Behaving Sequences

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Abstract

Let S be a sequence over an additively written abelian group. We denote by $\mathsf{h}(S)$ the maximum of the multiplicities of S, and by $\sum(S)$ the set of all subsums of S. In this paper, we prove that if S has no zero-sum subsequence of length in $[1, \mathsf{h}(S)]$, then either $|\sum(S)| \geq 2|S| - 1$, or S has a very special structure which implies in particular that $\sum(S)$ is an interval. As easy consequences of this result, we deduce several well-known results on zero-sum sequences.

Keywords: Zero-sum-free sequence; Behaving sequence; Maximum of the multiplicities

1 Introduction

Let G be an additive abelian group and $S = g_1 \cdot \ldots \cdot g_r$ be a sequence over G. As usual, $|S| = r \in \mathbb{N}_0$ denotes the length of S, $\sigma(S) = g_1 + \cdots + g_r$ the sum of S, $\mathsf{h}(S)$ the maximum of the multiplicities of S and $\Sigma(S) = \{\sum_{i \in I} g_i \colon \emptyset \neq I \subset [1, r]\}$ the set of subsums of S. We say that S is a zero-sum sequence if $\sigma(S) = 0$, and S is called zero-sum-free if $0 \notin \Sigma(S)$.

The following theorem is a fundamental result in zero-sum theory, which has been used in many papers, see e.g., [5], [7], [8].

Theorem A ([1], [4], [17]) *Let S be a sequence over a finite abelian group G. If $|S| \geq |G|$, then S has a zero-sum subsequence of length in* $[1, \mathsf{h}(S)]$.

Theorem A was first proved in [17] for G a cyclic group of prime order. A slightly weaker version of Theorem A for cyclic G was given in [1], and an equivalent version of Theorem A for any abelian group can be found in [4].

Let G be a finite abelian group, and let S be the sequence consisting of all non-zero group elements. Then $|S| = |G| - 1$, and S has no nonempty zero-sum subsequence of length $1 = \mathsf{h}(S)$. This example shows that the conclusion of Theorem A is not true if we relax the restriction imposed on the length of S.

In the spirits of inverse additive number theory, we ask for the structure of a sequence S which has no zero-sum subsequence of length in $[1, \mathsf{h}(S)]$. For that reason, we introduce the invariant $\rho(S)$ which is defined as the smallest length of a nonempty zero-sum subsequence of S. By definition, we set $\rho(S) = 0$ if S is zero-sum-free. We need the following definition (which is closely related to [9, Definition 5.1.3]).

Definition 1.1 *Let S be a sequence over an abelian group G. We say that S is a strictly g-behaving sequence (strictly behaving for short) if $S = (n_1 g) \cdot \ldots \cdot (n_r g)$ for some $g \in G$, where $n_i \in [1, \mathrm{ord}(g)]$ for every $i \in [1, r]$, $n_1 = 1$ and $n_t \le \sum_{i=1}^{t-1} n_i$ for every $t \in [2, r]$.*

Clearly, if S is a strictly g-behaving sequence, then $\sum(S) = \{g, 2g, \ldots, Ng\}$ where $N = \min(\mathrm{ord}(g), \sum_{i=1}^{r} n_i)$. Also note that if $r \ge 2$ then g occurs at least twice in S.

Here are the main results of the present paper.

Theorem 1.2 *Let S be a sequence over an abelian group. If $\rho(S) \notin [1, \mathsf{h}(S)]$, then S is a strictly behaving sequence or $|\sum(S)| \ge \min(|\langle \mathrm{supp}(S) \rangle|, 2|S| - 1)$, where $\mathrm{supp}(S)$ denotes the set that consists of all distinct elements which occur in S.*

If $0 \in \sum(S)$, Theorem 1.2 can be formulated as follows.

Theorem 1.3 *Let S be a sequence over an abelian group with $0 \in \sum(S)$. Then $\rho(S) \in [1, \mathsf{h}(S)]$ or $|\sum(S)| \ge \min(|\langle \mathrm{supp}(S) \rangle|, 2|S| - 1)$.*

Corollary 1.4 *Let S be a sequence over an abelian group with $|S| \ge \frac{|\langle \mathrm{supp}(S) \rangle| + 1}{2}$ and $0 \in \sum(S)$. Then $\rho(S) \in [1, \mathsf{h}(S)]$ or $\sum(S) = \langle \mathrm{supp}(S) \rangle$.*

As easy consequences of Theorem 1.2, we shall deduce the following well-known results.

Corollary 1.5 *([16], [19], [9, Theorem 5.1.8]) Let S be a zero-sum-free sequence over a cyclic group of order $n \ge 2$ with $|S| > \frac{n}{2}$. Then there is an element $g \in G$ with $\mathrm{ord}(g) = n$ such that $S = (n_1 g) \cdot \ldots \cdot (n_r g)$ with all $n_i \ge 1$ and $\sum_{i=1}^{r} n_i < n$.*

Corollary 1.6 *([14], [15], [20]) Let S be a zero-sum-free sequence over an abelian group. If $\langle \text{supp}(S) \rangle$ is not cyclic, then $|\sum(S)| \geq 2|S| - 1$.*

Corollary 1.7 *([3]) Let S be a zero-sum-free sequence over an abelian group G. Then the following two inequalities hold:*

(i) $|\sum(S)| \geq 2|S| - \mathsf{h}(S)$.

(ii) $|\sum(S)| \geq |S| + |\text{supp}(S)| - 1$.

Corollary 1.8 *([12]) Let S be a subset of an abelian group with $0 \notin S$. Then $|\sum(S)| \geq \min(|\langle S \rangle|, 2|S| - 1)$.*

2 Proofs of Theorem 1.2, Theorem 1.3 and Corollaries 1.4–1.8

Let G be an additive abelian group. For a subset $G_0 \subset G$ we denote by $\langle G_0 \rangle$ the subgroup generated by G_0. We fix the notation concerning sequences over G_0 (which is consistent with [6] and [10]). We write sequences multiplicatively and consider them as elements of the free abelian monoid $\mathcal{F}(G)$ over G. Thus we have all notions of abstract divisibility theory at our disposal. Let

$$S = g_1 \cdot \ldots \cdot g_r = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$$

be a sequence over G. Then $\text{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\} \subset G$ denotes the *support* of S, $\mathsf{v}_g(S)$ is called the *multiplicity* of S, and $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \colon g \in G\}$ is the *maximum of the multiplicities* of S. We say that S is *squarefree* if $\mathsf{v}_g(S) \leq 1$ for all $g \in G$. For convenience, we set $\Sigma_0(S) = \Sigma(S) \cup \{0\}$, Whenever we write a sequence in the form as in Definition 1.1, say $S = (n_1 g) \cdot \ldots \cdot (n_r g)$ for some $g \in G$, then we tacitly assume that $1 = n_1 \leq n_2 \leq \cdots \leq n_r$.

Now we collect some useful lemmas, after which we will prove Theorem 1.2 and Theorem 1.3.

Lemma 2.1 *[18] Let A and B be two finite subsets of an abelian group with $A \cap (-B) = \{0\}$. Then, $|A + B| \geq |A| + |B| - 1$, where $A + B = \{c = a + b : a \in A, b \in B\}$.*

Lemma 2.2 *[13, Theorem 4.3] Let A and B be two finite nonempty subsets of an abelian group with $\text{St}(A + B) = \{0\}$. Then, $|A + B| \geq |A| + |B| - 1$, where $\text{St}(A + B)$ denotes the maximal subgroup H of G such that $A + B + H = A + B$.*

Lemma 2.3 *Let S be a sequence over an abelian group. If $\rho(S) \notin [1, \mathsf{h}(S)]$ then $|\sum_0(S)| \geq |S| + 1$.*

Proof. Let $h = \mathsf{h}(S)$. Note that since $\rho(S) \neq 1$, we have $0 \notin \mathrm{supp}(S)$. Since no element occurs more than h times in S, we can write S as a product of h squarefree sequences, say $S = S_1 \cdot \ldots \cdot S_h$. Put $A_i = \mathrm{supp}(S_i) \cup \{0\}$ for every $i \in [1, h]$. Clearly, $\sum\limits_{i=1}^{h} A_i \subseteq \sum_0(S)$. Since $\rho(S) \notin [1, h]$, it follows that for any $(a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h$, $a_1 + \cdots + a_h = 0$ implies $(a_1, \ldots, a_h) = (0, \ldots, 0)$. Applying Lemma 2.1 recursively, we obtain $|\sum_0(S)| \geq |\sum\limits_{i=1}^{h} A_i| \geq |\sum\limits_{i=1}^{h-1} A_i| + |A_h| - 1 \geq \cdots \geq \sum\limits_{i=1}^{h} |A_i| - h + 1 = \sum\limits_{i=1}^{h}(|S_i| + 1) - h + 1 = \sum\limits_{i=1}^{h} |S_i| + 1 = |S| + 1$. □

Lemma 2.4 *Let $S = S_1 S_2$ be a sequence over an abelian group such that $\mathrm{St}(\sum(S)) = \{0\}$. Then $|\sum(S)| \geq |\sum(S_1)| + |\sum_0(S_2)| - 1$.*

Proof. If $\rho(S) = 0$, then the conclusion follows from [10, Proposition 5.3.1].

If $\rho(S) > 0$, it follows from Lemma 2.2 that $|\sum(S)| = |\sum_0(S)| = |\sum_0(S_1) + \sum_0(S_2)| \geq |\sum_0(S_1)| + |\sum_0(S_2)| - 1 \geq |\sum(S_1)| + |\sum_0(S_2)| - 1$. □

Lemma 2.5 *[10, Proposition 5.3.2] Let S be a zero-sum-free subset of an abelian group. Then $|\sum(S)| \geq 2|S| - 1$. In particular, if $|S| = 3$ and S contains no element of order two then $|\sum(S)| \geq 6$.*

We also need the following technical result.

Lemma 2.6 *Let S be a sequence over an abelian group such that every torsion element of S is of odd order. Then the following two statements hold.*

(i) If $\mathsf{h}(S) = 1$ and $|S| = \rho(S) = 3$ then $|\sum(S)| = 7$.

(ii) Suppose $S = a_1^2 \cdot a_2^2$ is not strictly behaving. If $\rho(S) \notin \{1, 2\}$ then $|\sum(S)| \geq 7$. In particular, if $\rho(S) \notin \{1, 2, 3\}$ then $|\sum(S)| \geq 8$.

Proof. (i). Let $S = a \cdot b \cdot c$. Since $|S| = \rho(S) = 3$, S is its own unique nonempty zero-sum subsequence. It is easy to see that $a, b, c, b+c(= -a), a+c(= -b), a+b(= -c), a+b+c(= 0)$ are pairwise distinct elements of $\sum(S)$. Hence, $|\sum(S)| = 7$.

(ii). Assume first $\rho(S) \notin \{1, 2\}$. Since S is not strictly behaving and every torsion element in S is of odd order, then

$$a_1, 2a_1, a_2, a_2 + a_1, a_2 + 2a_1, 2a_2, 2a_2 + a_1$$

are pairwise distinct elements of $\sum(S)$. Moreover, if $\rho(S) \neq 3$, then $2a_2 + 2a_1$ is an element of $\sum(S)$ distinct from all of the elements listed above. This completes the proof. $\square$

Lemma 2.7 *Let $T = (n_1 g) \cdot \ldots \cdot (n_k g)$ be a strictly g-behaving sequence with $\sum_{i=1}^{k} n_i < \mathrm{ord}(g)$. Let $m = 2k - |\sum(T)|$. If $m \in [2, k]$ then for every element $x \in \{g, 2g, \ldots, (m-1)g\}$, there exists a subsequence U of T such that $|U| < \mathsf{h}(T)$ and $\sigma(U) = x$.*

Proof. Let $h_1 = \mathsf{v}_g(T)$ and $h_2 = \mathsf{v}_{2g}(T)$ and $h_3 = k - h_1 - h_2$. We have $|\sum(T)| = \sum_{i=1}^{k} n_i \geq h_1 + 2h_2 + 3h_3$, which implies $h_1 \geq h_3 + m \geq m$. The lemma follows. $\square$

The following notation will be used often in the proof of Theorem 1.2. For $S \in \mathcal{F}(G)$ and $g \in G$, define

$$\lambda_S(g) = |\sum(S \cdot g) \setminus \sum(S)|.$$

Lemma 2.8 *$\lambda_S(g) \leq \lambda_{Sg^{-1}}(g)$ for every $g|S$.*

Proof. We have

$$\begin{aligned}
\lambda_S(g) &= |\textstyle\sum(S \cdot g) \setminus \sum(S)| \\
&= |(\textstyle\sum(S) \cup g + \sum_0(S)) \setminus \sum(S)| \\
&= |(g + \textstyle\sum_0(S)) \setminus \sum(S)| \\
&= |(g + \textstyle\sum_0(S)) \setminus (\sum(Sg^{-1}) \cup g + \sum_0(Sg^{-1}))| \\
&\leq |(g + \textstyle\sum_0(S)) \setminus (g + \sum_0(Sg^{-1}))| \\
&= |\textstyle\sum_0(S) \setminus \sum_0(Sg^{-1})| \\
&\leq |\textstyle\sum(S) \setminus \sum(Sg^{-1})| \\
&= \lambda_{Sg^{-1}}(g).
\end{aligned}$$

$\square$

Proof of Theorem 1.2. Let $H = \langle \mathrm{supp}(S) \rangle$. Assume that the theorem is false and let S be a minimal-length counterexample. Thus, $|\sum(S)| \leq \min(|H| - 1, 2|S| - 2)$. Let $S = a_1 \cdot \ldots \cdot a_r$, where $r = |S|$. Clearly, $r \geq 3$ and $0 \notin \mathrm{supp}(S)$.

Claim A. $\langle \mathrm{supp}(Sg^{-1}) \rangle = H$ for every $g|S$.

Proof of Claim A. Suppose to the contrary that there exists $g|S$ such that $\langle \mathrm{supp}(Sg^{-1}) \rangle$ is a proper subgroup of H. Thus, $g \notin \langle \mathrm{supp}(Sg^{-1}) \rangle$. Note that either $\rho(Sg^{-1}) \geq \rho(S) > \mathsf{h}(S) \geq \mathsf{h}(Sg^{-1})$ or $\rho(Sg^{-1}) = 0$ according to $0 \in \sum(S)$ or not. By Lemma 2.3, we have $|\sum_0(Sg^{-1})| \geq |Sg^{-1}| + 1 = r$. It follows that

$$\begin{aligned}
|\textstyle\sum_0(S)| &= |\textstyle\sum_0(Sg^{-1}) \cup g + \sum_0(Sg^{-1})| \\
&= |\textstyle\sum_0(Sg^{-1})| + |g + \sum_0(Sg^{-1})| \\
&= 2|\textstyle\sum_0(Sg^{-1})| \\
&\geq 2r,
\end{aligned}$$

a contradiction. This proves Claim A. $\square$

Now we distinguish two cases.

Case 1. There exists some $i \in [1, r]$ such that Sa_i^{-1} is strictly behaving.

We may assume that $Sa_r^{-1} = (n_1 g) \cdot \ldots \cdot (n_{r-1} g)$ is strictly g-behaving for some $g|S$. Let $A_{r-1} = \sum(Sa_r^{-1})$. Since S is not strictly behaving, we have $a_r \notin A_{r-1}$. By Claim A, we have that $a_r \in H = \langle \mathrm{supp}(Sa_r^{-1}) \rangle = \langle g \rangle$, i.e., $a_r = n_r g$ with $n_r \in [1 + \sum_{i=1}^{r-1} n_i, \mathrm{ord}(g) - 1]$. Clearly, $A_{r-1} = \{g, 2g, \ldots, (\sum_{i=1}^{r-1} n_i)g\}$. Let $m = 2(r-1) - \sum_{i=1}^{r-1} n_i$. Since $n_r g, (n_r+1)g \in \sum(S) \setminus A_{r-1}$, we have $m = 2(r-1) - \sum_{i=1}^{r-1} n_i \ge |\sum(S)| - |A_{r-1}| \ge 2$. Therefore,

$$m \in [2, r-1].$$

Let

$$\ell = \mathrm{ord}(g) - n_r.$$

Suppose $m > \ell$. Applying Lemma 2.7 with $T = Sa_r^{-1}$, there exists a subsequence U of T such that $|U| < \mathsf{h}(T) \le \mathsf{h}(S)$ and $\sigma(U) = \ell g$. It follows that $\sigma(U \cdot a_r) = \ell g + n_r g = 0$ and $|U \cdot a_r| \le \mathsf{h}(S)$, a contradiction. Hence,

$$m \le \ell.$$

It follows that A_{r-1} and $\{n_r g, (n_r+1)g, \ldots, (n_r+m)g\}$ are disjoint subsets of $\sum(S)$, which implies $|\sum(S)| \ge 2(r-1) - m + (m+1) = 2r-1$, a contradiction. This proves the theorem for this case.

Case 2. The length of each strictly behaving subsequence of S is less than $r-1$.

Case 2 is harder than Case 1, and we shall devote most of the rest of this section to prove it. Choose an arbitrary element g of S. From the minimality of S and Claim A we obtain that $|\sum(Sg^{-1})| \ge \min(|H|, 2|Sg^{-1}| - 1) = 2|Sg^{-1}| - 1 = 2r-3$. This together with $|\sum(S)| \le 2r-2$ shows that

$$\lambda_{Sg^{-1}}(g) \le 1. \tag{1}$$

Now, by Lemma 2.8, we get

$$\lambda_S(g) \le 1 \tag{2}$$

for every $g|S$.

If $\lambda_S(g) = 0$ for every $g|S$, then $g + \sum(S) = \sum(S)$ for every $g|S$. This shows that $H + \sum(S) = \langle \mathrm{supp}(S) \rangle + \sum(S) = \sum(S)$. It follows that $|\sum(S)| \ge |H|$, a contradiction. Therefore, $\lambda_S(f) = 1$ for some element $f|S$. By the minimality of S and Claim A, we have that $|\sum(Sf^{-1})| \geq \min(|H|, 2|Sf^{-1}|-1) = 2|Sf^{-1}|-1 = 2r-3$. It follows from Lemma 2.8 that $|\sum(S)| \geq |\sum(Sf^{-1})| + \lambda_S(f) \geq 2r-2$, and thus

$$|\sum(S)| = 2r-2. \tag{3}$$

Claim B. $|\sum(S)| = |H|-1$.

Proof of Claim B. Let a be an element of S such that $\mathsf{v}_a(S) = \mathsf{h}(S)$. Assume first that

$$\langle a\rangle = H.$$

By (2), we have that $\sum(S)$ is an arithmetic progression with difference a. Since $\rho(S) \notin [1, \mathsf{h}(S)]$ and $|S| \geq 3$, there exists an element b of $\operatorname{supp}(S)$ distinct from a and $-a$. By (2), $\lambda_S(b) \leq 1$, which implies $|\sum(S)| = |H|-1$.

Now assume that $\langle a\rangle$ is a proper subgroup of H. Then we have a decomposition of $\sum(S) = \bigcup\limits_{i=0}^{r_a} C_i^a$, where $C_0^a, C_1^a, \ldots, C_{r_a}^a$ are subsets of pairwise distinct cosets modulo $\langle a\rangle$ and $C_0^a \subseteq \langle a\rangle$. Since $\operatorname{supp}(S) \nsubseteq \langle a\rangle$, it follows that $\sum(S) \setminus \langle a\rangle \neq \emptyset$, and so $r_a \geq 1$. Note that

$$\begin{aligned} |(a+\sum(S)) \setminus \sum(S)| &= |(\bigcup_{i=0}^{r_a}(a+C_i^a)) \setminus (\bigcup_{i=0}^{r_a} C_i^a)| \\ &= |\bigcup_{i=0}^{r_a}((a+C_i^a) \setminus C_i^a)| \\ &= \sum_{i=0}^{r_a} |(a+C_i^a) \setminus C_i^a|. \end{aligned}$$

By (2), $|(a+\sum(S)) \setminus \sum(S)| \leq |\sum(S\cdot a) \setminus \sum(S)| = \lambda_S(a) \leq 1$. Therefore, we conclude that a is a torsion element, and there exists at most one index i of $[0, r_a]$ such that $|C_i^a| < \operatorname{ord}(a)$. In other words, there exists $k \in [0, r_a]$ such that C_i^a is a complete coset of $\langle a\rangle$ for every $i \in [0, r_a] \setminus \{k\}$. Denote by ϕ_a the canonical epimorphism of H onto $H/\langle a\rangle$.

We claim that

$$|C_k^a| \geq 2.$$

Assume $k \neq 0$. Taking an element $x \in C_k^a$, there is a subsequence W of S such that $x = \sigma(W)$. If $a|W$, since $x \neq a$, then $x - a = \sigma(Wa^{-1}) \in C_k^a$. Otherwise, $x + a = \sigma(Wa) \in C_k^a$. Hence, $|C_k^a| \geq 2$. Assume $k = 0$ and $0 \in \sum(S)$ then $\{0, a\} \subseteq C_0^a$. Otherwise, $0 \notin \sum(S)$. By Lemma 2.5, $\mathsf{v}_a(S) = \mathsf{h}(S) \geq 2$ and $\{a, 2a\} \subseteq C_k^a$. This proves that $|C_k^a| \geq 2$. Therefore,

$$|C_i^a| \geq 2 \tag{4}$$

for every $i \in [0, r_a]$.

By (2) and (4), we conclude that

$$\phi_a(g) + \phi_a(\sum(S)) = \phi_a(\sum(S)) \tag{5}$$

for every $g|S$, or equivalently, $g+\sum(S)+\langle a\rangle=\sum(S)+\langle a\rangle$. Therefore, $\langle \mathrm{supp}(S)\rangle+\sum(S)+\langle a\rangle=\sum(S)+\langle a\rangle$, which implies

$$\sum(S)+\langle a\rangle=H.$$

Choose $b|S$ such that $b\notin\langle a\rangle$. By (5), there exists $t\in[0,r_a]\setminus\{k\}$ such that $\phi_a(b)+\phi_a(C_t^a)=\phi_a(C_k^a)$. Therefore, $(b+C_t^a)\cap C_i^a=\emptyset$ for every $i\in[0,r_a]\setminus\{k\}$. Hence,

$$\begin{aligned}
|(b+C_t^a)\setminus C_k^a| &= |(b+C_t^a)\setminus \textstyle\sum(S)|\\
&\le |(b+\textstyle\sum(S))\setminus\sum(S)|\\
&\le |\textstyle\sum(S\cdot b)\setminus\sum(S)|\\
&= \lambda_S(b)\\
&\le 1.
\end{aligned}$$

It follows from $|C_t^a|=\mathrm{ord}(a)$ and $|C_k^a|\le\mathrm{ord}(a)$ that $|C_k^a|\in\{\mathrm{ord}(a)-1,\mathrm{ord}(a)\}$. Note that

$$\begin{aligned}
|\textstyle\sum(S)+\langle a\rangle| &= |\displaystyle\bigcup_{i=0}^{r_a}(C_i^a+\langle a\rangle)|\\
&= \sum_{i=0}^{r_a}|C_i^a+\langle a\rangle|\\
&= |C_k^a+\langle a\rangle|+\sum_{i\in[0,r_a]\setminus\{k\}}|C_i^a+\langle a\rangle|\\
&\le |C_k^a|+1+\sum_{i\in[0,r_a]\setminus\{k\}}|C_i^a|\\
&= |\textstyle\sum(S)|+1.
\end{aligned}$$

Therefore, $|\sum(S)|\ge|\sum(S)+\langle a\rangle|-1=|H|-1$. Hence, $\sum(S)=|H|-1$. This proves Claim B. □

It follows from (3) and Claim B that

$$r=\frac{|H|+1}{2}, \tag{6}$$

$$|H|\equiv 1 \pmod 2 \tag{7}$$

and

$$\mathrm{St}(\sum(S))=\{0\}. \tag{8}$$

By (2) and (8), we have that

$$\lambda_S(g)=1 \tag{9}$$

for every $g|S$.

We show next that

$$\mathsf{h}(S)\ge 2. \tag{10}$$

Assume to the contrary that, $a_1, \ldots, a_r$ are pairwise distinct. Take an arbitrary element $g \in H$. Since $|H|$ is odd, there exists at most one index $i \in [1, r]$ such that $2a_i = g$. By rearranging if necessary we assume that $2a_j \neq g$ holds for every $j \in [2, r]$. Consider two subsets $\{0, a_1, \ldots, a_r\}$ and $\{g - a_2, \ldots, g - a_r\}$ of H. Since $|\{0, a_1, \ldots, a_r\}| + |\{g - a_2, \ldots, g - a_r\}| = 2r = |H| + 1$, the two subsets cannot be disjoint. Therefore, $g - a_i = 0$ or $g - a_i = a_j$ for some $i \in [2, r]$ and some $j \in [1, r] \setminus \{i\}$. So, $g = a_i$ or $g = a_i + a_j$. It follows that $\sum(S) = H$, a contradiction. This proves (10).

Claim C. $|\sum(T)| < |\langle \mathrm{supp}(T) \rangle|$ for any nonempty proper subsequence T of S.

Proof of Claim C. Suppose to the contrary that there exists a nonempty proper subsequence T of S such that $|\sum(T)| = |\langle \mathrm{supp}(T) \rangle|$. Then $0 \in \sum(S)$ and $\sum(S) = \sum_0(T) + \sum_0(ST^{-1})$, which implies $\langle \mathrm{supp}(T) \rangle \subseteq \mathrm{St}(\sum(S))$, a contradiction with (8). This proves Claim C. □

By (10), $\rho(S) \notin \{1, 2\}$. We claim that

Claim D. $|\mathrm{supp}(S)| \geq 3$.

Proof of Claim D. Suppose to the contrary that $|\mathrm{supp}(S)| = 2$. We may rewrite $S = a_1^{\gamma} \cdot a_2^{\beta}$ where $\gamma \geq \beta \geq 2$.

Suppose $\beta \geq 3$. Then $\rho(S) \notin [1, 3]$ and $Sa_1^{-2} \cdot a_2^{-2}$ is not strictly behaving. It follows from the minimality of S and Claim C that $|\sum(Sa_1^{-2} \cdot a_2^{-2})| \geq 2|Sa_1^{-2} \cdot a_2^{-2}| - 1 = 2r - 9$. By (7), we have $\rho(a_1^2 \cdot a_2^2) = 0$. By Lemma 2.6 (ii), $|\sum_0(a_1^2 \cdot a_2^2)| \geq 9$. It follows from Lemma 2.4 that $|\sum(S)| \geq |\sum(Sa_1^{-2} \cdot a_2^{-2})| + |\sum_0(a_1^2 \cdot a_2^2)| - 1 \geq 2r - 1$, a contradiction. Hence,

$$\beta = 2.$$

Let $X_0 = \{a_1, 2a_1, \ldots, \gamma a_1\}$, $X_1 = a_2 + \{0, a_1, \ldots, \gamma a_1\}$ and $X_2 = 2a_2 + \{0, a_1, \ldots, (\gamma - 1)a_1\}$. It is easy to see that X_0, X_1 and X_2 are subsets of $\sum(S)$. Also, $X_0 \cap X_1 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. We show next that

$$|X_0 \cap X_2| \leq \gamma - 2.$$

The argument is as follows. If $2a_2 + (\gamma - 1)a_1 \in X_0$, then by $\rho(S) \notin [1, \gamma]$ we derive that $2a_2 + (\gamma - 1)a_1 = \gamma a_1$. Hence $2a_2 = a_1$ and S is a_2-strictly behaving, a contradiction. Therefore,

$$2a_2 + (\gamma - 1)a_1 \in X_2 \setminus X_0.$$

If $2a_2 + (\gamma - 2)a_1 \in X_0$, then by $\rho(S) \notin [1, \gamma]$ we derive that $2a_2 + (\gamma - 2)a_1 \in \{\gamma a_1, (\gamma - 1)a_1\}$. Therefore, $2a_1 = 2a_2$ or $a_1 = 2a_2$. If $2a_1 = 2a_2$, then by (7) we get $a_1 = a_2$, a contradiction. Hence, $a_1 = 2a_2$ and S is a_2-strictly behaving, also a contradiction. Therefore,

$$2a_2 + (\gamma - 2)a_1 \in X_2 \setminus X_0.$$

This proves $|X_0 \cap X_2| \le \gamma - 2$. Now we have

$$\begin{aligned} |\sum(S)| &\ge |X_0 \cup X_1 \cup X_2| \\ &= |X_1| + |X_0 \cup X_2| \\ &= |X_1| + |X_0| + |X_2| - |X_0 \cap X_2| \\ &\ge (\gamma+1) + \gamma + \gamma - (\gamma - 2) \\ &= 2\gamma + 3 \\ &= 2r - 1, \end{aligned}$$

a contradiction. This proves Claim D. □

By (10) and Claim D, we have

$$r \ge 4.$$

Claim E. There exists a squarefree subsequence $U|S$ of length three such that either $|SU^{-1}| = 1$ or SU^{-1} is not strictly behaving.

Proof of Claim E. Suppose that Claim E is false. By (10) and Claim D, we may assume that a_{r-2}, a_{r-1}, a_r are pairwise distinct and $\mathsf{v}_{a_{r-2}}(S) \ge 2$. Let $U_1 = a_{r-2}a_{r-1}a_r$. Then $|SU_1^{-1}| \ge 2$ and $SU_1^{-1} = (n_1 g) \cdot \ldots \cdot (n_{r-3} g)$ is strictly g-behaving. By Claim C, $\sum\limits_{i=1}^{r-3} n_i < \mathrm{ord}(g)$. Let $A_{r-3} = \{g, 2g, \ldots, (\sum\limits_{i=1}^{r-3} n_i)g\}$. By the choice of U_1 we have that

$$a_{r-2} = n_{r-2} g$$

with $n_{r-2} \le \sum\limits_{i=1}^{r-3} n_i$.

Suppose $\langle g \rangle \ne H$. By Claim C, we have $\mathrm{ord}(g) \ge 1 + \sum\limits_{i=1}^{r-2} n_i \ge r - 1$. It follows from Claim B and (7) that $|\sum(S)| \ge 3\,\mathrm{ord}(g) - 1 \ge 3(r-1) - 1 \ge 2r - 1$, a contradiction. Hence,

$$\langle g \rangle = H.$$

By Lemma 2.8, Claim B and (9),

$$|\sum(Sc^{-1})| \le |\sum(S)| - \lambda_S(c) = \mathrm{ord}(g) - 2 \qquad (11)$$

for every $c|S$.

Since $\langle g \rangle = H$, $a_i = n_i g$ for $i = r-1, r$. We may assume that $n_r > n_{r-1}$. Recalling that the maximal length of strictly behaving subsequence of S is less than $r-1$, it follows from $\rho(S) \notin \{1, 2\}$ that

$$\sum_{i=1}^{r-2} n_i < n_{r-1} < n_r < \mathrm{ord}(g) - 1.$$

If $n_{r-1} = 1 + \sum_{i=1}^{r-2} n_i$ then $\sum(Sa_r^{-1}) = \{g, 2g, \ldots, (\sum_{i=1}^{r-1} n_i)g\}$. It follows from (11) that $\sum_{i=1}^{r-1} n_i \leq \text{ord}(g) - 2$. Since $a_r \notin \{g, -g\}$, it is easy to see that $\lambda_{Sa_r^{-1}}(a_r) \geq 2$, a contradiction with (1). Hence,

$$n_{r-1} > 1 + \sum_{i=1}^{r-2} n_i.$$

By (6), we have $\sum_{i=1}^{r-2} n_i \geq r - 2 = \frac{\text{ord}(g)-3}{2}$, which implies $\text{ord}(g) - n_r \leq \text{ord}(g) - (3 + \sum_{i=1}^{r-2} n_i) \leq \sum_{i=1}^{r-2} n_i$. It follows that

$$\sum(Sa_{r-1}^{-1}) = \{n_r g, (n_r + 1)g, \ldots, \text{ord}(g)g\} \cup \{g, 2g, \ldots, (\sum_{i=1}^{r-2} n_i)g\}.$$

Hence, $\{(n_{r-1} - 1)g, n_{r-1}g\} \subseteq \sum(S) \setminus \sum(Sa_{r-1}^{-1})$. This gives us that $\lambda_{Sa_{r-1}^{-1}}(a_{r-1}) \geq 2$, a contradiction with (1). This proves Claim E. $\square$

Now we choose a squarefree subsequence U of S as in Claim E. It follows from the minimality of S and Claim C that $|\sum(SU^{-1})| \geq 2|SU^{-1}| - 1$. It follows from (7), Lemma 2.4, Lemma 2.5 and Lemma 2.6 (i) that $|\sum(S)| \geq |\sum(SU^{-1})| + |\sum_0(U)| - 1 \geq 2|S| - 1$, a contradiction. This completes the proof of Theorem 1.2 given Case 2, thereby finishing the proof of Theorem 1.2. $\square$

Remark 2.9 *The following example shows that the conclusion of Theorem 1.2 is, in a certain sense, best possible. Let G be an abelian group, and let $g \in G \setminus \{0\}$. Let $S = g^h \cdot (kg)$ where $2 \leq h + 1 \leq k \leq \text{ord}(g) - h$. It is easy to check that S is not strictly behaving, $\rho(S) \notin [1, h]$ and $|\sum(S)| = 2|S| - 1$.*

Proof of Theorem 1.3. Let $r = |S|$. If S is not strictly behaving, the conclusion follows immediately from Theorem 1.2. Hence, we may assume that $S = (n_1 g) \cdot \ldots \cdot (n_r g)$ is strictly g-behaving for some $g \neq 0$. Since $0 \in \sum(S)$, we have that $\sum_{i=1}^{r} n_i \geq \text{ord}(g)$, which implies $\langle \text{supp}(S) \rangle = \langle g \rangle = \sum(S)$. $\square$

Proof of Corollary 1.4. Assume $\rho(S) \notin [1, \mathsf{h}(S)]$. By Theorem 1.3, we have that $|\sum(S)| \geq \min(|\langle \text{supp}(S) \rangle|, 2|S| - 1) = |\langle \text{supp}(S) \rangle|$. $\square$

Proof of Corollary 1.5. Let $r = |S|$. Since $\rho(S) = 0$ and $r > n/2$ we infer that $\langle \text{supp}(S) \rangle = C_n$. Note that $|\sum(S)| \leq n - 1 \leq \min(|C_n| - 1, 2r - 2)$. By Theorem 1.2 $S = (n_1 g) \cdot \ldots \cdot (n_r g)$ is strictly g-behaving for some $g \in C_n$. Now the corollary follows from the obvious fact that $\sum_{i=1}^{r} n_i < \text{ord}(g)$. $\square$

Proof of Corollary 1.6. Since $\langle \mathrm{supp}(S)\rangle$ is not cyclic, then S is not strictly behaving. It follows from $\rho(S)=0$ and Theorem 1.2 that $|\sum(S)| \ge \min(|\langle \mathrm{supp}(S)\rangle|, 2|S|-1) = 2|S|-1$. □

Proof of Corollary 1.7. By Theorem 1.2, we need only to consider the case that $S=(n_1g)\cdot\ldots\cdot(n_rg)$ is strictly g-behaving for some $g\in G$. Now we have $|\sum(S)| = \sum_{i=1}^{r} n_i \ge \mathsf{v}_g(S)+2(|S|-\mathsf{v}_g(S)) = 2|S|-\mathsf{v}_g(S) \ge 2|S|-\mathsf{h}(S)$. □

Proof of Corollary 1.8. We may assume that $|S|\ge 2$. Since $\mathsf{h}(S)=1$, we have that S is not strictly behaving. From $0\notin S$ we know that $\rho(S)\notin[1,\mathsf{h}(S)]$. It follows from Theorem 1.2 that $|\sum(S)| \ge \min(|\langle S\rangle|, 2|S|-1)$. □

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References

[1] N. Alon, Subset sums, J. Number Theory 27 (1987) 196–205.

[2] M. DeVos, L. Goddyn, B. Mohar, A generalization of Kneser's addition theorem, Adv. Math. 220 (2009) 1531–1548.

[3] M. Freeze, W.W. Smith, Sumsets of zerofree sequence, Arab. J. Sci. Eng. Sect. C Theme 26 (2001) 97–105.

[4] W.D. Gao, Addition theorems for finite abelian groups, J. Number Theory 53 (1995) 241–246.

[5] W.D. Gao, A combinatorial problem on finite abelian groups, J. Number Theory 58 (1996) 100–103.

[6] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. 24 (2006) 337–369.

[7] W.D. Gao, I. Leader, Sums and k-sums in abelian groups of order k, J. Number Theory 120 (2006) 26–32.

[8] W.D. Gao, J.J. Zhuang, Sequences not containing long zero-sum subsequences, Eur. J. Comb. 26 (2005) 1053–1059.

[9] A. Geroldinger, *Additive group theory and non-unique factorizations*, Combinatorial Number Theory and Additive Group Theory (A. Geroldinger and I. Ruzsa, eds.), Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1–86.

[10] A. Geroldinger, F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[11] D.J. Grynkiewicz, O. Ordaz, M.T. Varela, F. Villarroel, On Erdős-Ginzburg-Ziv inverse theorems, Acta Arith. 129 (2007) 307–318.

[12] Y.O. Hamidoune, Adding distinct congruence classes, Combin. Probab. Comput. 7 (1998) 81–87.

[13] M.B. Nathanson, Additive number theory: Inverse problems and the geometry of sumsets, Vol. 165 of Graduate Texts in Mathematics, Springer, New York.

[14] J.E. Olson, E.T. White, Sums from a sequence of group elements, Number Theory and Algebra (H. Zassenhaus, ed.), Academic Press, 1977, pp. 215–222.

[15] A. Pixton, Sequences with small subsum sets, J. Number Theory 129 (2009) 806–817.

[16] S. Savchev, F. Chen, Long zero-free sequences in finite cyclic groups, Discrete Math. 307 (2007) 2671–2679.

[17] Z. Shan, A conjecture in elementary number theory, Adv. Math. (Beijing) 12 (1983) 299–301.

[18] P. Scherk, Distinct elements in a set of sums, Am. Math. Mon. 62 (1955) 46–47.

[19] P.Z. Yuan, On the index of minimal zero-sum sequences over finite cyclic groups, J. Combin. Theory Ser. A 114 (2007) 1545–1551.

[20] P.Z. Yuan, Subsequence sums of a zero-sumfree sequence, Eur. J. Combin. 30 (2009) 439–446.