The energy of random graphs *

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Dedicated to Prof. Dragos Cvetkovic on the occasion of his 70th birthday

Abstract

In 1970s, Gutman introduced the concept of the energy $\mathscr{E}(G)$ for a simple graph G, which is defined as the sum of the absolute values of the eigenvalues of G. This graph invariant has attracted much attention, and many lower and upper bounds have been established for some classes of graphs among which bipartite graphs are of particular interest. But there are only a few graphs attaining the equalities of those bounds. We however obtain an exact estimate of the energy for almost all graphs by Wigner's semi-circle law, which generalizes a result of Nikiforov. We further investigate the energy of random multipartite graphs by considering a generalization of Wigner matrix, and obtain some estimates of the energy for random multipartite graphs.

Keywords: graph, eigenvalues, graph energy, random graph, random matrix, empirical spectral distribution, limiting spectral distribution.

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1 Introduction

Throughout this paper, G denotes a simple graph of order n. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the adjacency matrix $\mathbf{A}(G) = (a_{ij})_{n\times n}$ of G are said to be the *eigenvalues of the graph* G. In chemistry, the eigenvalues of a molecular graph has a closed relation to the molecular orbital energy levels of π-electrons in conjugated hydrocarbons. For the Hüchkel molecular orbital approximation, the total π-electron energy in conjugated hydrocarbons is given by the sum of absolute values of the eigenvalues of the corresponding molecular graph in which the maximum degree is not more than 4 in general. In 1970s, Gutman [11] extended the concept of energy $\mathscr{E}(G)$ to all simple graphs G, and defined that

$$\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|,$$

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where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G. Evidently, one can immediately get the energy of a graph by computing the eigenvalues of the graph. It is rather hard, however, to compute the eigenvalues for a large matrix, even for a large symmetric (0,1)-matrix like $\mathbf{A}(G)$. So many researchers established a lot of lower and upper bounds to estimate the invariant for some classes of graphs among which the bipartite graphs are of particular interest. For further details, we refer the readers to the comprehensive survey [12]. But there is a common flaw for those inequalities that only a few graphs attain the equalities of those bounds. Thus we can hardly see the major behavior of the invariant $\mathscr{E}(G)$ for most graphs with respect to other graph parameters ($|V(G)|$, for instance). In this paper, however, we shall present an exact estimate of the energy for almost all graphs by Wigner's semi-circle law. Moreover, we investigate the energy of random multipartite graphs by employing the results on the spectral distribution of band matrix which is a generalization of Wigner matrix.

Similar results were obtained in [6] for Laplacian energy $LE(G)$ and in [7] for various other kinds of egergies, such as signless Laplacian energy $LE^+(G)$, incidence energy $IE(G)$, distance energy $DE(G)$ and Lapacian-energy like invariant $LEL(G)$. Actually, the idea of this paper came out earlier.

The structure of our article is as follows. In the next section, we shall consider the random graphs constructed from the classical Erdös–Rényi model. The second model is concerned with random multipartite graphs which will be defined and explored in the last section.

2 The energy of $G_n(p)$

In this section, we shall formulate an exact estimate of the energy for almost all graphs by Wigner's semi-circle law.

We start by recalling the Erdös–Rényi model $\mathcal{G}_n(p)$ (see [4]), which consists of all graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ in which the edges are chosen independently with probability $p = p(n)$. Apparently, the adjacency matrix $\mathbf{A}(G_n(p))$ of the random graph $G_n(p) \in \mathcal{G}_n(p)$ is a random matrix, and thus one can readily evaluate the energy of $G_n(p)$ once the spectral distribution of the random matrix $\mathbf{A}(G_n(p))$ is known.

In fact, the study on the spectral distributions of random matrices is rather abundant and active, which can be traced back to [17]. We refer the readers to [2, 5, 9] for an overview and some spectacular progress in this field. One important achievement in that field is Wigner's semi-circle law which characterizes the limiting spectral distribution of the empirical spectral distribution of eigenvalues for a sort of random matrix.

In order to characterize the statistical properties of the wave functions of quantum mechanical systems, Wigner in 1950s investigated the spectral distribution for a sort of random matrix, so-called *Wigner matrix*,

$$\mathbf{X}_n := (x_{ij}), \quad 1 \leq i, j \leq n,$$

which satisfies the following properties:

- x_{ij}'s are independent random variables with $x_{ij} = x_{ji}$;

- the x_{ii}'s have the same distribution F_1, while the x_{ij}'s $(i \neq j)$ have the same distribution F_2;
- $\mathbb{V}\mathrm{ar}(x_{ij}) = \sigma_2^2 < \infty$ for all $1 \leq i < j \leq n$.

We denote the eigenvalues of $\mathbf{X}_n$ by $\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n}$, and their empirical spectral distribution (ESD) by

$$\Phi_{\mathbf{X}_n}(x) = \frac{1}{n} \cdot \#\{\lambda_{i,n} \mid \lambda_{i,n} \leq x, \ i = 1, 2, \ldots, n\}.$$

Wigner [15, 16] considered the limiting spectral distribution (LSD) of $\mathbf{X}_n$, and obtained his semi-circle law.

Theorem 2.1. *Let $\mathbf{X}_n$ be a Wigner matrix. Then*

$$\lim_{n\to\infty} \Phi_{n^{-1/2}\mathbf{X}_n}(x) = \Phi(x) \ \text{a.s.}$$

i.e., with probability 1, the ESD $\Phi_{n^{-1/2}\mathbf{X}_n}(x)$ converges weakly to a distribution $\Phi(x)$ as n tends to infinity, where $\Phi(x)$ has the density

$$\phi(x) = \frac{1}{2\pi\,\sigma_2^2}\sqrt{4\,\sigma_2^2 - x^2}\ \mathbf{1}_{|x|\leq 2\sigma_2}.$$

Remark 2.1. One of classical methods to prove the theorem above is the moment approach. Employing the method, we can get more information about the LSD of Wigner matrix. Set $\mu_i = \int x\,dF_i$ $(i = 1, 2)$ and $\overline{\mathbf{X}}_n = \mathbf{X}_n - \mu_1\mathbf{I}_n - \mu_2(\mathbf{J}_n - \mathbf{I}_n)$, where $\mathbf{I}_n$ is the unit matrix of order n and $\mathbf{J}_n$ is the matrix of order n in which all entries equal 1. It is easily seen that the random matrix $\overline{\mathbf{X}}_n$ is a Wigner matrix as well. By means of Theorem 2.1, we have

$$\lim_{n\to\infty} \Phi_{n^{-1/2}\overline{\mathbf{X}}_n}(x) = \Phi(x) \text{ a.s.} \tag{1}$$

Evidently, each entry of $\overline{\mathbf{X}}_n$ has mean 0. Furthermore, one can show, using moment approach, that for each positive integer k,

$$\lim_{n\to\infty} \int x^k d\Phi_{n^{-1/2}\overline{\mathbf{X}}_n}(x) = \int x^k d\Phi(x) \text{ a.s.} \tag{2}$$

It is interesting that the existence of the second moment of the off-diagonal entries is the necessary and sufficient condition for the semi-circle law, but there is no moment requirement on the diagonal elements. For further comments on the moment approach and Wigner's semi-circle law, we refer the readers to the extraordinary survey by Bai [2].

We shall say that *almost every (a.e.)* graph in $\mathcal{G}_n(p)$ has a certain property Q (see [4]) if the probability that a random graph $G_n(p)$ has the property Q converges to 1 as n tends to infinity. Occasionally, we shall write *almost all* instead of almost every. It is easy to see that if F_1 is a *pointmass at 0, i.e.,* $F_1(x) = 1$ for $x \geq 0$ and $F_1(x) = 0$ for $x < 0$, and F_2 is the *Bernoulli distribution with mean p*, then the Wigner matrix $\mathbf{X}_n$ coincides with the adjacency matrix $\mathbf{A}(G_n(p))$ of the random graph $G_n(p)$. Obviously, $\sigma_2 = \sqrt{p(1-p)}$ in this case.

To establish the exact estimate of the energy $\mathscr{E}(G_n(p))$ for a.e. graph $G_n(p)$, we first present some notions. In what follows, we shall use $\mathbf{A}$ to denote the adjacency matrix $\mathbf{A}(G_n(p))$ for convenience. Set

$$\overline{\mathbf{A}} = \mathbf{A} - p(\mathbf{J}_n - \mathbf{I}_n).$$

It is easy to check that each entry of $\overline{\mathbf{A}}$ has mean 0. We define the *energy* $\mathscr{E}(\mathbf{M})$ *of a matrix* $\mathbf{M}$ as the sum of absolute values of the eigenvalues of $\mathbf{M}$. By virtue of the following two lemmas, we shall formulate an estimate of the energy $\mathscr{E}(\overline{\mathbf{A}})$, and then establish the exact estimate of $\mathscr{E}(\mathbf{A}) = \mathscr{E}(G_n(p))$ by using Lemma 2.4.

Let I be the interval $[-1, 1]$.

Lemma 2.2. *Let I^c be the set $\mathbb{R} \setminus I$. Then*

$$\lim_{n\to\infty} \int_{I^c} x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) = \int_{I^c} x^2 d\Phi(x) \ \textit{a.s.}$$

Proof. Suppose $\phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$ is the density of $\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$. According to Eq.(1), with probability 1, $\phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$ converges to $\phi(x)$ almost everywhere as n tends to infinity. Since $\phi(x)$ is bounded on I, it follows that with probability 1, $x^2\phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$ is bounded almost everywhere on I. Invoking bounded convergence theorem yields

$$\lim_{n\to\infty} \int_{I} x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) = \int_{I} x^2 d\Phi(x) \ \text{a.s.}$$

Combining the above fact with Eq.(2), we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{I^c} x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) &= \lim_{n\to\infty} \left(\int x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) - \int_I x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) \right) \\
&= \lim_{n\to\infty} \int x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) - \lim_{n\to\infty} \int_I x^2 d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) \\
&= \int x^2 d\Phi(x) - \int_I x^2 d\Phi(x) \ \text{a.s.} \\
&= \int_{I^c} x^2 d\Phi(x) \ \text{a.s.}
\end{aligned}$$

□

Lemma 2.3 (Billingsley [3] pp. 219). *Let μ be a measure. Suppose that functions a_n, b_n, f_n converges almost everywhere to functions a, b, f, respectively, and that $a_n \leq f_n \leq b_n$ almost everywhere. If $\int a_n d\mu \to \int a\, d\mu$ and $\int b_n d\mu \to \int b\, d\mu$, then $\int f_n d\mu \to \int f d\mu$.*

We now turn to the estimate of the energy $\mathscr{E}(\overline{\mathbf{A}})$. To this end, we first investigate the convergence of $\int |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$. According to Eq.(1) and the bounded convergence theorem, we can deduce, by an argument similar to the first part of the proof of Lemma 2.2, that

$$\lim_{n\to\infty} \int_I |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) = \int_I |x| d\Phi(x) \ \text{a.s.}$$

Obviously, $|x| \leq x^2$ if $x \in I^c := \mathbb{R} \setminus I$. Set $a_n(x) = 0, b_n(x) = x^2 \phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$, and $f_n(x) = |x|\phi_{n^{-1/2}\overline{\mathbf{A}}}(x)$. Employing Lemmas 2.2 and 2.3, we have

$$\lim_{n\to\infty} \int_{I^c} |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) = \int_{I^c} |x| d\Phi(x) \text{ a.s.}$$

Consequently,

$$\lim_{n\to\infty} \int |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) = \int |x| d\Phi(x) \text{ a.s.} \tag{3}$$

Suppose $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$ and $\overline{\lambda}'_1, \ldots, \overline{\lambda}'_n$ are the eigenvalues of $\overline{\mathbf{A}}$ and $n^{-1/2}\overline{\mathbf{A}}$, respectively. Clearly, $\sum_{i=1}^n |\overline{\lambda}_i| = n^{1/2} \sum_{i=1}^n |\overline{\lambda}'_i|$. By Eq.(3), we can deduce that

$$\begin{aligned}
\mathscr{E}\left(\overline{\mathbf{A}}\right)/n^{3/2} &= \frac{1}{n^{3/2}} \sum_{i=1}^n |\overline{\lambda}_i| \\
&= \frac{1}{n} \sum_{i=1}^n |\overline{\lambda}'_i| \\
&= \int |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}}(x) \\
&\to \int |x| d\Phi(x) \text{ a.s. } (n \to \infty) \\
&= \frac{1}{2\pi\,\sigma_2^2} \int_{-2\,\sigma_2}^{2\,\sigma_2} |x| \sqrt{4\,\sigma_2^2 - x^2}\; dx \\
&= \frac{8}{3\pi}\,\sigma_2 = \frac{8}{3\pi}\sqrt{p(1-p)}.
\end{aligned}$$

Therefore, the energy $\mathscr{E}\left(\overline{\mathbf{A}}\right)$ enjoys a.s. the equation as follows:

$$\mathscr{E}\left(\overline{\mathbf{A}}\right) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right).$$

We proceed to investigating $\mathscr{E}(\mathbf{A}) = \mathscr{E}(G_n(p))$ and present the following result due to Fan.

Lemma 2.4 (Fan [8]). *Let* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *be real symmetric matrices of order* n *such that* $\mathbf{X} + \mathbf{Y} = \mathbf{Z}$. *Then*

$$\sum_{i=1}^n |\lambda_i(\mathbf{X})| + \sum_{i=1}^n |\lambda_i(\mathbf{Y})| \geq \sum_{i=1}^n |\lambda_i(\mathbf{Z})|$$

where $\lambda_i(\mathbf{M})$ $(i = 1, \cdots, n)$ *are the eigenvalues of the matrix* $\mathbf{M}$.

It is not difficult to verify that the eigenvalues of the matrix $\mathbf{J}_n - \mathbf{I}_n$ are $n-1$ and -1 of $n-1$ times. Consequently $\mathscr{E}(\mathbf{J}_n - \mathbf{I}_n) = 2(n-1)$. One can readily see that $\mathscr{E}\left(p(\mathbf{J}_n - \mathbf{I}_n)\right) = p\,\mathscr{E}(\mathbf{J}_n - \mathbf{I}_n)$. Thus,

$$\mathscr{E}\left(p(\mathbf{J}_n - \mathbf{I}_n)\right) = 2p(n-1).$$

Since $\mathbf{A} = \overline{\mathbf{A}} + p(\mathbf{J}_n - \mathbf{I}_n)$, it follows from Lemma 2.4 that with probability 1,

$$\begin{aligned} \mathscr{E}(\mathbf{A}) &\leq \mathscr{E}\left(\overline{\mathbf{A}}\right) + \mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_n)) \\ &= n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right) + 2p(n-1). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \mathscr{E}(\mathbf{A})/n^{3/2} \leq \frac{8}{3\pi}\sqrt{p(1-p)} \text{ a.s.} \tag{4}$$

On the other hand, since $\overline{\mathbf{A}} = \mathbf{A} + p\big(-(\mathbf{J}_n - \mathbf{I}_n)\big)$, we can deduce by Lemma 2.4 that with probability 1,

$$\begin{aligned} \mathscr{E}(\mathbf{A}) &\geq \mathscr{E}\left(\overline{\mathbf{A}}\right) - \mathscr{E}\left(p\big(-(\mathbf{J}_n - \mathbf{I}_n)\big)\right) \\ &= \mathscr{E}\left(\overline{\mathbf{A}}\right) - \mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_n)) \\ &= n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right) - 2p(n-1). \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \mathscr{E}(\mathbf{A})/n^{3/2} \geq \frac{8}{3\pi}\sqrt{p(1-p)} \text{ a.s.} \tag{5}$$

Combining Ineq.(4) with Ineq.(5), we have

$$\mathscr{E}(\mathbf{A}) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right) \text{ a.s.}$$

Recalling that $\mathbf{A}$ is the adjacency matrix of $G_n(p)$, we thus obtain that a.e. random graph $G_n(p)$ enjoys the equation as follows:

$$\mathscr{E}(G_n(p)) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right).$$

Remark 2.2. Note that for $p = \frac{1}{2}$, Nikiforov in [14] got the above equation. Here, our result is for any probability p, which could be seen as a generalization of his result.

3 The energy of the random multipartite graph

We begin with the definition of the random multipartite graph. We use $K_{n;\nu_1,\ldots,\nu_m}$ to denote the complete m-partite graph with vertex set $[n]$ whose parts $V_1, \ldots, V_m$ ($m = m(n) \geq 2$) are such that $|V_i| = n\nu_i = n\nu_i(n)$, $i = 1, \ldots, m$. Let $\mathcal{G}_{n;\nu_1\ldots\nu_m}(p)$ be the set of random m-partite graphs with vertex set $[n]$ in which the edges are chosen independently with probability p from the set of edges of $K_{n;\nu_1,\ldots,\nu_m}$. We further introduce two classes of random m-partite graphs. Denote by $\mathcal{G}_{n,m}(p)$ and $\mathcal{G}'_{n,m}(p)$, respectively, the sets of random m-partite graphs satisfy, respectively, the following conditions:

$$\lim_{n\to\infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} > 0 \text{ and } \lim_{n\to\infty} \frac{\nu_i(n)}{\nu_j(n)} = 1. \tag{6}$$

and

$$\lim_{n\to\infty} \max\{\nu_1(n), \ldots, \nu_m(n)\} = 0. \quad (7)$$

One can easily see that to obtain the estimate of the energy for the random multipartite graph $G_{n;\nu_1\ldots\nu_m}(p) \in \mathcal{G}_{n;\nu_1\ldots\nu_m}(p)$, we need to investigate the spectral distribution of the random matrix $\mathbf{A}(G_{n;\nu_1\ldots\nu_m}(p))$. It is not difficult to verify that $\mathbf{A}(G_{n;\nu_1\ldots\nu_m}(p))$ would be a special case of a random matrix $\mathbf{X}_n(\nu_1,\ldots,\nu_m)$ (or $\mathbf{X}_{n,m}$ for short) called a *random multipartite matrix* which satisfies the following properties:

- x_{ij}'s are independent random variables with $x_{ij} = x_{ji}$;
- the x_{ij}'s have the same distribution F_1 if i and $j \in V_k$, while the x_{ij}'s have the same distribution F_2 if $i \in V_k$ and $j \in [n]\setminus V_k$, where $V_1,\ldots,V_m$ are the parts of $K_{n;\nu_1,\ldots,\nu_m}$ and k is an integer with $1 \le k \le m$;
- $|x_{ij}| \le K$ for some constant K.

Apparently, if F_1 is a pointmass at 0 and F_2 is a Bernoulli distribution with mean p, then the random matrix $\mathbf{X}_{n,m}$ coincides with the adjacency matrix $\mathbf{A}(G_{n;\nu_1\ldots\nu_m}(p))$. Thus, we can readily evaluate the energy $\mathscr{E}(G_{n;\nu_1\ldots\nu_m}(p))$ once we obtain the spectral distribution of $\mathbf{X}_{n,m}$. In fact, the random matrix $\mathbf{X}_{n,m}$ is a special case of the random matrix considered by Anderson and Zeitouni [1] in a rather general setting called the band matrix model which can be regarded as one of generalization of the Wigner matrix, and we shall employ their results to deal with the spectral distribution of $\mathbf{X}_{n,m}$.

The rest of this section will be divided into two parts. In the first part, we shall present, respectively, exact estimates of the energies for random graphs $G_{n,m}(p) \in \mathcal{G}_{n,m}(p)$ and $G'_{n,m}(p) \in \mathcal{G}'_{n,m}(p)$ by exploring the spectral distribution of the band matrix. We establish lower and upper bounds of the energy for the random multipartite graph $G_{n;\nu_1\ldots\nu_m}(p)$, and moreover we obtain an exact estimate of the energy for the random bipartite graph $G_{n;\nu_1,\nu_2}(p)$ in the second part.

3.1 The energy of $G_{n,m}(p)$ and $G'_{n,m}(p)$

In this part, we shall formulate exact estimates of the energies for random graphs $G_{n,m}(p)$ and $G'_{n,m}(p)$, respectively. For this purpose, we shall establish the following theorem. To state our result, we first present some notations. Let $\mathbf{I}_{n,m} = (i_{p,q})_{n\times n}$ be a *quasi-unit matrix* such that

$$i_{p,q} = \begin{cases} 1 & \text{if } p, q \in V_k, \\ 0 & \text{if } p \in V_k \text{ and } q \in [n] \setminus V_k, \end{cases}$$

where $V_1,\ldots,V_m$ are the parts of $K_{n;\nu_1,\ldots,\nu_m}$ and k is an integer with $1 \le k \le m$. Set $\mu_i = \int x\, dF_i$ $(i = 1, 2)$ and

$$\overline{\mathbf{X}}_{n,m} = \mathbf{X}_{n,m} - \mu_1 \mathbf{I}_{n,m} - \mu_2(\mathbf{J}_n - \mathbf{I}_{n,m}).$$

Evidently, $\overline{\mathbf{X}}_{n,m}$ is a random multipartite matrix as well in which each entry has mean 0. To make our statement concise, we define $\Delta^2 = (\sigma_1^2 + (m-1)\,\sigma_2^2)/m$.

Theorem 3.1.

(i) *If condition (6) holds, then*

$$\Phi_{n^{-1/2}\overline{\mathbf{X}}_{n,m}}(x) \to_P \Psi(x) \text{ as } n \to \infty$$

i.e., the ESD $\Phi_{n^{-1/2}\overline{\mathbf{X}}_{n,m}}(x)$ converges weakly to a distribution $\Psi(x)$ in probability as n tends to infinity where $\Psi(x)$ has the density

$$\psi(x) = \frac{1}{2\pi\Delta^2}\sqrt{4\Delta^2 - x^2}\ \mathbf{1}_{|x|\le 2\Delta}.$$

(ii) *If condition (7) holds, then $\Phi_{n^{-1/2}\overline{\mathbf{X}}_{n,m}}(x) \to_P \Phi(x)$ as $n \to \infty$.*

Our theorem can be proved by a result established by Anderson and Zeitouni [1]. We begin with a succinct introduction of the band matrix model defined by Anderson and Zeitouni in [1], from which one can readily see that a random multipartite matrix is a band matrix.

We fix a non-empty set $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ which is finite or countably infinite. The elements of $\mathcal{C}$ are called *colors*. Let κ be a surjection from $[n]$ to the color set $\mathcal{C}$, and we say that $\kappa(i)$ is the color of i. Naturally, we can obtain a partition $V_1, \ldots, V_m$ of $[n]$ according to the colors of its elements, *i.e.*, two elements i and i' in $[n]$ belong to the same part V_j if and only if their colors are identical. We next define the probability measure θ_m on the color set as follows:

$$\theta_m(C) = \theta_{m(n)}(C) = |\kappa^{-1}(C)|/n, 1 \le i \le m = m(n),$$

where $C \subseteq \mathcal{C}$ and $\kappa^{-1}(C) = \{x \in [n] : \kappa(x) \in C\}$. Evidently, the probability space $(\mathcal{C}, 2^{\mathcal{C}}, \theta_m)$ is a discrete probability space. Set

$$\theta = \lim_{n\to\infty} \theta_m.$$

For each positive integer k we fix a bounded nonnegative function $d^{(k)}$ on color set and a symmetric bounded nonnegative function $s^{(k)}$ on the product of two copies of the color set. We make the following assumptions:

- $d^{(k)}$ is constant for $k \neq 2$;
- $s^{(k)}$ is constant for $k \notin \{2, 4\}$;

Let $\{\xi_{ij}\}_{i,j=1}^n$ be a family of independent real-valued mean zero random variables. We suppose that for all $1 \le i, j \le n$ and positive integers k,

$$\mathbb{E}(|\xi_{ij}|^k) \le \begin{cases} s^{(k)}(\kappa(i), \kappa(j)) & \text{if } i \neq j, \\ d^{(k)}(\kappa(i)) & \text{if } i = j, \end{cases}$$

and moreover we assume that equality holds above whenever one of the following conditions holds:

- $k = 2$,
- $i \neq j$ and $k = 4$.

In other words, the rule is to enforce equality whenever the not-necessarily-constant functions $d^{(2)}$, $s^{(2)}$ or $s^{(4)}$ are involved, but otherwise merely to impose a bound.

We are now ready to present the random symmetric matrix $\mathbf{Y}_n$ called *band matrix* in which the entries are the r.v. ξ_{ij}. Evidently, $\mathbf{Y}_n$ is the same as $\overline{\mathbf{X}}_{n,m}$ providing

$$s^{(2)}(\kappa(i),\kappa(j)) = \begin{cases} \sigma_1^2 & \text{if } \kappa(i) = \kappa(j) \\ \sigma_2^2 & \text{if } \kappa(i) \neq \kappa(j) \end{cases} \quad \text{and } d^{(2)}(\kappa(i)) = \sigma_1^2, 1 \leq i,j \leq n. \tag{8}$$

So the random multipartite matrix $\overline{\mathbf{X}}_{n,m}$ is a special case of the band matrix $\mathbf{Y}_n$.

Define the standard semi-circle distribution $\Phi_{0,1}$ of zero mean and unit variance to be the measure on the real set of compact support with density $\phi_{0,1}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{|x|\leq 2}$. Anderson and Zeitouni investigated the LSD of $\mathbf{Y}_n$ and proved the following result (Theorem 3.5 in [1]).

Lemma 3.2 (Anderson and Zeitouni [1]). *If $\int s^{(2)}(c,c')\theta(dc') \equiv 1$, then $\Phi_{n^{-1/2}\mathbf{Y}_n}(x)$ converges weakly to the standard semi-circle distribution $\Phi_{0,1}$ in probability as n tends to infinity.*

Remark 3.1. The main approach employed by Anderson and Zeitouni to prove the assertion is a combinatorial enumeration scheme for the different types of terms that contribute to the expectation of products of traces of powers of the matrices. It is worthwhile to point out that by an analogous method called moment approach one can readily obtain a stronger assertion for $\overline{\mathbf{X}}_{n,m}$ that the convergence could be valid with probability 1. Moreover, one can show that for each positive integer k,

$$\lim_{n\to\infty} \int x^k \Phi_{n^{-1/2}\overline{\mathbf{X}}_n}(x) = \begin{cases} \displaystyle\int x^k \Psi(x) \text{ a.s.} & \text{if condition (6) holds,} \\ \displaystyle\int x^k \Phi(x) \text{ a.s.} & \text{if condition (7) holds.} \end{cases} \tag{9}$$

However, we shall not present the proof of Eq.(9) here since the arguments of the two methods are similar and the calculation of the moment approach is rather tedious. We refer the readers to Bai's survey [2] for details.

Using Lemma 3.2, to prove Theorem 3.1, we just need to verify $\int s^{(2)}(c,c')\theta(dc') \equiv 1$. For Theorem 3.1(i), we consider the matrix $\Delta^{-1}\overline{\mathbf{X}}_{n,m}$ where $\Delta^2 = (\sigma_1^2 + (m-1)\,\sigma_2^2)/m$. Note that condition (6) implies that $\theta_m(c_i) \to 1/m$ as $n \to \infty$, $1 \leq i \leq m$. By means of condition (8), one can readily see that for the random matrix $\Delta^{-1}\overline{\mathbf{X}}_{n,m}$,

$$\int s^{(2)}(c,c')\theta(dc') = \frac{1}{\Delta^2}\left(\frac{\sigma_1^2}{m} + \frac{(m-1)\,\sigma_2^2}{m}\right) \equiv 1.$$

Consequently, Lemma 3.2 implies that

$$\Phi_{n^{-1/2}\Delta^{-1}\overline{\mathbf{X}}_{n,m}} \to_P \Phi_{0,1} \text{ as } n \to \infty.$$

Therefore,

$$\Phi_{n^{-1/2}\overline{\mathbf{X}}_{n,m}} \to_P \Psi(x) \text{ as } n \to \infty,$$

and thus the first part of Theorem 3.1 follows.

For the second part of Theorem 3.1, we consider the matrix $\sigma_2^{-1}\overline{\mathbf{X}}_{n,m}$. Note that condition (7) implies that $\theta(c_i) = \lim_{n\to\infty} \theta_m(c_i) = \lim_{n\to\infty} \nu_i(n) = 0$, $1 \le i \le m$. By virtue of condition (8), if $c \neq c'$ then $s^{(2)}(c, c') = 1$. Consequently, for the random matrix $\sigma_2^{-1}\overline{\mathbf{X}}_{n,m}$, we have

$$\begin{aligned} \int s^{(2)}(c,c')\theta(dc') &= \int s^{(2)}(c,c')\chi_{\mathcal{C}\setminus\{c\}}\theta(dc') \\ &= \int \chi_{\mathcal{C}\setminus\{c\}}\theta(dc') \\ &= \theta(\mathcal{C}\setminus\{c\}) \equiv 1. \end{aligned}$$

As a result, Lemma 3.2 implies that

$$\Phi_{n^{-1/2}\sigma_2^{-1}\overline{\mathbf{X}}_{n,m}} \to_P \Phi_{0,1} \text{ as } n \to \infty.$$

Therefore,

$$\Phi_{n^{-1/2}\overline{\mathbf{X}}_{n,m}} \to_P \Phi(x) \text{ as } n \to \infty,$$

and thus the second part follows.

We now employ Theorem 3.1 to estimate the energy of $G_{n;\nu_1\ldots\nu_m}(p)$ under condition (6) or (7). For convenience, we shall use $\mathbf{A}_{n,m}$ to denote the adjacency matrix $\mathbf{A}(G_{n,m}(p))$. One can readily see that if a random multipartite matrix $\mathbf{X}_{n,m}$ satisfies condition (6), and F_1 is a pointmass at 0 and F_2 is a Bernoulli distribution with mean p, then $\mathbf{X}_{n,m}$ coincides with the adjacency matrix $\mathbf{A}_{n,m}$. Set

$$\overline{\mathbf{A}}_{n,m} = \mathbf{A}_{n,m} - p(\mathbf{J}_n - \mathbf{I}_{n,m}) \tag{10}$$

where $\mathbf{I}_{n,m}$ is the quasi-unit matrix whose parts are the same as $\mathbf{A}_{n,m}$. Evidently, each entry of $\overline{\mathbf{A}}_{n,m}$ has mean 0. It follows from the first part of Theorem 3.1 that

$$\Phi_{n^{-1/2}\overline{\mathbf{A}}_{n,m}} \to_P \Psi(x) \text{ as } n \to \infty.$$

Since the density of $\Psi(x)$ is bounded with the finite support, we can use a similar method for showing Eq.(3) to prove that

$$\int |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}_{n,m}}(x) \to_P \int |x| d\Psi(x) \text{ as } n \to \infty.$$

Consequently,

$$\begin{aligned} \mathscr{E}\left(\overline{\mathbf{A}}_{n,m}\right)/n^{3/2} &= \int |x| d\Phi_{n^{-1/2}\overline{\mathbf{A}}_{n,m}}(x) \\ &\to_P \int |x| d\Psi(x) \text{ as } n \to \infty \\ &= \frac{m}{2\pi(m-1)\sigma_2^2} \int_{-2\sqrt{\frac{m-1}{m}}\sigma_2}^{2\sqrt{\frac{m-1}{m}}\sigma_2} |x| \sqrt{4\frac{(m-1)\sigma_2^2}{m} - x^2}\, dx \\ &= \frac{8}{3\pi}\sqrt{\frac{m-1}{m}}\sigma_2 = \frac{8}{3\pi}\sqrt{\frac{m-1}{m}p(1-p)}. \end{aligned}$$

Therefore, a.e. random matrix $\overline{\mathbf{A}}_{n,m}$ enjoys the equation as follows:

$$\mathscr{E}\left(\overline{\mathbf{A}}_{n,m}\right) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{\frac{m-1}{m}p(1-p)} + o(1)\right).$$

We now turn to the estimate of the energy $\mathscr{E}(\mathbf{A}_{n,m}) = \mathscr{E}(G_{n,m}(p))$. Evidently,

$$\mathbf{J}_n - \mathbf{I}_{n,m} = (\mathbf{J}_n - \mathbf{I}_n) + (\mathbf{I}_n - \mathbf{I}_{n,m}).$$

By virtue of Lemma 2.4, we have

$$\mathscr{E}(\mathbf{J}_n - \mathbf{I}_{n,m}) \leq \mathscr{E}(\mathbf{J}_n - \mathbf{I}_n) + \mathscr{E}(\mathbf{I}_n - \mathbf{I}_{n,m}).$$

Recalling the definition of the quasi-unit matrix $\mathbf{I}_{n,m}$ and the fact that $\mathscr{E}(\mathbf{J}_n - \mathbf{I}_n) = 2(n-1)$, we have $\mathscr{E}(\mathbf{J}_n - \mathbf{I}_{n,m}) \leq O(n)$. According to Eq.(10), we can use a similar argument for the estimate of the energy $\mathscr{E}(\mathbf{A})$ from $\mathscr{E}(\overline{\mathbf{A}})$ to show that a.e. random matrix $\mathbf{A}_{n,m}$ enjoys the equation as follows:

$$\mathscr{E}(\mathbf{A}_{n,m}) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{\frac{m-1}{m}p(1-p)} + o(1)\right).$$

Since the random matrix $\mathbf{A}_{n,m}$ is the adjacency matrix of $G_{n,m}(p)$, we thus show that a.e. random graph $G_{n,m}(p)$ enjoys the following equation:

$$\mathscr{E}(G_{n,m}(p)) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{\frac{m-1}{m}p(1-p)} + o(1)\right).$$

In what follows, we shall use $\mathbf{A'}_{n,m}$ to denote the adjacency matrix $\mathbf{A}(G'_{n,m}(p))$. It is easily seen that if a random multipartite matrix $\mathbf{X}_{n,m}$ satisfies condition (7), and F_1 is a pointmass at 0 and F_2 is a Bernoulli distribution with mean p, then $\mathbf{X}_{n,m}$ coincides with the adjacency matrix $\mathbf{A'}_{n,m}$. Set

$$\overline{\mathbf{A'}}_{n,m} = \mathbf{A}'_{n,m} - p(\mathbf{J}_n - \mathbf{I}'_{n,m})$$

where $\mathbf{I}'_{n,m}$ is the quasi-unit matrix whose parts are the same as $\mathbf{A}'_{n,m}$. One can readily check that each entry in $\overline{\mathbf{A'}}_{n,m}$ has mean 0. It follows from the second part of Theorem 3.1 that

$$\Phi_{n^{-1/2}\overline{\mathbf{A'}}_{n,m}}(x) \to_P \Phi(x) \text{ as } n \to \infty.$$

Employing the argument analogous to the estimate of $\mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_{n,m}))$, $\mathscr{E}(\overline{\mathbf{A}}_{n,m})$ and $\mathscr{E}(\mathbf{A}_{n,m})$, one can evaluate, respectively, $\mathscr{E}(p(\mathbf{J}_n - \mathbf{I}'_{n,m}))$, $\mathscr{E}(\overline{\mathbf{A'}}_{n,m})$ and $\mathscr{E}(\mathbf{A}'_{n,m})$, and finally show that a.e. random graph $G'_{n,m}(p)$ satisfying condition (7) enjoys the following equation:

$$\mathscr{E}(G'_{n,m}(p)) = n^{3/2}\left(\frac{8}{3\pi}\sqrt{p(1-p)} + o(1)\right). \tag{11}$$

3.2 The energy of $G_{n;\nu_1\ldots\nu_m}(p)$

In this part, we shall give an estimate of energy for the random multipartite graph $G_{n;\nu_1\ldots\nu_m}(p)$ satisfying the following condition:

$$\lim_{n\to\infty}\max\{\nu_1(n),\ldots,\nu_m(n)\}>0 \text{ and there exist } \nu_i \text{ and } \nu_j\ ,\ \lim_{n\to\infty}\frac{\nu_i(n)}{\nu_j(n)}<1. \tag{12}$$

Moreover, for random bipartite graphs $G_{n;\nu_1,\nu_2}(p)$ satisfying $\lim_{n\to\infty}\nu_i(n)>0$ $(i=1,2)$, we shall formulate an exact estimate of its energy.

Anderson and Zeitouni [1] established the existence of the LSD of $\mathbf{X}_{n,m}$ with partitions satisfying condition (12). Unfortunately, they failed to get the exact form of the LSD, which appears to be much hard and complicated. However, we can establish the lower and upper bounds for the energy $\mathscr{E}(G_{n;\nu_1\ldots\nu_m}(p))$ via another way.

Here, we still denote the adjacency matrix of multipartite graph satisfying condition (12) by $\mathbf{A}_{n,m}$. Without loss of generality, we assume, for some $r\geq 1$, $|V_1|,\ldots,|V_r|$ are of order $O(n)$ while $|V_{r+1}|,\cdots,|V_m|$ of order $o(n)$. Let $\mathbf{A}'_{n,m}$ be a random symmetric matrix such that

$$\mathbf{A}'_{n,m}(ij)=\begin{cases}\mathbf{A}_{n,m}(ij) & \text{if } i \text{ or } j\notin V_s, 1\leq s\leq r,\\ t_{ij} & \text{if } i,j\in V_s, 1\leq s\leq r \text{ and } i>j,\\ 0 & \text{if } i,j\in V_s\ (r+1\leq s\leq m) \text{ or } i=j,\end{cases}$$

where t_{ij}'s are independent Bernulli r.v. with mean p. Evidently, $\mathbf{A}'_{n,m}$ is a random multipartite matrix. By means of Eq.(11), we have $\mathscr{E}(\mathbf{A}'_{n,m})=\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)n^{3/2}$.

Set

$$\mathbf{D}_n=\mathbf{A}'_{n,m}-\mathbf{A}_{n,m}=\begin{pmatrix}\mathbf{K}_1 &&&&&& \\ & \mathbf{K}_2 &&&&& \\ && \ddots &&&& \\ &&& \mathbf{K}_r &&& \\ &&&& \mathbf{0} && \\ &&&&& \ddots & \\ &&&&&& \mathbf{0}\end{pmatrix}_{n\times n} \tag{13}$$

One can readily see that $\mathbf{K}_i$ $(i=1,\ldots,r)$ is a Wigner matrix and thus a.e. $\mathbf{K}_i$ enjoys the following:

$$\mathscr{E}(\mathbf{K}_i)=\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)(\nu_i n)^{3/2}.$$

Consequently, a.e. matrix $\mathbf{D}_n$ satisfies the following:

$$\mathscr{E}(\mathbf{D}_n)=\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)\left(\nu_1^{\frac{3}{2}}+\cdots+\nu_r^{\frac{3}{2}}\right)n^{\frac{3}{2}}.$$

By Eq.(13), we have $\mathbf{A}_{n,m}+\mathbf{D}_n=\mathbf{A}'_{n,m}$ and $\mathbf{A}'_{n,m}+(-\mathbf{D}_n)=\mathbf{A}_{n,m}$. Employing Lemma 2.4, we deduce

$$\mathscr{E}(\mathbf{A}'_{n,m})-\mathscr{E}(\mathbf{D}_n)\leq\mathscr{E}(\mathbf{A}_{n,m})\leq\mathscr{E}(\mathbf{A}'_{n,m})+\mathscr{E}(\mathbf{D}_n).$$

Recalling that $\mathbf{A}_{n,m}$ is the adjacency matrix of $G_{n;\nu_1\ldots\nu_m}(p)$, the following result is relevant.

Theorem 3.3. *Almost every random graph $G_{n;\nu_1\ldots\nu_m}(p)$ satisfies the inequality below*

$$\left(1-\sum_{i=1}^{r}\nu_i^{\frac{3}{2}}\right)n^{3/2} \le \mathscr{E}(G_{n;\nu_1\ldots\nu_m}(p))\left(\frac{8}{3\pi}\sqrt{p(1-p)}+o(1)\right)^{-1} \le \left(1+\sum_{i=1}^{r}\nu_i^{\frac{3}{2}}\right)n^{3/2}.$$

Remark 3.2. Since $\nu_1,\ldots,\nu_r$ are positive real numbers with $\sum_{i=1}^r \nu_i \le 1$, we have $\sum_{i=1}^r \nu_i(1-\nu_i^{1/2}) > 0$. Therefore, $\sum_{i=1}^r \nu_i > \sum_{i=1}^r \nu_i^{3/2}$, and thus $1 > \sum_{i=1}^r \nu_i^{3/2}$. Hence, we can deduce, by the theorem above, that a.e. random graph $G_{n;\nu_1\ldots\nu_m}(p)$ enjoys the following

$$\mathscr{E}(G_{n;\nu_1\ldots\nu_m}(p)) = O(n^{3/2}).$$

In what follows, we investigate the energy of random bipartite graphs $G_{n;\nu_1,\nu_2}(p)$ satisfying $\lim_{n\to\infty}\nu_i(n) > 0$ $(i=1,2)$, and present the precise estimate of $\mathscr{E}(G_{n;\nu_1,\nu_2}(p))$ via Marčenko-Pastur Law.

For convenience, set $n_1 = \nu_1 n$ and $n_2 = \nu_2 n$. Let $\mathbf{I}_{n,2}$ be a quasi-unit matrix with the same partition as $\mathbf{A}_{n,2}$. Set

$$\overline{\mathbf{A}}_{n,2} = \mathbf{A}_{n,2} - p(\mathbf{J}_n - \mathbf{I}_{n,2}) = \begin{bmatrix} \mathbf{O} & \mathbf{X}^T \\ \mathbf{X} & \mathbf{O} \end{bmatrix}, \tag{14}$$

where $\mathbf{X}$ is a random matrix of order $n_2 \times n_1$ in which the entries $\mathbf{X}(ij)$ are iid. with mean zero and variance $\sigma^2 = p(1-p)$. By the equation

$$\begin{pmatrix} \lambda\mathbf{I}_{n_1} & \mathbf{0} \\ -\mathbf{X} & \lambda\mathbf{I}_{n_2} \end{pmatrix}\begin{pmatrix} \lambda\mathbf{I}_{n_1} & -\mathbf{X}^T \\ \mathbf{0} & \lambda\mathbf{I}_{n_2} - \lambda^{-1}\mathbf{X}\mathbf{X}^T \end{pmatrix} = \lambda\begin{pmatrix} \lambda\mathbf{I}_{n_1} & -\mathbf{X}^T \\ -\mathbf{X} & \lambda\mathbf{I}_{n_2} \end{pmatrix},$$

we have

$$\lambda^n \cdot \lambda^{n_1}|\lambda\mathbf{I}_{n_2} - \lambda^{-1}\mathbf{X}\mathbf{X}^T| = \lambda^n|\lambda\mathbf{I}_n - \overline{\mathbf{A}}_{n,2}|,$$

and consequently,

$$\lambda^{n_1}|\lambda^2\mathbf{I}_{n_2} - \mathbf{X}\mathbf{X}^T| = \lambda^{n_2}|\lambda\mathbf{I}_n - \overline{\mathbf{A}}_{n,2}|.$$

Thus, the eigenvalues of $\overline{\mathbf{A}}_{n,2}$ are symmetric, and moreover $\overline{\lambda}$ is the eigenvalue of $\frac{1}{\sqrt{n_1}}\overline{\mathbf{A}}_{n,2}$ if and only if $\overline{\lambda}^2$ is the eigenvalue of $\frac{1}{n_1}\mathbf{X}\mathbf{X}^T$. Therefore, we can characterize the spectral of $\overline{\mathbf{A}}_{n,2}$ by the spectral of $\mathbf{X}\mathbf{X}^T$. Bai formulated the LSD of $\frac{1}{n_1}\mathbf{X}\mathbf{X}^T$ (Theorem 2.5 in [2]) by moment approach.

Lemma 3.4 (Marčenko-Pastur Law [2]). *Suppose that $\mathbf{X}(ij)$'s are iid. with mean zero and variance $\sigma^2 = p(1-p)$, and $\nu_2/\nu_1 \to y \in (0,\infty)$. Then, with probability 1, the ESD $\Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}$ converges weakly to the Marčenko-Pastur Law F_y as $n\to\infty$ where F_y has the density*

$$f_y(x) = \frac{1}{2\pi p(1-p)xy}\sqrt{(b-x)(x-a)}\;\mathbf{1}_{a\le x\le b}$$

and has a point mass $1-1/y$ at the origin if $y>1$ where $a = p(1-p)(1-\sqrt{y})^2$ and $b = p(1-p)(1+\sqrt{y})^2$.

By the symmetry of the eigenvalues of $\frac{1}{\sqrt{n_1}}\overline{\mathbf{A}}_{n,2}$, to evaluate the energy $\mathscr{E}(\frac{1}{\sqrt{n_1}}\overline{\mathbf{A}}_{n,2})$, we just need to consider the positive eigenvalues. Define $\Theta_{n_2}(x) = \frac{\sum 1_{\overline{\lambda}<x}}{n_2}$. One can see that the sum of the positive eigenvalues of $\frac{1}{\sqrt{n_1}}\overline{\mathbf{A}}_{n,2}$ equals $n_2 \int_0^\infty x d\Theta_{n_2}(x)$. Suppose $0 < x_1 < x_2$, we have

$$\Theta_{n_2}(x_2) - \Theta_{n_2}(x_1) = \Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x_2^2) - \Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x_1^2).$$

It follows that

$$\int_0^\infty x d\Theta_{n_2}(x) = \int_0^\infty \sqrt{x} d\Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x).$$

Note that all the eigenvalues of $\frac{1}{n_1}\mathbf{X}\mathbf{X}^T$ are nonnegative. By the moment approach (see [2] for instance), we have

$$\begin{aligned}\int x^2 d\Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x) &= \int_0^\infty x^2 d\Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x) \\ &\to \int_0^\infty x^2 dF_y(x) \text{ a.s. } (n \to \infty) \\ &= \int x^2 dF_y(x)\end{aligned}$$

Analogous to the proof of Eq.(3), we can deduce that

$$\lim_{n\to\infty} \int_0^\infty \sqrt{x} d\Phi_{\frac{1}{n_1}\mathbf{X}\mathbf{X}^T}(x) = \int_0^\infty \sqrt{x} dF_y(x) \text{ a.s.}$$

Therefore,

$$\lim_{n\to\infty} \int_0^\infty x d\Theta_{n_2}(x) = \int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\pi p(1-p)y} \sqrt{(b-x^2)(x^2-a)}\, dx \text{ a.s.}$$

Let

$$\Lambda = \int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\pi p(1-p)y} \sqrt{(b-x^2)(x^2-a)}\, dx.$$

We obtain that for a.e. $\overline{\mathbf{A}}_{n,2}$ the sum of the positive eigenvalues is $(\Lambda + o(1))n_2\sqrt{n_1}$. Thus, a.e. $\mathscr{E}(\overline{\mathbf{A}}_{n,2})$ enjoys the equation as follows:

$$\mathscr{E}(\overline{\mathbf{A}}_{n,2}) = (2\Lambda + o(1))n_2\sqrt{n_1}.$$

Furthermore, we can get

$$\Lambda = \frac{\sqrt{b}[(a+b)\,\mathrm{Ep}(1-a/b) - 2a\,\mathrm{Ek}(1-a/b)]}{3\pi p(1-p)y},$$

where Ek is the complete elliptic integral of the first kind and Ep is the complete elliptic integral of the second kind. Let $t \in [0,1]$, the two kinds of complete elliptic integral are defined as follows

$$\mathrm{Ek}(t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-t\sin^2\theta}} \text{ and } \mathrm{Ep}(t) = \int_0^{\frac{\pi}{2}} \sqrt{1-t\sin^2\theta}\, d\theta.$$

The value can be got by mathematical software for every parameter t.

Employing Eq.(14) and Lemma 2.4, we have

$$\mathscr{E}(\overline{\mathbf{A}}_{n,2}) - \mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_{n,2})) \leq \mathscr{E}(\mathbf{A}_{n,2}) \leq \mathscr{E}(\overline{\mathbf{A}}_{n,2}) + \mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_{n,2})).$$

Together with the fact that $\mathscr{E}(p(\mathbf{J}_n - \mathbf{I}_{n,2})) = 2p\sqrt{\nu_1\nu_2}n$ and $n_2\sqrt{n_1} = \nu_2\sqrt{\nu_1}n^{3/2}$, we get

$$\mathscr{E}(\mathbf{A}_{n,2}) = (2\nu_2\sqrt{\nu_1}\Lambda + o(1))n^{3/2}.$$

Therefore, the following theorem is relevant.

Theorem 3.5. *Almost every random bipartite graph $G_{n;\nu_1,\nu_2}(p)$ with $\nu_2/\nu_1 \to y$ enjoys*

$$\mathscr{E}(G_{n;\nu_1,\nu_2}(p)) = (2\nu_2\sqrt{\nu_1}\Lambda + o(1))n^{3/2}.$$

We now compare the above estimate of the energy $\mathscr{E}(G_{n;\nu_1,\nu_2}(p))$ with bounds obtained in Theorem 3.3 for $p = 1/2$. In fact, Koolen and Moulton [13] established the following upper bound of the energy $\mathscr{E}(G)$ for simple graphs G:

$$\mathscr{E}(G) \leq \frac{n}{2}(\sqrt{n} + 1).$$

Consequently, for $p = 1/2$, this upper bound is better than ours. So we turn our attention to compare the estimate of $\mathscr{E}(G_{n;\nu_1,\nu_2}(1/2))$ in Theorem 3.5 with the lower bound in Theorem 3.3. By the numerical computation (see the table below), the energy $\mathscr{E}(G_{n;\nu_1,\nu_2}(1/2))$ of a.e. random bipartite $G_{n;\nu_1,\nu_2}(1/2)$ is close to our lower bound.

y	$\mathscr{E}(G_{n;\nu_1,\nu_2}(p))$	lower bound of $\mathscr{E}(G_{n;\nu_1,\nu_2}(p))$
1	$(0.3001 + o(1))n^{3/2}$	$(0.1243 + o(1))n^{3/2}$
2	$(0.2539 + o(1))n^{3/2}$	$(0.1118 + o(1))n^{3/2}$
3	$(0.2071 + o(1))n^{3/2}$	$(0.0957 + o(1))n^{3/2}$
4	$(0.1731 + o(1))n^{3/2}$	$(0.0828 + o(1))n^{3/2}$
5	$(0.1482 + o(1))n^{3/2}$	$(0.0727 + o(1))n^{3/2}$
6	$(0.1294 + o(1))n^{3/2}$	$(0.06470 + o(1))n^{3/2}$
7	$(0.1148 + o(1))n^{3/2}$	$(0.05828 + o(1))n^{3/2}$
8	$(0.1031 + o(1))n^{3/2}$	$(0.05301 + o(1))n^{3/2}$
9	$(0.09353 + o(1))n^{3/2}$	$(0.04862 + o(1))n^{3/2}$
10	$(0.08558 + o(1))n^{3/2}$	$(0.04491 + o(1))n^{3/2}$

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