Hypergeometric Series Solutions of Linear Operator Equations

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**Abstract**

Let $K$ be a field and $L\colon K[x] \to K[x]$ be a linear operator acting on the ring of polynomials in $x$ over the field $K$. We provide a method to find a suitable basis $\{b_k(x)\}$ of $K[x]$ and a hypergeometric term $c_k$ such that $y(x) = \sum_{k=0}^{\infty} c_k b_k(x)$ is a formal series solution to the equation $L(y(x)) = 0$. This method is applied to construct hypergeometric representations of orthogonal polynomials from the differential/difference equations or recurrence relations they satisfied. Both the ordinary cases and the $q$-cases are considered.

*AMS Classifications*: 33C45, 33D45, 47E05.

*Keywords*: hypergeometric series, orthogonal polynomials, differential/difference equations, three term recurrence relations

# 1. Introduction

A *hypergeometric series* $\sum_{k\geq 0} t_k$ is a series in which the ratio of two consecutive terms is a rational function of the summation index $k$. Consequently, $t_k$ is called a *hypergeometric term*. When the series contains only finitely many

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non-zero summands, it is called a *hypergeometric polynomial*. Hypergeometric series and polynomials appear frequently in the theory of orthogonal polynomials. For instant, all orthogonal polynomials in the Askey-scheme are hypergeometric polynomials or their $q$-analogue [7].

It is well-known that one can solve linear differential equations by means of power series. Abramov and Petkovšek [2] considered general polynomial sequences besides the powers and presented an algorithm to find nice power series solutions of linear differential equations. Abramov, Paule and Petkovšek [1] presented an algorithm for finding formal power series solutions and basic hypergeometric series solutions of $q$-difference equations. In this paper, we provide a method to find hypergeometric series solutions to the equation $L(y(x)) = 0$ where $L\colon K[x] \to K[x]$ is a linear operator acting on $K[x]$, the ring of polynomials in $x$ over the field $K$. The key idea is to search for a suitable basis $\{b_k(x)\}$ of $K[x]$ such that the solution $y(x)$ can be expressed as $\sum_{k=0}^{\infty} c_k b_k(x)$ with $c_k$ being a hypergeometric term.

Our main motivation comes from finding the hypergeometric representations of orthogonal polynomials. As pointed by Koepf [10], starting from the hypergeometric representations, one can compute the corresponding differential/difference equations, the recurrence relations and the structure relations. For more details, see Koepf's book [9] which covers Zeilberger's and Petkovšek's algorithms and many variants such as $h$-hypergeometric series. Chen and the authors [5] used the extended Zeilberger's algorithm to provide a unified treatment of these tasks. Here we consider the inverse problem, that is, finding hypergeometric representations from the differential/difference equations. We know that all orthogonal polynomials $P_n(x)$ in the Askey-scheme satisfy certain differential/difference equations. Rewriting the equation as $L(P_n(x)) = 0$ with $L$ being a linear operator, we see that finding a hypergeometric representation is precisely finding a hypergeometric polynomial solution to the equation $L(y(x)) = 0$.

Koepf and Schmersau [13] discussed the conversions between the differential/difference equations, the hypergeometric representations and the recurrence relations for the continuous and discrete cases. Koepf and Masjed-Jamei [12] provided generic hypergeometric polynomial solutions for the continuous case. Atakishiyev and Suslov [4] discussed difference equations on the lattice with non-uniform steps [17]. Foupouagnigni [6] further studied the difference equations satisfied by classical orthogonal polynomials and their modifications. Our approach provides a uniform and algorithmic treatment for classical orthogonal polynomials. Moreover, in a similar way we can also find the hypergeometric representations of orthogonal polynomials directly from the three term recurrence relations.

The paper is organized as follows. In section 2, we provide a heuristic method to find out suitable bases $\{b_k(x)\}$ of $K[x]$ for a given linear operator $L$, which satisfy

$$L(b_k(x)) = A_k b_k(x) + B_k b_{k-h}(x), \quad k = 0, 1, \ldots, \tag{1.1}$$

where $A_k, B_k \in K$ and $h$ is a positive integer. For this purpose, we solve non-linear equations to get the explicit $b_k(x)$ for small $k$ and guess the general form. Then in Section 3, we present an algorithm to check whether a given basis $\{b_k(x)\}$ satisfies (1.1). Moreover, the algorithm computes $A_k$ and $B_k$ for general $k$ and thus leads to a formal series solution $y(x)$ to the equation $L(y(x)) = 0$. In Section 4, we apply the method to derive hypergeometric representations of orthogonal polynomials from their differential/difference equations. Finally in Section 5, we use the method to find hypergeometric representations of orthogonal polynomials from their recurrence relations. Both the ordinary cases and the $q$-cases are visited.

We have implemented the algorithms in `Maple`, which can be obtained from `http://www.combinatorics.net.cn/homepage/hou/basis.html`.

## 2. Searching for suitable bases

Let $L$ be a linear operator acting on the ring $K[x]$ of polynomials in variable $x$ over the field $K$. Denoting the set of nonnegative integers by $\mathbb{N}$, we aim to find a basis $\{b_k(x)\}$ of $K[x]$ such that

$$L(b_k(x)) = A_k b_k(x) + B_k b_{k-h}(x), \quad \forall\, k \in \mathbb{N}, \tag{2.1}$$

where $A_k, B_k \in K$ and $h$ is a fixed positive integer. Here and in the remainder part of the paper, we always set $b_i(x) = 0$ for $i < 0$.

Without loss of generality, we assume that $b_k(x), k = 0, 1, \ldots$ are monic polynomials of degree $k$. For convenience, we further require that $b_{k-1}(x)$ divides $b_k(x)$. Under these assumptions, we may write $b_k(x)$ as

$$b_k(x) = (x - x_1)(x - x_2) \cdots (x - x_k), \tag{2.2}$$

where $x_1, \ldots, x_k \in K$. A basis $\{b_k(x)\}$ of form (2.2) is called a *suitable basis* (with respect to the operator $L$) if (2.1) holds.

Now fix a positive integer $k$ and regard $x_1, \ldots, x_k$ as undeterminates. By comparing the coefficients of powers of $x$ on both sides of (2.1), we derive that

$$A_k = [x^k]L(b_k(x)) \quad \text{and} \quad B_k = [x^{k-h}]\big(L(b_k(x)) - A_k b_k(x)\big), \tag{2.3}$$

where $[x^m]p(x)$ denotes the coefficient of $x^m$ in the polynomial $p(x)$. Thus $A_k, B_k$ are expressed in terms of $x_1, \ldots, x_k$. Substituting (2.3) into (2.1) and equating each power of $x$ of both sides, we obtain a system of polynomial equations on $x_1, \ldots, x_k$.

Starting from $k = 1$, we iteratively set up and solve the equations on $x_1, \ldots, x_k$ until reaching a certain degree $k_0$. In each iteration, we obtain either the explicit values of $x_i$ or some relations among them. The number of equations can be roughly estimated as follows. Suppose that the degree of $L(b_k(x))$ is less than or equal to $k$. Then (2.1) leads to $k+1$ equations on $x_1, \ldots, x_k$ and $A_k, B_k$. Expressing $A_k, B_k$ in terms of $x_1, \ldots, x_k$ by (2.3), we still have $k-1$ equations. All together there are $0 + 1 + \cdots + (k_0 - 1) = \binom{k_0}{2}$ equations on $x_1, \ldots, x_{k_0}$. Therefore when $k_0$ is large enough, we will obtain $x_1, \ldots, x_{k_0}$ explicitly. In fact, for all examples appearing in this paper, $k_0 = 5$ is enough. Finally, we guess the general form of $x_k$ from the pattern, which is often straightforward.

**Example 2.1** Let $L$ be given by

$$L(p(x)) = (1 - x^2)p''(x) - xp'(x) + n^2 p(x). \tag{2.4}$$

Take $h = 1$ and set $b_0(x) = 1$, $b_1(x) = x - x_1$. For $k = 1$, (2.1) becomes

$$(n^2 - 1)x - n^2 x_1 = A_1(x - x_1) + B_1. \tag{2.5}$$

By (2.3), we derive that $A_1 = (n^2 - 1)$ and $B_1 = -x_1$.

Now consider $k = 2$ and set $b_2(x) = (x - x_1)(x - x_2)$. Then (2.1) becomes

$$\begin{aligned}(n^2 - 4)x^2 - (n^2 - 1)(x_1 + x_2)x + 2 + n^2 x_1 x_2 \\ = A_2(x - x_1)(x - x_2) + B_2(x - x_1),\end{aligned}$$

which leads to

$$A_2 = n^2 - 4, \quad B_2 = -3(x_1 + x_2), \quad \text{and} \quad x_1 x_2 = 3x_1^2 - 2.$$

Substituting $(3x_1^2 - 2)/x_1$ for $x_2$ and solving the equations corresponding to $k = 3$, we derive that $x_1 = x_2 = x_3 = 1$ or $x_1 = x_2 = x_3 = -1$. Continuing this process, we obtain that $x_1 = \cdots = x_{k_0} = 1$ or $x_1 = \cdots = x_{k_0} = -1$ for any $k_0 \geq 3$. This leads us to guess $b_k(x) = (x+1)^k$or $b_k(x) = (x-1)^k$. ∎

# 3. Hypergeometric polynomial solutions

Once we have guessed the form of $b_k(x)$, we can then check whether $\{b_k(x)\}$ forms a suitable basis, i.e., (2.1) holds for arbitrary non-negative integer $k$.

**Theorem 3.1** *Let $L\colon K[x] \to K[x]$ be a linear operator and $\{b_k(x)\}$ be a suitable basis satisfying* (2.1). *Then for any $k \in \mathbb{N}$, $L(b_k(x))/b_{k-h}(x)$ is a polynomial in $x$ of degree no more than $h$. Furthermore, we have*

$$A_k = [x^h]\frac{L(b_k(x))}{b_{k-h}(x)}, \quad \textit{and} \quad B_k = [x^0]\left(\frac{L(b_k(x))}{b_{k-h}(x)} - A_k\frac{b_k(x)}{b_{k-h}(x)}\right). \tag{3.1}$$

*Proof.* Dividing both sides of (2.1) by $b_{k-h}(x)$, we obtain

$$\frac{L(b_k(x))}{b_{k-h}(x)} = A_k\frac{b_k(x)}{b_{k-h}(x)} + B_k.$$

Since $b_k(x)/b_{k-h}(x)$ is a polynomial of degree $h$, we immediately derive that $L(b_k(x))/b_{k-h}(x)$ is a polynomial of degree less than or equal to $h$. Comparing the coefficients of $x^h$ and $x^0$, we obtain (3.1). ∎

Theorem 3.1 provides us an algorithm to verify whether $\{b_k(x)\}$ forms a suitable basis. Moreover, we solve out $A_k$ and $B_k$ simultaneously.

*Algo-Verify*

1. Check whether $L(b_k(x))/b_{k-h}(x)$ is a polynomial in $x$ of degree no more than $h$. If not, return "$\{b_k(x)\}$ is not a suitable basis" and stop.

2. Compute $A_k, B_k$ according to (3.1).

3. Check whether

$$\frac{L(b_k(x))}{b_{k-h}(x)} = A_k \frac{b_k(x)}{b_{k-h}(x)} + B_k \tag{3.2}$$

holds for the above $A_k, B_k$. If yes, return $A_k$ and $B_k$; otherwise, return "$\{b_k(x)\}$ is not a suitable basis".

**Example 3.1** Let $L$ be the operator given in Example 2.1:

$$L(p(x)) = (1-x^2)p''(x) - xp'(x) + n^2 p(x).$$

For $h = 1$, we guess that $b_k(x) = (x-1)^k$. Applying *Algo-Verify*, we verify that $\{b_k(x)\}$ is indeed a suitable basis with

$$A_k = n^2 - k^2 \quad \text{and} \quad B_k = k - 2k^2. \qquad \blacksquare$$

Let $L\colon K[x] \to K[x]$ be a linear operator and $\{b_k(x)\}$ be a suitable basis. As done by Abramov and Petkovšek [2], $L$ can be extended to formal series of the form $\sum_{k=0}^{\infty} c_k b_k(x)$ by setting

$$L\left(\sum_{k=0}^{\infty} c_k b_k(x)\right) = \sum_{k=0}^{\infty} (c_k A_k + c_{k+h} B_{k+h}) b_k(x).$$

Suppose that $\{c_k\}$ is a sequence satisfying

$$c_k A_k + c_{k+h} B_{k+h} = 0, \quad \forall\, k \in \mathbb{N}.$$

Then we immediately derive that $\sum_{k=0}^{\infty} c_k b_k(x)$ is a formal series solution to the equation $L(y(x)) = 0$. When the series $\{c_k\}$ contains only finitely many non-zero entries, $\sum_{k=0}^{\infty} c_k b_k(x)$ becomes a polynomial and hence is a polynomial solution to the equation $L(y(x)) = 0$. We thus derive the following theorem.

**Theorem 3.2** *Let $L\colon K[x] \to K[x]$ be a linear operator and $\{b_k(x)\}$ be a suitable basis satisfying* (2.1)*. Suppose that $y(x) = \sum_{k=0}^{\infty} c_k b_k(x)$ is a polynomial solution to the equation $L(y(x)) = 0$. Then*

$$c_{k+h} B_{k+h} = -c_k A_k, \quad \forall\, k \in \mathbb{N}. \tag{3.3}$$

*Conversely, let $c_0, c_1, \ldots$ be a sequence satisfying* (3.3) *and containing only finitely many non-zero entries. Then $L(\sum_{k=0}^{\infty} c_k b_k(x)) = 0$.*

Let $t_k = c_k b_k(x)$. When $A_k, B_k$ and $x_k$ are all rational functions of $k$, $t_k$ is an $h$-fold hypergeometric term defined by Koepf [8]. Especially when $h = 1$, $t_k$ becomes a hypergeometric term and $y(x) = \sum_{k=0}^{\infty} t_k$ becomes a hypergeometric series. More precisely, suppose that

$$\frac{t_{k+h}}{t_k} = -\frac{A_k \cdot b_{k+h}(x)}{B_{k+h} \cdot b_k(x)} = \frac{(k+u_1)\cdots(k+u_r)}{(k+v_1)\cdots(k+v_s)} z$$

Then

$$t_{kh+i} = t_i \frac{(\frac{u_1+i}{h})_k \cdots (\frac{u_r+i}{h})_k}{(\frac{v_1+i}{h})_k \cdots (\frac{v_s+i}{h})_k} (zh^{r-s})^k, \quad i = 0, 1, \ldots, h-1,$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the raising factorial. Therefore we can express $y(x)$ in terms of the standard notation of hypergeometric series:

$${}_rF_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}.$$

**Example 3.2.** Let $n$ be a nonnegative integer and

$$L(p(x)) = (1-x^2)p''(x) - xp'(x) + n^2 p(x).$$

As shown in Example 3.1, a suitable basis with respect to $L$ is $\{(x-1)^k\}$ and the corresponding $A_k = n^2 - k^2$, $B_k = k - 2k^2$. By direct computation, we derive that

$$\frac{t_{k+1}}{t_k} = -\frac{A_k}{B_{k+1}}(x-1) = \frac{(k-n)(k+n)}{(k+1)(k+1/2)} \cdot \frac{1-x}{2},$$

and hence

$$y(x) = c \cdot {}_2F_1\left(\begin{array}{c} -n, n \\ 1/2 \end{array} \middle| \frac{1-x}{2}\right)$$

is a polynomial solution to the equation $L(y(x)) = 0$. In fact, it is a multiple of the Chebyshev polynomial of the first kind. ∎

When $A_k, B_k$ and $x_k$ are rational functions of $q^k$, $t_k$ becomes the $q$-analogue of $h$-hypergeometric terms. In this case, we can express $y(x) = \sum_{k=0}^{\infty} t_k$ in terms of the standard notation of $q$-hypergeometric series:

$${}_r\phi_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s+1-r},$$

where $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ is the $q$-shifted factorial.

**Example 3.3.** Let $n$ be a nonnegative integer and $L$ be given by

$$L(p(x)) = xp(qx) + (1 - xq^n)p(x) - p(x/q).$$

For $h = 1$, we find a suitable basis $\{x^k\}$ and

$$-\frac{A_k}{B_{k+1}} = \frac{1 - q^{-n}q^k}{1 - qq^k} q^k q^{n+1}.$$

Therefore,

$$y(x) = c \cdot {}_1\phi_1 \left( \begin{array}{c} q^{-n} \\ 0 \end{array} \middle| q; -q^{n+1}x \right),$$

which is a multiple of the Stieltjes-Wigert polynomial.

# 4. Differential/difference equations

As we know, the classical orthogonal polynomials can be characterized by satisfying a certain differential, difference or $q$-difference equation. Rewriting the equation as an operator $L$ acting on polynomials, we see that the orthogonal polynomials are precisely the polynomial solutions to the equation $L(y(x)) = 0$. Therefore, we may use the above method to derive hypergeometric representations of orthogonal polynomials from the corresponding differential/difference equations. The method is feasible for all orthogonal polynomials in the Askey-scheme. We take the Jacobi polynomials, the Hahn polynomials, the Racah polynomials and the Askey-Wilson polynomials as examples.

**4.1. The differential cases.** Consider the Jacobi polynomials. The corresponding linear operator $L$ is given by

$$L(p(x)) = (1-x^2)p''(x) + (\beta - \alpha - (\alpha+\beta+2)x)p'(x) + n(n+\alpha+\beta+1)p(x).$$

By the first few terms, we find two candidates for $b_k(x)$: $(x-1)^k$ and $(x+1)^k$. By *Algo-Verify*, both of them are suitable bases, leading to two representations for Jacobi polynomials:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}\middle|\,\frac{1-x}{2}\right) \\ &= (-1)^n\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \beta+1 \end{array}\middle|\,\frac{1+x}{2}\right). \end{aligned}$$

The constant $\frac{(\alpha+1)_n}{n!}$ is determined by the condition $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ and $(-1)^n\frac{(\beta+1)_n}{n!}$ is determined by comparing the coefficient of $x^n$.

In the special case when $\alpha = \beta$, we find one more suitable basis $\{x^k\}$ with $h = 2$. This leads to another representation for ultraspherical polynomials:

$$P_n^{(\alpha,\alpha)}(x) = c_n \cdot x^\delta\ {}_2F_1\left(\begin{array}{c} \frac{-n+\delta}{2}, \frac{n+1+\delta}{2}+\alpha \\ \frac{1}{2}+\delta \end{array}\middle|\, x^2\right),$$

where $\delta = 0$ for $n$ even and $\delta = 1$ for $n$ odd. Reversing the summation index, we obtain a uniform representation

$$P_n^{(\alpha,\alpha)}(x) = \frac{(n+2\alpha+1)_n}{2^n n!} x^n\ {}_2F_1\left(\begin{array}{c} -n/2, (-n+1)/2 \\ -n+\frac{1}{2}-\alpha \end{array}\middle|\,\frac{1}{x^2}\right).$$

**4.2. The difference cases.** Consider the Hahn polynomials $Q_n(x)$. The corresponding linear operator $L$ is given by

$$L(p(x)) = B(x)y(x+1)-(n(n+\alpha+\beta+1)+B(x)+D(x))y(x)+D(x)y(x-1),$$

where $B(x) = (x+\alpha+1)(x-N)$ and $D(x) = x(x-\beta-N-1)$. We find four suitable bases:

$$\{(x+\alpha+1)_k\}, \quad \{(-1)^k(-x+N+\beta+1)_k\}, \quad \{(x-N)_k\}, \quad \{(-1)^k(-x)_k\}.$$

These bases lead to four hypergeometric representations of Hahn polynomials. For example, taking $b_k(x) = (x+\alpha+1)_k$, we derive

$$Q_n(x) = c_n \cdot {}_3F_2\left(\begin{array}{c} -n, n+\alpha+\beta+1, x+\alpha+1 \\ \alpha+1, \alpha+\beta+N+2 \end{array}\middle|\, 1\right).$$

**4.3. Non-uniform lattice cases.** In general, the classical orthogonal polynomials $P_n(x)$ on non-uniform lattices satisfy [16, Equation (3.15)]

$$\tilde{\sigma}[x(s)]\frac{\Delta}{\Delta x(s-1/2)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\frac{\tilde{\tau}[x(s)]}{2}\left[\frac{\Delta y(s)}{\Delta x(s)}+\frac{\nabla y(s)}{\nabla x(s)}\right]+\lambda y(s)=0, \quad (4.1)$$

where $x(s)$ is the lattice function, $y(s) = P_n(x(s))$ and $\Delta f(s) = f(s+1) - f(s), \nabla f(s) = f(s) - f(s-1)$. Moreover, the lattice function $x(s)$ has one of the following forms:

$$x(s) = c_1 s^2 + c_2 s + c_3, \quad \text{or} \quad x(s) = c_1 q^s + c_2 q^{-s} + c_3.$$

By linear transformation, we may assume without loss of generality that $x(s) = s(s + \mu)$ or $x(s) = q^s + \mu q^{-s}$.

**I. Lattice** $x(s) = s(s+\mu)$. In this case, Equation (4.1) is a linear difference equation with respect to the variable $s$. Let $L$ be a linear difference operator which maps polynomials in variable $s$ to rational functions of $s$ (instead of polynomials in $s$). We firstly use the fact that $L(x(s))$ is a polynomial in $x(s)$ to determine the parameter $\mu$ and hence the lattice function $x(s)$. Then we apply our method to find out $x_1, \ldots, x_k$ such that

$$b_k(s) = (x(s) - x_1)(x(s) - x_2)\cdots(x(s) - x_k)$$

satisfy

$$L(b_k(s)) = A_k b_k(s) + B_k b_{k-h}(s).$$

Consequently, we obtain hypergeometric polynomial solutions to the equation $L(y(s)) = 0$, which are precisely the hypergeometric representations of the corresponding orthogonal polynomials.

Let us consider the Racah polynomials. The corresponding linear operator is given by

$$L(p(s)) = B(s)p(s+1) - (n(n+\alpha+\beta+1) + B(s) + D(s))p(s) + D(s)p(s-1),$$

where

$$B(s) = \frac{(s+\alpha+1)(s+\beta+\delta+1)(s+\gamma+1)(s+\gamma+\delta+1)}{(2s+\gamma+\delta+1)(2s+\gamma+\delta+2)},$$

and

$$D(s) = \frac{s(s-\alpha+\gamma+\delta)(s-\beta+\gamma)(s+\delta)}{(2s+\gamma+\delta)(2s+\gamma+\delta+1)}.$$

We see that $L(s(s+\mu))$ is a rational function of $s$ whose denominator is $(2s+\gamma+\delta)(2s+\gamma+\delta+1)(2s+\gamma+\delta+2)$. To ensure that $L(s(s+\mu))$ is a polynomial, the numerator must equal to zero when we substitute $s$ with $-(\gamma+\delta)/2$. This leads to $\mu=\gamma+\delta+1$, and hence $x(s)=s(s+\gamma+\delta+1)$. We find four candidates for $x_k=x(s_k)$

$$s_k=k+\alpha-\gamma-\delta-1,\quad s_k=k-\delta-1,\quad s_k=k-1,\quad \text{or}\quad s_k=k+\beta-\gamma-1.$$

The corresponding $b_k(s)$' are

$$(-1)^k(-s+\alpha-\gamma-\delta)_k(s+\alpha+1)_k,\quad (-1)^k(-s-\delta)_k(s+\gamma+1)_k$$

and

$$(-1)^k(-s)_k(s+\gamma+\delta+1)_k,\quad (-1)^k(-s+\beta-\gamma)_k(s+\beta+\delta+1)_k.$$

Each of these bases leads to a hypergeometric representation of Racah polynomials. For example, the first basis leads to

$$R_n(x(s))=c_n\cdot {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-s+\alpha-\gamma-\delta,s+\alpha+1\\ \alpha+1,\alpha-\delta+1,\alpha+\beta-\gamma+1\end{matrix}\,\middle|\,1\right).$$

**II. Lattice** $x(s)=q^s+\mu q^{-s}$. In this case, we write $z=q^s$ and $\tilde{x}(z)=z+\mu/z$. Then Equation (4.1) becomes a $q$-difference equation with respect to the variable $z$. Given a $q$-difference operator $L$, we search for

$$b_k(z)=(\tilde{x}(z)-x_1)(\tilde{x}(z)-x_2)\cdots(\tilde{x}(z)-x_k)$$

such that

$$L(b_k(z))=A_kb_k(z)+B_kb_{k-h}(z).$$

Consequently, we will obtain $q$-hypergeometric polynomial solutions to the equation $L(y(z))=0$, which are precisely the $q$-hypergeometric representations of the corresponding orthogonal polynomials.

Consider the Askey-Wilson polynomials. The linear operator is given by

$$\begin{aligned}L(p(z))&=A(z)p(zq)\\&-[q^{-n}(1-q^n)(1-abcdq^{n-1})+A(z)+A(z^{-1})]p(z)+A(z^{-1})p(z/q)\end{aligned}\tag{4.2}$$

where

$$A(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

The requirement that $L(z+\mu/z)$ is a Laurent polynomial of $z$ forces $\mu = 1$, and hence $\tilde{x}(z) = z + 1/z$. We find four candidates for $x_k = \tilde{x}(z_k)$

$$z_k = aq^{k-1}, \quad z_k = bq^{k-1}, \quad z_k = cq^{k-1}, \quad \text{or} \quad z_k = dq^{k-1},$$

which coincides with the symmetry of $a, b, c, d$ in (4.2). Taking, for example, $z_k = aq^{k-1}$, we have

$$b_k(z) = \prod_{j=1}^{k}(z - aq^{j-1})\left(1 - \frac{1}{azq^{j-1}}\right) = \frac{(-1)^k}{a^k} q^{-\binom{k}{2}}(az;q)_k(a/z;q)_k,$$

where $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ is the $q$-shifted factorial. Finally, we derive a $q$-hypergeometric representation of the Askey-Wilson polynomials:

$$P_n(\tilde{x}(z)) = c_n \cdot {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix} \,\middle|\, q; q\right).$$

# 5. Recurrence relations

An important characterization of the orthogonal polynomials is the three term recurrence relations that they satisfy. Let $P_n(x)$ be orthogonal polynomials. Then there are constants $\alpha_n, \beta_n$ and $\gamma_n$ such that

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x). \tag{5.1}$$

Koornwinder and Swarttouw [15] implemented the package `rec2ortho` to recover the type of orthogonal polynomials from their three term recurrence relations. The algorithm is based on case by case checking. Koepf provided algorithms [14] to find the corresponding differential/difference equations from the three term recurrence relations. Combining the method given in Section 4 with Koepf's algorithms, we can find out the hypergeometric representations of $P_n(x)$. Here we provide another approach, which solves Equation (5.1) directly. This approach is feasible for almost all orthogonal polynomials in the Askey-scheme.

**5.1. The ordinary cases.** We firstly consider the ordinary cases in which $\alpha_n, \beta_n$ and $\gamma_n$ are rational functions of $n$. Notice that $P_n(x)$ is not a polynomial in the variable $n$ so that we can not apply Theorem 3.2. Fortunately, we have

**Theorem 5.1** *Let $L\colon K[x] \to K[x]$ be a linear operator of the form*

$$L(p(x)) = \sum_{i=u}^{v} a_i(x)p(x+i),$$

*where $u, v$ are integers and $a_i(x)$ are rational functions of $x$. Let $\{b_k(x)\}$ be a suitable basis with respect to $L$. Assume further that for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $b_k(n) = 0$ for all $k > k_n$. Suppose that $c_0, c_1, \ldots$ is a sequence satisfying $c_k A_k + c_{k+h} B_{k+h} = 0$. Then*

$$\sum_{i=u}^{v} a_i(n)y(n+i) = 0, \quad \forall\, n \geq -u,$$

*where $y(n) = \sum_{k=0}^{\infty} c_k b_k(n)$.*

*Proof.* For each $n \geq -u$, let $K = \max\{k_n + h, k_{n+u}, k_{n+u+1}, \ldots, k_{n+v}\}$. Then

$$\begin{aligned}
\sum_{i=u}^{v} a_i(n)y(n+i) &= \sum_{i=u}^{v} a_i(n) \sum_{k=0}^{K} c_k b_k(n+i) \\
&= \sum_{k=0}^{K} c_k L(b_k(x))|_{x=n} \\
&= \sum_{k=0}^{K} c_k (A_k b_k(n) + B_k b_{k-h}(n)) \\
&= \sum_{k=K-h+1}^{K} c_k A_k b_k(n) = 0,
\end{aligned}$$

as desired. ∎

We see that all ordinary orthogonal polynomials in the Askey-scheme are of the form $P_n = a_n \sum_{k=0}^{\infty} c_k b_k(n)$ with $a_n$ being a hypergeometric term of $n$ and $b_k(n)$ being a polynomial in $n$ which satisfies $b_k(n) = 0$ for $k > n$.

Therefore, we can recover $P_n$ from Equation (5.1) by searching for the basis $\{b_k(n)\}$. Suppose that $a_{n+1}/a_n = r(n)$, a rational function of $n$. Then $\{b_k(n)\}$ is a suitable basis with respect to the linear operator $L$ defined by

$$L(p(n)) = \alpha_n r(n) p(n+1) + (\beta_n - x) p(n) + \frac{\gamma_n}{r(n-1)} p(n-1). \tag{5.2}$$

Suppose that we are given Equation (5.1). We aim to find an $r(n)$ such that the operator $L$ given by (5.2) has a suitable basis. Let

$$r(n) = \frac{f(n)}{g(n)} \frac{u(n+1)/v(n+1)}{u(n)/v(n)}$$

be its rational normal form [3]. We restrict ourself on those $r(n)$ with $u(n) = v(n) = 1$. Let $D(n)$ be the least common denominator of $\alpha_n, \beta_n$ and $\gamma_n$. By the condition that $L(1)$ is a constant, we derive that

$$(D(n)\alpha_n) f(n) f(n-1) + D(n)(\beta_n - x) f(n-1) g(n) + (D(n)\gamma_n) g(n) g(n-1)$$

is divisible by $D(n) f(n-1) g(n)$. Therefore,

$$f(n-1) \mid (D(n)\gamma_n) g(n) g(n-1) \quad \text{and} \quad g(n) \mid (D(n)\alpha_n) f(n) f(n-1).$$

From the definition of rational normal form, we know that $\gcd(f(n), g(n+h)) = 1$ for any integer $h$. Therefore $f(n-1) \mid (D(n)\gamma_n)$ and $g(n) \mid (D(n)\alpha_n)$. Thus $r(n)$ can be chosen by the following algorithm.

*Algo-Ratio*

1. Choose a monic factor $f(n)$ of $D(n)\gamma_n$ and a monic factor $g(n)$ of $D(n)\alpha_n$.
2. Set $r(n) = \lambda f(n+1)/g(n)$.
3. Solve $\lambda$ by the condition that $L(1)$ is a constant.

Once $r(n)$ is chosen, we can then search for a suitable basis with respect to the linear operator $L$ given by (5.2), and hence obtain the explicit formula of $P_n(x)$. There are two cases.

1. $L$ maps polynomials in $n$ to polynomials in $n$. Then we set

$$b_k(n) = (n - x_1)(n - x_2) \cdots (n - x_k).$$

2. $L$ maps polynomials in $n$ to rational functions of $n$. We firstly find $\mu$ such that $L(n(n+\mu))$ is a polynomial in $n$ and then set

$$b_k(n) = (u(n) - u_1)(u(n) - u_2) \cdots (u(n) - u_k),$$

where $u(n) = n(n+\mu)$.

By the method given in Sections 2 and 3, we can find a formal series solution $y(n) = \sum_{k=0}^{\infty} c_k b_k(n)$ to the equation $L(y(n)) = 0$. Finally, we have

$$P_n(x) = a_0 \left( \prod_{k=0}^{n-1} r(k) \right) y(n).$$

**Example 5.1.** Consider the example given in [11, Section 8]. Suppose that

$$P_{n+1}(x) + (n - x)P_n(x) + \alpha n^2 P_{n-1}(x) = 0.$$

We aim to find an explicit formula of $P_n$.

By *Algo-Ratio*, we find that

$$r(n) = \frac{-1 \pm \sqrt{1 - 4\alpha}}{2}(n+1).$$

For convenience, we write $u^2 = 1 - 4\alpha$ and hence $\alpha = (1 - u^2)/4$ and $r(n) = (u-1)(n+1)/2$. Thus the linear operator $L$ is given by

$$L(p(n)) = \frac{u-1}{2}(n+1)p(n+1) + (n-x)p(n) - \frac{u+1}{2}np(n-1).$$

We find two suitable bases: $b_k(n) = (-1)^k(-n)_k$ or $b_k(n) = (n+1)_k$. Only the first one satisfies $b_k(n) = 0$ for $k$ large enough. Thus we finally derive that

$$P_n(x) = a_0 \left( \frac{u-1}{2} \right)^n n! \, {}_2F_1 \left( \begin{matrix} -n, (-2x+u-1)/2u \\ 1 \end{matrix} \middle| \frac{2u}{u-1} \right), \quad u \neq 0,$$

and

$$P_n(x) = a_0 \frac{(-1)^n n!}{2^n} \, {}_1F_1\left( \begin{matrix} -n \\ 1 \end{matrix} \,\middle|\, 2x+1 \right), \quad u = 0. \qquad \blacksquare$$

**Example 5.2.** Consider the Jacobi polynomials which satisfy

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

where

$$\alpha_n = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad \beta_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

and

$$\gamma_n = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

One choice for $r(n)$ is

$$r(n) = (n+\alpha+1)/(n+1).$$

Now define

$$L(p(n)) = r(n)\alpha_n p(n+1) + (\beta_n - x)p(n) + \frac{\gamma_n}{r(n-1)} p(n-1).$$

We see that $L(n)$ is a rational function of $n$. By the requirement that $L(n(n+\mu))$ is a polynomial in $n$, we find that $\mu = \alpha+\beta+1$. Setting

$$b_k(n) = (u(n)-u_1)\cdots(u(n)-u_k),$$

with $u(n) = n(n+\alpha+\beta+1)$, we find a suitable basis with $u_k = (k-1)(\alpha+\beta+k)$. By *Algo-Verify*, we obtain

$$P_n^{(\alpha,\beta)}(x) = a_0 \frac{(\alpha+1)_n}{n!} \, {}_2F_1\left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right). \qquad \blacksquare$$

**5.2. The $q$-cases.** Now consider the $q$-cases, in which the $\alpha_n, \beta_n$ and $\gamma_n$ in Equation (5.1) are rational functions of $q^n$. We write $t = q^{-n}$ so that $\alpha_n, \beta_n$ and $\gamma_n$ become $\alpha(t), \beta(t)$ and $\gamma(t)$ which are rational functions of $t$. Similar to the ordinary case, we choose a suitable rational function $r(t)$ and define the $q$-difference operator $L$ as follows

$$L(p(t)) = \alpha(t)r(t)p(t/q) + (\beta(t)-x)p(t) + \frac{\gamma(t)}{r(tq)} p(tq).$$

Then by finding suitable bases with respect to $L$, we will derive the $q$-hypergeometric representations of the orthogonal polynomials $P_n(x)$.

**Example 5.3.** Consider the Al-Salam-Chihara polynomials whose three term recurrence relation is given by

$$2xQ_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})Q_{n-1}(x).$$

One choice for $r(t)$ is $(t-ab)/at$ and hence

$$L(p(t)) = \frac{t-ab}{2at}p(t/q) + \left(\frac{a+b}{2t} - x\right)p(t) + \frac{a(t-1)}{2t}p(tq).$$

We find a suitable basis $b_k(t) = (t-1)(t-q^{-1})\cdots(t-q^{-k+1})$. By *Algo-Verify* we finally obtain

$$Q_n(x) = a_0\frac{(ab;q)_n}{a^n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}\,\middle|\, q;q\right), \quad x = \cos\theta. \qquad \blacksquare$$

**Acknowledgments.** We are grateful to the reviewer for valuable comments, especially the suggestion on the choice of $r(n)$ in Section 5. This work was supported by the PCSIRT Project of the Ministry of Education and the National Natural Science Foundation of China (Projects 10731040 and 10826038).

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