Singletons and Adjacencies of Set Partitions of Type B

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Abstract

We show that the joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over set partitions of type B_n without zero-block, in analogy with the result of Callan for ordinary partitions.

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1 Introduction

The main objective of this paper is to give a type B analogue of a property of set partitions discovered by Bernhart [1], that is, the number s_n of partitions of $[n] = \{1, 2, \ldots, n\}$ without singletons is equal to the number a_n of partitions of $[n]$ for which no block contains two adjacent elements i and $i+1$ modulo n. In fact, it is easy to show that s_n and a_n have the same formula by the principle of inclusion-exclusion. Bernhart gave a recursive proof of the fact that $s_n = a_n$ by showing that $s_n + s_{n+1} = B_n$ and $a_n + a_{n+1} = B_n$, where B_n denotes the Bell number, namely, the number of partitions of $[n]$. As noted by Bernhart, there may be no simple way to bring the set of partitions of $[n]$ without singletons and the set of partitions of $[n]$ without adjacencies into a one-to-one correspondence.

From a different perspective, Callan [4] found a bijection in terms of an algorithm that interchanges singletons and adjacencies. Indeed, Callan has established a stronger statement that the joint distribution of the number of singletons and the number of adjacencies is symmetric over the set of partitions of $[n]$. While the proof of Callan is purely combinatorial, we feel that there is still some truth in the remark of Bernhart.

The study of singletons and adjacencies of partitions goes back to Kreweras [9] for noncrossing partitions. Kreweras has shown that the number of noncrossing partitions of $[n]$ without singletons equals the number of noncrossing partitions of $[n]$ without adjacencies. Bernhart [1] found a combinatorial proof of this assertion. Deutsch and Shapiro [6] considered noncrossing partitions of $[n]$ without visible singletons and showed that such partitions are enumerated by the Fine number. Here a visible singleton of a partition means a singleton not covered by any arc in the linear representation. Canfield [5] has shown that the average number of singletons in a partition of $[n]$ is an increasing function of n. Biane [2] has derived a bivariate generating function for the number of partitions of $[n]$ containing a given number of blocks but no singletons. Knuth [8] proposed the problem of finding the generating function for the number of partitions of $[n]$ with a given number of blocks but no adjacencies. The generating function has been found by several problem solvers, see Lossers [10] for example. The sequence of the numbers s_n is listed as the entry A000296 in Sloane [12].

The lattice of ordinary set partitions can be regarded as the intersection lattice for the hyperplane arrangement corresponding to the root system of type A, see Björner and Brenti [3] or Humphreys [7]. Type B set partitions are a generalization of ordinary partitions from this point of view, see Reiner [11]. To be more precise, ordinary set partitions encode the intersections of hyperplanes in the hyperplane arrangement for the type A root system, while the intersections of subsets of hyperplanes from the type B hyperplane arrangement can be encoded by type B set partitions. A partition of type B_n is a partition π of the set

$$[\pm n] = \{1, 2, \ldots, n, -1, -2, \ldots, -n\}$$

such that for any block B of π, $-B$ is also a block of π, and that π has at most one block, called the zero-block, which is of the form $\{i_1, i_2, \ldots, i_k, -i_1, -i_2, \ldots, -i_k\}$.

It is natural to ask whether there exists a type B analogue of Bernhart's theorem and a type B analogue of Callan's algorithm. We give the peeling and patching algorithm which implies the symmetric distribution of the number of singleton pairs and the number of adjacency pairs for type B partitions without zero-block. Moreover, we can transform the bijection into an involution.

2 The peeling and patching algorithm

In this section, we give a type B analogue of Callan's symmetric distribution of singletons and adjacencies over set partitions. We find that the algorithm of Callan can be extended into the type B case. This type B algorithm will be called the peeling and patching algorithm.

Let π be a B_n-partition. We call $\pm i$ a singleton pair of π if π contains a block $\{i\}$, and call $\pm(j, j+1)$ an adjacency pair of π if j and $j+1$ (modulo n) lie in the same block of π. Denote the number of singleton pairs (resp. adjacency pairs) of π by s_π (resp. a_π). For example, let $n = 12$ and

$$\pi = \big\{\pm\{1\},\ \pm\{2\},\ \pm\{3, 11, 12\},\ \pm\{4, -7, 9, 10\},\ \pm\{5, 6, -8\}\big\}. \tag{2.1}$$

Then we have $s_\pi = 2$ and $a_\pi = 3$.

Denote by V_n the set of B_n-partitions without zero-block. The following theorem is the main result of this paper.

Theorem 2.1 *The joint distribution of the number of singleton pairs and the number of adjacency pairs is symmetric over B_n-partitions without zero-block. In other words, let*

$$P_n(x, y) = \sum_{\pi \in V_n} x^{s_\pi} y^{a_\pi},$$

then we have $P_n(x, y) = P_n(y, x)$.

For example, there are three B_2-partitions without zero-block:

$$\big\{\pm\{1\},\ \pm\{2\}\big\}, \quad \big\{\pm\{1, 2\}\big\}, \quad \big\{\pm\{1, -2\}\big\}.$$

So $P_2(x, y) = x^2 + y^2 + 1$. Moreover,

$$\begin{aligned}
P_3(x, y) &= (x^3 + y^3) + 3xy + 3(x + y),\\
P_4(x, y) &= (x^4 + y^4) + 4(x^2y + xy^2) + 8(x^2 + y^2) + 8xy + 4(x + y) + 7.
\end{aligned}$$

Recall that Bernhart showed that the number of partitions of $[n]$ without singletons equals the number of partitions of $[n]$ without adjacencies. As a type B analogue of this result, we have the following consequence of Theorem 2.1.

Corollary 2.2 *The number of B_n-partitions containing no zero-block and no singleton pairs equals the number of B_n-partitions containing no zero-block and no adjacency pairs.*

To prove Theorem 2.1, we shall construct a map $\psi\colon V_n \to V_n$, called the peeling and patching algorithm, such that for any B_n-partition π without zero-block, $s_\pi = a_{\psi(\pi)}$ and $a_\pi = s_{\psi(\pi)}$. To describe this algorithm, we need a more general setting. Let

$$S = \{\pm t_1,\ \pm t_2,\ \ldots,\ \pm t_r\}$$

be a subset of $[\pm n]$, where $0 < t_1 < t_2 < \cdots < t_r$. Let π be a partition of S. We call π a *symmetric partition* if for any block B of π, $-B$ is also a block of π. Similarly, we call $\pm t_i$ a *singleton pair* of π if π contains a block $\{t_i\}$, and call $\pm(t_j, t_{j+1})$ an *adjacency pair* of π if t_j and t_{j+1} are contained in the same block. Moreover, we identify t_{r+1} with t_1. We call $\pm t_j$ (resp. $\pm t_{j+1}$) a *left-point-pair* (resp. *right-point-pair*) if $\pm(t_j, t_{j+1})$ is an adjacency pair. For example, for the case $r = 1$, the partition $\pi = \big\{\pm\{t_1\}\big\}$ contains exactly one singleton pair $\pm t_1$ and one adjacency pair $\pm(t_1, t_1)$.

The peeling and patching algorithm ψ consists of the peeling procedure α and the patching procedure β. For the peeling procedure, at each step we take out the singleton pairs and the left-point-pairs, until there exists neither singleton pairs nor adjacency pairs. Then we use the patching procedure. By interchanging the roles of singleton pairs and adjacency pairs, we put the singleton pairs and left-point-pairs back to the partition. It should be mentioned that the patching procedure is not exactly the reverse of the peeling procedure.

The peeling procedure α. Given a type B partition π without zero-block, let $\pi_0 = \pi$. We extract the set S_1 of singleton pairs and the set L_1 of left-point-pairs (of adjacency pairs) from π_0. Let π_1 be the remaining partition of the set $[\pm n]\backslash(S_1 \cup L_1)$. Note that π_1 is again a type B partition without zero-block. So we can extract the set S_2 of singleton pairs and the set L_2 of left-point-pairs from π_1. Denote by π_2 the remaining partition. Repeating this process, we eventually obtain a partition π_k that has neither singleton pairs nor adjacency pairs. It should noticed that π_k may be the empty partition.

For example, consider the partition π in (2.1), that is,

$$\pi = \big\{\pm\{1\},\ \pm\{2\},\ \pm\{3, 11, 12\},\ \pm\{4, -7, 9, 10\},\ \pm\{5, 6, -8\}\big\}.$$

The peeling procedure is illustrated by Table 2.1. In this example, we have $k = 4$.

j	S_j	L_j	π_j
1	±1, ±2	±5, ±9, ±11	$\pm\{3, 12\},\ \pm\{4, -7, 10\},\ \pm\{6, -8\}$
2	$\emptyset$	±12	$\pm\{3\},\ \pm\{4, -7, 10\},\ \pm\{6, -8\}$
3	±3	$\emptyset$	$\pm\{4, -7, 10\},\ \pm\{6, -8\}$
4	$\emptyset$	±10	$\pm\{4, -7\},\ \pm\{6, -8\}$

Table 2.1: The peeling procedure.

The patching procedure β. Let $\sigma_k = \pi_k$. We shall interchange the roles of the singleton-sets and the adjacency-sets in the process to reconstruct a type B partition. For each i from k to 1, we put the elements of S_i and L_i back into the partition σ_i, so that S_i (resp. L_i) is the right-point-set (resp. singleton-set) of the resulting partition σ_{i-1}.

To be more precise, we start the patching procedure by putting the elements of S_k and L_k back to σ_k in such a way that the resulting partition σ_{k-1} contains S_k (resp. L_k) as its right-point-set (resp. singleton-set). The existence of such a partition σ_{k-1} will be confirmed later. Next, in the same manner we put the elements of S_{k-1} and L_{k-1} back into σ_{k-1} to get σ_{k-2}. Repeating this process, we finally arrive at a type B partition σ_0, which is defined to be the output of the patching procedure.

Now let us describe the process of constructing σ_{k-1} in detail. In the case that S_k is empty, we define σ_{k-1} to be the partition obtained by adding to σ_k the singleton blocks that each singleton block consists of a number in L_k. Suppose that S_k is not empty. Let

$$T_{k-1} = \{\pm t_1,\, \pm t_2,\, \ldots,\, \pm t_r\}$$

be the underlying set of π_{k-1}, where $0 < t_1 < t_2 < \cdots < t_r$. Consider another extremal case that σ_k is empty. Since S_k is not empty, it follows from the last step of the peeling procedure that L_k is empty. In this case, we define σ_{k-1} to be the partition $\big\{\pm\{t_1, t_2, \ldots, t_r\}\big\}$.

Now we can assume that both S_k and σ_k are not empty. According to the adjacent relation imposed on the set T_{k-1}, we can uniquely decompose the set $S_k (\subset T_{k-1})$ into maximal subsets of consecutive elements, which are of the following form

$$\{\pm t_{i+1},\, \pm t_{i+2},\, \ldots,\, \pm t_{i+h}\}. \tag{2.2}$$

In other words, the element $t_i \in T_{k-1}$ does not appear in S_k since σ_k is not empty. On the other hand, $t_i \notin L_k$ by the definition of L_k. Thus t_i is contained in σ_k. This observation allows us to put the elements $t_{i+1},\, t_{i+2},\, \ldots,\, t_{i+h}$ into the block of σ_k that contains t_i. Accordingly, we put the elements $-t_{i+1},\, -t_{i+2},\, \ldots,\, -t_{i+h}$ into the block that contains $-t_i$. After having processed all maximal subsets of consecutive elements of S_k, we put each element in L_k as a singleton block into the partition σ_k. The resulting partition is defined to be σ_{k-1}. Since σ_k contains neither singleton pairs nor adjacency pairs, it is easy to check that L_k (resp. S_k) is the set of singleton pairs (right-point-pairs) of σ_{k-1}.

For example, Table 2.2 is an illustration of the patching procedure for the partition generated in Table 2.1. In the last step, putting S_1 and L_1 back to σ_1, we finally obtain

j	S_j	L_j	σ_j
4	$\emptyset$	± 10	$\pm\{4,-7\},\ \pm\{6,-8\}$
3	± 3	$\emptyset$	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{10\}$
2	$\emptyset$	± 12	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{3,10\}$
1	$\pm 1,\ \pm 2$	$\pm 5,\ \pm 9,\ \pm 11$	$\pm\{4,-7\},\ \pm\{6,-8\},\ \pm\{3,10\},\pm\{12\}$

Table 2.2: The patching procedure.

$$\sigma_0 = \big\{\pm\{1,2,12\},\ \pm\{3,10\},\ \pm\{4,-7\},\ \pm\{5\},\ \pm\{6,-8\},\ \pm\{9\},\ \pm\{11\}\big\}. \tag{2.3}$$

The peeling and patching algorithm ψ is defined by

$$\psi(\pi) = \beta(\alpha(\pi))$$

for any B_n-partition π without zero-block. Keep in mind that the roles of singleton pairs and adjacency pairs are interchanged in the patching procedure. We are now ready to give a proof of Theorem 2.1.

Proof of Theorem 2.1. We proceed to show that the peeling and patching algorithm ψ gives a bijection on B_n-partitions without zero-block, which interchanges the number of singleton pairs and the number of adjacency pairs.

It is easy to see that the inverse algorithm can be described as follows. In fact, the inverse algorithm of ψ is the composition of another peeling procedure and another patching procedure. To be precise, let σ be the input partition. Let $\sigma_0 = \sigma$. We first peel the singleton pairs and right-point-pairs at each step, until we obtain a partition σ_k which has neither singleton pairs nor adjacency pairs. As the second step, based on the partition $\pi_k = \sigma_k$, we recursively put back the elements that have been taken out before. Meanwhile, we also interchange the roles of the singleton-sets and right-point-sets. Finally, we get a type B partition, as the output of the inverse algorithm. Therefore, ψ is a bijection which induces a bijection on B_n-partitions without zero-block. ∎

An illustration of the peeling and patching algorithm is given by the partition (2.1), Table 2.1, Table 2.2, and (2.3). To conclude this section, we give the generating function for the number s_n^B of B_n-partitions without zero-block and singleton pairs, that is,

$$\sum_{n\geq 0} s_n^B \frac{x^n}{n!} = \exp\left(\sinh(x)e^x - x\right). \tag{2.4}$$

In fact, by the principle of inclusion-exclusion, we obtain

$$s_n^B = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{j=0}^{k} 2^{k-j} S(k,j), \tag{2.5}$$

where $S(k,j)$ is the Stirling number of the second kind. Note that $2^{k-j}S(k,j)$ is the number of partitions in V_k with $2j$ blocks. The formula (2.4) can be easily derived from (2.5).

3 From bijection to involution

The bijection given in the previous section is not an involution although it interchanges the number of singleton pairs and the number of adjacency pairs. In this section, we show that the peeling and patching algorithm can be turned into an involution. Such an involution for ordinary partitions has been given by Callan [4].

For any $i \in [n]$, we define the *complement* of i to be $n+1-i$. The complement of $-i$ is defined to be $-(n+1-i)$. This notion can be extended to any symmetric partition π of $[\pm n]$ by taking the complement of each element in the partition. The complement of π is denoted by $\omega(\pi)$. It is clear that ω is an involution. Assume that σ_0 is given by (2.3). We have

$$\omega(\sigma_0) = \big\{\pm\{1,11,12\},\ \pm\{2\},\ \pm\{3,10\},\ \pm\{4\},\ \pm\{5,-7\},\ \pm\{6,-9\},\ \pm\{8\}\big\}. \tag{3.1}$$

In light of the complementation operation, we get an involution based on the peeling and patching algorithm.

Theorem 3.1 *The mapping $\omega\circ\psi$ is an involution on B_n-partitions without zero-block, which interchanges the number of singleton pairs and the number of adjacency pairs.*

The proof is straightforward and hence is omitted. We give an example to demonstrate that $\omega \circ \psi$ is involution, that is,

$$\omega(\psi(\pi)) = \psi^{-1}(\omega(\pi)). \tag{3.2}$$

Consider the partition π in (2.1). In this case, the left hand side of (3.2) is $\omega(\sigma_0)$, which is computed as in (3.1). On the other hand,

$$\omega(\pi) = \big\{\pm\{1,2,10\},\ \pm\{3,4,-6,9\},\ \pm\{5,-7,-8\},\ \pm\{11\},\ \pm\{12\}\big\}.$$

Applying the procedure β^{-1}, we obtain Table 3.3, where R_j (resp. S_j) denotes the set of right-point-pairs (singleton pairs). Next, by the procedure α^{-1}, we get Table 3.4. Finally, putting R_1 and S_1 back to π_1, we obtain the partition π_0 which is in agreement with (3.1).

j	R_j	S_j	σ_j
1	$\pm2,\ \pm4,\ \pm8$	$\pm11,\ \pm12$	$\pm\{1,10\},\ \pm\{3,-6,9\},\ \pm\{5,-7\}$
2	±1	$\emptyset$	$\pm\{10\},\ \pm\{3,-6,9\},\ \pm\{5,-7\}$
3	$\emptyset$	±10	$\pm\{3,-6,9\},\ \pm\{5,-7\}$
4	±3	$\emptyset$	$\pm\{5,-7\},\ \pm\{6,-9\}$

Table 3.3: The procedure β^{-1}.

j	R_j	S_j	π_j
4	±3	$\emptyset$	$\pm\{5,-7\},\ \pm\{6,-9\}$
3	$\emptyset$	±10	$\pm\{3\},\ \pm\{5,-7\},\ \pm\{6,-9\}$
2	±1	$\emptyset$	$\pm\{3,10\},\ \pm\{5,-7\},\ \pm\{6,-9\}$
1	$\pm2,\ \pm4,\ \pm8$	$\pm11,\ \pm12$	$\pm\{1\},\ \pm\{3,10\},\ \pm\{5,-7\},\ \pm\{6,-9\}$

Table 3.4: The procedure α^{-1}.

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