Congruences for Bipartitions with Odd Parts Distinct

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Abstract. Hirschhorn and Sellers studied arithmetic properties of the number of partitions with odd parts distinct. In another direction, Hammond and Lewis investigated arithmetic properties of the number of bipartitions. In this paper, we consider the number of bipartitions with odd parts distinct. Let this number be denoted by $pod_{-2}(n)$. We obtain two Ramanujan type identities for $pod_{-2}(n)$, which imply that $pod_{-2}(2n+1)$ is even and $pod_{-2}(3n+2)$ is divisible by 3. Furthermore, we show that for any $\alpha \geq 1$ and $n \geq 0$, $pod_{-2}\left(3^{2\alpha+1}n + \frac{23\times 3^{2\alpha}-7}{8}\right)$ is a multiple of 3 and $pod_{-2}\left(5^{\alpha+1}n + \frac{11\times 5^{\alpha}+1}{4}\right)$ is divisible by 5. We also find combinatorial interpretations for the two congruences modulo 2 and 3.

Keywords: partition, bipartition, congruence, birank

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1 Introduction

A partition λ of a positive integer n is any non-increasing sequence of positive integers whose sum is n. The weight of λ is the sum of its parts, denoted by $|\lambda|$. A bipartition π of n is a pair of partitions (π_1, π_2) with $|\pi_1| + |\pi_2| = n$. Let $p_{-2}(n)$ denote the number of bipartitions of n. The generating function for $p_{-2}(n)$ equals

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q;q)_\infty^2}.$$

In this paper, we shall employ the standard q-series notation [1]

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{for } n \geq 1,$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty = \lim_{n\to\infty} \prod_{j=1}^{m} (a_j;q)_n, \quad |q| < 1.$$

The function $p_{-2}(n)$ has drawn much interest, see, for example, [3, 9, 11–13, 16]. Ramanathan [16] established the following congruences:

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

which are analogous to the classical congruences of Ramanujan, namely,

$$p(5n+4) \equiv 0 \pmod 5 \tag{1.2}$$

and

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$

where $p(n)$ is the number of partitions of n.

Dyson [10] defined the rank of a partition as the largest part minus the number of parts. Let $N(r,t,n)$ denote the number of partitions of n whose rank is congruent to r modulo t. Aktin and Swinnerton-Dyer [4] proved the following conjecture of Dyson [10]

$$N(r,5,5n+4) = \frac{p(5n+4)}{5} \quad 0 \le r \le 4,$$

and

$$N(r,7,7n+5) = \frac{p(7n+5)}{7} \quad 0 \le r \le 6.$$

For a bipartition $\pi = (\pi_1, \pi_2)$, Hanmmond and Lewis [13] defined the birank $b(\pi)$ as

$$b(\pi) = n(\pi_1) - n(\pi_2), \tag{1.4}$$

where $n(\lambda)$ denotes the number of parts of λ. It has been shown that the birank $b(\pi)$ can be used to give combinatorial interpretations of the congruences in (1.1). Recently, Garvan [11] defined two biranks. One can be utilized to explain all three congruences in (1.1), while the other is valid for two of the three congruences.

We wish to consider bipartitions with odd parts distinct. Recall that Andrews, Hirschhorn and Sellers [2] have investigated arithmetic properties of partitions with even parts distinct. Hirschhorn and Sellers [14] considered arithmetic properties of partitions with odd parts distinct. To be precise, by a bipartition with odd parts distinct we mean a bipartition $\pi = (\pi_1, \pi_2)$ for which the odd parts of π_1 are distinct and the odd parts of π_2 are also distinct. Notice that π_1 and π_2 are allowed to have an odd part in common. For example, there are 11 bipartitions of 4:

$$((4),\emptyset)\ ((3,1),\emptyset)\ ((2,2),\emptyset)\ ((3),(1))\ ((2,1),(1))\ ((2),(2))$$
$$((1),(2,1))\ ((1),(3))\ (\emptyset,(2,2))\ (\emptyset,(3,1))\ (\emptyset,(4)).$$

Let $pod_{-2}(n)$ denote the number of bipartitions of n with odd parts distinct. It is easy to derive the generating function for $pod_{-2}(n)$, that is,

$$\sum_{n=0}^{\infty} pod_{-2}(n)q^n = \frac{(-q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{1.5}$$

The main objective of this paper is to study arithmetic properties of $pod_{-2}(n)$ in the spirit of Ramanujan's congruences for the partition function $p(n)$. We shall prove that

$$\sum_{n=0}^{\infty} pod_{-2}(2n+1)q^n = \frac{2(q^8;q^8)_\infty^2}{(q;q)_\infty^3(q^4;q^4)_\infty} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} pod_{-2}(3n+2)q^n = 3\frac{(q^2;q^2)_\infty^4(q^6;q^6)_\infty^6}{(q;q)_\infty^6(q^4;q^4)_\infty^6}, \tag{1.7}$$

which implies that for all $n \geq 0$,

$$pod_{-2}(2n+1) \equiv 0 \pmod{2} \tag{1.8}$$

and

$$pod_{-2}(3n+2) \equiv 0 \pmod{3}. \tag{1.9}$$

We also give three infinite families of congruences modulo 3 and two infinite families of congruences modulo 5. For example, for $\alpha \geq 1$ and $n \geq 0$,

$$pod_{-2}\left(3^{2\alpha+1}n + \frac{23 \times 3^{2\alpha} - 7}{8}\right) \equiv 0 \pmod{3} \tag{1.10}$$

and

$$pod_{-2}\left(5^{\alpha+1}n + \frac{11 \times 5^{\alpha} + 1}{4}\right) \equiv 0 \pmod{5}. \tag{1.11}$$

Furthermore, we show that the birank $b(\pi)$ defined by Hammond and Lewis can be used to explain the congruence (1.9). Furthermore, we introduce another birank to give a combinatorial explanation of (1.9). Our birank $c(\pi)$ of a bipartition $\pi = (\pi_1, \pi_2)$ is defined by

$$c(\pi) = l(\pi_1) - l(\pi_2), \tag{1.12}$$

where $l(\lambda)$ denotes the largest part of λ. It is worth mentioning that neither of the two biranks $b(\pi)$ and $c(\pi)$ leads to a combinatorial interpretation of the congruence (1.8). It should be noted that the birank $c(\pi)$ is not the conjugate of $b(\pi)$ for bipartitions with odd parts distinct because the conjugation of such a bipartition no longer preserves this property.

This paper is organized as follows. In Section 2, two identities of Ramanujan type are obtained. In Section 3, three infinite families of congruences modulo 3 for $pod_{-2}(n)$ are established. In section 4, we obtain two infinite families of congruences modulo 5 for $pod_{-2}(n)$. In Section 5, we prove that both biranks $b(\pi)$ and $c(\pi)$ can be applied to give a combinatorial interpretation of the fact that $pod_{-2}(3n+2)$ is a multiple of 3. We also give a simple combinatorial explanation of the fact that $pod_{-2}(2n+1)$ is even for any n.

2 Two Ramanujan-type identities

In this section, we shall prove the following two Ramanujan-type identities for the number of bipartitions with odd parts distinct.

Theorem 2.1. *We have*

$$\sum_{n=0}^{\infty} pod_{-2}(2n+1)q^n = \frac{2(q^8;q^8)_\infty^2}{(q;q)_\infty^3 (q^4;q^4)_\infty}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} pod_{-2}(3n+2)q^n = 3\frac{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^6}{(q;q)_\infty^6 (q^4;q^4)_\infty^6}. \tag{2.2}$$

We need some properties of the function $\psi(q)$, namely,

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.3}$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}, \quad |ab| < 1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows

$$f(a,b) = (-a, ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \tag{2.4}$$

Thus,

$$\psi(-q) = f(-q, -q^3) = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty}. \tag{2.5}$$

Combining (1.5) and (2.5), we obtain that

$$\sum_{n=0}^{\infty} pod_{-2}(n) q^n = \frac{1}{\psi(-q)^2}. \tag{2.6}$$

It should be noted that Bringmann and Lovejoy [7] have studied arithmetic properties of the numbers $\overline{pp}(n)$, which are the coefficients of q^n in $1/\varphi(-q)^2$, namely,

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{1}{\varphi(-q)^2},$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{2.7}$$

Lemma 2.1. *We have*

$$\begin{aligned} \frac{1}{\psi(-q)} &= \frac{1}{(q^2; q^2)_\infty (q^4; q^4)_\infty} \left(f(q^6, q^{10}) + q f(q^2, q^{14}) \right) &(2.8)\\ &= \frac{\psi(-q^9)}{\psi(-q^3)^4} \left(A(-q^3)^2 + q A(-q^3) \psi(-q^9) + q^2 \psi(-q^9)^2 \right), &(2.9) \end{aligned}$$

where

$$A(q) = \frac{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (q^6; q^6)_\infty}.$$

Proof. It is easily checked that

$$\psi(q)\psi(-q) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \cdot \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \frac{(q^2; q^2)_\infty^2}{(q^2; q^4)_\infty} = (q^2; q^2)_\infty (q^4; q^4)_\infty. \tag{2.10}$$

From [5, Corollary (ii), p.49], it follows that

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}). \tag{2.11}$$

Dividing (2.11) by (2.10), we are led to the 2-dissection (2.8) of $1/\psi(-q)$. The proof of (2.9) is a little more involved; See [14, Lemma 2.2] for the details. ∎

In view of the above lemma, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. By the 2-dissection (2.8) of $1/\psi(-q)$ and the generating function (2.6) for $pod_{-2}(n)$, we see that

$$\sum_{n=0}^{\infty} pod_{-2}(n)q^n = \frac{1}{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2} \left(f(q^6, q^{10}) + qf(q^2, q^{14})\right)^2.$$

Considering the coefficients of q^{2n+1} on both sides, we observe that

$$\sum_{n=0}^{\infty} pod_{-2}(2n+1)q^n = \frac{2}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} f(q^3, q^5) f(q, q^7).$$

Consequently, we get (2.1), since

$$\begin{aligned} f(q^3, q^5) f(q, q^7) &= (-q, -q^3, -q^5, -q^7; q^8)_\infty (q^8;q^8)_\infty^2 \\ &= (-q;q^2)_\infty (q^8;q^8)_\infty^2 \\ &= \frac{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty}. \end{aligned}$$

This completes the proof of (2.1).

By the 3-dissection (2.9) of $1/\psi(-q)$, we find that

$$\sum_{n=0}^{\infty} pod_{-2}(n)q^n = \frac{\psi(-q^9)^2}{\psi(-q^3)^8} \left(A(-q^3)^2 + qA(-q^3)\psi(-q^9) + q^2\psi(-q^9)^2\right)^2.$$

Extracting the terms q^{3n+2} on both sides, we obtain

$$\sum_{n=0}^{\infty} pod_{-2}(3n+2)q^{3n+2} = 3q^2 \frac{\psi(-q^9)^2}{\psi(-q^3)^8} A(-q^3)^2 \psi(-q^9)^2.$$

By dividing both sides by q^2 and replacing q^3 by q, arrive at (2.2). This completes the proof. ∎

As consequences of Theorem 2.1, we obtain the following congruences.

Corollary 2.1. *For each nonnegative integer n,*

$$pod_{-2}(2n+1) \equiv 0 \pmod 2 \quad \textit{and} \quad pod_{-2}(3n+2) \equiv 0 \pmod 3.$$

3 Three infinite families of congruences modulo 3

In this section, we wish to establish the following three infinite families of Ramanujan-like congruences modulo 3 satisfied by $pod_{-2}(n)$ by two different approaches. The proof of Theorem 3.1 needs the formula for the number of ways to represent an integer n as a sum of two triangular numbers as well as a characterization of integers that cannot be written as a sum of two squares. On the other hand, Theorem 3.2 follows from the generating function for the numbers $pod_{-2}(3n+1)$. For notational convenience, we assume that all the congruences in this section are modulo 3.

Theorem 3.1. *For all* $\alpha \geq 0$ *and* $n \geq 0$,

$$pod_{-2}\left(3^{2\alpha+1}n + \frac{23 \times 3^{2\alpha} - 7}{8}\right) \equiv 0 \pmod 3. \tag{3.1}$$

Theorem 3.2. *For all* $\alpha \geq 1$ *and* $n \geq 0$,

$$pod_{-2}\left(3^{2\alpha+1}n + \frac{7 \times 3^{2\alpha} + 1}{4}\right) \equiv 0 \pmod 3 \tag{3.2}$$

and

$$pod_{-2}\left(3^{2\alpha+1}n + \frac{11 \times 3^{2\alpha} + 1}{4}\right) \equiv 0 \pmod 3. \tag{3.3}$$

To prove the above congruences, the following lemma is useful.

Lemma 3.1.

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{3.4}$$

$$\psi(q^3) \equiv \psi(q)^3. \tag{3.5}$$

Proof. From [5, Corollary (ii), p.49] it is clear that the identity (3.4) holds. Since

$$(1-q^n)^3 \equiv (1-q^{3n}) \pmod 3$$

and

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \frac{\prod_{n\geq 1}(1-q^{2n})^2}{\prod_{n\geq 1}(1-q^n)},$$

we obtain (3.5). This completes the proof. ∎

Proof of Theorem 3.1. By Lemma 3.1, we have

$$\sum_{n=0}^{\infty} pod_{-2}(n)(-q)^n = \frac{\psi(q)}{\psi(q)^3} \equiv \frac{f(q^3,q^6) + q\psi(q^9)}{\psi(q^3)}.$$

Extracting the terms $q, q^4, q^7, \ldots$ on both sides of the above identity, dividing by q, and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} (-1)^{n+1} pod_{-2}(3n+1)q^n \equiv \frac{\psi(q^3)}{\psi(q)} \equiv \psi(q)^2. \tag{3.6}$$

Let the numbers $t_2(n)$ be defined by

$$\psi(q)^2 = \sum_{n=0}^{\infty} t_2(n) q^n.$$

By comparing the coefficients of q^n on both sides of (3.6), we find that for each $n \geq 0$,

$$pod_{-2}(3n+1) \equiv (-1)^{n+1} t_2(n). \tag{3.7}$$

From [6, Theorem 3.6.2], it follows that for all integers $n \geq 0$,

$$t_2(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1),$$

where $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \pmod{k}$. Moreover, by [15, Theorem 2.15], we have that $d_{1,4}(n) - d_{3,4}(n) = 0$ if and only if there exists a prime p congruent to 3 modulo 4 in the canonical factorization of n appears with an odd exponent.

It is clear that for $\alpha \geq 1$ and $n \geq 0$, the integer $s = 4 \times 3^{2\alpha} n + \frac{23 \times 3^{2\alpha-1} - 3}{2}$ is a multiple of 3 but not divisible by 9. This implies that

$$t_2\left(\frac{s-1}{4}\right) = d_{1,4}(s) - d_{3,4}(s) = 0.$$

Substituting $n = \frac{s-1}{4}$ into (3.7), we obtain that

$$pod_{-2}\left(3^{2\alpha+1} n + \frac{23 \times 3^{2\alpha} - 7}{8}\right) \equiv 0 \pmod{3}.$$

The case $\alpha = 0$ has been considered in Corollary 2.1. This completes the proof. ∎

Proof of Theorem 3.2. Invoking the identity (3.6) in the proof of Theorem 3.1, we deduce that

$$\sum_{n=0}^{\infty} pod_{-2}(3n+1) q^n \equiv -\psi(-q)^2. \tag{3.8}$$

Applying (2.9) and (3.5) to (3.8), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} pod_{-2}(3n+1) q^n &= -\frac{\psi(-q)^3}{\psi(-q)} \equiv -\frac{\psi(-q^3)}{\psi(-q)} \\ &\equiv -\frac{\psi(-q^9)}{\psi(-q^3)^3} \left(A(-q^3)^2 + qA(-q^3)\psi(-q^9) + q^2 \psi(-q^9)^2\right). \end{aligned}$$

Extracting the terms q^{3n+2} for $n \geq 0$, we find that

$$\sum_{n=0}^{\infty} pod_{-2}(9n+7) q^{3n+2} \equiv -q^2 \frac{\psi(-q^9)^3}{\psi(-q^3)^3}.$$

Dividing both sides of the above identity by q^2 and replacing q^3 by q, we see that

$$\sum_{n=0}^{\infty} pod_{-2}(9n+7)q^n \equiv -\frac{\psi(-q^3)^3}{\psi(-q)^3} \equiv -\psi(-q^3)^2.$$

Similarly, it can be shown that

$$\sum_{n=0}^{\infty} pod_{-2}(27n+7)q^n \equiv -\psi(-q)^2 \tag{3.9}$$

and for $n \geq 0$,

$$pod_{-2}(27n+16) \equiv pod_{-2}(27n+25) \equiv 0.$$

So the proof is complete for the case $\alpha = 1$. Combining (3.8) and (3.9), it can be seen that for $n \geq 0$,

$$pod_{-2}(3n+1) \equiv pod_{-2}(27n+7). \tag{3.10}$$

By induction on α, it is easy to establish the congruences (3.2) and (3.3) based on the relation (3.10). ∎

4 Two infinite families of congruences modulo 5

In this section, we give two infinite families of Ramanujan-like congruences modulo 5 satisfied by $pod_{-2}(n)$ from a modular equation of degree 5 due to Ramanujan. For notational convenience, we assume that all the congruences in this section are modulo 5.

Theorem 4.1. *For all $\alpha \geq 1$ and $n \geq 0$,*

$$pod_{-2}\left(5^{\alpha+1}n + \frac{11 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod 5 \tag{4.1}$$

and

$$pod_{-2}\left(5^{\alpha+1}n + \frac{19 \times 5^{\alpha}+1}{4}\right) \equiv 0 \pmod 5. \tag{4.2}$$

To prove the above congruences, we need the following lemma.

Lemma 4.1. *Let $1 \leq r \leq 4$. Let the numbers $a(n)$ be given by*

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{n=0}^{\infty} \frac{q^{5n+r}}{1-q^{10n+2r}}.$$

Then

$$\sum_{n=0}^{\infty} a(5n)q^n = \sum_{n=0}^{\infty} \frac{q^{5n+r}}{1-q^{10n+2r}}.$$

Proof. Clearly,

$$\sum_{n=0}^{\infty} \frac{q^{5n+r}}{1-q^{10n+2r}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{(5n+r)(2k+1)}. \tag{4.3}$$

Since for $1 \le r \le 4$ and $k \ge 0$, $(5n+r)(2k+1)$ is a multiple of 5 if and only if $k \equiv 2 \pmod 5$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a(5n)q^{5n} &= \sum_{n=0}^{\infty} \sum_{k\equiv 2 \pmod 5} q^{(5n+r)(2k+1)} \\ &= \sum_{n=0}^{\infty}\sum_{t=0}^{\infty} q^{(5n+r)(10t+5)}.\end{aligned}$$

Replacing q^5 by q and using (4.3), we complete the proof. ∎

Proof of Theorem 4.1. It is easy to deduce the following relation

$$\psi(q^5) \equiv \psi(q)^5. \tag{4.4}$$

From the generating function (2.6) for $pod_{-2}(n)$ and (4.4) it follows that

$$\begin{aligned} q\sum_{n=0}^{\infty} pod_{-2}(n)(-q)^n &= \frac{q}{\psi(q)^2} \equiv \frac{q\psi(q)^3\psi(q^5)}{\psi(q^5)^2} \\ &\equiv \frac{q\psi(q)^3\psi(q^5) - 5q^2\psi(q)\psi(q^5)^3}{\psi(q^5)^2}. \end{aligned} \tag{4.5}$$

From [5, Entry 8(i), p.249], we see that

$$\begin{aligned} q\psi(q)^3\psi(q^5) - 5q^2\psi(q)\psi(q^5)^3 &= \sum_{n=0}^{\infty} \frac{(5n+1)q^{5n+1}}{1-q^{10n+2}} - \sum_{n=0}^{\infty} \frac{(5n+2)q^{5n+2}}{1-q^{10n+4}} \\ &\quad - \sum_{n=0}^{\infty} \frac{(5n+3)q^{5n+3}}{1-q^{10n+6}} + \sum_{n=0}^{\infty} \frac{(5n+4)q^{5n+4}}{1-q^{10n+8}}. \end{aligned}$$

This implies that

$$\begin{aligned} q\psi(q)^3\psi(q^5) - 5q^2\psi(q)\psi(q^5)^3 &\equiv \sum_{n=0}^{\infty} \frac{q^{5n+1}}{1-q^{10n+2}} - \sum_{n=0}^{\infty} \frac{2q^{5n+2}}{1-q^{10n+4}} \\ &\quad - \sum_{n=0}^{\infty} \frac{3q^{5n+3}}{1-q^{10n+6}} + \sum_{n=0}^{\infty} \frac{4q^{5n+4}}{1-q^{10n+8}}. \end{aligned}$$

Write the above series modulo 5 as

$$\sum_{n=0}^{\infty} A(n)q^n.$$

Applying Lemma 4.1 yields

$$\sum_{n=0}^{\infty} A(5n)q^n \equiv q\psi(q)^3\psi(q^5) - 5q^2\psi(q)\psi(q^5)^3.$$

Extracting the terms q^{5n} from (4.5) and replacing q^5 by q, we have

$$-\sum_{n=0}^{\infty} pod_{-2}(5n+4)(-q)^{n+1} \equiv \frac{\sum\limits_{n=0}^{\infty} A(5n)q^n}{\psi(q)^2}.$$

Combining the above two equations, we find that

$$\begin{aligned}
-\sum_{n=1}^{\infty} pod_{-2}(5n-1)(-q)^n &\equiv \frac{q\psi(q)^3\psi(q^5) - 5q^2\psi(q)\psi(q^5)^3}{\psi(q)^2} \\
&\equiv \frac{q\psi(q)^3\psi(q^5)}{\psi(q)^2} \\
&\equiv q\psi(q)\psi(q^5) \qquad (4.6)\\
&= \psi(q^5)\left(qf(q^{10},q^{15}) + q^2 f(q^5,q^{20}) + q^4\psi(q^{25})\right). \qquad (4.7)
\end{aligned}$$

Note that the last equation follows from [5, Corollary (ii), p.49]. Comparing coefficients of $q^{5n+a}(a=0,3,4)$ in (4.7), we see that for $n \geq 0$,

$$pod_{-2}(25n+14) \equiv pod_{-2}(25n+24) \equiv 0 \qquad (4.8)$$

and

$$\sum_{n=0}^{\infty} pod_{-2}(25n+19)(-q)^{n+1} \equiv q\psi(q)\psi(q^5).$$

In view of the above identity and (4.6), we deduce that

$$\sum_{n=0}^{\infty} pod_{-2}(25n+19)(-q)^{n+1} \equiv -\sum_{n=0}^{\infty} pod_{-2}(5n+4)(-q)^{n+1},$$

which implies that for $n \geq 0$,

$$pod_{-2}(25n+19) \equiv -pod_{-2}(5n+4).$$

Using the above relation and (4.8), it is easily checked by induction that for $\alpha \geq 1$,

$$pod_{-2}\left(5^{\alpha+1}n + \frac{11\times 5^\alpha + 1}{4}\right) \equiv pod_{-2}\left(5^{\alpha+1}n + \frac{19\times 5^\alpha + 1}{4}\right) \equiv 0.$$

This completes the proof. ∎

It should be noted that Chan [8] has used modular forms to establish infinite families of congruences modulo 5 for Andrews-Paule's broken 2-diamond partitions. His approach can also be used to prove the congruences in this section.

5 Combinatorial interpretations

In this section, we show that both the biranks $b(\pi)$ and $c(\pi)$ can be used to give a combinatorial interpretation of the fact that $pod_{-2}(3n+2)$ is divisible by 3. We conclude this paper with a simple explanation of the parity of $pod_{-2}(2n+1)$.

Let $R(m,n)$ denote the number of bipartitions π of n with odd parts distinct such that birank $b(\pi)=m$. By using the transformation that interchanges π_1 and π_2 in (1.4), we see that

$$R(m,n)=R(-m,n). \tag{5.1}$$

Let $R(r,t,n)$ be the number of bipartitions π of n with odd parts distinct such that birank $b(\pi)$ is congruent to r modulo t, i.e.,

$$R(r,t,n)=\sum_{m\equiv r \pmod{t}} R(m,n).$$

Then we have $R(r,t,n)=R(t-r,t,n)$. Moreover, it is easy to derive the following generating function for $R(m,n)$,

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R(m,n)z^m q^n=\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty}{(q^2z;q^2)_\infty(q^2/z;q^2)_\infty}. \tag{5.2}$$

The above formula enables us to obtain generating functions for the numbers of the form $R(r,t,n)-R(s,t,n)$.

Theorem 5.1.

$$\sum_{n=0}^{\infty}\left(R(0,2,n)-R(1,2,n)\right)q^n = \frac{\varphi(-q)}{\psi(q^2)}, \tag{5.3}$$

$$\sum_{n=0}^{\infty}\left(R(0,3,n)-R(1,3,n)\right)q^n = \frac{\psi(-q)}{\psi(-q^3)}, \tag{5.4}$$

$$\sum_{n=0}\left(R(0,4,n)-R(2,4,n)\right)q^n = \frac{\varphi(q^2)}{\psi(q^2)}. \tag{5.5}$$

Proof. Taking $z=-1$ in the generating function (5.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(R(0,2,n)-R(1,2,n)\right)q^n &= \frac{(q;q^2)_\infty^2}{(-q^2;q^2)_\infty^2}=\frac{(q;q)_\infty^2}{(q^4;q^4)_\infty^2}\\ &= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\times\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\\ &= \frac{\varphi(-q)}{\psi(q^2)}.\end{aligned}$$

Note that the last equation holds since

$$\varphi(q)=f(q,q)=(-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

Substituting $z=\xi=e^{2\pi i/3}$ into both sides of (5.2) and applying the relation $R(1,3,n)=$

$R(2,3,n)$, we find that

$$\begin{aligned}\frac{(-q\xi;q^2)_\infty(-q\xi^2;q^2)_\infty}{(q^2\xi;q^2)_\infty(q^2\xi^2;q^2)_\infty} &= \sum_{n=0}^{\infty}\left(R(0,3,n)+R(1,3,n)\xi+R(2,3,n)\xi^2\right)q^n \\ &= \sum_{n=0}^{\infty}\left(R(0,3,n)-R(1,3,n)\right)q^n.\end{aligned}$$

Since $1-x^3=(1-x)(1-x\xi)(1-x\xi^2)$, we see that

$$\frac{(-q\xi;q^2)_\infty(-q\xi^2;q^2)_\infty}{(q^2\xi;q^2)_\infty(q^2\xi^2;q^2)_\infty}=\frac{(-q^3;q^6)_\infty(q^2;q^2)_\infty}{(q^6;q^6)_\infty(-q;q^2)_\infty}=\frac{\psi(-q)}{\psi(-q^3)}.$$

Hence we arrive at the relation (5.4). Similarly, setting $z=i$ in (5.2) and using the fact that $R(1,4,n)=R(3,4,n)$, we get

$$\sum_{n=0}^{\infty}\left(R(0,4,n)-R(2,4,n)\right)q^n=\frac{(-qi,qi;q^2)_\infty}{(q^2i,-q^2i;q^2)_\infty}=\frac{(-q^2;q^4)_\infty}{(-q^4;q^4)_\infty}.$$

It remains to show that

$$\begin{aligned}\frac{\varphi(q^2)}{\psi(q^2)} &= (-q^2;q^4)_\infty^2(q^4;q^4)_\infty\times\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty} \\ &= (-q^2;q^4)_\infty(q^4;q^8)_\infty=\frac{(-q^2;q^4)_\infty}{(-q^4;q^4)_\infty}.\end{aligned}$$

This completes the proof. ∎

Based on the relation (5.4), we see that the birank given by Hammond and Lewis leads to a classification of the bipartitions of $3n+2$ with odd parts distinct into three equinumerous sets. Thus we deduce the following theorem.

Theorem 5.2. *For $0\le r\le 2$,*

$$R(r,3,3n+2)=\frac{pod_{-2}(3n+2)}{3}.$$

Proof. By (3.4) and (5.4), we find

$$\sum_{n=0}^{\infty}\left(R(0,3,n)-R(1,3,n)\right)q^n=\frac{f(-q^3,q^6)-q\psi(-q^9)}{\psi(-q^3)}.$$

Since the term q^{3n+2} does not appear on the right-hand side of above identity, it follows that for $n\ge 0$,

$$R(0,3,3n+2)=R(1,3,3n+2).$$

Combining the fact that $R(1,3,n)=R(2,3,n)$ and the relation $\sum_{r=0}^{2}R(r,3,n)=pod_{-2}(n)$, we conclude that for $0\le r\le 2$,

$$R(r,3,3n+2)=\frac{pod_{-2}(3n+2)}{3}.$$

This completes the proof. ∎

We now use the new birank $c(\pi)$ to give another interpretation of the congruence relation (1.9). As above, we need to consider $R_2(m,n)$ as the number of bipartitions π of n with odd parts distinct and whose birank $c(\pi)$ equals m. The following theorem gives the generating function for $R_2(m,n)$.

Theorem 5.3.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R_2(m,n)z^m q^n = (1+q/z)(1+qz)\frac{(-q^3z^2;q^2)_\infty}{(q^2z^2;q^2)_\infty}\frac{(-q^3/z^2;q^2)_\infty}{(q^2/z^2;q^2)_\infty}. \tag{5.6}$$

Proof. Let $A_k(n)$ (resp. $B_k(n)$) denote the number of partitions of n such that the odd parts are distinct and the largest part equals $2k$ (resp. $2k+1$). It is easy to see that

$$A(z,q) := \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)z^{2k}q^n = \sum_{m=0}^{\infty}\frac{q^{2m}z^{2m}(-q;q^2)_m}{(q^2;q^2)_m}$$

and

$$B(z,q) := \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)z^{2k+1}q^n = \sum_{m=0}^{\infty}\frac{q^{2m+1}z^{2m+1}(-q;q^2)_m}{(q^2;q^2)_m}.$$

By the q-binomial theorem [6, Theorem 1.3.1], we find that

$$A(z,q) = \frac{(-q^3z^2;q^2)_\infty}{(q^2z^2;q^2)_\infty}$$

and

$$B(z,q) = qz\frac{(-q^3z^2;q^2)_\infty}{(q^2z^2;q^2)_\infty}.$$

Let $\pi=(\lambda,\mu)$ be a bipartition. Consider the parities of the largest parts of λ and μ. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R_2(m,n)z^m q^n &= A(z,q)A(1/z,q)+A(z,q)B(1/z,q)\\
&\quad +B(z,q)A(1/z,q)+B(z,q)B(1/z,q)\\
&= (1+q/z+qz+q^2)\frac{(-q^3z^2;q^2)_\infty}{(q^2z^2;q^2)_\infty}\frac{(-q^3/z^2;q^2)_\infty}{(q^2/z^2;q^2)_\infty}\\
&= (1+q/z)(1+qz)\frac{(-q^3z^2;q^2)_\infty}{(q^2z^2;q^2)_\infty}\frac{(-q^3/z^2;q^2)_\infty}{(q^2/z^2;q^2)_\infty}.
\end{aligned}$$

This completes the proof. ∎

Let $R_2(r,t,n)$ denote the number of bipartitions π of n with odd parts distinct such that $c(\pi)\equiv r \pmod t$. We are now ready to show that the birank $c(\pi)$ implies a combinatorial explanation of congruence (1.9).

Theorem 5.4. *For $0 \le r \le 2$,*

$$R_2(r, 3, 3n+2) = \frac{pod_{-2}(3n+2)}{3}.$$

Proof. Let $\xi = e^{2\pi i/3}$. Substituting $z = \xi$ into (5.6), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{r=0}^{2} R(r,3,n)\xi^r q^n &= (1+q\xi)(1+q\xi^2)\frac{(-q^3\xi^2;q^2)_\infty}{(q^2\xi^2;q^2)_\infty}\frac{(-q^3\xi;q^2)_\infty}{(q^2\xi;q^2)_\infty} \\
&= \frac{(-q\xi^2;q^2)_\infty}{(q^2\xi^2;q^2)_\infty}\frac{(-q\xi;q^2)_\infty}{(q^2\xi;q^2)_\infty} \\
&= \frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty}\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}.
\end{aligned}$$

Since no triangular numbers $n(n+1)/2$ are congruent to 2 modulo 3, equating coefficients of q^n on both sides, we find that, for each integer $n \ge 0$,

$$\sum_{r=0}^{2} R(r,3,3n+2)\xi^r = 0.$$

Consequently,

$$R_2(0,3,3n+2) = R_2(1,3,3n+2) = R_2(2,3,3n+2),$$

since ξ is one of the roots of the irreducible polynomial $1 + z + z^2 = 0$. This completes the proof. ∎

Here is a simple combinatorial explanation of the congruence (1.8). Let us define the rank $d(\pi)$ of a bipartition $\pi = (\pi_1, \pi_2)$ as the number of parts of π_1. Let $R_3(m, n)$ denote the number of bipartitions π of n with odd parts distinct and whose rank $d(\pi) = m$. Let $R_3(r, t, n)$ denote the number of bipartitions π of n with odd parts distinct such that $d(\pi) \equiv r \pmod{t}$. The generating function for $R_3(m, n)$ equals

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} R_3(m,n)z^m q^n = \frac{(-qz;q^2)_\infty}{(q^2z;q^2)_\infty}\cdot\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Setting $z = -1$, we get

$$\sum_{n=0}^{\infty}\left(R_3(0,2,n) - R_3(1,2,n)\right)q^n = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}, \tag{5.7}$$

which immediately implies that

$$R_3(0,2,2n+1) = R_3(1,2,2n+1) \tag{5.8}$$

and

$$\sum_{n=0}^{\infty}\left(R_3(0,2,2n) - R_3(1,2,2n)\right)(-q)^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{5.9}$$

In light of (5.8), we see that the rank $d(\pi)$ leads to a combinatorial interpretation of the congruence (1.8). It is worth mentioning that the right-hand side of (5.9) is the generating function for partitions with odd parts distinct [14].

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