

Graph tilings in incompatibility systems

Jie Hu^a Hao Li^b Yue Wang^c Donglei Yang^{d*}

^a Center for Combinatorics and LPMC,
Nankai University, Tianjin 300071, China

^b Laboratoire Interdisciplinaire des Sciences du Numérique,
CNRS–Université Paris-Saclay, Orsay 91405, France

^c School of Mathematics and Statistics,
Shandong Normal University, Jinan 250358, China

^d Data Science Institute,
Shandong University, Jinan 250100, China

Abstract

Given two graphs H and G , an H -tiling of G is a collection of vertex-disjoint copies of H in G and an H -factor is an H -tiling that covers all vertices of G . Kühn and Osthus managed to characterize, up to an additive constant, the minimum degree threshold which forces an H -factor in a host graph G . In this paper we study a similar tiling problem in a system that is locally bounded. An *incompatibility system* \mathcal{F} over G is a family $\mathcal{F} = \{F_v\}_{v \in V(G)}$ with $F_v \subseteq \{\{e, e'\} \in \binom{E(G)}{2} : e \cap e' = \{v\}\}$. We say that two edges $e, e' \in E(G)$ are *incompatible* if $\{e, e'\} \in F_v$ for some $v \in V(G)$, and otherwise *compatible*. A subgraph H of G is *compatible* if every pair of edges in H are compatible. An incompatibility system \mathcal{F} is Δ -bounded if for any vertex v and any edge e incident with v , there are at most Δ members of F_v containing e . This notion was partly motivated by a concept of transition system introduced by Kotzig in 1968, and first formulated by Krivelevich, Lee and Sudakov to study the robustness of Hamiltonicity of Dirac graphs.

We prove that for any $\alpha > 0$ and any graph H with h vertices, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, if G is an n -vertex graph with $\delta(G) \geq (1 - \frac{1}{\chi^*(H)} + \alpha)n$ and \mathcal{F} is a μn -bounded incompatibility system over G , then there exists a compatible H -factor in G , where the value $\chi^*(H)$ is either the chromatic number $\chi(H)$ or the critical chromatic number $\chi_{cr}(H)$ and we provide a dichotomy as in the Kühn–Osthus result. Moreover, we give examples for which the error term αn cannot be replaced by any sublinear function. Unlike in the previous work of Kühn and Osthus on embedding H -factors, our proof uses the lattice-based absorption method.

1 Introduction

All the graphs considered here are finite, undirected and simple. Let H be an h -vertex graph and G be an n -vertex graph. An H -tiling is a collection of vertex-disjoint copies of H in G . An H -factor (or *perfect H -tiling*) is an H -tiling which covers all vertices of G . Note that $n \in h\mathbb{N}$ is a necessary condition for G containing an H -factor.

1.1 Perfect graph tilings

One of the most fundamental research topics in extremal graph theory is to determine sufficient conditions forcing spanning structures, such as perfect matchings, Hamilton cycles, H -factors,

*Data Science Institute, Shandong University, Shandong, China. Email: dlyang@sdu.edu.cn. Supported by the China Postdoctoral Science Foundation (2021T140413), Natural Science Foundation of China (12101365) and Natural Science Foundation of Shandong Province (ZR2021QA029).

etc. Textbook results of Hall and Tutte give a sufficient condition for the existence of a perfect matching (see e.g. [20]). A classical theorem of Dirac [21] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. For H -factors, when $H = K_k$, the seminal Hajnal–Szemerédi theorem [25] characterizes the minimum degree forcing a K_k -factor and the bound is best possible.

Theorem 1.1 (Hajnal and Szemerédi [25]). *Let G be an n -vertex graph with $n \in k\mathbb{N}$. If $\delta(G) \geq (1 - 1/k)n$, then G contains a K_k -factor.*

Up to an error term, the following theorem of Alon and Yuster [4] generalizes Theorem 1.1.

Theorem 1.2 (Alon and Yuster [4]). *For every $\alpha > 0$ and every graph H there exists an integer n_0 such that every graph G whose order $n \geq n_0$ is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi(H) + \alpha)n$ contains an H -factor.*

Later, it was proved by Komlós, Sárközy and Szemerédi [35] that the $o(n)$ term in Theorem 1.2 can be replaced with a constant $C = C(H)$, which resolves a conjecture of Alon and Yuster [4]. As given in [15, 31], there are graphs H for which the term $1 - 1/\chi(H)$ in the minimum degree condition can be improved significantly. To illustrate this, Komlós [34] introduced the concept of critical chromatic number. The *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as $(\chi(H) - 1)|H|/(|H| - \sigma(H))$, where $\sigma(H)$ denotes the minimum size of the smallest color class in a proper coloring of H with $\chi(H)$ colors. Note that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$. Komlós [34] proved that one can replace $\chi(H)$ with $\chi_{cr}(H)$ at the price of obtaining an H -tiling covering all but εn vertices, and he also conjectured that the error term εn can be replaced with a constant that only depends on H . This was confirmed by Shokoufandeh and Zhao [54].

Theorem 1.3 (Shokoufandeh and Zhao [54]). *For any H there is an integer n_0 so that if G is a graph on $n \geq n_0$ vertices and minimum degree at least $(1 - 1/\chi_{cr}(H))n$, then G contains an H -tiling that covers all but at most $5|H|^2$ vertices.*

Further, a significant result of Kühn and Osthus [45] managed to characterize, up to an additive constant, the best possible minimum degree condition which forces an H -factor. To state their result, we need more definitions here. Let $\chi(H) = k$ and $\mathcal{C}(H)$ be the family of all proper k -colorings of H . Given a k -coloring c in $\mathcal{C}(H)$, let $h_1^c \leq \dots \leq h_k^c$ be the sizes of the color classes of c and $D(c) := \{h_{i+1}^c - h_i^c \mid i \in [k-1]\}$. Let

$$D(H) := \bigcup_{c \in \mathcal{C}(H)} D(c).$$

Then we denote by $\text{hcf}_\chi(H)$ the highest common factor of all the integers in $D(H)$. In particular, if $D(H) = \{0\}$, then we set $\text{hcf}_\chi(H) := \infty$. Also, we write $\text{hcf}_c(H)$ for the highest common factor of all the orders of components of H . If $\chi(H) > 2$, then define $\text{hcf}(H) = 1$ if $\text{hcf}_\chi(H) = 1$. If $\chi(H) = 2$, then define $\text{hcf}(H) = 1$ if both $\text{hcf}_c(H) = 1$ and $\text{hcf}_\chi(H) \leq 2$. Then let

$$\chi^*(H) = \begin{cases} \chi_{cr}(H) & \text{if } \text{hcf}(H) = 1, \\ \chi(H) & \text{otherwise.} \end{cases}$$

Theorem 1.4 (Kühn and Osthus [45]). *For any H there exist integers $C = C(H)$ and $n_0 = n_0(H)$ such that every graph G whose order $n \geq n_0$ is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi^*(H))n + C$ contains an H -factor.*

The degree condition in Theorem 1.4 is best possible up to the constant C and there are also graphs H for which the constant C cannot be omitted entirely [45]. There are some generalisations of Theorem 1.4. For example, Kühn, Osthus and Treglown [46] determined, asymptotically, the Ore-type degree condition forcing an H -factor; Hyde and Treglown [30] proved a degree sequence version of Theorem 1.4. Very recently, Hurley, Joos and Lang [29] recovered and extended the work of Kühn and Osthus by replacing the H -factor with bounded degree graphs with components of sublinear order.

1.2 Motivation

As mentioned above, many typical Dirac-type results in extremal graph theory are of the form “under certain degree conditions, G has property \mathcal{P} ”. Once such a result is established, it is natural to ask how strongly does G possess \mathcal{P} ? In particular, this has motivated recent trends in the study of robustness of graph properties (see [55]), aiming to strengthen classical results in extremal graph theory and probabilistic combinatorics. What measures of robustness can one utilize? In this direction, several different measures of robustness have been explored in

1. (sparse) random graphs $G(n, p)$ in the setting of so-called resilience (see e.g. [3, 5, 7, 19, 40, 41, 56]);
2. edge-colored graphs (see e.g. [1, 2, 8, 13, 16, 17, 18, 23, 24, 33, 47, 50, 53, 57]);
3. Maker-Breaker games (see e.g. [6, 9, 14, 39, 44]).

For further works in this area the reader is referred to a comprehensive survey by Sudakov [55].

Finding spanning subgraphs in edge-colored graphs with certain constraints has also been widely and well studied. An edge-coloring c of a graph is (*globally*) k -*bounded* if every color appears at most k times in the coloring, while c is *locally* k -*bounded* if every color appears at most k times at any given vertex (so c is a proper edge-coloring when $k = 1$). A fundamental line of research is to find *rainbow subgraphs* (all edges have distinct colors) in g -bounded edge-colorings of graphs and *properly colored subgraphs* (any adjacent edges receive different colors) in locally ℓ -bounded edge-colorings of graphs. For perfect matchings, Erdős and Spencer [23] proved that any $\frac{n-1}{16}$ -bounded edge-coloring of $K_{n,n}$ admits a rainbow perfect matching. Recently, Coulson and Perarnau [17] considered the sparse version and obtained that there exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and G is a *Dirac bipartite graph* on $2n$ vertices, which is a balanced bipartite graph with minimum degree at least a quarter of its order, then any μn -bounded edge-coloring of G contains a rainbow perfect matching. For Hamilton cycles, a conjecture of Bollobás and Erdős [8] states that every locally $(\lfloor \frac{n}{2} \rfloor - 1)$ -bounded edge-colored K_n contains a properly colored Hamilton cycle. There is a series of partial results toward this conjecture (see e.g. [1, 13, 53]). In [47], Lo proved that the Bollobás–Erdős conjecture is true asymptotically. For sparse version, Coulson and Perarnau [18] derived that there exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and G is a *Dirac graph* on n vertices, which is a graph with minimum degree at least half of its order, then any μn -bounded edge-coloring of G contains a rainbow Hamilton cycle. For H -factors, Coulson, Keevash, Perarnau and Yepremyan [16] showed that there is a constant μ such that any μn -bounded edge-coloring of G with $\delta(G) \geq (1 - \frac{1}{\chi^*(H)} + o(1))n$ contains a rainbow H -factor.

In this paper, we consider a more general setting of incompatibility systems, which was first proposed by Krivelevich, Lee and Sudakov [43].

Definition 1.5 (Incompatibility system). Let $G = (V, E)$ be a graph.

- An *incompatibility system* \mathcal{F} over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of 2-subsets of edges incident with v , i.e. $F_v \subseteq \{\{e, e'\} \in \binom{E}{2} : e \cap e' = \{v\}\}$.
- For any two edges e and e' , if there exists some vertex v such that $\{e, e'\} \in F_v$, then we say that e and e' are *incompatible* at v . Otherwise, they are *compatible*. A subgraph $H \subseteq G$ is *compatible* if all pairs of edges are compatible.
- For a positive integer Δ , an incompatibility system \mathcal{F} is Δ -*bounded* if for any vertex v and any edge e incident with v , there are at most Δ other edges incident with v that are incompatible with e .

Locally ℓ -bounded edge-colorings can be viewed as transitive $(\ell - 1)$ -bounded incompatibility systems in which two adjacent edges are incompatible if only they have the same color, and compatible subgraphs generalize the concept of properly colored subgraphs. Note that there is also a similar generalization of g -bounded edge-colorings, called *systems of conflicts* (see [17]). Definition 1.5 was firstly introduced in [43] as a measure of robustness and motivated by two concepts in graph theory. First, it generalizes *transition systems* introduced by Kotzig [38] in 1968. In our terminology, a transition system is simply a 1-bounded incompatibility system. Kotzig's work was motivated by a problem of Nash–Williams on cycle covering of Eulerian graphs (see, e.g. Section 8.7 in [10]).

In [43], Krivelevich, Lee and Sudakov studied the robustness of Hamiltonicity of Dirac graphs with respect to the incompatibility system and derived Theorem 1.6.

Theorem 1.6 ([43]). *There exists a constant $\mu > 0$ such that the following holds for large enough n . For every n -vertex Dirac graph G and a μn -bounded incompatibility system \mathcal{F} defined over G , there exists a compatible Hamilton cycle.*

Their proof is based on Pósa's rotation-extension technique. Theorem 1.6 settled in a very strong form, a conjecture of Häggkvist from 1988 (see Conjecture 8.40 in [10]). They further studied compatible Hamilton cycles in random graphs in [42] and proved that there is a constant $\mu > 0$ such that if $p \gg \frac{\log n}{n}$, then w.h.p. $G = G(n, p)$ contains a compatible Hamilton cycle for every μnp -bounded incompatibility system defined over G , which strengthens the result about Hamilton cycles in $G(n, p)$ without restrictions of incompatibility systems.

The concept of incompatibility system appears to provide a new and interesting take on robustness of graph properties. Krivelevich, Lee and Sudakov [43] also suggested looking at how various extremal results can be strengthened in this context.

1.3 Main result and discussion

It is natural to consider other compatible spanning structures. We study minimum degree conditions for H -factors in incompatibility system. Note that throughout this paper, we always assume $\chi(H) \geq 2$, since $\chi(H) = 1$ is a trivial case. For convenience, given constants μ, δ and $n \in \mathbb{N}$, an (n, δ, μ) -incompatibility system (G, \mathcal{F}) consists of an n -vertex graph G with $\delta(G) \geq \delta n$ and a μn -bounded incompatibility system \mathcal{F} over G . Our main result is formally stated as follows.

Theorem 1.7. *Let $h \in \mathbb{N}$ and H be any h -vertex graph. For any $\alpha > 0$, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, every $(n, 1 - \frac{1}{\chi^*(H)} + \alpha, \mu)$ -incompatibility system (G, \mathcal{F}) contains a compatible H -factor. In particular, the term of α in the minimum degree condition cannot be omitted when H is a complete r -partite graph for $r \geq 3$.*

Towards the ‘in particular’ part in the statement, an obvious question is for which graphs the error term αn in the minimum degree condition can be replaced by a constant term.

1.3.1 A space barrier

For $r \geq 3$ and $h_1, \dots, h_r \geq 1$, we use $K_r(h_1, h_2, \dots, h_r)$ to denote the complete r -partite graph with each part of size h_i . In particular, when $h_i = s$ for any $i \in [r]$, we write K_r^s for $K_r(s, s, \dots, s)$. Now we show that the minimum degree condition is in some sense tight by the following result.

Proposition 1.8. *Let $H = K_r(h_1, h_2, \dots, h_r)$, $h = \sum_{i \in [r]} h_i$ and $n \in h\mathbb{N}$. For any $0 < \mu < \frac{\chi_{cr}(H) - r + 1}{\chi_{cr}(H)} - \frac{r-1}{n}$, there exists an $(n, 1 - \frac{1}{\chi^*(H)} + \frac{\mu}{2}, \mu)$ -incompatibility system (G, \mathcal{F}) which contains no compatible $K_r(h_1, h_2, \dots, h_r)$ -factor.*

To prove Proposition 1.8, we need two results as follows which provide lower bound constructions for the minimum degree threshold forcing an H -factor.

Proposition 1.9. [34] *For every graph H with $2 \leq \chi(H) =: r$ and every integer n that is divisible by $|H|$, there exists a complete r -partite graph G of order n with minimum degree $(1 - \frac{1}{\chi_{cr}(H)})n - 1$ which does not contain an H -factor.*

Since the above n -vertex graph G is complete r -partite with minimum degree $(1 - \frac{1}{\chi_{cr}(H)})n - 1$, the size of the largest vertex class of G is $\frac{n}{\chi_{cr}(H)} + 1$, while the size of the smallest vertex class is at least $n - (r - 1)(\frac{n}{\chi_{cr}(H)} + 1) = \frac{\chi_{cr}(H) - r + 1}{\chi_{cr}(H)}n - r + 1$.

Proposition 1.10. [45] *Let H be a graph with $3 \leq \chi(H) =: r$ and let $h \in \mathbb{N}$. Let G be the complete r -partite graph of order $k|H|$ whose vertex classes V_1, \dots, V_r satisfy $|V_1| = \lfloor k|H|/r \rfloor + 1, |V_2| = \lceil k|H|/r \rceil - 1$ and $\lfloor k|H|/r \rfloor \leq |V_i| \leq \lceil k|H|/r \rceil$ for all $i \geq 3$. So $\delta(G) = \lceil (1 - \frac{1}{\chi(H)})|G| \rceil - 1$. If $\text{hcf}(H) \neq 1$, then G does not contain an H -factor.*

Proof of Proposition 1.8. Let G_0 be an n -vertex complete r -partite graph with $\delta(G_0) \geq (1 - \frac{1}{\chi^*(H)})n - 1$ such that G_0 contains no H -factors. Indeed such G_0 can be obtained from Proposition 1.9 or Proposition 1.10. Write V_1, \dots, V_r for the r parts of G_0 and inside every part V_i of G_0 , we add a spanning bipartite subgraph with minimum degree at least $\frac{\mu n}{2} + 1$ and maximum degree at most μn . Denote the resulting graph by G . Hence, $\delta(G) \geq (1 - \frac{1}{\chi^*(H)} + \frac{\mu}{2})n$ and for every $i \in [r]$, $G[V_i]$ is a triangle-free graph with $\delta(G[V_i]) \geq \frac{\mu n}{2} + 1$ and $\Delta(G[V_i]) \leq \mu n$. Now we define an incompatibility system \mathcal{F} over G . For any two different parts V_i, V_j of G , let v be any vertex in V_i and u, w be any two different vertices in V_j . If uw is an edge in $G[V_j]$, then let vu and vw be incompatible at v . Since $\Delta(G[V_j]) \leq \mu n$, \mathcal{F} is μn -bounded.

Now it remains to verify that there is no compatible $K_r(h_1, h_2, \dots, h_r)$ -factor. Before that, for any subgraph $F \subseteq G$ we define the *index vector* $\mathbf{i}_F = (x_1, \dots, x_r)$ by choosing $x_i = |V(F) \cap V_i|, i \in [r]$. We claim that for every compatible copy of $K_r(h_1, h_2, \dots, h_r)$ in G , its index vector is an r -tuple of h_1, h_2, \dots, h_r . In fact, assume that there is a compatible copy of $K_r(h_1, h_2, \dots, h_r)$ in G , denoted by K , whose index vector is not a permutation of h_1, h_2, \dots, h_r . Then K has an edge in $G[V_i]$ for some $i \in [r]$, say uw . Since $r \geq 3$, uw is contained in a triangle of K , say vuw . As $G[V_i]$ is triangle-free, one can observe that v is in some V_j for $j \neq i$ and by assumption vu and vw are compatible at v , which contradicts the definition of \mathcal{F} . Hence, since G_0 contains no $K_r(h_1, h_2, \dots, h_r)$ -factor, it follows from the aforementioned claim that G contains no compatible $K_r(h_1, h_2, \dots, h_r)$ -factor as well. \square

The rest of the paper is organised as follows. In Section 2, we set up some basic notation and outline the proof of Theorem 1.7. Then we state two crucial results (Lemmas 2.3 and 2.4) that decode the proof of Theorem 1.7. Sections 3 and 4 are devoted to proving Lemmas 2.3 and 2.4, respectively.

2 Notation and preliminaries

For a graph G , we use $e(G)$ to denote the number of edges of G . For a graph $G = (V, E)$ and k pairwise disjoint vertex subsets $U_1, \dots, U_k \subseteq V$, we use $G[U_1, \dots, U_k]$ to denote the r -partite subgraph induced by U_1, \dots, U_r . In the proofs, if we choose $a \ll b$, then this means that for any $b > 0$, there exists $a_0 > 0$ such that for any $a < a_0$ the subsequent statement holds. Hierarchies of other lengths are defined similarly.

2.1 Proof strategy and main tools

Our proof uses the absorption method, pioneered by the work of Rödl, Ruciński and Szemerédi [52] on perfect matchings in hypergraphs. In recent years, the method has become an extremely important tool for studying the existence of spanning structures in graphs, digraphs and hypergraphs. Before its appearance a well-known systematic way in this area is the blow-up lemma due to Komlós, Sárközy and Szemerédi [36], which has successfully been adopted by many scholars for various classical results [4, 45]. It is worth noting that Glock and Joos [24] developed a rainbow version of the blow-up lemma, which enables the systematic study of rainbow embeddings of bounded degree spanning subgraphs [22, 33]. However, there has been no such variant for incompatibility systems. Instead, we utilize the absorption framework and develop a different counting argument for embedding compatible graphs under certain pseudorandomness conditions (see Lemma 3.6).

The general idea of absorption is to split the problem of finding perfect tilings into two subproblems. The first major task is to define and find an absorbing set in the host graph which can ‘absorb’ left-over vertices. We will start with the notion of absorbers needed in our proof.

Definition 2.1 (Absorber). Let H be an h -vertex graph, G be an n -vertex graph and \mathcal{F} be an incompatibility system over G . For any h -set $S \subseteq V(G)$ and integer t , we say that a set $A_S \subseteq V(G) \setminus S$ is an (H, t) -*absorber* for S if $|A_S| \leq h^2 t$ and both $G[A_S]$ and $G[A_S \cup S]$ contain compatible H -factors.

The first task is handled as follows.

Definition 2.2 (Absorbing set). Let H be an h -vertex graph, G be an n -vertex graph, \mathcal{F} be an incompatibility system over G and ξ be a constant. A set $A \subseteq V(G)$ is called a ξ -*absorbing set* (for $V(G)$) if for any set $R \subseteq V(G) \setminus A$ with $|R| \leq \xi n$ and $|A \cup R| \in h\mathbb{N}$, the graph $G[A \cup R]$ contains a compatible H -factor.

Lemma 2.3 (Absorbing lemma). *Let H be an h -vertex graph with $\chi(H) \geq 2$. For any $\alpha, \sigma > 0$, there exist $\mu, \xi > 0$ such that for any sufficiently large n , if (G, \mathcal{F}) is an $\left(n, 1 - \frac{1}{\chi^*(H)} + \alpha, \mu\right)$ -incompatibility system, then G contains a ξ -absorbing set A of size at most σn .*

The second major task in absorption arguments for perfect tilings is to find a tiling that covers most of the vertices, leaving just a small portion of vertices uncovered. We will sometimes call such a tiling an *almost perfect tiling* or an *almost cover*.

Lemma 2.4 (Almost cover). *Let H be an h -vertex graph with $\chi(H) \geq 2$. For any $\alpha, \tau > 0$, there exists $\mu > 0$ such that for any sufficiently large n , if (G, \mathcal{F}) is an $\left(n, 1 - \frac{1}{\chi_{cr}(H)} + \alpha, \mu\right)$ -incompatibility system, then there exists a compatible H -tiling covering all but at most τn vertices of G .*

Now we are ready to prove Theorem 1.7 to end this section.

Proof of Theorem 1.7. Given $\alpha > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \tau \ll \xi \ll \sigma \ll \alpha.$$

Applying Lemma 2.3 with $\sigma \leq \frac{\alpha}{2}$, G contains a ξ -absorbing set A of size at most σn . Since $\delta(G - A) \geq \left(1 - \frac{1}{\chi^*(H)} + \frac{\alpha}{2}\right)n$, we can apply Lemma 2.4 on $G - A$ to obtain a compatible H -tiling \mathcal{H}_1 covering all but at most τn vertices in $G - A$. Denote by R the set of uncovered vertices in $G - A$. Since $\tau \ll \xi$, $G[A \cup R]$ contains a compatible H -factor \mathcal{H}_2 . Thus, $\mathcal{H}_1 \cup \mathcal{H}_2$ is a compatible H -factor of G . \square

3 Almost compatible H -factor

The main tools in the proof of Lemma 2.4 are Szemerédi's Regularity Lemma and an embedding lemma (see Lemma 3.6). We first give some definitions and lemmas.

3.1 Regularity

Definition 3.1 (Regular pair). Given a graph G and disjoint vertex subsets $X, Y \subseteq V(G)$, the *density* of the pair (X, Y) is defined as

$$d(X, Y) := \frac{e(X, Y)}{|X||Y|},$$

where $e(X, Y) := e(G[X, Y])$. For $\varepsilon > 0$, the pair (X, Y) is ε -regular if for any $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon|X|, |B| \geq \varepsilon|Y|$, we have

$$|d(A, B) - d(X, Y)| < \varepsilon.$$

Additionally, if $d(X, Y) \geq d$ for some $d \geq 0$, then we say that (X, Y) is (ε, d) -regular.

Fact 3.2. Let (X, Y) be an (ε, d) -regular pair, and $B \subseteq Y$ with $|B| \geq \varepsilon|Y|$. Then all but $\varepsilon|X|$ vertices in X have degree at least $(d - \varepsilon)|B|$ in B .

Fact 3.3 (Slicing lemma, [37]). Let (X, Y) be an (ε, d) -regular pair, and for some $\eta > \varepsilon$, let $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \eta|X|, |Y'| \geq \eta|Y|$. Then (X', Y') is an ε' -regular pair with $\varepsilon' = \max\{\varepsilon/\eta, 2\varepsilon\}$, and for its density d' we have $|d' - d| < \varepsilon$.

Lemma 3.4 (Degree form of the Regularity Lemma, [37]). *For every $\varepsilon > 0$, there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition $V = V_0 \cup V_1 \cup \dots \cup V_k$ and a spanning subgraph $G' \subseteq G$ with the following properties:*

- $1/\varepsilon \leq k \leq M$,
- $|V_0| \leq \varepsilon|V|$,
- $|V_i| = m$ for all $1 \leq i \leq k$ with $m \leq \varepsilon|V|$,
- $d_{G'}(v) > d_G(v) - (d + \varepsilon)|V|$ for all $v \in V$,
- $e(G'[V_i]) = 0$ for all $i \geq 1$,
- all pairs (V_i, V_j) ($1 \leq i < j \leq k$) are ε -regular in G' with density 0 or at least d .

Moreover, we usually call V_0, V_1, \dots, V_k *clusters* and call the cluster V_0 *exceptional set*.

Definition 3.5 (Reduced graph). Given an arbitrary graph $G = (V, E)$, a partition $V = V_1 \cup \dots \cup V_k$, and two parameters $\varepsilon, d > 0$, the *reduced graph* $R = R(\varepsilon, d)$ of G is defined as follows:

- $V(R) = [k]$,
- $ij \in E(R)$ if and only if (V_i, V_j) is (ε, d) -regular.

As remarked in [37], a typical application of degree form of the Regularity Lemma begins with a graph $G = (V, E)$ and appropriate parameters $\varepsilon, d > 0$, and then obtains a partition $V = V_0 \cup V_1 \cup \dots \cup V_k$ and a subgraph G' with above-mentioned properties. Then we usually drop the exceptional set V_0 to get a pure graph $G' - V_0$ and study the properties of reduced graph $R = R(\varepsilon, d)$. By Lemma 3.4,

$$\delta(R) \geq \frac{\delta(G) - (d + \varepsilon)|V| - |V_0|}{m} \geq \frac{\delta(G) - (d + 2\varepsilon)|V|}{m}.$$

In particular, if $\delta(G) \geq c|V|$, then $\delta(R) \geq (c - d - 2\varepsilon)k$.

3.2 Proof of Lemma 2.4

To find an almost perfect cover, we adopt the approach of Komlós in [34], making use of Szemerédi's Regularity Lemma and a result of Shokoufandeh and Zhao [54] (see Theorem 1.3). In [34], due to the graph counting lemma (e.g, see [37]), the arguments often reduce to greedily picking vertex-disjoint copies of H within a bunch of clusters under certain pseudorandom properties. To adopt this approach in the context of incompatibility systems which boils down to embedding compatible copies of H , we develop a 'compatible' variant of the graph counting lemma as follows.

Lemma 3.6. *For constant $\eta, d > 0$ and positive integers r, h_1, \dots, h_r with $\sum_{i=1}^r h_i =: h$, there exist positive constants $\varepsilon^* = \varepsilon^*(r, d, h), c = c(r, d, h)$ and $\mu = \mu(r, d, h, \eta)$ such that the following holds for sufficiently large n . Let (G, \mathcal{F}) be a μn -bounded incompatibility system with $|G| = n$ and U_1, \dots, U_r be pairwise vertex-disjoint sets in $V(G)$ with $|U_i| \geq \eta n, i \in [r]$ and every pair (U_i, U_j) being (ε^*, d) -regular. Then there exist at least $c \prod_{i=1}^r |U_i|^{h_i}$ compatible copies of $K_r(h_1, \dots, h_r)$ in $G[U_1, \dots, U_r]$, each containing exactly h_i vertices in U_i for every $i \in [r]$.*

We also give a notion of bottle-graphs by Komlós in [34].

Definition 3.7. Given integers h, r and an h -vertex graph H with $\chi(H) = r$, we define a *bottle-graph* B of H as a complete r -partite graph with part sizes $(r-1)\sigma(H), h-\sigma(H), \dots, h-\sigma(H)$.

It is easy to see that $\chi_{cr}(B) = \chi_{cr}(H)$ and B contains an H -factor consisting of $r-1$ copies of H . Based on this, our proof strategy is to find an almost perfect cover with compatible copies of B .

Proof of Lemma 2.4. Given $\alpha, \tau > 0$ and an h -vertex graph H , we let $\chi(H) =: r$ and choose

$$\frac{1}{n} \ll \mu \ll \varepsilon \ll \rho \ll \alpha, \tau, \frac{1}{r}, \frac{1}{h}.$$

We first apply Lemma 3.4 on G with $d = \rho$ to obtain a spanning subgraph G' and a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_k$ for some $1/\varepsilon \leq k \leq M$ with $|V_i| = m \geq \frac{(1-\varepsilon)n}{k}$ for every $i \in [k]$. Let $R = R(\varepsilon, \rho)$ be the reduced graph for this partition. Since $\varepsilon \ll \rho \ll \alpha$, we have $\delta(R) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \alpha - \rho - 2\varepsilon\right)k \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \frac{\alpha}{2}\right)k$. Since $\chi_{cr}(B) = \chi_{cr}(H)$, by Theorem 1.3, R has a B -tiling \mathcal{B} that covers all but at most $5|B|^2$ vertices in R .

Recall that B has part sizes $(r-1)\sigma(H), h-\sigma(H), \dots, h-\sigma(H)$. Here we write $\ell := h-\sigma(H), y := (r-1)\sigma(H)$ and $r' = |B| = (r-1)h$ for simplicity, and thus $r' = y + (r-1)\ell$. Given a copy of B in \mathcal{B} , without loss of generality, we may assume that its vertex set is $\{V_1, \dots, V_{r'}\}$ with the r -partition of $V(B)$ denoted as

$$\mathcal{W}_1 = \{V_1, \dots, V_y\} \text{ and } \mathcal{W}_{s+1} = \{V_{y+1+(s-1)\ell}, \dots, V_{y+s\ell}\} \text{ for } s \in [r-1].$$

Note that every pair of clusters V_i, V_j from distinct parts forms an (ε, ρ) -regular pair.

We shall greedily embed in G' vertex-disjoint compatible copies of B that cover almost all the vertices in $\cup_{i=1}^{r'} V_i$. Now for every $i \in [y]$ we divide V_i arbitrarily into ℓ subclusters $V_{i,1}, \dots, V_{i,\ell}$ of (almost) equal size. For every $j \in [y+1, r']$ we divide V_j into y subclusters $V_{j,1}, \dots, V_{j,y}$ of (almost) equal size. Here for simplicity we may further assume that $|V_{i,i'}| = \frac{m}{\ell}$ for $i \in [y], i' \in [\ell]$ and $|V_{j,j'}| = \frac{m}{y}$ for every $j \in [y+1, r'], j' \in [y]$. We call a family $\{V_{i_s, j_s}\}_{s=1}^r$ of r subclusters *legal* if $V_{i_s} \in \mathcal{W}_s$ for every $s \in [r]$, i.e., $\{V_{i_s}\}_{s=1}^r$ forms a copy of K_r in R . Note that every \mathcal{W}_s ($s \in [r]$) contains exactly $y\ell$ subclusters in total. Therefore we can greedily partition the set of all subclusters into $y\ell$ pairwise disjoint legal families.

Now if for every legal family $\{V_{i_s, j_s}\}_{s=1}^r$ we have a B -tiling in $G'[\cup_{s=1}^r V_{i_s, j_s}]$ that covers all but at most $\frac{\tau}{4y\ell} r' m$ vertices of $\cup_{s=1}^r V_{i_s, j_s}$, then putting them together would give a B -tiling

covering all but at most $\frac{\tau}{4}r'm$ vertices of $V_1 \cup V_2 \cup \dots \cup V_{r'}$. Applying this to every copy of B from \mathcal{B} would give us a B -tiling covering all but at most

$$|V_0| + 5|B|^2m + |\mathcal{B}|\frac{\tau}{4}r'm < \varepsilon n + 5(r')^2\varepsilon n + \frac{\tau}{4}n < \tau n$$

vertices in G . So to complete the proof of Lemma 2.4, it is sufficient to prove the following claim.

Claim 3.8. Given any legal family $\{V_{i_s, j_s}\}_{s=1}^r$, $G[V_{i_1, j_1}, \dots, V_{i_r, j_r}]$ admits a B -tiling covering all but at most $\frac{\tau}{4y\ell}r'm$ vertices of $\bigcup_{s=1}^r V_{i_s, j_s}$.

For convenience, we write $Y_s := V_{i_s, j_s}$ with $s \in [r]$. Recall that $|Y_1| = \frac{m}{\ell}$ and $|Y_s| = \frac{m}{y}$ for $s \in [2, r]$. Now it suffices to show that for any $Y'_s \subseteq Y_s$ with $s \in [r]$, each of size at least $\frac{\tau}{4y\ell}m$, there exists a compatible copy of B with exactly y vertices inside Y'_1 and ℓ vertices inside every Y'_s , $s \in [2, r]$.

For any distinct $s, t \in [r]$, the pair (V_{i_s}, V_{i_t}) is ε -regular with density at least ρ . Then Fact 3.3 implies that every two sets from Y'_1, \dots, Y'_r forms an ε' -regular pair with density at least $\rho - \varepsilon$, where $\varepsilon' = \frac{4y\ell}{\tau}\varepsilon$. Therefore by the fact that $\frac{1}{n} \ll \mu \ll \varepsilon \ll \rho, \tau, \frac{1}{r}, \frac{1}{h}$, Lemma 3.6 applied on G with $U_i = Y'_i, d = \rho/2, \varepsilon^* = \varepsilon', \eta = \frac{\tau}{4y\ell} \frac{1-\varepsilon}{k}, h_1 = y, h_i = \ell$ for $i \in [2, r]$, gives a desired compatible copy of B . This completes the proof of Claim 3.8. \square

3.3 Proof of Lemma 3.6: counting compatible subgraphs

In the rest of this section, we will focus on the proof of Lemma 3.6. Before that, we give a definition.

Definition 3.9 (Good pair). Given an incompatibility system (G, \mathcal{F}) , a vertex $v \in V(G)$ and a subgraph $H \subseteq G$, we say that (v, H) is a *good pair* if

- H is compatible;
- v is adjacent to all vertices of H ;
- all edges from v to $V(H)$ are mutually compatible at v .

Moreover, the vertex v is called the *center* of the good pair (v, H) .

We also state an obvious fact as follows, which is frequently used throughout the proofs.

Fact 3.10. Let (G, \mathcal{F}) be a μn -bounded incompatibility system with $|G| = n$. Then for every $v \in V(G)$, there are at most $2\mu n^2$ pairs $\{v_1, v_2\}$ of vertices from $N(v)$ that satisfy either of the following properties:

- (1) vv_1 and vv_2 are incompatible at v ;
- (2) vv_1 and v_1v_2 are incompatible at v_1 .

Proof of Lemma 3.6. We prove this lemma by induction on r . Given $\eta, d > 0$, we choose

$$\frac{1}{n} \ll \varepsilon_r \ll \varepsilon_{r-1} \ll \dots \ll \varepsilon_1 \ll d, \frac{1}{r}, \frac{1}{h} \text{ and additionally } \frac{1}{n} \ll \mu \ll \eta, d, \frac{1}{r}, \frac{1}{h}.$$

The base case $r = 1$ is trivial since we can find $\binom{|U_1|}{h_1}$ h_1 -subsets in U_1 .

Suppose that Lemma 3.6 is true for $r - 1$. Now we consider the case r , that is, U_1, \dots, U_r are pairwise (ε_r, d) -regular and $|U_i| \geq \eta n$ for any $i \in [r]$. We want to find $c \prod_{i=1}^r |U_i|^{h_i}$ compatible copies of $K_r(h_1, \dots, h_r)$ in $G[U_1, \dots, U_r]$ for some $c = c(r, d, h) > 0$.

We choose a subset $U'_r \subseteq U_r$ such that every vertex in U'_r has degree at least $(d - \varepsilon_r)|U_i|$ in U_i for every $i \in [r - 1]$. By Fact 3.2, there are at least $(1 - \varepsilon_r)|U_r|$ vertices of U_r with degree at least $(d - \varepsilon_r)|U_i|$ in U_i for any fixed $i \in [r - 1]$. Hence, $|U'_r| \geq (1 - (r - 1)\varepsilon_r)|U_r|$.

Let v be an arbitrary vertex in U'_r . Denote by U'_i the neighborhood of v in U_i for every $i \in [r - 1]$. Then $|U'_i| \geq (d - \varepsilon_r)|U_i|$. By Fact 3.3, for any $1 \leq i < j \leq r - 1$, U'_i and U'_j is an (ε', d') -regular pair with $\varepsilon' = \max\{\varepsilon_r/(d - \varepsilon_r), 2\varepsilon_r\}$ and $|d' - d| < \varepsilon_r$, which is also an $(\varepsilon_{r-1}, \frac{d}{2})$ -regular pair by the fact that $\varepsilon_r \ll \varepsilon_{r-1}$ and $d' > d - \varepsilon_r \geq \frac{d}{2}$. By the induction hypothesis on $r - 1$ with $h' = \sum_{i=1}^{r-1} h_i$, there is a family \mathcal{K}_v of at least $c(r - 1, \frac{d}{2}, h') \prod_{i=1}^{r-1} |U'_i|^{h_i}$ compatible copies of $K_{r-1}(h_1, \dots, h_{r-1})$ in $G[U'_1, \dots, U'_{r-1}]$. In \mathcal{K}_v , we know from Fact 3.10 that there are at most $2\mu n^2 \cdot n^{\sum_{i=1}^{r-1} h_i - 2} = 2\mu n^{\sum_{i=1}^{r-1} h_i}$ compatible copies of $K_{r-1}(h_1, \dots, h_{r-1})$, each containing two vertices v_1, v_2 such that vv_1 and vv_2 are incompatible at v or vv_1 and v_1v_2 are incompatible at v_1 . Thus it follows by definition and the fact that $\frac{1}{n} \ll \mu \ll \eta, d, \frac{1}{r}, \frac{1}{h}$, there exists a subfamily \mathcal{K}'_v of at least

$$c(r - 1, \frac{d}{2}, h') \prod_{i=1}^{r-1} |U'_i|^{h_i} - 2\mu n^{\sum_{i=1}^{r-1} h_i} \geq \frac{1}{2} c(r - 1, \frac{d}{2}, h') \prod_{i=1}^{r-1} |U'_i|^{h_i}$$

copies of $K_{r-1}(h_1, \dots, h_{r-1})$ that can form good pairs with v . Let $\mathcal{K}' := \bigcup_{v \in U'_r} \mathcal{K}'_v$. Since $|U'_r| \geq (1 - (r - 1)\varepsilon_r)|U_r|$, there are at least

$$\begin{aligned} |U'_r| \cdot \min\{|\mathcal{K}'_v| : v \in U'_r\} &\geq (1 - (r - 1)\varepsilon_r)|U_r| \cdot \frac{1}{2} c(r - 1, \frac{d}{2}, h') \prod_{i=1}^{r-1} |U'_i|^{h_i} \\ &\geq \frac{1}{4} c(r - 1, \frac{d}{2}, h') |U_r| \prod_{i=1}^{r-1} |U'_i|^{h_i} = c_1 |U_r| \prod_{i=1}^{r-1} |U_i|^{h_i} \end{aligned} \quad (1)$$

good pairs (v, K) with center $v \in U'_r$ and $K \in \mathcal{K}'$, where $c_1 = \frac{1}{4} c(r - 1, \frac{d}{2}, h') \cdot (d - \varepsilon_r)^{\sum_{i=1}^{r-1} h_i}$.

Now we shall greedily embed compatible copies of $K_r(h_1, \dots, h_r)$ by extending every element inside \mathcal{K}' . Here we define

$$\mathcal{X}_1 := \{K \in \mathcal{K}' \mid K \text{ is contained in at least } c_1 |U_r|/2 \text{ good pairs with centers in } U'_r\}.$$

Claim 3.11. The following statements hold.

- (1) $|\mathcal{X}_1| \geq \frac{c_1}{2} \prod_{i=1}^{r-1} |U_i|^{h_i}$;
- (2) every compatible copy $K \in \mathcal{X}_1$ is contained in at least $c_2 |U_r|^{h_r}$ compatible copies of $K_r(h_1, \dots, h_r)$ in $G[U_1, \dots, U_r]$, where c_2 will be determined later.

Proof. We first show (2). For every $K \in \mathcal{X}_1$, we denote by $C(K)$ the set of vertices $v \in U_r$ which together with K forms a good pair. Then $|C(K)| \geq c_1 |U_r|/2$. To extend a fixed $K \in \mathcal{X}_1$ into a compatible copy of $K_r(h_1, \dots, h_r)$, we can iteratively choose h_r proper vertices v_1, \dots, v_{h_r} from $C(K)$ in the following way:

- choose $v_1 \in C(K)$ such that $V(K) \cup \{v_1\}$ induces a compatible copy of $K_r(h_1, \dots, h_{r-1}, 1)$;
- for $2 \leq i \leq h_r$, suppose that v_1, \dots, v_{i-1} have been given, then choose $v_i \in C(K)$ such that $V(K) \cup \{v_1, \dots, v_i\}$ induces a compatible copy of $K_r(h_1, \dots, h_{r-1}, i)$.

Note that there are at least $c_1 |U_r|/2 - 2\mu n \cdot e(K_{r-1}(h_1, \dots, h_{r-1})) \geq c_1 |U_r|/4$ choices for v_1 , where the term $2\mu n \cdot e(K_{r-1}(h_1, \dots, h_{r-1}))$ bounds the number of vertices $u \in C(K)$ for which there exists an incompatible pair (uv, vw) of edges with $vw \in E(K)$. Similarly, there are at least

$$c_1 |U_r|/2 - (i - 1) - 2\mu n \cdot e(K_{r-1}(h_1, \dots, h_{r-1})) - \mu n \cdot (i - 1) \sum_{i=1}^{r-1} h_i \geq c_1 |U_r|/4$$

choices for v_i . Hence, there are at least $(c_1|U_r|/4)^{h_r}/h_r! = c_2|U_r|^{h_r}$ compatible copies of $K_r(h_1, \dots, h_r)$ in $G[U_1, \dots, U_r]$ containing K , where $c_2 = (c_1/4)^{h_r}/h_r!$.

The proof of (1) is easy by the double counting and we write $\mathcal{X}_2 = \mathcal{K}' \setminus \mathcal{X}_1$. By inequality (1), the number N of good pairs (v, K) with centers $v \in U'_r$ and $K \in \mathcal{K}'$ satisfy

$$c_1|U_r| \prod_{i=1}^{r-1} |U_i|^{h_i} \leq N \leq |\mathcal{X}_2| \cdot c_1|U_r|/2 + |\mathcal{X}_1| \cdot |U_r|,$$

which implies that $|\mathcal{X}_1| \geq \frac{c_1}{2} \prod_{i=1}^{r-1} |U_i|^{h_i}$. \square

Thus, the proof easily follows from Claim 3.11 by letting $c(r, d, h) = c_1 c_2/2$ and $\varepsilon^* = \varepsilon_r$ as there are at least

$$|\mathcal{X}_1| \cdot c_2|U_r|^{h_r} \geq \frac{c_1 c_2}{2} \prod_{i=1}^r |U_i|^{h_i} = c(r, d, h) \prod_{i=1}^r |U_i|^{h_i}$$

compatible copies of $K_r(h_1, \dots, h_r)$ in $G[U_1, \dots, U_r]$, each containing exactly h_i vertices in U_i for every $i \in [r]$. \square

4 Building an absorbing set

The construction of an absorbing set has now been known via a novel idea of Montgomery [49] in a series of works [11], provided that (almost) every set of h vertices has linearly many vertex-disjoint absorbers. In this paper we make use of a ‘compatible’ variant as follows, whose proof follows from that of Lemma 2.2 in [51] and will be included in Appendix A.

Lemma 4.1. *Let $t \in \mathbb{N}$, H be an h -vertex graph and $\gamma > 0$. Then there exists $\xi = \xi(t, h, \gamma) > 0$ such that the following holds. If (G, \mathcal{F}) is an incompatibility system with $|V(G)| = n$ such that for every $S \in \binom{V(G)}{h}$ there is a family of at least γn vertex-disjoint (H, t) -absorbers, then G contains a ξ -absorbing set of size at most γn .*

4.1 Finding absorbers

As aforementioned, the first main task is to find for every h -set of vertices linearly many vertex-disjoint absorbers. To achieve this, we first state a crucial notion of H -reachability as follows, which is slightly different from the original version by Lo and Markström [48].

Definition 4.2 (Reachability & Closedness). Let $m, t \in \mathbb{N}$, H be an h -vertex graph and G be an n -vertex graph. We say that two vertices $u, v \in V(G)$ are (H, m, t) -reachable if for any set W of m vertices, there is a set $S \subseteq V(G) \setminus W$ of size at most $ht - 1$ such that both $G[S \cup \{u\}]$ and $G[S \cup \{v\}]$ have compatible H -factors, where we call such S an H -connector for u, v . We say $V(G)$ is (H, m, t) -closed if every two vertices u, v in $V(G)$ are (H, m, t) -reachable.

The following result guarantees that every h -vertex set has a family of linearly many vertex-disjoint absorbers as long as $V(G)$ is $(H, \beta n, t)$ -closed for some $\beta > 0$ and $t \in \mathbb{N}$. Its proof follows from that of Lemma 3.9 in [27] and will be included in Appendix A for completeness.

Lemma 4.3 (Lemma 3.9 in [27]). *Given $h, t \in \mathbb{N}$ with $h \geq 3$ and $\beta > 0$, the following holds for any h -vertex graph H and sufficiently large $n \in \mathbb{N}$. If (G, \mathcal{F}) is an incompatibility system with $|V(G)| = n$ such that $V(G)$ is $(H, \beta n, t)$ -closed, then every $S \in \binom{V(G)}{h}$ has at least $\frac{\beta}{h^3 t} n$ vertex-disjoint (H, t) -absorbers.*

Note that the original versions of Lemmas 4.1 and 4.3 deal with absorbing sets and absorbers without the context of incompatibility systems. By going through their proofs, the two lemmas are also valid for compatible version. Based on Lemmas 4.1 and 4.3, it suffices to show that $V(G)$ is closed.

Lemma 4.4. *Let H be an h -vertex graph and $\chi(H) \geq 2$. For any $\alpha > 0$, there exist $\mu, \beta > 0$ and $t \in \mathbb{N}$ such that for any sufficiently large n , if (G, \mathcal{F}) is an $\left(n, 1 - \frac{1}{\chi^*(H)} + \alpha, \mu\right)$ -incompatibility system, then $V(G)$ is $(H, \beta n, t)$ -closed.*

Proof of Lemma 2.3. Given $\alpha, \sigma > 0$ and an h -vertex graph H , we choose

$$\frac{1}{n} \ll \mu, \xi \ll \beta, \frac{1}{t} \ll \alpha, \sigma, \frac{1}{h},$$

and let (G, \mathcal{F}) be an $\left(n, 1 - \frac{1}{\chi^*(H)} + \alpha, \mu\right)$ -incompatibility system. By Lemma 4.4, $V(G)$ is $(H, \beta n, t)$ -closed. By applying Lemma 4.3 on G , every $S \in \binom{V(G)}{h}$ has at least $\frac{\beta}{h^3 t} n$ vertex-disjoint (H, t) -absorbers. Thus Lemma 4.1 applied on G with $\gamma = \min\{\frac{\beta}{h^3 t}, \sigma\}$ immediately gives a ξ -absorbing set A of size at most γn . □

4.2 Lattice-based absorbing method

The proof of Lemma 4.4 makes use of the lattice-based absorbing method developed by Han [26], Keevash, Knox and Mycroft [32]. To illustrate this, we first need some notions introduced in [32]. For any vector $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k$, let $|\mathbf{x}| := \sum_{i=1}^k x_i$. We say that $\mathbf{x} \in \mathbb{Z}^k$ is an r -vector if it has non-negative coordinates and $|\mathbf{x}| = r$.

Definition 4.5. Let (G, \mathcal{F}) be an incompatibility system with $|G| = n$ and $\mathcal{P} = \{V_1, \dots, V_k\}$ be a vertex partition of $V(G)$ for some integer $k \geq 1$.

1. The *index vector* $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^k$ of a subset $S \subseteq V(G)$ with respect to \mathcal{P} is the vector whose i th coordinate is the size of the intersection of S with V_i , i.e. $|S \cap V_i|$, for every $i \in [k]$.
2. Given an h -vertex graph H and a constant $\beta > 0$, an h -vector $\mathbf{i} \in \mathbb{Z}^k$ is called (H, β) -robust if for any set $W \subseteq V(G)$ of size βn , $G - W$ contains a compatible copy of H whose vertex set has the index vector \mathbf{i} with respect to \mathcal{P} .
3. Denote by $I_{\mathcal{P}}^{\beta}(G)$ the set of all (H, β) -robust h -vectors $\mathbf{i} \in \mathbb{Z}^k$. Let $L_{\mathcal{P}}^{\beta}(G)$ denote the lattice (additive subgroup) in \mathbb{Z}^k generated by the elements of $I_{\mathcal{P}}^{\beta}(G)$.
4. For every $j \in [k]$, let $\mathbf{u}_j \in \mathbb{Z}^k$ be the j th unit vector, i.e. \mathbf{u}_j has 1 on the j th coordinate and 0 on the other coordinates.
5. A *transferral* is a vector of the form $\mathbf{u}_i - \mathbf{u}_j$ for some distinct $i, j \in [k]$.

As stated in Lemma 4.4, we shall prove that $V(G)$ is $(H, \beta n, t)$ -closed for some $\beta > 0$ and $t \in \mathbb{N}$. Our proof adopts a less direct approach as follows and builds on the merging techniques in [27]:

- (1) **1-reachability:** We first show that every vertex in $V(G)$ is 1-reachable to linearly many vertices (see Lemma 4.6);
- (2) **Reachability partition:** Then construct a partition $\mathcal{P} = \{V_1, \dots, V_p\}$ of $V(G)$ such that every V_i is closed (see Lemma 4.7);

- (3) **Merging process:** Lemma 4.8 allows us to merge two parts V_i, V_j into a larger (still closed) part, provided that there exists a transferral $\mathbf{u}_i - \mathbf{u}_j$ in the lattice $L_{\mathcal{P}}^{\beta}(G)$ for some $\beta > 0$. Then the arguments often reduce to iteratively detecting the existence of a transferral with respect to the up-to-date partition.

Here we collect a couple of lemmas as sketched above which together decode the proof of Lemma 4.4.

Lemma 4.6. *Let H be an h -vertex graph and $\chi(H) = r \geq 2$. For any $\alpha > 0$, there exist $\mu, \beta_1, \gamma_1 > 0$ such that for any sufficiently large n , if (G, \mathcal{F}) is an $\left(n, 1 - \frac{1}{r-1} + \alpha, \mu\right)$ -incompatibility system, then every vertex $u \in V(G)$ is $(H, \beta_1 n, 1)$ -reachable to at least $\gamma_1 n$ other vertices.*

Lemma 4.7 (see Lemma 4.1 in [27]). *For positive constants γ_1, β_1 , an integer $h \geq 2$ and an h -vertex graph H , there exist $\beta_2 = \beta_2(\gamma_1, \beta_1, h) > 0$ and $t \in \mathbb{N}$ such that the following holds for sufficiently large n . Let (G, \mathcal{F}) be an incompatibility system with $|G| = n$ such that every vertex in $V(G)$ is $(H, \beta_1 n, 1)$ -reachable to at least $\gamma_1 n$ other vertices. Then there is a partition $\mathcal{P} = \{V_1, \dots, V_p\}$ of $V(G)$ with $p \leq \lceil \frac{1}{\gamma_1} \rceil$ such that for every $i \in [p]$, V_i is $(H, \beta_2 n, t)$ -closed and has size at least $\frac{\gamma_1}{2}n$.*

Lemma 4.8. *Given any positive integers ℓ, h, t with $h \geq 3$, constants $\beta, \mu > 0$ and an h -vertex graph H , there exist $\beta' > 0$ and $t' \in \mathbb{N}$ such that the following holds for sufficiently large n . Let (G, \mathcal{F}) be an incompatibility system with $|G| = n$ and $\mathcal{P} = \{V_1, \dots, V_{\ell}\}$ be a partition of $V(G)$ such that every V_i is $(H, \beta n, t)$ -closed. If $\mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}}^{\mu}(G)$ for some distinct $i, j \in [\ell]$, then $V_i \cup V_j$ is $(H, \beta' n, t')$ -closed.*

We shall present the proofs of Lemmas 4.6 and 4.8 in Sections 4.4 and 4.5, respectively, whilst the proof of Lemma 4.7 follows that of Lemma 4.1 in [27] and will be included in Appendix B for completeness.

4.3 $V(G)$ is closed

Now we are ready to prove Lemma 4.4, taking for granted Lemmas 4.6-4.8.

Proof of Lemma 4.4. Given $\alpha > 0$ and an h -vertex graph H with $\chi(H) = r \geq 2$, we choose

$$\frac{1}{n} \ll \mu, \beta, \frac{1}{t} \ll \alpha, \frac{1}{h}$$

and let (G, \mathcal{F}) be an $\left(n, 1 - \frac{1}{\chi^*(H)} + \alpha, \mu\right)$ -incompatibility system. By Lemma 4.6, we obtain that every vertex in $V(G)$ is $(H, \beta_1 n, 1)$ -reachable to at least $\gamma_1 n$ other vertices in $V(G)$. Then applying Lemma 4.7 on G , we obtain a partition $\mathcal{P}_0 = \{V_1, \dots, V_p\}$ of $V(G)$ for some integer $p \leq \lceil \frac{1}{\gamma_1} \rceil$, where every V_i is $(H, \beta_2 n, t_2)$ -closed and $|V_i| \geq \frac{\gamma_1 n}{2}$.

We shall iteratively merge as many distinct parts as possible. As an intermediate step, we choose positive constants

$$\beta = \beta_{p+1} \ll \beta'_p \ll \beta_p \ll \dots \ll \beta'_3 \ll \beta_3 \ll \beta'_2 \ll \beta_2 \text{ and } \frac{1}{t} = \frac{1}{t_{p+1}} \ll \frac{1}{t_p} \dots \ll \frac{1}{t_3} \ll \frac{1}{t_2}.$$

Our merging procedure is stated as follows: at the step s ($s \geq 1$), based on the current partition $\mathcal{P}_{s-1} = \{V_1, \dots, V_{p'}\}$ with $p' = p - s + 1 \in \mathbb{N}$ and every V_i being $(H, \beta_{s+1} n, t_{s+1})$ -closed, if there exist distinct $i, j \in [p']$ such that $\mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}_{s-1}}^{\beta'_{s+1}}(G)$, then we merge V_i, V_j as a new part. In fact, by applying Lemma 4.8 with $\beta = \beta_{s+1}, \mu = \beta'_{s+1}$ and $t = t_{s+1}$, we obtain that $V_i \cup V_j$ is $(H, \beta_{s+2} n, t_{s+2})$ -closed. By renaming if necessary, we end the s th step with a new partition

$\mathcal{P}_s = \{V_1, \dots, V_{p'-1}\}$ such that every V_i is $(H, \beta_{s+2}n, t_{s+2})$ -closed. In this way, we continue until the procedure terminates after at most $p - 1$ steps.

Suppose we end up with a final partition $\mathcal{P} = \mathcal{P}_s = \{V_1, \dots, V_{p'}\}$ for some integer $p' = p - s$ and $0 \leq s < p$, where every V_i is $(H, \beta_{s+2}n, t_{s+2})$ -closed. Now we claim that $p' = 1$, and thus the proof is completed as $V(G)$ is $(H, \beta n, t)$ -closed by taking $\beta = \beta_{p+1}$ and $t = t_{p+1}$. Assume for a contradiction that $p' > 1$, and recall that every V_i is $(H, \beta_{s+2}n, t_{s+2})$ -closed with $|V_i| \geq \frac{\gamma_1}{2}n, i \in [p']$. We additionally choose

$$\beta'_{s+2} \ll \frac{1}{M} \ll \varepsilon \ll d \ll \alpha, \gamma_1, \frac{1}{h}.$$

Then by applying Lemma 3.4 on G with constants ε and $d := \frac{\alpha}{4}$, we obtain, as a refinement of \mathcal{P} , a partition $\mathcal{P}^* = \{V_0\} \cup \{V_{i,j} \subseteq V_i : i \in [p'], j \in [k_i]\}$ and a spanning subgraph $G' \subseteq G$ such that $|V_0| \leq \varepsilon n$ and $\frac{1}{\varepsilon} \leq \sum_{i=1}^{p'} k_i \leq M$. We write $m := |V_{i,j}|$ for all $i \in [p'], j \in [k_i]$ and $k := \sum_{i=1}^{p'} k_i$. Let $R := R(\varepsilon, d)$ be the reduced graph for the partition \mathcal{P}^* . By the choice $\varepsilon \ll d \ll \alpha$ and the fact that $\delta(G) \geq (1 - \frac{1}{\chi^*(H)} + \alpha)n$, we have $\delta(R) \geq (1 - \frac{1}{\chi^*(H)} + \alpha - d - 2\varepsilon)k \geq (1 - \frac{1}{\chi^*(H)} + \frac{\alpha}{2})k$.

We call a subgraph in R *crossing* with respect to \mathcal{P} if it has two vertices in R whose corresponding clusters lie in different parts of \mathcal{P} . Recall that $p' > 1$. Then the following claims allow us to further merge V_i, V_j into a new part, a contrary to the minimality of p' . We divide the proof into two cases depending on the dichotomy of $\chi^*(H)$. Before that, we state a simple fact as follows which would be frequently used in the proof.

Fact 4.9. Let G be an n -vertex graph and v_1, \dots, v_k be k distinct vertices in G . Then

$$|\cap_{i=1}^k N(v_i)| \geq \sum_{i=1}^k |N(v_i)| - (k-1)n.$$

Claim 4.10. If $\chi^*(H) = \chi(H) = r$, then there exist distinct $i, j \in [p']$ such that $\mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

Proof. In this case, we have that $\delta(R) \geq (1 - \frac{1}{r} + \frac{\alpha}{2})k > \frac{k}{2}$. Thus we can easily find in R a crossing edge, say $V_{i,1}V_{j,1}$ for distinct $i, j \in [p']$. Next we show that such a pair $\{i, j\}$ is as desired by finding two vectors

$$\mathbf{s}, \mathbf{t} \in I_{\mathcal{P}}^{\beta'_{s+2}}(G) \text{ with } \mathbf{s} - \mathbf{t} = \mathbf{u}_i - \mathbf{u}_j.$$

By Fact 4.9, we can find a copy of K_{r+1} containing $V_{i,1}, V_{j,1}$ in R , where the other $r-1$ vertices, without loss of generality, are denoted as $V_{a_1,1}, \dots, V_{a_{r-1},1}$.

Let $h_1 \leq \dots \leq h_r$ be the color-class sizes of H under a proper r -coloring of $V(H)$. Clearly, H is a subgraph of $K_r(h_1, \dots, h_r)$. By Lemma 3.6 applied on G with

$$\eta = \frac{1-\varepsilon}{k}, U_s = V_{a_s,1} \text{ for } s \in [r-1] \text{ and } U_r = V_{j,1}$$

and the fact that $\frac{1}{n} \ll \mu \ll \varepsilon \ll d$, we obtain that there exist at least $c(r, d, h)m^h$ compatible copies of $K_r(h_1, \dots, h_r)$ (also H) in $G[V_{a_1,1}, \dots, V_{a_{r-1},1}, V_{j,1}]$, each containing exactly h_s vertices in $V_{a_s,1}$ for every $s \in [r-1]$ and h_r vertices in $V_{j,1}$. Note that all those compatible copies of H have the same index vector with respect to \mathcal{P} , denoted by \mathbf{t} . Similarly, Lemma 3.6 applied on G with

$$\eta = \frac{1-\varepsilon}{k}, U_s = V_{a_s,1} \text{ for } s \in [r-1] \text{ and } U_r = V_{i,1}, U_{r+1} = V_{i,1}$$

gives at least $c(r+1, d, h)m^h$ compatible copies of $K_{r+1}(h_1, \dots, h_r - 1, 1)$ (also H and here we may assume $h_r > 1$ as the case $h_r = 1$ can be done in the same way) each containing exactly h_s vertices in $V_{a_s,1}$ for every $s \in [r-1]$, $h_r - 1$ vertices in $V_{j,1}$ and one vertex in $V_{i,1}$, where the corresponding index vector is denoted as \mathbf{s} . Then it is easy to see that $\mathbf{s} - \mathbf{t} = \mathbf{u}_i - \mathbf{u}_j$ and we claim that $\mathbf{s}, \mathbf{t} \in I_{\mathcal{P}}^{\beta'_{s+2}}(G)$. Indeed, this easily follows by the choice $\beta'_{s+2} \ll \frac{1}{M} \ll \varepsilon \ll d \ll \alpha, \frac{1}{h}$ and

the fact that there are at least $c'm^h$ compatible copies of H for $c' = \min\{c(r, d, h), c(r+1, d, h)\}$, whose vertex sets all have index vector \mathbf{s} (or \mathbf{t}). \square

Before diving into the case $\chi^*(H) = \chi_{cr}(H)$ (i.e. $\text{hcf}(H) = 1$), we first deal with a case which would be frequently used in the proof of the case $\chi^*(H) = \chi_{cr}(H)$.

Claim 4.11. If $\text{hcf}_\chi(H)$ is finite and R has a crossing copy of K_r , then there exist distinct $i, j \in [p']$ such that $\text{hcf}_\chi(H)(\mathbf{u}_i - \mathbf{u}_j) \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

Proof. Without loss of generality, let $V_{x_1,1}, \dots, V_{x_r,1}$ be the vertices of a crossing copy of K_r in R , where $x_1 = 1, x_2 = 2$ and $1 \leq x_3 \leq \dots \leq x_r \leq p'$ (if $r \geq 3$). Next we show that such a pair $\{1, 2\}$ is as desired.

For any r -coloring ϕ in $\mathcal{C}(H)$ and its color-class sizes $h_1^\phi \leq \dots \leq h_r^\phi$, by Lemma 3.6 applied on G' with $U_i = V_{x_i,1}$ for every $i \in [r]$, we obtain for every permutation $\sigma \in \mathbf{S}_r$, a family \mathcal{F}_σ of at least cm^h compatible copies of H each containing exactly $h_{\sigma(i)}^\phi$ vertices in $V_{x_i,1}$ for $i \in [r]$. It is easy to see by definition that every \mathcal{F}_σ yields a (H, β'_{s+2}) -robust h -vector, denoted as \mathbf{i}_σ^ϕ , whose i th coordinate is

$$\mathbf{i}_\sigma^\phi(i) = \sum_{s: x_s = i} h_{\sigma(s)}^\phi, \quad i \in [p'].$$

Thus for every $i \in [r-1]$, we choose $\sigma, \sigma' \in \mathbf{S}_r$ which have $\sigma(1) = i = \sigma'(2), \sigma(2) = i+1 = \sigma'(1)$ and $\sigma(j) = \sigma'(j)$ for $j \neq 1, 2$. Then it follows that

$$\mathbf{d}_i^\phi := (h_{i+1}^\phi - h_i^\phi, h_i^\phi - h_{i+1}^\phi, 0, \dots, 0) = \mathbf{i}_{\sigma'}^\phi - \mathbf{i}_\sigma^\phi \in L_{\mathcal{P}}^{\beta'_{s+2}}(G).$$

Recall that $D(H) = \bigcup_{\phi \in \mathcal{C}(H)} \{h_{i+1}^\phi - h_i^\phi : i \in [r-1]\}$ and all the integers in $D(H)$ have the highest common factor $\text{hcf}_\chi(H)$. Then there exist integers a_i^ϕ for any $\phi \in \mathcal{C}(H), i \in [r-1]$ such that

$$\sum_{\phi \in \mathcal{C}(H), i \in [r-1]} a_i^\phi (h_{i+1}^\phi - h_i^\phi) = \text{hcf}_\chi(H).$$

Therefore

$$\sum_{\phi \in \mathcal{C}(H), i \in [r-1]} a_i^\phi \mathbf{d}_i^\phi = (\text{hcf}_\chi(H), -\text{hcf}_\chi(H), 0, \dots, 0)$$

and by definition we have $\text{hcf}_\chi(H)(\mathbf{u}_1 - \mathbf{u}_2) \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$. This completes the proof. \square

Now we are ready to study the case $\chi^*(H) = \chi_{cr}(H)$. If $r \geq 3$, then by definition, it follows that $\text{hcf}_\chi(H) = 1$. Moreover, since $\chi_{cr}(H) > r-1$, it holds that $\delta(R) \geq \left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right)k > \frac{k}{2}$ and we can easily find in R a crossing edge, say $V_{1,1}V_{2,1}$ for example. By Fact 4.9 with $k = r-1$, we can similarly find a crossing copy of K_r containing $V_{1,1}, V_{2,1}$, and we are done by Claim 4.11. Hence, it remains to consider the case when H is bipartite with $\text{hcf}_c(H) = 1$ and $\text{hcf}_\chi(H) \leq 2$.

Claim 4.12. If $\chi(H) = 2$, $\text{hcf}_c(H) = 1$ and $\text{hcf}_\chi(H) \leq 2$, then there exist distinct $i, j \in [p']$ such that $\mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

Proof. Since $\text{hcf}_c(H) = 1$, one can easily observe that H is disconnected. Let C_1, C_2, \dots, C_ℓ be the components of H and $c_i := |C_i|, i \in [\ell]$. We further split the argument into two subcases: $\text{hcf}_\chi(H) = 1, 2$.

If $\text{hcf}_\chi(H) = 1$, then we may further assume that R contains no crossing edge with respect to \mathcal{P} as otherwise we are done by Claim 4.11. Since $p' \geq 2$, we can pick two edges from different parts, say $V_{1,1}V_{1,2}$ and $V_{2,1}V_{2,2}$. Clearly, every component C_i is a subgraph of $K_{h,h}$. By Lemma 3.6 on $V_{1,1}, V_{1,2}$ (resp. $V_{2,1}, V_{2,2}$), we obtain for every subset $S \subseteq [\ell]$, a family \mathcal{F}_S of at least c^2m^h compatible copies of H each containing exactly $\sum_{i \in S} c_i$ vertices in V_1 and $h - \sum_{i \in S} c_i$

vertices in V_2 . It is easy to see by definition that every \mathcal{F}_S yields an (H, β'_{s+2}) -robust h -vector, denoted as $\mathbf{i}_S := (\sum_{i \in S} c_i, h - \sum_{i \in S} c_i, 0, \dots, 0)$. Thus for every $i \in [\ell]$, we choose $S = \{i\}$, $S' = \emptyset$ and it follows that

$$\mathbf{d}_i := (c_i, -c_i, 0, \dots, 0) = \mathbf{i}_S - \mathbf{i}_{S'} \in L_{\mathcal{P}}^{\beta'_{s+2}}(G).$$

Recall that all the integers in $\{c_1, c_2, \dots, c_\ell\}$ have the highest common factor $\text{hcf}_c(H) = 1$. Then there exist integers a_i , $i \in [\ell]$ such that $\sum_{i \in [\ell]} a_i c_i = 1$. Therefore

$$\sum_{i \in [\ell]} a_i \mathbf{d}_i = (1, -1, 0, \dots, 0)$$

and thus by definition we have $\mathbf{u}_1 - \mathbf{u}_2 \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

For the case $\text{hcf}_\chi(H) = 2$, we may instead assume that there exists a crossing edge in R . As otherwise we can pick two edges from different parts since $p' \geq 2$, one can thus follow the same arguments in the paragraph above. Let $V_{1,1}V_{2,1}$ be such a crossing edge. Then Claim 4.11 immediately implies that $2(\mathbf{u}_1 - \mathbf{u}_2) \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

Since $\chi(H) = 2$, $\text{hcf}_\chi(H) = 2$ and $\text{hcf}_c(H) = 1$, there exists an odd component of H , say C_1 , i.e. c_1 is odd. Given a 2-coloring ϕ in $\mathcal{C}(H)$, we write x_1, x_2 as the color-class sizes for C_1 , and y_1, y_2 for $H - C_1$ under ϕ . Recall that $V_{1,1}V_{2,1}$ is a crossing edge. Then by Lemma 3.6 applied on G with $r = 2$, $U_1 = V_{1,1}$, $U_2 = V_{2,1}$ and $h_1 = x_1 + y_1$, $h_2 = x_2 + y_2$, we obtain at least cm^h compatible copies of H each containing exactly $x_i + y_i$ vertices in $V_{i,1}$ for $i \in [2]$. It is easy to see by definition that every such family yields an (H, β'_{s+2}) -robust h -vector, denoted as $\mathbf{i}_{\text{even}} := (x_1 + y_1, x_2 + y_2, 0, \dots, 0)$. Similarly we obtain $\mathbf{i}_{\text{odd}} := (x_2 + y_1, x_1 + y_2, 0, \dots, 0) \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$.

Recall that $2(\mathbf{u}_1 - \mathbf{u}_2) \in L_{\mathcal{P}}^{\beta'_{s+2}}(G)$. Since x_1, x_2 have distinct parity, by letting $s = \frac{x_1 - x_2 - 1}{2}$, we obtain that

$$\mathbf{i}_{\text{even}} - \mathbf{i}_{\text{odd}} - 2s(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{u}_1 - \mathbf{u}_2 \in L_{\mathcal{P}}^{\beta'_{s+2}}(G).$$

This completes the proof. □

□

4.4 1-reachability

In this section, we shall focus on proving Lemma 4.6.

Proof of Lemma 4.6. Given $\alpha > 0$, we choose

$$\frac{1}{n} \ll \mu, \beta_1 \ll c, \gamma_1 \ll \frac{1}{M} \ll \varepsilon \ll d \ll \alpha, \frac{1}{h}.$$

Apply Lemma 3.4 on G and obtain a partition $V(G) = V_0 \cup V_1 \cup \dots \cup V_k$ for some $1/\varepsilon \leq k \leq M$ such that $|V_0| \leq \varepsilon n$, $|V_i| = m$ for every $i \in [k]$. Let $R = R(\varepsilon, d)$ be the reduced graph for this partition. Then $\delta(R) \geq \left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right)k$.

Choose an arbitrary vertex u from $V(G)$. Let $N' = \{V_i : i \in [k] \text{ and } |N_G(u) \cap V_i| \geq \alpha m/2\}$. By double counting, we have

$$(1 - 1/(r-1) + \alpha)n - |V_0| \leq d_{G-V_0}(u) \leq |N'| \cdot m + (k - |N'|) \cdot \alpha m/2,$$

that is, $|N'| \geq \left(1 - \frac{1}{r-1} + \frac{\alpha}{4}\right)k$. Let $\ell = \left(1 - \frac{1}{r-1} + \frac{\alpha}{4}\right)k$. We choose arbitrary ℓ clusters from N' . Suppose that the corresponding vertices in R of these ℓ clusters are V_1, \dots, V_ℓ . Write $L = \{V_1, \dots, V_\ell\}$. Then we shall find a copy of K_{r-1} in $R[L]$. Indeed, the case when $r = 2$ is trivial and thus we assume that $r \geq 3$. Since $\delta(R) \geq \left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right)k$, we have $\delta(R[L]) \geq \left(1 - \frac{2}{r-1} + \frac{3\alpha}{4}\right)k$. Since $(r-2)\left(1 - \frac{2}{r-1} + \frac{3\alpha}{4}\right)k - (r-3)\ell > 1$, by Fact 4.9 with $k = r-2$,

we can greedily pick a copy of K_{r-1} as every $r-2$ vertices in L have a common neighbor in L . Suppose that the corresponding clusters of this copy of K_{r-1} are V_1, \dots, V_{r-1} . Again by Fact 4.9 applied on R with $k = r-1$ and the fact that $(r-1) \left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right) k - (r-2)k > 1$, we obtain that V_1, \dots, V_{r-1} have a common neighbor in $V(R)$, say V_r . It is worth noting that here the cluster V_r may not be in L . Let $V'_i = V_i \cap N_G(u)$ for every $i \in [r-1]$. Then $|V'_i| \geq \frac{\alpha m}{2} \geq \frac{\alpha(1-\varepsilon)}{2k}n$ for every $i \in [r-1]$.

Let $V'_r := \{v \in V_r \mid d_{V'_i}(v) \geq (d-\varepsilon)|V'_i| \text{ for each } i \in [r-1]\}$. Then by Fact 3.2, we have that $|V'_r| \geq (1 - (r-1)\varepsilon)m \geq \frac{1}{2}m$. The following claim would finish the proof as $|V'_r| \geq \gamma_1 n$.

Claim 4.13. Every vertex $v \in V'_r$ is $(H, \beta_1 n, 1)$ -reachable to u .

To prove this, for every $v \in V'_r$, we write $U_i = V'_i \cap N(v)$ for every $i \in [r-1]$ and $U_r := V'_r \setminus \{v\}$. Then $|U_i| \geq (d-\varepsilon)|V'_i| \geq \frac{d\alpha}{4}m \geq \frac{d\alpha}{8M}n$, $i \in [r]$ and Lemma 3.3 implies that U_1, \dots, U_r are pairwise (ε', d') -regular with $\varepsilon' = \max\{4\varepsilon/d\alpha, 2\varepsilon\}$ and $|d' - d| < \varepsilon$. By Lemma 3.6 on G with all subsets U_i as given above, $\eta = \frac{d\alpha}{8M}$ and the fact that $\mu \ll \frac{1}{M} \ll \varepsilon \ll d \ll \alpha, \frac{1}{h}$, we obtain at least cn^{rh-1} compatible copies of $K_r(h, \dots, h, h-1)$ in $G[U_1, \dots, U_r]$ each containing h vertices in U_i for $i \in [r-1]$ and $h-1$ vertices in U_r . Note that by Fact 3.10, there are at most $4\mu n^{rh-1} \leq \frac{\varepsilon}{2}n^{rh-1}$ compatible copies of $K_r(h, \dots, h, h-1)$, each of which fails to form a compatible copy of K_r^h with u or v . Therefore we can find a family \mathcal{K} of at least $\frac{\varepsilon}{2}n^{rh-1}$ compatible copies of $K_r(h, \dots, h, h-1)$ which together with u (also for v) form compatible copies of K_r^h . Moreover, from the choice $\beta_1 \ll c$, one can observe that deleting an arbitrary set of $\beta_1 n$ vertices would destroy at most $\beta_1 n^{rh-1} < |\mathcal{K}|$ copies of $K_r(h, \dots, h, h-1)$ in \mathcal{K} . Thus, u and v are $(K_r^h, \beta_1 n, 1)$ -reachable and also are $(H, \beta_1 n, 1)$ -reachable. \square

4.5 Transferral

Following similar arguments as in the proof of Lemma 3.9 in [28], we give a proof of Lemma 4.8.

Proof of Lemma 4.8. Let G , $\beta, \mu > 0$ and $\mathcal{P} = \{V_1, V_2, \dots, V_\ell\}$ be a partition of $V(G)$ as given in the assumption. Recall that $I = I_{\mathcal{P}}^\mu(G) \subseteq \mathbb{Z}^\ell$ is a base for $L_{\mathcal{P}}^\mu(G)$, that is, every vector in $L_{\mathcal{P}}^\mu(G)$ can be written as linear combinations of the h -vectors in I . Thus given $\mathbf{u}_i - \mathbf{u}_j \in L_{\mathcal{P}}^\mu(G)$, there exist $a_{\mathbf{v}} \in \mathbb{Z}$ such that $\mathbf{u}_i - \mathbf{u}_j = \sum_{\mathbf{v} \in I} a_{\mathbf{v}} \mathbf{v}$. Without loss of generality, we may take $i = 1, j = 2$ for instance. For every $\mathbf{v} \in I$, let $p_{\mathbf{v}} = \max\{a_{\mathbf{v}}, 0\}$ and $q_{\mathbf{v}} = \max\{-a_{\mathbf{v}}, 0\}$. Hence

$$\mathbf{u}_1 - \mathbf{u}_2 = \sum_{\mathbf{v} \in I} (p_{\mathbf{v}} - q_{\mathbf{v}}) \mathbf{v}, \quad \text{i.e.} \quad \sum_{\mathbf{v} \in I} q_{\mathbf{v}} \mathbf{v} + \mathbf{u}_1 = \sum_{\mathbf{v} \in I} p_{\mathbf{v}} \mathbf{v} + \mathbf{u}_2. \quad (2)$$

By comparing the sums of all the coordinates from two sides of either equation in (2), we obtain that $\sum_{\mathbf{v} \in I} p_{\mathbf{v}} = \sum_{\mathbf{v} \in I} q_{\mathbf{v}} =: C$.

Next, by choosing $\frac{1}{n} \ll \beta', \frac{1}{t'} \ll \beta, \mu, \frac{1}{C}, \frac{1}{t}, \frac{1}{h}$, we shall show that every two vertices $x \in V_1$ and $y \in V_2$ are $(H, \beta' n, t')$ -reachable. Let x and y be given as above and $W \subset V(G) \setminus \{x, y\}$ be any vertex set of size $\beta' n$. Since every h -vector $\mathbf{v} \in I$ is $(H, \mu n)$ -robust and $\beta' \ll \mu$, we can greedily select, outside $W \cup \{x, y\}$, a collection of $p_{\mathbf{v}} + q_{\mathbf{v}}$ vertex-disjoint compatible copies of H with index vector \mathbf{v} for every $\mathbf{v} \in I$. This is possible as during the process we need to avoid at most $|W| + 2 + h \sum_{\mathbf{v} \in I} (p_{\mathbf{v}} + q_{\mathbf{v}}) < \frac{\mu}{2}n$ vertices. This gives rise to two disjoint families \mathcal{F}^p and \mathcal{F}^q , where \mathcal{F}^p consists of $p_{\mathbf{v}}$ vertex-disjoint compatible copies of H with index vector \mathbf{v} for every $\mathbf{v} \in I$, and \mathcal{F}^q consists of $q_{\mathbf{v}}$ vertex-disjoint compatible copies of H with index vector \mathbf{v} for every $\mathbf{v} \in I$.

By equality (2), we have $|V(\mathcal{F}^p)| = |V(\mathcal{F}^q)| = hC$ and $\mathbf{i}(V(\mathcal{F}^q)) + \mathbf{u}_1 = \mathbf{i}(V(\mathcal{F}^p)) + \mathbf{u}_2$. This implies that we may write $V(\mathcal{F}^p) = \{x_1, \dots, x_{hC}\}$, $V(\mathcal{F}^q) = \{y_1, \dots, y_{hC}\}$ such that $x_1 \in V_1$, $y_1 \in V_2$, and for $i \geq 2$, x_i and y_i are from the same part of \mathcal{P} . Since each V_i is $(H, \beta n, t)$ -closed for each $i \in [\ell]$, we greedily pick a collection $\{S_2, S_3, \dots, S_{hC}\}$ of vertex-disjoint subsets in $V(G) \setminus (W \cup V(\mathcal{F}^p \cup \mathcal{F}^q) \cup \{x, y\})$ such that every S_j is an H -connector for x_j, y_j with

$|S_j| \leq ht - 1$ for $2 \leq j \leq hC$. Indeed, by the choice that $\frac{1}{n} \ll \beta' \ll \beta, \frac{1}{C}, \frac{1}{h}$, we need to avoid at most

$$\left| \left(\bigcup_{j=2}^{hC} S_j \right) \cup W \cup V(\mathcal{F}^p \cup \mathcal{F}^q) \cup \{x, y\} \right| \leq \frac{\beta}{2}n$$

vertices and therefore we are able to select every S_j as x_j and y_j are $(H, \beta n, t)$ -reachable for $2 \leq j \leq hC$. Similarly, we additionally choose two H -connectors vertex-disjoint from each other and all other previously chosen vertices, say S_0 and S_1 for x, x_1 and y, y_1 , respectively. At this point, we shall verify that the set $\hat{S} := \bigcup_{j=0}^{hC} S_j \cup V(\mathcal{F}^p \cup \mathcal{F}^q)$ is an H -connector for x, y of size at most $h(t + C + thC) - 1$. In fact, to build a compatible H -factor for $G[\hat{S} \cup \{x\}]$ (leaving y uncovered), we can take H -tilings in $G[S_0 \cup \{x\}]$, $G[V(\mathcal{F}^p)]$ and $G[S_j \cup \{y_j\}]$ for $j \in [hC]$, which altogether form an H -tiling as desired. Similarly, one can observe that $G[\hat{S} \cup \{y\}]$ also admits a compatible H -factor. Hence x and y are $(H, \beta'n, t')$ -reachable by taking $t' = t + C + thC$. \square

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Appendix A Proofs of Lemma 4.1 and Lemma 4.3

The proof of Lemma 4.1 can be easily derived by the following result.

Lemma A.1. [12] *Let F be a k -graph with b vertices. Let $\gamma > 0$ and $0 < \varepsilon < \min\{1/3, \gamma/2\}$. Then there exists $\xi = \xi(b, \gamma)$ such that the following holds for every sufficiently large n . Suppose \mathcal{G} is an n -vertex k -graph such that, there exists $V_0 \subset V(\mathcal{G})$ of size at most εn such that for every b -subset S of $V(\mathcal{G}) \setminus V_0$, there are at least γn vertex-disjoint S -absorbers, and for every vertex $v \in V(\mathcal{G})$, there are at least γn copies of F containing v , where each pair of these copies only intersects at vertex v . Then \mathcal{G} contains a subset $A \subseteq V(\mathcal{G})$ of size at most γn such that, for every subset $R \subseteq V(\mathcal{G}) \setminus A$ with $|R| \leq \xi n$ such that b divides $|A| + |R|$, the k -graph $\mathcal{G}[A \cup R]$ contains an F -factor.*

Proof of Lemma 4.1. Given $h, t \in \mathbb{N}$, an h -vertex graph H and $\gamma > 0$, we shall choose $\frac{1}{n} \ll \xi \ll \frac{1}{t}, \frac{1}{h}, \gamma$. Let (G, \mathcal{F}) be an incompatibility system with $|V(G)| = n$ such that for every $S \in \binom{V(G)}{h}$ there is a family of at least γn vertex-disjoint (H, t) -absorbers. We construct an h -graph \mathcal{G} on vertex set $V(G)$ where $E(\mathcal{G})$ consists of all compatible copies of H in (G, \mathcal{F}) . Let F be an h -graph with exactly one edge. Then by definition, every $S \in \binom{V(G)}{h}$ has at least γn vertex-disjoint (F, t) -absorbers. By applying Lemma A.1 on \mathcal{G} with $b = h$ and $V_0 = \emptyset$, we obtain a subset $A \subseteq V(\mathcal{G})$ of size at most γn such that, for every subset $R \subseteq V(\mathcal{G}) \setminus A$ with $|R| \leq \xi n$ such that h divides $|A| + |R|$, the k -graph $\mathcal{G}[A \cup R]$ contains an F -factor. Then A is a desired ξ -absorbing set in (G, \mathcal{F}) . \square

Proof of Lemma 4.3. For positive integers h, t with $h \geq 3$ and $\beta > 0$, let G, H, \mathcal{F} be given as in the assumption such that $V(G)$ is $(H, \beta n, t)$ -closed. Then for every h -set $S \subset V(G)$, we shall greedily construct as many pairwise disjoint (H, t) -absorbers for S as possible. Let $\mathcal{A} = \{A_1, A_2, \dots, A_\ell\}$ be a maximal family of vertex-disjoint (H, t) -absorbers as above. Suppose to the contrary that $\ell < \frac{\beta}{h^{3t}}n$. Then $|\cup_{\mathcal{A}} A_i| \leq \frac{\beta}{h}n$ as each such A_i has size at most h^2t (see Definition 2.1).

As $V(G)$ is $(H, \beta n, t)$ -closed and $|\cup_{\mathcal{A}} A_i \cup S| \leq \beta n$, we can pick a compatible copy of H in $V(G) \setminus (\cup_{\mathcal{A}} A_i \cup S)$ whose vertex set is denoted by T . Write $S = \{s_1, s_2, \dots, s_h\}$ and $T = \{t_1, t_2, \dots, t_h\}$. We now greedily pick a collection $\{S_1, S_2, \dots, S_h\}$ of vertex-disjoint subsets in $V(G) \setminus (\cup_{\mathcal{A}} A_i \cup S \cup T)$ such that each S_i is an H -connector for s_i, t_i with $|S_i| \leq ht - 1$. Indeed, since

$$\left| \bigcup_{i=1}^{\ell} A_i \cup \left(\bigcup_{i=1}^{k'} S_i \right) \cup S \cup T \right| \leq \beta n,$$

for any $0 \leq k' \leq h$ (using that n is sufficiently large), it is possible to pick each S_i one by one because s_i and t_i are $(H, \beta n, t)$ -reachable. At this point, it is easy to verify that $\bigcup_{i=1}^h S_i \cup T$ is actually an (H, t) -absorber for S , contrary to the maximality of ℓ . \square

Appendix B Proof of Lemma 4.7

We begin with a simple result as follows.

Lemma B.1. *Let $s_1, s_2 \in \mathbb{N}, \beta_1, \beta_2 > 0$ and H be an h -vertex graph. The following holds for any sufficiently large n . Let (G, \mathcal{F}) be an incompatibility system with $|V(G)| = n$, $u, v \in V(G)$ and $Z \subseteq V(G)$ with $|Z| = cn$ for some positive constant c . If u is $(H, \beta_1 n, s_1)$ -reachable to every vertex in Z , while v is $(H, \beta_2 n, s_2)$ -reachable to every vertex in Z , then u, v are $(H, \beta_3 n, s_1 + s_2)$ -reachable, where $\beta_3 n = \min\{cn - 1, \beta_1 n/2, \beta_2 n/2\}$.*

Proof. Let W be an arbitrary subset of G with $|W| \leq \beta_3 n$. Since $|W| \leq \min\{cn - 1, \beta_1 n/2\}$, we can find a vertex $u_1 \in Z \setminus W$ and an H -connector S_1 for u, u_1 in $V(G) \setminus (W \cup \{u, u_1, v\})$ with $|S_1| \leq ht_1 - 1$. Meanwhile, since $|W| \leq \beta_2 n/2$, we can also find an H -connector S_2 for u_1, v in $V(G) \setminus (W \cup \{u, u_1, v\} \cup S_1)$ with $|S_2| \leq ht_2 - 1$. Hence, u, v are $(H, \beta_3 n, t_1 + t_2)$ -reachable as both $G[S_1 \cup S_2 \cup \{u_1, u\}]$ and $G[S_1 \cup S_2 \cup \{u_1, v\}]$ contain compatible H -factors. \square

Proof of Lemma 4.7. Write $r_0 = \lceil 1/\gamma_1 \rceil + 1$, $t_j = 2^{r_0+1-j}$ for each $j \in [r_0 + 1]$ and choose constants $\frac{1}{n} \ll \lambda_1 \ll \lambda_2 \ll \lambda_3 \cdots \ll \lambda_{r_0+1} = \beta_1, \gamma_1, \frac{1}{r_0}$. Let (G, \mathcal{F}) be an incompatibility system with $|G| = n$ such that every vertex in $V(G)$ is $(H, \beta_1 n, 1)$ -reachable to at least $\gamma_1 n$ other vertices.

Assume that there are two vertices that are not $(H, \lambda_2 n, t_2)$ -reachable, as otherwise we just output $\mathcal{P} = \{V(G)\}$ as the desired partition. Observe also that every set of r_0 vertices must contain two vertices that are $(H, \lambda_{r_0} n, t_{r_0})$ -reachable to each other. Indeed, by the inclusion-exclusion principle we obtain a pair of vertices, say u, v , which are $(H, \beta_1 n, 1)$ -reachable to a set of at least $\delta' n$ vertices for some $\delta' \geq \gamma_1 / \binom{r_0}{2}$. Thus, by Lemma B.1 with $s_1 = s_2 = 1$, we deduce that u and v are $(H, \lambda_{r_0} n, t_{r_0})$ -reachable as $\lambda_{r_0} \ll \beta_1, \gamma_1, \frac{1}{r_0}$ and $t_{r_0} = 2$.

Let d be the largest integer such that there are d vertices v_1, \dots, v_d in G which are pairwise not $(H, \lambda_d n, t_d)$ -reachable. Note that d exists and $2 \leq d \leq r_0 - 1$. Indeed, we assumed above that there are 2 vertices that are not $(H, \lambda_2 n, t_2)$ -reachable and if $d = r_0$, then there would be r_0 vertices in G which are pairwise not $(H, \lambda_{r_0} n, t_{r_0})$ -reachable, contrary to the observation above. Let $S = \{v_1, \dots, v_d\}$. Note that v_1, \dots, v_d are also pairwise not $(H, \lambda_{d+1} n, t_{d+1})$ -reachable.

For a vertex v and every $j \in [r_0 + 1]$, we write $\tilde{N}_j(v)$ for the set of vertices which are $(H, \lambda_j n, t_j)$ -reachable to v . For the sets $\tilde{N}_{d+1}(v_i)$, $i \in [d]$, we conclude that

- (1) any vertex $v \in V(G) \setminus \{v_1, \dots, v_d\}$ must be in $\tilde{N}_{d+1}(v_i)$ for some $i \in [d]$. Otherwise, v, v_1, \dots, v_d are pairwise not $(H, \lambda_{d+1} n, t_{d+1})$ -reachable, contradicting the maximality of d .
- (2) $|\tilde{N}_{d+1}(v_i) \cap \tilde{N}_{d+1}(v_j)| < \lambda_{d+1} n$ for any distinct $i, j \in [d]$. Otherwise Lemma B.1 applied with $s_1 = s_2 = t_{d+1}$ implies that v_i, v_j are $(H, \lambda_d n, t_d)$ -reachable to each other, a contradiction to the choice of vertices $v_i, i \in [d]$ (using that $\lambda_d \ll \lambda_{d+1}$ and $2t_{d+1} = t_d$).

For $i \in [d]$, let

$$U_i = (\tilde{N}_{d+1}(v_i) \cup \{v_i\}) \setminus \bigcup_{j \neq i} \tilde{N}_{d+1}(v_j).$$

Then we claim that each U_i is $(H, \lambda_{d+1}n, t_{d+1})$ -closed. Indeed otherwise, there exist $u_1, u_2 \in U_i$ that are not $(H, \lambda_{d+1}n, t_{d+1})$ -reachable to each other. Then $\{u_1, u_2\} \cup \{v_1, \dots, v_d\} \setminus \{v_i\}$ contradicts the maximality of d .

Let $U_0 = V(G) \setminus (U_1 \cup \dots \cup U_d)$. We have $|U_0| < d^2 \lambda_{d+1}n$ due to **(2)** above. In order to obtain the desired reachability partition, we now drop every vertex of U_0 back into some U_i for $i \in [d]$ as follows. Since each $v \in U_0$ is $(H, \beta_1 n, 1)$ -reachable (i.e. $(H, \lambda_{r_0+1}n, t_{r_0+1})$ -reachable) to at least $\gamma_1 n$ vertices, it follows from the fact $\lambda_{d+1} \ll \gamma_1, \frac{1}{r_0}$ that

$$|\tilde{N}_{r_0+1}(v) \setminus U_0| \geq \gamma_1 n - |U_0| \geq \gamma_1 n - d^2 \lambda_{d+1}n > d \lambda_{d+1}n.$$

Therefore by the pigeonhole principle there exists some $i \in [d]$ such that v is $(H, \lambda_{r_0+1}n, t_{r_0+1})$ -reachable to at least $\lambda_{d+1}n + 1$ vertices in U_i . Lemma B.1 (using that $\lambda_d \ll \lambda_{d+1} \ll \lambda_{r_0+1}$) implies that v is $(H, \lambda_d n, t_d)$ -reachable to every vertex in U_i .

Recall that $2 \leq d \leq r_0 - 1$. Now partition U_0 as $U_0 = \cup_{i \in [d]} R_i$ where for each $i \in [d]$, R_i denotes a set of vertices $v \in U_0$ that are $(H, \lambda_d n, t_d)$ -reachable to every vertex in U_i . Again by Lemma B.1 with $s_1 = s_2 = t_d$, for every $i \in [d]$, every two vertices in R_i (if any) are $(H, \lambda_{d-1}n, t_{d-1})$ -reachable to each other. Let $\mathcal{P} = \{V_1, \dots, V_d\}$ be the final partition by setting $V_i = R_i \cup U_i$. Then each V_i is $(H, \lambda_{d-1}n, t_{d-1})$ -closed. Also, for each $i \in [d]$, it holds that

$$|V_i| \geq |U_i| \geq |\tilde{N}_{d+1}(v_i)| - d^2 \lambda_{d+1}n \geq |\tilde{N}_{r_0+1}(v_i)| - \frac{\gamma_1}{2}n \geq \frac{\gamma_1}{2}n.$$

This completes the proof by taking $\beta_2 = \lambda_{d-1}$ and $t = t_{d-1}$. □