Anti-lecture Hall Compositions and Overpartitions

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Abstract. We show that the number of anti-lecture hall compositions of n with the first entry not exceeding $k-2$ equals the number of overpartitions of n with non-overlined parts not congruent to $0, \pm 1$ modulo k. This identity can be considered as a finite version of the anti-lecture hall theorem of Corteel and Savage. To prove this result, we find two Rogers–Ramanujan type identities for overpartitions which are analogous to the Rogers–Ramanjan type identities due to Andrews. When k is odd, we give another proof by using the bijections of Corteel and Savage for the anti-lecture hall theorem and the generalized Rogers–Ramanujan identity also due to Andrews.

Keywords. Anti-lecture hall composition, Rogers–Ramanujan type identity, overpartition, Durfee dissection

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1 Introduction

The objective of this paper is to establish a connection between anti-lecture hall compositions with an upper bound on the first entry and overpartitions under a congruence condition on non-overlined parts.

Corteel and Savage [6] introduced the notion of anti-lecture hall compositions and obtained a formula for the generating function by constructing a bijection. An anti-lecture hall composition of length k is defined to be an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_{k-1}}{k-1} \geq \frac{\lambda_k}{k} \geq 0.$$

The set of anti-lecture hall compositions of length k is denoted by A_k. Corteel and Savage have shown that

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}}. \tag{1.1}$$

Let A denote the set of anti-lecture hall compositions. Since any anti-lecture hall composition can be written as an infinite vector ending with zeros, we have $A = A_\infty$ and

$$\sum_{\lambda \in A} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^{i+1}}. \tag{1.2}$$

In view of the above generating function, one sees that anti-lecture hall compositions are connected to overpartitions. An overpartition of n is defined by a non-increasing sequence of positive integers whose sum is n in which the first occurrence of a part may be overlined, see, Corteel and Lovejoy [7]. In the language of overpartitions, the right-hand side of (1.2) is the generating function for overpartitions of n with non-overlined parts greater than 1.

We use the common notation on q-series. Let

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and

$$(a_1, \ldots, a_k; q)_\infty = (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

We also write

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The main result of this paper is the following refinement of the anti-lecture hall theorem of Corteel and Savage [6]:

Theorem 1.1 *For $k \geq 3$,*

$$\sum_{\lambda \in A,\, \lambda_1 \leq k-2} q^{|\lambda|} = \frac{(-q)_\infty}{(q)_\infty}(q, q^{k-1}, q^k; q^k)_\infty. \tag{1.3}$$

We shall establish a connection between anti-lecture hall compositions and the overpartitions with congruence restrictions. Let $F_k(n)$ be the set of anti-lecture hall compositions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n with $\lambda_1 \leq k$. Let $H_k(n)$ be the set of overpartitions of n for which the non-overlined parts are not congruent to $0, \pm 1$ modulo k. Therefore, Theorem 1.1 can be stated in the following equivalent form.

Theorem 1.2 *For $k \geq 3$ and any positive integer n, we have*

$$|F_{k-2}(n)| = |H_k(n)|. \tag{1.4}$$

To prove the above relation, we need to compute the generating functions of the anti-lecture hall compositions λ with $\lambda_1 \leq k$, depending on the parity of k. Moreover, we shall show that these two generating functions of the anti-lecture hall compositions in $F_{2k-2}(n)$ and $F_{2k-3}(n)$ are equal to the generating functions of overpartitions in $H_{2k}(n)$ and $H_{2k-1}(n)$ respectively. To this end, we shall establish two Rogers–Ramanujan type identities (2.9) and (2.12) for overpartitions which are analogous to the following Rogers–Ramanujan type identity obtained by Andrews [1, 2]:

$$\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_a+\cdots+N_{k-1}}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_{k-1}-N_k}}=\frac{(q^a,q^{2k+1-a},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty} \tag{1.5}$$

where $1\leq a\leq k$. For $k=2$ and $a=1,2$, (1.5) implies the classical Rogers–Ramanujan identities [8]:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{n=0}^{\infty}(1-q^{5n+1})^{-1}(1-q^{5n+4})^{-1}, \tag{1.6}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}=\prod_{n=0}^{\infty}(1-q^{5n+2})^{-1}(1-q^{5n+3})^{-1}. \tag{1.7}$$

It is worth mentioning that Andrews' multiple series transformation [2] can be employed to derive the overpartition analogues of (1.5).

When the upper bound k is even, the weighted counting of anti-lecture hall compositions leads to the left-hand side of the first Rogers–Ramanujan type identity (2.9), whereas the generating function for the number of overpartitions equals the right-hand side of the first Rogers–Ramanujan type identity (2.9). The case when k is odd can be dealt with in the same way.

For the case of an odd upper bound, we provide an alternative proof based on a refined version of a bijection of Corteel and Savage [6] for the anti-lecture hall theorem and the generalized Rogers–Ramanujan identity (1.5) of Andrews.

This paper is organized as follows. In Section 2, we give two Rogers–Ramanujan type identities for overpartitions. Section 3 is concerned with the case the upper bound k is even. Two proofs for the case when the upper bound is odd will be given in Section 4.

2 Rogers–Ramanujan type identities for overpartitions

In this section, we give two Rogers–Ramanujan type identities (2.9) and (2.12) for overpartitions. It can be seen that the right-hand side of (2.9) is the generating function for overpartitions in $H_{2k}(n)$. In the next section, we shall show that the left-hand side of (2.9) equals the generating function for anti-lecture hall compositions in $F_{2k-2}(n)$. Similarly, the right-hand side of (2.12) equals the generating function for overpartitions in $H_{2k-1}(n)$. In Section 4, we shall show that the left-hand side of (2.12) equals the generating function for anti-lecture hall compositions in $F_{2k-3}(n)$.

Let us recall Andrews' multiple series transformation [2]:

$$
\begin{aligned}
&{}_{2k+4}\phi_{2k+3}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, \ldots, b_k, c_k, q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, aq/b_2, aq/c_2, \ldots, aq/b_k, aq/c_k, aq^{N+1} \end{matrix}; q, \frac{a^k q^{k+N}}{b_1 \cdots b_k c_1 \cdots c_k}\right] \\
&= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{m_1,\ldots,m_{k-1}\geq 0} \frac{(aq/b_1c_1)_{m_1}(aq/b_2c_2)_{m_2}\cdots(aq/b_{k-1}c_{k-1})_{m_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}} \\
&\cdot \frac{(b_2)_{m_1}(c_2)_{m_1}(b_3)_{m_1+m_2}(c_3)_{m_1+m_2}\cdots(b_k)_{m_1+\cdots+m_{k-1}}}{(aq/b_1)_{m_1}(aq/c_1)_{m_1}(aq/b_2)_{m_1+m_2}(aq/c_2)_{m_1+m_2}\cdots(aq/b_{k-1})_{m_1+\cdots+m_{k-1}}} \\
&\cdot \frac{(c_k)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1})_{m_1+\cdots+m_{k-1}}} \cdot \frac{(q^{-N})_{m_1+m_2+\cdots+m_{k-1}}}{(b_k c_k q^{-N}/a)_{m_1+m_2+\cdots+m_{k-1}}} \\
&\cdot \frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+m_2+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_3c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+m_2+\cdots+m_{k-2}}}. \qquad (2.8)
\end{aligned}
$$

The following summation formula is a consequence of the above transformation formula. It can be considered as a Rogers–Ramanujan type identity for overpartitions.

Theorem 2.1 *For $k \geq 2$, we have*

$$
\begin{aligned}
&\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} \\
&= \frac{(-q)_\infty (q, q^{2k-1}, q^{2k}; q^{2k})_\infty}{(q)_\infty}. \qquad (2.9)
\end{aligned}
$$

Proof. Applying the above transformation formula by setting all variables to infinity except for c_k, a and q, we get

$$
\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{(c_k)_{N_1} a^{N_1+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-c_k)^{N_1}} \\
&= \frac{(aq/c_k)_\infty}{(a)_\infty} \sum_{n\geq 0} \frac{(1-aq^{2n})(a, c_k; q)_n a^{kn} q^{kn^2}}{(q, aq/c_k; q)_n c_k^n}.
\end{aligned}
$$

Setting $a = q$ and $c_k = -q$, we find that

$$
\begin{aligned}
&\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} \\
&= \frac{(-q)_\infty}{(q)_\infty} \sum_{n\geq 0} (-1)^n (1-q^{2n+1}) q^{kn^2+(k-1)n}. \qquad (2.10)
\end{aligned}
$$

Using Jacobi's triple product identity, we get

$$\begin{aligned}
&(q, q^{2k-1}, q^{2k}; q^{2k})_\infty \\
&\quad = \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+(k-1)n} \\
&\quad = \sum_{n=0}^{\infty} (-1)^n (1-q^{2n+1}) q^{kn^2+(k-1)n}. \qquad (2.11)
\end{aligned}$$

In view of (2.10) and (2.11), we obtain (2.9). This completes the proof. ∎

The second Rogers–Ramanujan type identity for overpartitions can be stated as follows.

Theorem 2.2 *For $k \geq 2$, we have*

$$\begin{aligned}
&\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}} \\
&\quad = \frac{(-q)_\infty (q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty}. \qquad (2.12)
\end{aligned}$$

Proof. Applying Andrews' transformation formula by setting all variables except for a, c_1, c_k and q to infinity, we find

$$\begin{aligned}
&\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{(c_k)_{N_1} a^{N_1+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-c_k)^{N_1}(aq/c_1)_{N_{k-1}}} \\
&\quad = \frac{(aq/c_k)_\infty}{(a)_\infty} \sum_{n \geq 0} \frac{(-1)^n (1-aq^{2n})(a, c_k; q)_n (c_1)_n a^{kn} q^{kn^2-(n-1)n/2}}{(q, aq/c_k; q)_n (aq/c_1)_n c_1^n c_k^n}.
\end{aligned}$$

Moreover, setting $a = q$, $c_k = -q$ and $c_1 = -q$ yields

$$\begin{aligned}
&\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_2+\cdots+N_{k-1}} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}} \\
&\quad = \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 0} (-1)^n (1-q^{2n+1}) q^{kn^2+kn-n^2/2-3n/2}. \qquad (2.13)
\end{aligned}$$

Using Jacobi's triple product identity, we have

$$\begin{aligned}
&(q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty \\
&\quad = \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2+kn-n^2/2-3n/2} \\
&\quad = \sum_{n=0}^{\infty} (-1)^n (1-q^{2n+1}) q^{kn^2+kn-n^2/2-3n/2}. \qquad (2.14)
\end{aligned}$$

Combining (2.13) and (2.14), we deduce (2.12). This completes the proof. ∎

3 The case of an even upper bound

In this section, we shall give a proof of Theorem 1.2 for an even upper bound $2k-2$. More precisely, we have the following relation.

Theorem 3.1 *For $k \geq 2$ and $n \geq 1$, we have*

$$|F_{2k-2}(n)| = |H_{2k}(n)|. \tag{3.15}$$

Recall that the generating function for overpartitions in $H_{2k}(n)$ equals

$$\sum_{n\geq 0} |H_{2k}(n)|q^n = \frac{(-q)_\infty (q, q^{2k-1}, q^{2k}; q^{2k})_\infty}{(q)_\infty}. \tag{3.16}$$

Because of (2.9), Theorem 3.1 can be deduced from the following generating function for anti-lecture hall compositions in $F_{2k-2}(n)$.

Theorem 3.2 *The generating function of anti-lecture hall compositions in $F_{2k-2}(n)$ is given by*

$$\sum_{n=0}^{\infty} |F_{2k-2}(n)|q^n = \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \tag{3.17}$$

In order to prove Theorem 3.2, we introduce a triangular representation $T(\lambda) = (t_{ij})_{1\leqslant i\leqslant j}$ of an anti-lecture hall composition λ which is similar to the T-triangles introduced by Bousquet-Mélou and Eriksson[5].

It should be noted that Corteel and Savage [6] used a representation of a composition λ as a pair of vectors $(l, r) = ((l_1, l_2, \ldots), (r_1, r_2, \ldots))$, where $\lambda_i = il_i + r_i$, with $0 \leq r_i \leq i-1$. Then $l = \lfloor\lambda\rfloor = (\lfloor\lambda_1/1\rfloor, \lfloor\lambda_2/2\rfloor, \ldots)$. It can be checked that a composition λ is an anti-lecture hall composition if and only if

(1) $l_1 \geq l_2 \geq \cdots \geq 0$, and

(2) $r_i \geq r_{i+1}$ whenever $l_i = l_{i+1}$.

Definition 3.3 *The A-triangular representation $T(\lambda) = (t_{i,j})_{1\leqslant i\leqslant j}$ of an anti-lecture hall composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined to be a triangular array $(t_{i,j})_{1\leqslant i\leqslant j}$ of nonnegative integers satisfying the following conditions:*

(1) A diagonal entry $t_{j,j}$ in $T(\lambda)$ equals $l_j = \lfloor\lambda_j/j\rfloor$.

(2) The first r_j entries of the j-th column are equal to $t_{j,j}+1$, while the other entries in the j-th column are equal to $t_{j,j}$.

It is easily seen that the sum of all entries of $T(\lambda)$ is equal to $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. It is also easy to check that the A-triangular representation $T(\lambda)$ of an anti-lecture hall composition possesses the following properties:

(1) The diagonal entries of T are weakly decreasing, that is, $t_{1,1} \geq t_{2,2} \geq \cdots \geq 0$.

(2) The entries in the j-th column are non-increasing, and they are equal to $t_{j,j}$ or $t_{j,j} + 1$.

(3) If $t_{j,j} = t_{j+1,j+1}$, then $t_{i,j} \geq t_{i,j+1}$ for $1 \leq i \leq j$.

Conversely, a triangular array satisfying the above conditions must be the A-triangular representation of an anti-lecture hall composition.

For example, let $\lambda = (4, 8, 11, 14, 16, 15, 11, 10, 5, 2)$. The A-triangular representation $T(\lambda)$ of λ is given below:

$$\begin{array}{cccccccccc}
4 & 4 & 4 & 4 & 4 & 3 & 2 & 2 & 1 & 1 \\
 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
 & & 3 & 3 & 3 & 3 & 2 & 1 & 1 & 0 \\
 & & & 3 & 3 & 2 & 2 & 1 & 1 & 0 \\
 & & & & 3 & 2 & 1 & 1 & 1 & 0 \\
 & & & & & 2 & 1 & 1 & 0 & 0 \\
 & & & & & & 1 & 1 & 0 & 0 \\
 & & & & & & & 1 & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}$$

We are now ready to give a proof of Theorem 3.2 by using the A-triangular representation.

Proof of Theorem 3.2. Let λ be an anti-lecture hall composition with $\lambda_1 \leq 2k-2$, let $T(\lambda)$ be the A-triangular representation of λ, and let N_i denote the number of diagonal entries $t_{j,j}$ in $T(\lambda)$ which are greater than or equal to $2i-1$ for $1 \leq i \leq k-1$. Then we have $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$. Now we use $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$ to denote the set of anti-lecture hall compositions λ such that there are N_i diagonal entries in $T(\lambda)$ that are greater than or equal to $2i-1$ and $\lambda_1 \leq 2k-2$. We aim to compute the generating function of the anti-lecture hall compositions in $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$, which can be summed up to derive the generating function for $F_{2k-2}(n)$.

Let λ be an anti-lecture hall composition in $F_{2k-2}(N_1, \ldots, N_{k-1}; n)$, and let $\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{N_1})$, $\lambda^{(2)} = (\lambda_{N_1+1}, \ldots, \lambda_l)$. Since $\lfloor \lambda_{N_1+1}/(N_1+1) \rfloor = \cdots = \lfloor \lambda_l / l \rfloor = 0$, we see that $\lambda_l \leq \cdots \leq \lambda_{N_1+1} \leq N_1$. Evidently, $\lambda^{(2)}$ is a partition whose first part is less than $N_1 + 1$, and the generating function for possible choices of $\lambda^{(2)}$ equals $1/(q)_{N_1}$.

Examining the A-triangular representation of the composition $\lambda^{(1)}$, we find that the triangular array $T(\lambda^{(1)})$ can be split into k triangular arrays so that we can compute the generating function for possible choices of $\lambda^{(1)}$.

Step 1. Let $T^{(1)} = T(\lambda^{(1)})$. Extract 1 from each entry in the first N_1 columns of $T^{(1)}$ to form a triangular array of size N_1 with all entries equal to 1, denoted by $R(N_1, 1)$.

Step 2. For $2 \le i \le k-1$, extract 2 from each entry in the first N_i columns of the remaining triangular array $T^{(1)}$ to generate a triangular array of size N_i with all entries equal to 2, denoted by $R(N_i, 2)$.

Step 3. Let S denote the remaining triangular array $T^{(1)}$.

After the above operations, $T(\lambda^{(1)})$ is decomposed into k triangular arrays, including an A-triangle $R(N_1, 1)$ of size N_1 with entries equal to 1, $k-2$ A-triangular arrays $R(N_i, 2)$ of sizes $N_2, \ldots, N_{k-1}$ respectively with entries equal to 2 where $i = 2, \ldots, k-1$, and a triangular array $S = (s_{i,j})_{1 \le i \le j \le N_1}$ of size N_1. It is easy to see that the generating function for triangular arrays in $R(N_1, 1)$ is $q^{(N_1+1)N_1/2}$ and the generating function of triangular arrays in $R(N_i, 2)$ is $q^{N_i^2+N_i}$.

We give an example to illustrate the above decomposition. Let $\lambda = (4, 8, 11, 14, 16, 15, 11, 10, 5, 2)$ as given before, let $k = 3$. Then $\lambda^{(2)} = (5, 2)$, $N_1 = 8$, $N_2 = 5$. The decomposition of $T(\lambda)$ is given below:

$$
\underset{T(\lambda)}{\begin{array}{cccccccccc}
4&4&4&4&4&3&2&2&1&1\\
&4&4&4&3&3&2&2&1&1\\
&&3&3&3&3&2&1&1&0\\
&&&3&3&2&2&1&1&0\\
&&&&3&2&1&1&1&0\\
&&&&&2&1&1&0&0\\
&&&&&&1&1&0&0\\
&&&&&&&1&0&0\\
&&&&&&&&0&0\\
&&&&&&&&&0
\end{array}}
\longrightarrow
\underset{R(8,1)}{\begin{array}{cccccccc}
1&1&1&1&1&1&1&1\\
&1&1&1&1&1&1&1\\
&&1&1&1&1&1&1\\
&&&1&1&1&1&1\\
&&&&1&1&1&1\\
&&&&&1&1&1\\
&&&&&&1&1\\
&&&&&&&1
\end{array}}
+
\underset{R(5,2)}{\begin{array}{ccccc}
2&2&2&2&2\\
&2&2&2&2\\
&&2&2&2\\
&&&2&2\\
&&&&2
\end{array}}
$$

$$
+
\underset{S}{\begin{array}{cccccccc}
1&1&1&1&1&2&1&1\\
&1&1&1&0&2&1&1\\
&&0&0&0&2&1&0\\
&&&0&0&1&1&0\\
&&&&0&1&0&0\\
&&&&&1&0&0\\
&&&&&&0&0\\
&&&&&&&0
\end{array}}
+
\underset{T(\lambda^{(2)})}{\begin{array}{cccccccccc}
0&0&0&0&0&0&0&0&1&1\\
&0&0&0&0&0&0&0&1&1\\
&&0&0&0&0&0&0&1&0\\
&&&0&0&0&0&0&1&0\\
&&&&0&0&0&0&1&0\\
&&&&&0&0&0&0&0\\
&&&&&&0&0&0&0\\
&&&&&&&0&0&0\\
&&&&&&&&0&0\\
&&&&&&&&&0
\end{array}}
$$

It can be checked that S possesses the following properties by the definition of the A-triangular representation:

(1) The entries on the diagonal of S are equal to 1 or 0. Notice that S has N_1 diagonal elements $s_{1,1}, s_{2,2}, \ldots, s_{N_1,N_1}$. These diagonal elements can be divided into $k-1$ segments such that the first segment contains $n_1 = N_1 - N_2$ elements $s_{N_2+1,N_2+1}, \ldots, s_{N_1,N_1}$, the second segment contains $n_2 = N_2 - N_3$ elements $s_{N_3+1,N_3+1}, \ldots, s_{N_2,N_2}$, and so on,

whereas the last segment contains $n_{k-1} = N_{k-1}$ elements $s_{1,1}, \ldots, s_{N_{k-1},N_{k-1}}$. Moreover, the i-th segment contains some 1's followed by some 0's. We define m_i to be the number of 1's contained in the i-th segment.

(2) The entries in the j-th column are non-increasing, and they are equal to $s_{j,j}$ or $s_{j,j}+1$.

(3) If $s_{j,j} = s_{j+1,j+1}$, then $s_{i,j} \geq s_{i,j+1}$ for $1 \leq i \leq j$.

We denote the set of triangular arrays possessing the above three properties by $S(N_1, N_2, \ldots, N_{k-1})$. Now we proceed to compute the generating function of triangular arrays in $S(N_1, N_2, \ldots, N_{k-1})$. As the first step, we may partition a triangular array $S \in S(N_1, N_2, \ldots, N_{k-1})$ into $k-1$ blocks of columns, where the i-th block consists of the $(N_{i+1}+1)$-th column to the N_i-th column of S. We denote the i-th block by S_i. According to the above three properties, we deduce that the first m_i diagonal entries of S_i must be 1 and the entries in the first m_i columns of S_i are either 1 or 2.

To compute the generating function for S_i, we shall split S_i into three trapezoidal arrays $S_i^{(1)}$, $S_i^{(2)}$ and $S_i^{(3)}$. First, we may form a trapezoidal array $S_i^{(1)}$ of the same size as S_i and with the entries in the first m_i columns equal to 1 and the other entries equal to 0. Let $S_i^{'}$ denote the trapezoidal array obtained from S_i by subtracting 1 from every entry in the first m_i columns. Observe that every entry in $S_i^{'}$ is either 1 or 0, and $S_i^{(1)}$ can be regarded as the Ferrers diagram of the conjugate of the partition

$$\alpha^{(1)} = (N_{i+1}+m_i, N_{i+1}+m_i-1, \ldots, N_{i+1}+1).$$

Furthermore, $S_i^{'}$ satisfies the following conditions:

(1) All entries in $S_i^{'}$ are equal to 0 or 1, but the diagonal entries must be 0.

(2) The entries in the same column must be non-increasing.

(3) The first m_i entries in the j-th row must be non-increasing, and the remaining entries in the j-th row are also non-increasing.

We continue to consider the trapezoidal array formed by the first m_i columns of $S_i^{'}$, and denote it by $S_i^{(2)}$. Similarly, we see that $S_i^{(2)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(2)}$, where

$$\alpha_1^{(2)} \leq N_{i+1} \quad \text{and} \quad l(\alpha^{(2)}) \leq m_i.$$

Define $S_i^{(3)}$ to be the trapezoidal array obtained from S_i' by extracting from the (m_i+1)-th column to the $(N_i - N_{i+1})$-th column. Again, $S^{(3)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(3)}$, where

$$\alpha_1^{(3)} \leq N_{i+1} + m_i \quad \text{and} \quad l(\alpha^{(3)}) \leq N_i - N_{i+1} - m_i.$$

So the generating function for possible choices of the i-th block S_i is given by

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2} \frac{(q)_{N_{i+1}+m_i}}{(q)_{m_i}(q)_{N_{i+1}}} \frac{(q)_{N_i}}{(q)_{N_{i+1}+m_i}(q)_{N_i-N_{i+1}-m_i}} \tag{3.18}$$

which equals

$$\frac{(q)_{N_i}}{(q)_{N_{i+1}}(q)_{N_i-N_{i+1}}}\sum_{m_i=0}^{N_i-N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2}\frac{(q)_{N_i-N_{i+1}}}{(q)_{m_i}(q)_{N_i-N_{i+1}-m_i}}. \tag{3.19}$$

Observe that the sum

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2}\frac{(q)_{N_i-N_{i+1}}}{(q)_{m_i}(q)_{N_i-N_{i+1}-m_i}}$$

is the generating function for partitions with distinct parts between $N_{i+1}+1$ and N_i. Therefore,

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2}\frac{(q)_{N_i-N_{i+1}}}{(q)_{m_i}(q)_{N_i-N_{i+1}-m_i}} = (-q^{N_{i+1}+1})_{N_i-N_{i+1}}. \tag{3.20}$$

By (3.20), the generating function (3.18) can be simplified to

$$\frac{(q)_{N_i}}{(q)_{N_{i+1}}(q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1})_{N_i-N_{i+1}}. \tag{3.21}$$

Thus the generating function for triangular arrays in S can be written as

$$\prod_{i=1}^{k-1}\frac{(q)_{N_i}}{(q)_{N_{i+1}}(q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1})_{N_i-N_{i+1}} = \frac{(q)_{N_1}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}.$$

Recall that the generating function for possible choices of $T(\lambda^{(2)})$ equals $1/(q)_{N_1}$ and the generating functions for $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ are equal to $q^{(N_1+1)N_1/2}$, $q^{N_2^2+N_2}$,…, $q^{N_{k-1}^2+N_{k-1}}$ respectively. We also note that the generating function for anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$ is the product of the generating functions for $T(\lambda^{(2)})$, $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ and S, and hence it equals

$$\begin{aligned}&\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}}{(q)_{N_1}}\frac{(q)_{N_1}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}\\ &=\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}.\end{aligned}$$

Summing up the generating functions of anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$, we get the generating function for $F_{2k-2}(n)$,

$$\sum_{n\geq 0}|F_{2k-2}(n)|q^n = \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}. \tag{3.22}$$

This completes the proof. ∎

4 The case of an odd upper bound

In this section, we provide two proofs of Theorem 1.2 for the case when the upper bound on the first component of the compositions is an odd number $2k-3$. The first proof is analogous to the proof of the even case, whereas the second proof requires a Rogers–Ramanujan type identity of Andrews, a bijection of Corteel and Savage, as well as a refined version of a bijection also due to Corteel and Savage. However, the approach of the second proof does not seem to apply to the case when on the upper bound is an even number.

Theorem 4.1 *For $k \geq 2$ and a positive integer n, we have*

$$|F_{2k-3}(n)| = |H_{2k-1}(n)|. \tag{4.23}$$

The first proof relies on the following formula for anti-lecture hall compositions in $F_{2k-3}(n)$.

Theorem 4.2 *For $k \geq 2$,*

$$\sum_{n=0}^{\infty} |F_{2k-3}(n)|q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}. \tag{4.24}$$

Proof of Theorem 4.2. Let λ be an anti-lecture hall composition with $\lambda_1 \leq 2k-3$. We consider the A-Triangular representation $T(\lambda)$ of λ. For $1 \leq i \leq k-1$, let N_i be the number of diagonal entries $t_{j,j}$ in $T(\lambda)$ which are greater than or equal to $2i-1$. Thus we have $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$. We now use $F_{2k-3}(N_1, \ldots, N_{k-1}; n)$ to denote the set of anti-lecture hall compositions λ with $\lambda_1 \leq 2k-3$ such that there are N_i diagonal entries in $T(\lambda)$ that are greater than or equal to $2i-1$.

In order to compute the generating function for $F_{2k-3}(N_1, \ldots, N_{k-1}; n)$, we split λ into two parts depending on whether $t_{j,j} = 0$. To this end, we set $\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{N_1})$, $\lambda^{(2)} = (\lambda_{N_1+1}, \ldots, \lambda_l)$. It is easily checked that $\lambda^{(2)}$ is a partition whose first part does not exceed N_1. Hence the generating function for possible choices of $\lambda^{(2)}$ equals $1/(q)_{N_1}$.

We now consider $\lambda^{(1)}$ and its A-Triangular representation $T(\lambda^{(1)})$. In this case, we can split $T(\lambda^{(1)})$ into k triangular arrays so that we can compute the generating function for possible choices of $\lambda^{(1)}$.

Step 1. Let $T^{(1)} = T(\lambda^{(1)})$. Extract 1 from each entry in the first N_1 columns of $T^{(1)}$ to form a triangular array of size N_1 with all entries equal to 1, denoted by $R(N_1, 1)$.

Step 2. For $i = 2, \ldots, k-1$, extract 2 from each entry in the first N_i columns of the remaining array $T^{(1)}$ to form a triangular array of size N_i with all entries equal to 2, denoted by $R(N_i, 2)$.

Step 3. Let S be the remaining triangular array $T^{(1)}$.

After the above procedures, $T(\lambda^{(1)})$ is decomposed into k triangular arrays, including an A-Triangle $R(N_1, 1)$ of size N_1 with all entries being 1, $(k-2)$ A-Triangles $R(N_i, 2)$ of sizes $N_2, \ldots, N_{k-1}$ respectively with all entries being 2 and a triangular array $S = (s_{i,j})$ of size N_1 possessing the following properties:

(1) All the entries on the diagonal of S are equal to 1 or 0. Note that S has N_1 diagonal elements $s_{1,1}, s_{2,2}, \ldots, s_{N_1,N_1}$. These diagonal elements can be divided into $k-1$ segments such that the first segment contains $n_1 = N_1 - N_2$ elements $s_{N_2+1,N_2+1}, \ldots, s_{N_1,N_1}$, the second segment contains $n_2 = N_2 - N_3$ elements $s_{N_3+1,N_3+1}, \ldots, s_{N_2,N_2}$, and so on, while the last segment contains $n_{k-1} = N_{k-1}$ elements $s_{1,1}, \ldots, s_{N_{k-1},N_{k-1}}$. Moreover, the i-th segment contains some 1's followed by some 0's. We denote m_i to be the number of 1's in the i-th segment.

(2) The entries in the j-th column are non-increasing, and they are equal to either $s_{j,j}$ or $s_{j,j}+1$.

(3) If $s_{j,j} = s_{j+1,j+1}$, then $s_{i,j} \geq s_{i,j+1}$ for $1 \leq i \leq j$.

(4) The entries in the first N_{k-1} columns of S are equal to 0, that is, $m_{k-1} = 0$, since $t_{ij} = 2k-3$ for $1 \leq i \leq j \leq N_{k-1}$.

For example, let $\mu = (5, 10, 14, 17, 18, 20, 18, 15, 12, 3)$ and $k = 4$. We can decompose $T(\mu)$ into the following triangular arrays

$$
\underset{T(\mu)}{\begin{array}{cccccccccc}
5&5&5&5&4&4&3&2&2&1\\
&5&5&4&4&4&3&2&2&1\\
&&4&4&4&3&3&2&2&1\\
&&&4&3&3&3&2&1&0\\
&&&&3&3&2&2&1&0\\
&&&&&3&2&2&1&0\\
&&&&&&2&2&1&0\\
&&&&&&&1&1&0\\
&&&&&&&&1&0\\
&&&&&&&&&0
\end{array}}
\longrightarrow
\underset{R(9,1)}{\begin{array}{ccccccccc}
1&1&1&1&1&1&1&1&1\\
&1&1&1&1&1&1&1&1\\
&&1&1&1&1&1&1&1\\
&&&1&1&1&1&1&1\\
&&&&1&1&1&1&1\\
&&&&&1&1&1&1\\
&&&&&&1&1&1\\
&&&&&&&1&1\\
&&&&&&&&1
\end{array}}
+
\underset{R(6,2)}{\begin{array}{cccccc}
2&2&2&2&2&2\\
&2&2&2&2&2\\
&&2&2&2&2\\
&&&2&2&2\\
&&&&2&2\\
&&&&&2
\end{array}}
+
\underset{R(2,2)}{\begin{array}{cc}
2&2\\
&2
\end{array}}
$$

$$
+
\underset{S}{\begin{array}{ccccccccc}
0&0&2&2&1&1&2&1&1\\
&0&2&1&1&1&2&1&1\\
&&1&1&1&0&2&1&1\\
&&&1&0&0&2&1&0\\
&&&&0&0&1&1&0\\
&&&&&0&1&1&0\\
&&&&&&1&1&0\\
&&&&&&&0&0\\
&&&&&&&&0
\end{array}}
+
\underset{\mu^{(2)}}{\begin{array}{cccccccccc}
0&0&0&0&0&0&0&0&0&1\\
&0&0&0&0&0&0&0&0&1\\
&&0&0&0&0&0&0&0&1\\
&&&0&0&0&0&0&0&0\\
&&&&0&0&0&0&0&0\\
&&&&&0&0&0&0&0\\
&&&&&&0&0&0&0\\
&&&&&&&0&0&0\\
&&&&&&&&0&0\\
&&&&&&&&&0
\end{array}}
$$

Let us write $\overline{S}(N_1, N_2, \ldots, N_{k-1})$ for the set of triangular arrays possessing the above four properties. We proceed to compute the generating function for the triangular arrays in $\overline{S}(N_1, N_2, \ldots, N_{k-1})$. We may partition a triangular array $S \in \overline{S}(N_1, N_2, \ldots, N_{k-1})$ into $k-1$ blocks of columns, where the i-th block starting from the $(N_{i+1}+1)$-th column and ending with the N_i-th column of S. We denote the i-th block by S_i. According to the above four properties, we infer that the first m_i diagonal entries of S_i must be 1, and for $i = 1, \ldots, k-2$, the entries in the first m_i columns of S_i are either 1 or 2. Moreover, S_{k-1} is a triangular array of size N_{k-1} with all entries equal to 0.

We continue to split S_i into three trapezoidal arrays $S_i^{(1)}$, $S_i^{(2)}$ and $S_i^{(3)}$ for $i = 1, \ldots, k-2$. First, we may form a trapezoidal array $S_i^{(1)}$ of the same size as S_i and with the entries in the first m_i columns equal to 1 and the other entries equal to 0. Let $S_i^{'}$ denote the trapezoidal array obtained from S_i by subtracting 1 from every entry in the first m_i columns. It is easily seen that every entry in $S_i^{'}$ is either 1 or 0, and that $S_i^{(1)}$ can be regarded as the Ferrers diagram of the conjugate of the partition

$$\alpha^{(1)} = (N_{i+1} + m_i, N_{i+1} + m_i - 1, \ldots, N_{i+1} + 1).$$

Furthermore, we see that for $i = 1, \ldots, k-2$, the trapezoidal array $S_i^{'}$ possesses the following properties:

(1) All the entries in $S_i^{'}$ equal 0 or 1, but the diagonal entries must be 0.

(2) The entries in the j-th column must be non-increasing.

(3) The first m_i entries in the j-th row are non-increasing, and the remaining entries in the j-th row are also non-increasing.

We now consider the trapezoidal array formed by the first m_i columns of $S_i^{'}$, and denote it by $S_i^{(2)}$. Again, we see that $S_i^{(2)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(2)}$, where

$$\alpha_1^{(2)} \leq N_{i+1} \quad \text{and} \quad l(\alpha^{(2)}) \leq m_i.$$

Notice that there are still some columns left. Define $S_i^{(3)}$ to be the trapezoidal array as a block of $S_i^{'}$ starting with the (m_i+1)-th column and ending with the $(N_i - N_{i+1})$-th column. Once more, $S_i^{(3)}$ can be regarded as the Ferrers diagram of the conjugate of a partition $\alpha^{(3)}$, where

$$\alpha_1^{(3)} \leq N_{i+1} + m_i \quad \text{and} \quad l(\alpha^{(3)}) \leq N_i - N_{i+1} - m_i.$$

Based on the above analysis, for $i = 1, \ldots, k-2$, the generating function for possible choices of the i-th block S_i equals

$$\sum_{m_i=0}^{N_i - N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2} \frac{(q)_{N_{i+1}+m_i}}{(q)_{m_i}(q)_{N_{i+1}}} \frac{(q)_{N_i}}{(q)_{N_{i+1}+m_i}(q)_{N_i-N_{i+1}-m_i}}$$

which can be rewritten as

$$\frac{(q)_{N_i}}{(q)_{N_{i+1}}(q)_{N_i-N_{i+1}}} \sum_{m_i=0}^{N_i - N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2} \frac{(q)_{N_i-N_{i+1}}}{(q)_{m_i}(q)_{N_i-N_{i+1}-m_i}}.$$

Evidently, the sum in the above expression is the generating function for partitions with distinct parts between $N_{i+1}+1$ and N_i. So we deduce that

$$\sum_{m_i=0}^{N_i-N_{i+1}} q^{m_i(2N_{i+1}+1+m_i)/2}\frac{(q)_{N_i-N_{i+1}}}{(q)_{m_i}(q)_{N_i-N_{i+1}-m_i}} = (-q^{N_{i+1}+1})_{N_i-N_{i+1}}.$$

Since the generating function for S_{k-1} equals 1, the generating function for possible choices of S is the product of the generating functions for S_i for $i=1,\ldots,k-2$, that is,

$$\prod_{i=1}^{k-2}\frac{(q)_{N_i}}{(q)_{N_{i+1}}(q)_{N_i-N_{i+1}}}(-q^{N_{i+1}+1})_{N_i-N_{i+1}} = \frac{(q)_{N_1}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}.$$

Recall that the generating function for possible choices of $T(\lambda^{(2)})$ equals $1/(q;q)_{N_1}$ and the generating functions for $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ are equal to $q^{(N_1+1)N_1/2}$, $q^{N_2^2+N_2},\ldots$ $,q^{N_{k-1}^2+N_{k-1}}$ respectively. We also observe that the generating function for anti-lecture hall compositions in $F_{2k-2}(N_1,\ldots,N_{k-1},n)$ is the product of the generating functions for $T(\lambda^{(2)})$, $R(N_1,1)$, $R(N_2,2),\ldots,R(N_{k-1},2)$ and S. Hence it equals

$$\begin{aligned}&\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}}{(q)_{N_1}}\frac{(q)_{N_1}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}\\ =\quad&\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}.\end{aligned}$$

Summing up the generating functions for anti-lecture hall compositions in $F_{2k-3}(N_1,\ldots,N_{k-1},n)$ yields the generating function for $F_{2k-3}(n)$,

$$\sum_{n\geq 0}|F_{2k-3}(n)|q^n = \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{(N_1+1)N_1/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}(-q)_{N_{k-1}}}. \tag{4.25}$$

This completes the proof. ∎

In virtue of (2.12), Theorem 4.1 immediately follows from Theorem 4.2, since the generating function for overpartitions in $H_{2k-1}(n)$ is given by

$$\frac{(-q)_\infty(q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty}{(q)_\infty}. \tag{4.26}$$

We now turn to the second proof of Theorem 4.1. In their proof of the anti-lecture hall theorem, Corteel and Savage [6] established two bijections. The first is a bijection between the set $E(n)$ of anti-lecture hall compositions μ of n such that $\lfloor \mu_i/i\rfloor$ is even and the set $P(n)$ of partitions of n with each part greater than one. The second bijection is between the set $A(n)$ of anti-lecture hall compositions of n and the set $D\times E(n)$ of pairs (λ,μ) such that $|\lambda|+|\mu|=n$ and $\lambda\in D$, $\mu\in E$, where D is the set of partitions into distinct parts. Then the anti-lecture hall theorem is a consequence of the correspondence between $A(n)$ and $D\times P(n)$.

To give a proof of Theorem 4.1, we shall present a bijection between a subset of $P(n)$ and a subset of $E(n)$. To be more specific, let $Q_k(n)$ be the subset of $E(n)$ consisting of anti-lecture hall compositions λ such that $\lambda_1\leq k$ and let $R_k(n)$ be the subset of $P(n)$ consisting of partitions having at most $k-1$ successive $N \times (N+1)$ Durfee rectangles such that there is no part below the last Durfee rectangle. Then we have the following correspondence, which can be considered as a refined version of the first bijection of Corteel and Savage.

Theorem 4.3 *There is a bijection between the set $R_k(n)$ and the set $Q_{2k-2}(n)$.*

Proof. We proceed to give a bijection θ from $R_k(n)$ to $Q_{2k-2}(n)$. Consider the A-triangular representation $T(\mu)$ of an anti-lecture hall composition μ of n such that $\lfloor \frac{\mu_i}{i} \rfloor$ are even for all i and $\mu_1 \leq 2k-2$. By definition, we have $t_{1,1} \leq 2k-2$ and all the diagonal entries of $T(\mu)$ are even.

Now we define the map θ from a partition λ in P with exactly $k-1$ successive Durfee rectangles to an anti-lecture hall composition μ of n.

Step 1. We break the Ferrers diagram of λ into $k-1$ blocks such that the i-th block contains the i-th Durfee rectangle and the dots on the right of the i-th Durfee rectangle.

Step 2. Change the i-th Durfee rectangle in the i-th block into a triangular array with all entries equal to 2, and the rest of the dots in the i-th block into entries equal to 1. So these $k-1$ blocks become $k-1$ A-triangles with all the diagonal entries equal to 0 or 2.

Step 3. Put the $k-1$ A-triangles obtained in Step 2 together to form an A-triangle T.

The resulting A-triangle corresponds to an anti-lecture hall composition μ such that $\mu_1 = 2k-2$ and $\lfloor \lambda_i / i \rfloor$ are even for all i. It is easy to check that the map θ is reversible. This completes the proof. ∎

For example, let

$$\lambda = (10, 10, 9, 8, 7, 7, 7, 7, 5, 4, 3)$$

be a partition in $R_4(77)$. Then the corresponding anti-lecture hall composition in $Q_6(77)$ equals

$$\mu = (6, 12, 13, 11, 12, 14, 4, 3, 2).$$

The successive Durfee rectangles of λ are given below, where the zeros in the arrays are omitted.

$$\begin{array}{cccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & & & & & \\
\circ & \circ & \circ & \circ & & & & & & \\
\circ & \circ & \circ & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
 & & 2 & 2 & 2 & 2 & 1 & 1 & \\
 & & & 2 & 2 & 2 & 1 & & \\
 & & & & 2 & 2 & & & \\
 & & & & & 2 & & & \\
 & & & & & & & & \\
2 & 2 & 2 & 1 & 1 & 1 & & & \\
 & 2 & 2 & 1 & 1 & 1 & & & \\
 & & 2 & 1 & & & & & \\
 & & & & & & & & \\
2 & 2 & 1 & & & & & & \\
 & 2 & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccccccc}
6 & 6 & 5 & 3 & 3 & 3 & 1 & 1 & 1 \\
 & 6 & 4 & 3 & 3 & 3 & 1 & 1 & 1 \\
 & & 4 & 3 & 2 & 2 & 1 & 1 & \\
 & & & 2 & 2 & 2 & 1 & & \\
 & & & & 2 & 2 & & & \\
 & & & & & 2 & & &
\end{array}$$

Second Proof of Theorem 4.1. Examining Corteel and Savage's second bijection γ from A to $D \times E$, we see that it maps an anti-lecture hall composition λ of n in A to a pair (α, β) in

$D \times E$. If λ_1 is odd, then $\beta_1 = \lambda_1 - 1$; otherwise $\beta_1 = \lambda_1$. So it can be checked that γ maps an anti-lecture hall composition of n in A with the first part not exceeding $2k-1$ to a pair (α, β) in $D \times E$ such that β is an anti-lecture hall composition in E with the first part β_1 not exceeding $2k-2$ and the sum of parts of α and β equals n. On the other hand if $\lambda = \gamma^{-1}(\alpha, \beta)$, then it can be checked that $\lambda_1 \leq \beta_1 + 1$. In other words, γ is a bijection between F_{2k-1} and $D \times Q_{2k-2}$. Together with Theorem 4.3, we are led to a bijection between F_{2k-1} and $D \times R_k$.

Recall that there is a combinatorial interpretation of the left-hand side of (1.5) in terms of the Durfee dissection, or the Durfee square dissection, to be precise, of a partition, as given by Andrews [3], see also, Andrews and Eriksson [4]. We observe that the idea of Andrews easily extends to the Durfee rectangle dissection of a partition. In this way, we find that the generating function of partitions in $R_k(n)$ is given by

$$\sum_{n=0}^{\infty} |R_k(n)| q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_1 + \cdots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}} (q)_{N_{k-1}}}. \tag{4.27}$$

Setting $a = 1$ in the generalization of the Rogers–Ramanujan identity (1.5) gives

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_1 + \cdots + N_{k-1}}}{(q)_{N_1 - N_2} \ldots (q)_{N_{k-2} - N_{k-1}} (q)_{N_{k-1}}} = \frac{(q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}.$$

Hence the generating function of partitions in $R_k(n)$ can be expressed as follows

$$\sum_{n=0}^{\infty} |R_k(n)| q^n = \frac{(q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{4.28}$$

By the bijection between $F_{2k-1}(n)$ and $D \times R_k(n)$ we conclude that

$$\sum_{n=0}^{\infty} |F_{2k-1}(n)| q^n = \frac{(-q)_\infty (q, q^{2k}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{4.29}$$

It is easy to see that the right-hand side of the above identity is the generating function of overpartitions in $H_{2k+1}(n)$. This completes the proof. ∎

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