eulerian subgraph of $L(G)$, any component of $L(G) - V(C)$ is of order at most two. Then any branch in $\mathfrak{B}(L(G)) \setminus \mathfrak{B}_C(L(G))$ has length at most three and any branch in $\mathfrak{B}_1(L(G))$ has length at most two. By Lemma 7, any branch in $\mathfrak{B}(G) \setminus \mathfrak{B}_{H'}(L(G))$ has length at most four and any branch in $\mathfrak{B}_1(G)$ has length at most three. Hence H' satisfies (IV*) and (V*). $\square$

3 Characterization of graphs with iterated line graphs that have dominating induced cycles

Berge's Strong Perfect Graph Conjecture is a longstanding conjecture in graph theory that relates to induced cycles of graphs, which states that a graph is perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph. Some forty years after Berge proposed this conjecture, it was proved by Chudnovsky et al.[8]. In this section, we will investigate the existence of dominating induced cycles of graphs.

We first introduce some special graphs used in this section. Let $\mathcal{G}_1$ be the set of those simple connected graphs G with $\Delta(G) = 3$ such that there exists a cycle C of G with $\Delta(G - E(C)) = 1$, that is, the deletion of edges in $E(C)$ would result in a graph with maximum degree 1, and $E(G)$ can be partitioned into an edge set of a cycle and a matching.

Let $\mathcal{G}_2$ be the set of those simple connected graphs G with $\Delta(G) = 3$ such that there exists a cycle C of G containing all vertices of degree three in G and satisfying the following conditions:

(i) each branch in $\mathcal{B}_1(G)$ is of length at most two,

(ii) each branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with edges in C is of length at least two,

(iii) each branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with edges not in C is of length exactly two.

Let $T_{i,j,k}$ be a tree obtained from three disjoint paths of length $i, j, k \geq 1$ by identifying one endvertex of each of them. $T_{i,j,k}$ has three branches of length i, j, k. Let $Z_1 = L(T_{1,1,2})$, $Z_2 = L(T_{1,1,3})$, $B_{1,1} = L(T_{1,2,2})$ and $N_{1,1,1} = L(T_{2,2,2})$. We use $Z_{i,j}$ to denote the graph obtained from a triangle by identifying a vertex of the triangle with each endvertex of two path of length i and j. Let $K_{1,4}^{(i)}$ be the graph obtained from $K_{1,4}$ by subdividing i edges of $K_{1,4}$ once for $1 \leq i \leq 4$, where subdividing an edge e is replacing it by the path of length two. Some of the graphs and their iterated line graphs are shown in Table 1, where one dominating induced cycle of a graph is marked with thick lines if such cycle exists.

Since the line graph of a cycle is still a cycle, in the discussions of dominating induced cycles of iterated line graph problems, we assume that graphs under considerations are connected other than a cycle.

Theorem 9. Let G be a connected simple graph other than a cycle. Then

(1) $L(G)$ has a dominating induced cycle if and only if $G \in \mathcal{G}_1 \cup \{Z_{1,1}, Z_{1,2}, K_{1,4}, K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)}, K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}, Z_2\}$;

(2) $L^2(G)$ has a dominating induced cycle if and only if $G \in \mathcal{G}_2 \cup \{K_{1,3}, T_{1,1,2}, T_{1,1,3}, T_{1,2,2}, T_{2,2,2}, K_{1,4}\}$;

(3) $L^3(G)$ has a dominating induced cycle if and only if $G \in \{K_{1,3}, T_{1,1,2}, T_{1,1,3}\}$;

(4) for $i \geq 4$, $L^i(G)$ has a dominating induced cycle if and only if $G = K_{1,3}$.

Corollary 10. Let G be a connected simple graph with at least eight edges other than a cycle. Then

(1) if $L(G)$ has a dominating induced cycle, then $G \in \mathcal{G}_1$;

(2) if $L^2(G)$ has a dominating induced cycle, then $G \in \mathcal{G}_2$;

(3) for $i \geq 3$, $L^i(G)$ has no dominating induced cycle.

Lemma 11. Let G be a connected simple graph with $\Delta(G) \geq 4$. Then $L(G)$ has a dominating induced cycle if and only if $G \in \{Z_{1,1}, Z_{1,2}, K_{1,4}, K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)}\}$.

Proof. It is not difficult to verify the sufficient part. The line graphs of graphs $Z_{1,1}, Z_{1,2}, K_{1,4}, K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)}$ have dominating induced cycles.

We now present the necessity. Let G be a connected simple graph of maximum degree at least four whose line graph has a dominating induced cycle. We claim that $\Delta(G) = 4$. Otherwise suppose u is a vertex of degree at least five and let $e_1, \ldots, e_5$ be five edges of G incident to u. Then $L(G)[\{e_1, \ldots, e_5\}]$ is a complete graph of order five. However, any induced cycle of $L(G)$ cannot dominate all edges of K_5, a contradiction.

Let v be a vertex of degree four of G. We have the following observation.

Claim 1. *$G - v$ has no P_3 as subgraph.*

Proof. Suppose to the contrary that $u_1e_1u_2e_2u_3$ is a P_3 of $G - v$. Let $e_3, \ldots, e_6$ be four edges of G incident to v. Then $L(G)[\{e_3, \ldots, e_6\}]$ is a complete graph of order four. Suppose C is a dominating induced cycle of $L(G)$. Then C contains exactly three vertices of $L(G)[\{e_3, \ldots, e_6\}]$ and it is a dominating triangle. However, the edge e_1e_2 of $L(G)$ cannot be dominated by such triangle, a contradiction. □

We further claim that $|V_4(G)| = 1$, that is, v is the unique vertex of degree four of G. Suppose otherwise that v_1 and v_2 are two distinct vertices of degree four in G. Then by Claim 1, $G - v_1$ has a subgraph P_3 with internal vertex v_2, a contradiction. By similar reasoning, we have $|V_3(G)| = 0$. Next we divide the proof into two cases according to whether G is a tree or not.

If G is a tree, then it has four branches since $|V_4(G)| = 1$ and $|V_3(G)| = 0$. Having a dominating cycle is obviously a necessary condition for a graph to have a dominating induced cycle. By Theorem 2, each branch of G is of length at most two. Since $L(K_{1,4}^{(4)})$ has no dominating induced cycle. We have $G \in \{K_{1,4}, K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)}\}$.

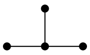
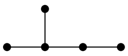
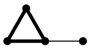
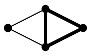
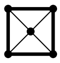
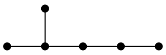
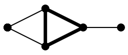
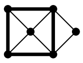
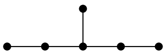
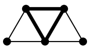
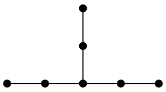
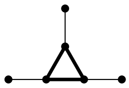
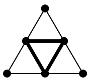
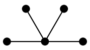
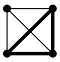
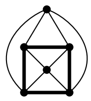
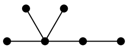
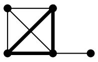
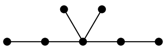
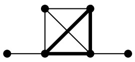
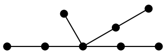
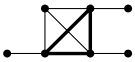
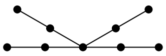
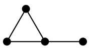
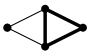
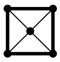
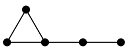
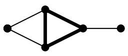
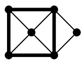
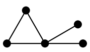
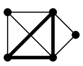
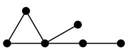
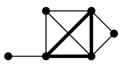
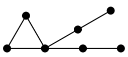
G		$L(G)$	$L^2(G)$	$L^3(G)$	$L^i(G)(i \geq 4)$
$K_{1,3}$					
$T_{1,1,2}$					$\times$
$T_{1,1,3}$		$\times$			$\times$
$T_{1,2,2}$				$\times$	$\times$
$T_{2,2,2}$				$\times$	$\times$
$K_{1,4}$				$\times$	$\times$
$K_{1,4}^{(1)}$			$\times$	$\times$	$\times$
$K_{1,4}^{(2)}$			$\times$	$\times$	$\times$
$K_{1,4}^{(3)}$			$\times$	$\times$	$\times$
$K_{1,4}^{(4)}$		$\times$	$\times$	$\times$	$\times$
Z_1				$\times$	$\times$
Z_2				$\times$	$\times$
$Z_{1,1}$			$\times$	$\times$	$\times$
$Z_{1,2}$			$\times$	$\times$	$\times$
$Z_{2,2}$		$\times$	$\times$	$\times$	$\times$

Table 1: The existence of dominating induced cycles of some iterated line graphs.

If G is not a tree, then v is a cut vertex of G since other vertices are of degree at most two. Moreover, since $G - v$ has no P_3 as subgraph, we have $c(G) = 3$. It is not difficult to verify that G has a unique triangle. Together with Theorem 2, only graphs $Z_{1,1}, Z_{1,2}, Z_{2,2}$ satisfy these conditions. Since $L(Z_{2,2})$ has no dominating induced cycle, we have $G \in \{Z_{1,1}, Z_{1,2}\}$. $\Box$

Lemma 12. Let G be a connected simple graph with $\Delta(G) = 3$. Then $L(G)$ has a dominating induced cycle if and only if $G \in \mathcal{G}_1 \cup \{K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}, Z_2\}$.

Proof. It is not difficult to verify the sufficient part. Suppose $C = u_1e_1u_2 \ldots u_ie_iu_1$ is a cycle of a graph G in $\mathcal{G}_1$ such that $\Delta(G - E(C)) = 1$. Then $L(C) = e_1e_2 \ldots e_ie_1$ is a dominating induced cycle of $L(G)$ since $L(G) - L(C)$ is edgeless. And the line graphs of graphs $K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}, Z_2$ have dominating induced cycles.

We now present the necessity. By Theorem 2, $L(G)$ has a dominating cycle, which implies that G has a D_3-eulerian subgraph. Since $\Delta(G) = 3$, such D_3-eulerian subgraph is actually a D_3-cycle or a D_3-vertex.

Suppose first that C is a longest D_3-cycle of G. Then branches of G with both endvertices on C are of length at most three and branches of G with exactly one endvertex on C are of length at most two.

Let C' be a dominating induced cycle of $L(G)$. If $|V(C)| \geq 4$, then $E(C) \subset V(C')$. Actually we have $V(C') = E(C)$, otherwise C' has a chord with endvertices in $E(C)$, a contradiction. And note that consecutive edges not on C will result in an edge not dominated by C' in $L(G)$. Hence branches of G with edges not on C are of length one. That is, $G \in \mathcal{G}_1$. If $|V(C)| = 3$, then $G \in \{Z_1, B_{1,1}, N_{1,1,1}\}$ or $G = Z_2$, where $\{Z_1, B_{1,1}, N_{1,1,1}\} \subset \mathcal{G}_1$.

Suppose then that G has no D_3-cycle but it has a D_3-vertex. Then G is a tree and the D_3-vertex is the unique vertex of degree three in G. Hence $G \in \{K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}\}$. Therefore $G \in \mathcal{G}_1 \cup \{K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}, Z_2\}$. $\Box$

One easily derives from Lemma 11 the following corollary, which is useful in the proof of Theorem 9.

Corollary 13. Let G be a connected simple graph.

(1) If G is a line graph with $\Delta(G) \geq 4$, then $L(G)$ has no dominating induced cycle.

(2) If $d(x) + d(y) \geq 6$ holds for some $xy \in E(G)$, then $L^2(G)$ has no dominating induced cycle.

(3) If G is a graph with $\Delta(G) \geq 4$ other than $K_{1,4}$, then $L^2(G)$ has no dominating induced cycle.

Proof of Theorem 9.

Since the i-iterated line graph of a path is still a path or empty, we suppose that G is neither a path nor a cycle.

(1) Using Lemmas 11 and 12, one easily derives (1) of Theorem 9.

(2) It is not difficult to verify the sufficient part. Suppose C is a cycle of a graph G in $\mathcal{G}_2$ containing all vertices of degree three and satisfying conditions(i), (ii) and (iii). Then there exists a cycle C' of $L(G)$ having $\overline{E}(C)$ as its vertex set, $V(C') = \overline{E}(C)$. Since each branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with edges in C is of length at least two, we have $\Delta(L(G)) = 3$ and C' contains all vertices of degree three of $L(G)$. Since G satisfies conditions (i) and (iii), C' is actually a cycle such that

$\Delta(L(G) - E(C')) = 1$, that is, $L(G) \in \mathcal{G}_1$. By (1) of Theorem 9, $L^2(G)$ has a dominating induced cycle. And the 2-iterated line graphs of graphs $K_{1,3}, T_{1,1,2}, T_{1,1,3}, T_{1,2,2}, T_{2,2,2}, K_{1,4}$ are depicted in Table 1. They all have dominating induced cycles.

We now present the necessity.

Suppose $L^2(G)$ has a dominating induced cycle. Note that graphs $Z_{1,1}, Z_{1,2}, K_{1,4}, K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)}$, $K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}$ are not line graphs. Then by (1) of Theorem 9, $L(G) \in \mathcal{G}_1$ or $L(G) = Z_2$. Hence $G = T_{1,1,3}$ or G is a graph whose line graph is in $\mathcal{G}_1$.

Note that graphs in $\mathcal{G}_1$ have dominating cycles, then $L(G)$ has dominating cycles. By Theorem 2, G has a D_3-eulerian subgraph. Suppose that G is not $K_{1,4}$. Then by (3) of Corollary 13, $\Delta(G) = 3$. The D_3-eulerian subgraphs of G are actually D_3-cycles or D_3-vertices. Note also that by (2) of Corollary 13, vertices of degree three of G are pairwise nonadjacent.

Suppose first that G has a D_3-cycle. We choose a longest D_3-cycle of G, denoted C_0. We claim that C_0 contains all vertices of degree three of G. Otherwise suppose v is a vertex of degree three not on C_0, then v is at distance at least two to C_0, contradicting that C_0 is D_3-dominating.

Since C_0 is D_3-dominating, each branch in $\mathcal{B}_1(G)$ is of length at most two, satisfying condition(i). Since vertices of degree three of G are pairwise nonadjacent, each branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with edges in C is of length at least two, satisfying condition(ii).

Each branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ with edges not in C_0 is of length at least two and at most three. However, C_0 is a longest D_3-cycle of G, which implies that the length of branch in $\mathcal{B}(G) \setminus \mathcal{B}_1(G)$ cannot be three, otherwise $L(G)$ is not in $\mathcal{G}_1$, a contradiction. Hence G satisfies condition(iii). Therefore $G \in \mathcal{G}_2$.

Suppose then that G has no D_3-cycle but a D_3-vertex. Then this D_3-vertex is the unique vertex of degree greater than two of G. Together with $\Delta(G) = 3$, we have $G \in \{K_{1,3}, T_{1,1,2}, T_{1,2,2}, T_{2,2,2}\}$. Therefore $G \in \mathcal{G}_2 \cup \{K_{1,3}, T_{1,1,2}, T_{1,1,3}, T_{1,2,2}, T_{2,2,2}, K_{1,4}\}$.

(3) Suppose that G is a star. Then $G = K_{1,3}$ since $L^3(K_{1,4})$ has no dominating induced cycles. Suppose that G is not a star. Then the maximum degree of $L^i(G)$ is nondecreasing with respect to i, that is, $\Delta(L^{i+1}(G)) \geq \Delta(L^i(G))$. Hence by (1) of Corollary 13, $\Delta(G) = 3$.

Suppose that G is a tree. Then it has a unique vertex of degree three, and hence it has three branches. Since any vertices of degree three in $L(G)$ cannot be adjacent, at least two branches of G is of length one. Hence $G = T_{1,1,2}$ or $G = T_{1,1,3}$. Suppose that G is not a tree, then there exists a triangle that has two adjacent vertices of degree three in $L(G)$. By (2) of Corollary 13, $L^3(G)$ has no dominating induced cycle. Therefore $G \in \{K_{1,3}, T_{1,1,2}, T_{1,1,3}\}$.

(4) G has a vertex of degree at least 3. Suppose that G is not $K_{1,3}$. Then $L^{i-2}(G)$ contains a triangle with two adjacent vertices of degree at least three. Hence by (2) of Corollary 13, $L^i(G)$ has no dominating induced cycle for $i \geq 4$. □

4 Analysis of the complexity for the existence of dominating (induced) cycles in iterated line graphs

It was showed in [21] that the problem to decide whether the hamiltonian index of a given graph is less than or equal to a given constant is NP-complete, while it has a polynomial time algorithm to determine a graph has a 2-factor and an even factor in iterated line graphs [27]. However, both Theorem 3 and the result in [21] may imply that it is NP-complete to determine whether a iterated line graph has a dominating cycle. In [29], it was showed that it is also NP-hard to determine the length of a longest induced cycle in line graphs.

Theorem 9(3)(4) implies that there is a polynomial time algorithm to determine whether $L^i(G)(i \geq 3)$ has a dominating induced cycle. Now we consider those line graphs and 2-iterated line graphs.

Theorem 14. Let H be a cubic graph and H^* be the graph obtained from H by subdividing each edge of H once. Then $L^2(H^*)$ has a dominating induced cycle if and only if H is hamiltonian.

Proof. Suppose that H is hamiltonian. Then H^* has a dominating cycle containing all vertices of degree three in H^*, say C. Moreover, all branches of H^* with edges not in C are of length exactly two, and these branches are pairwise at distance at least two, satisfying conditions (i)(ii)(iii) of graphs in $\mathcal{G}_2$. Hence $H^* \in \mathcal{G}_2$. By Theorem 9(2), $L^2(H^*)$ has a dominating induced cycle.

Suppose that $L^2(H^*)$ has a dominating induced cycle. By Theorem 9(2), $H^* \in \mathcal{G}_2$ since graphs $K_{1,3}, T_{1,1,2}, T_{1,1,3}, T_{1,2,2}, T_{2,2,2}, K_{1,4}$ are not cubic. By the definition of $\mathcal{G}_2$, H^* has a cycle containing all vertices of degree three in H^*. Then H also has a cycle containing all vertices of degree three in H, that is, H is hamiltonian. $\square$

Note that $\mathcal{G}_2$ contains those graphs by subdividing each edge of a hamiltonian cubic graph once. It was showed in [11] that it is NP-complete to decide whether a given cubic graph is hamiltonian. Therefore, by Theorem 14, we have the following result.

Theorem 15. It is NP-complete to decide whether a given graph, particularly, a 2-iterated line graph, has a dominating induced cycle.

Observe that 2-iterated line graphs are a subclass of line graphs. Then it is NP-complete to decide whether a given line graph has a dominating induced cycle.

5 Concluding remarks

(1) Comparing Theorem 3 with Theorem 2, it turns out that the characterizations are different. One might think that there may be a unified characterization for Theorem 2 in terms of branches. However, Fig. 1 shows that for each eulerian subgraph H of G_0 containing $V_3(G_0)$, there exists a long branch in $\mathcal{B}(G_0) \setminus \mathcal{B}_H(G_0)$. But $L(G_0)$ is hamiltonian. Therefore Theorem 2 is not a special case of Theorem 3.

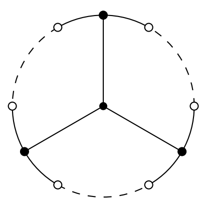


Figure 1: The graph G_0

(2) Comparing Theorem 1(1) with Theorem 2, they can be unified as Corollary 16. However, Corollary 16 cannot be directly extended to $i \geq 3$, as shown by the graph obtained from a long cycle of length $n - i$ and a complete graph of order i by adding an edge between them.

Corollary 16. Let G be a connected simple graph that is not a path. Then $L(G)$ has a D_i-cycle if and only if G has a D_{i+1}-eulerian subgraph for $i = 1, 2$.

Comparing Theorem 1(2) with Theorem 3, it turns out that the existence of Hamilton cycles and dominating cycles of iterated line graphs depend on the existence of $EU_k(G)$ and $EDU_k(G)$, respectively.

(3) Note that the property "every longest cycle of a graph G is dominating " is stronger than"G has a dominating cycle". However, the former characterizations of Theorems 2 and 3 that guarantee $L^i(G)$ to have a dominating cycle cannot force any longest cycle of $L^i(G)$ to be a dominating cycle.

We look at a family of examples:

Let F be a graph of vertex set $V(F) = \{u_1, u_2, v_1, v_2, w_1, w_2\}$ with edge set $E(F) = \{u_iv_i, u_iw_i, v_iw_i : i = 1 \text{ or } 2\} \cup \{u_1u_2, v_1v_2, w_1w_2\}$.

For any nonegative integer $i \geq 0$, let F_i be a graph derived from F by subdividing edges $u_1u_2, v_1v_2, u_1w_1, v_1w_1$ i times, and subdividing edges u_1v_1, u_2v_2, w_1w_2 $i+1$ times, and subdividing edges u_2w_2, v_2w_2 $i+2$ times. Then F_i has branches $b(u_1, u_2), b(v_1, v_2), b(u_1, w_1), b(v_1, w_1)$ of length $i+1$, $b(u_1, v_1), b(u_2, v_2), b(w_1, w_2)$ of length $i+2$, and $b(u_2, w_2), b(v_2, w_2)$ of length $i+3$.

The length of a longest cycle of a graph G is called its *circumference*, denoted $c(G)$. Note that $u_1b(u_1, v_1)v_1b(v_1, v_2)v_2b(v_2, u_2)u_2b(u_2, w_2)w_2b(w_2, w_1)w_1b(w_1, u_1)u_1$ is a longest cycle of F_0 of length 11, and $u_1b(u_1, w_1)w_1b(w_1, v_1)v_1b(v_1, v_2)v_2b(v_2, w_2)w_2(w_2, u_2)u_2b(u_2, u_1)u_1$ is a dominating cycle of length 10. Therefore F_0 is a graph of order 13 with $c(F_0) = 13 - 2 = 11$, while all dominating cycles of F_0 are of length $13 - 3 = 10$. F_1 is a graph of order 22, $L(F_1)$ is of order 25 with $c(L(F_1)) = 22$, while all dominating cycles of F_1 are of length 21. F_2 is a graph of order 31, $L(F_2)$ is of order 34, and $L^2(F_2)$ is of order 43 with $c(L^2(F_1)) = 41$, while all dominating cycles of $L^2(F_2)$ are of length 40.

F_i is a graph of order $9i + 13$ with $c(L^i(F_i)) = n(L^i(F_i)) - 2$, while all dominating cycles of $L^i(F_i)$ are of length $n(L^i(F_i)) - 3$. Therefore, the i-iterated line graph of $L^i(F_i)$ has dominating cycles while its longest cycles are not dominating.

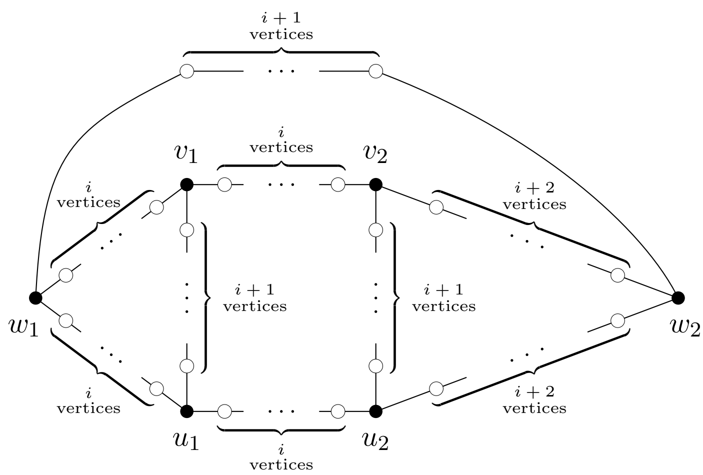


Figure 2: The graph F_i

(4) As we discussed in Section 2 of this paper, it would be interesting to consider the following question: Does there exist a characterization of those graphs G such that each longest cycle of $L^i(G)$ is dominating for any $i \geq 1$?

Acknowledgements The authors thanks three anonymous referees for their nice suggestions to improve the original presentation. The work is supported by the Natural Science Funds of China (Nos: 12131013 and 11871099).

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