A note on the von Neumann entropy of random graphs

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In this note, we consider the von Neumann entropy of a density matrix obtained by normalizing the combinatorial Laplacian of a graph by its degree sum. We prove that the von Neumann entropy of the typical Erdös-Rényi random graph saturates its upper bound. Since connected regular graphs saturate this bound as well, our result highlights a connection between randomness and regularity. A general interpetation of the von Neumann entropy of a graph is an open problem.

I. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph with set of vertices $V(G) = \{1, 2, ..., n\}$ and set of edges $E(G) \subseteq V(G) \times V(G) - \{\{v, v\} : v \in V(G)\}$. The graph G is then without self-loops or multiple edges. The *adjacency matrix* of G is denoted by $A(G)$: $[A(G)]_{u,v} = 1$ if $\{u, v\} \in E(G)$ and $[A(G)]_{u,v} = 0$, otherwise. The number of edges adjacent to a vertex $v \in V(G)$ is the *degree* of a v; this is denoted by $d(v)$. Let us define by $d_G = \sum_{v \in V(G)} d(v)$ the *degree-sum* of G. The *degree matrix* of G is denoted by $\Delta(G)$ and defined by $[\Delta(G)]_{u,v} = d(v)$ if $u = v$ and $[\Delta(G)]_{u,v} = 0$, otherwise. The *combinatorial Laplacian matrix* of a graph G (for short, *Laplacian*) is the matrix $L(G) = \Delta(G) - A(G)$. The Laplacian has a number of important roles and its properties have extensively studied in the algebraic graph theory literature [5, 7]. Importantly, as a consequence of the Geršgorin disc theorem, $L(G)$ is positive semidefinite, *i.e.*, its eigenvalues are nonnegative. We shall consider the *density matrix* of G: $\rho_G := \frac{1}{d_G} L(G) = \frac{L(G)}{\mathrm{Tr}(\Delta(G))}$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the eigenvalues of ρ_G. Introduced by Braunstein *et al.* [3], *the von Neumann entropy of a graph* G is defined by $S(G) = -\sum_{i=1}^{n} \lambda_i \log_2 \lambda_i = -\mathrm{Tr}(\rho_G \log_2 \rho_G)$, where $0 \log_2 0 = 0$, by convention. The terminology is justified by the fact that ρ_G is a density matrix, or, equivalently, it is symmetric, positive semidefinite, and its trace is one. Each density matrix axiomatically represents the state of a quantum n-level system [8]. Rovelli and Vidotto [10] have shown that $S(G)$ has a role in quantum gravity, specifically, in the treatment of non-relativistic particles interacting with a quantum gravitational field. Passerini and Severini [9] have pointed out that, for a graph on n vertices, $S(G) \leq \log_2(n-1)$, where the bound is saturated by the complete graph K_n. Moreover, in [9] it was shown that the von Neumann entropy of regular graphs tends to this maximum in the large n limit. Anand and Bianconi [1] observed that the average von Neumann entropy of a canonical power-law network ensemble is linearly related to the Shannon entropy of the ensemble. In this note, we consider the von Neumann entropy of Erdös-Rényi random graphs (see [2] for a background reference on this family of graphs). Let us denote by $\mathcal{G}_n(p)$ the set all graphs with vertex-set $[n] = \{1, 2, \ldots, n\}$, in which the edges are chosen independently with probability p, where $0 < p < 1$. We prove the following theorem:

Theorem 1 *Let $G = G_n(p) \in \mathcal{G}_n(p)$ be a random graph. Then almost surely $S(G) = (1 + o(1)) \log_2 n$, independently of $0 < p < 1$.*

It is remarkable how the von Neumann entropy of a random graph does not depend on the probability of choice. Together with [9] (Theorem 3), this statement shows that both randomness and regularity have similar roles with respect to the von Neumann entropy, at least for sufficiently large graphs. The proof of the theorem is in the next section.

II. PROOF OF THE THEOREM

The adjacency matrix $A := A(G_n(p))$ of the random graph $G := G_n(p) \in \mathcal{G}_n(p)$ is a random matrix.The entries $[A]_{i,j}$ $(i > j)$ are then *i.i.d.* random variables satisfying a Bernoulli distribution with mean p and $[A]_{i,j}$ $(i = j)$ are zeros. Following [2], we shall say that *almost every* graph $G_n(p)$ satisfies the property Q if $G_n(p)$ has the property with probability converging to 1 as n tends to infinity. If the probability of the property Q converges to 1 then we can say that *almost surely* (for short, *a.s.*) every graph in $\mathcal{G}_n(p)$ satisfies Q. In what follows, we study the eigenvalues of ρ_G for random graphs. To this end, we firstly concentrate on the eigenvalues of $L := L(G_n(p))$. Define the following auxiliary matrix: $\overline{L} = L - p(n-1)I_n + p(J_n - I_n)$, where I_n is the identity matrix and J_n is the all-ones matrix. Obviously, $\overline{L} = (\Delta - p(n-1)I_n) - (A - p(J_n - I_n))$, where $\Delta := \Delta(G_n(p))$. We need two lemmata. The first one is due to Bryc *et al.* [4]. The second one is the Weyl inequality [11]. Let us denote by $\|X\|$ the spectral radius of a matrix X.

Lemma 2 (Bryc *et al.* [4]) *Let X be a symmetric random matrix. The entries $[X]_{i,j}$, with $1 \leq i < j$, are a collection of* i.i.d. *random variables with*

$EX_{1,2} = 0$, $\mathrm{Var}(X_{1,2}) = 1$ *and* $EX_{1,2}^4 < \infty$. *Let* $S := diag\left(\sum_{i \neq j} X_{i,j}\right)_{1 \leq i \leq n}$. *Set* $M = S - X$. *Then*

$$\lim_{n \to \infty} \frac{\|M\|}{\sqrt{2n \log n}} = 1 \quad a.s.$$

Lemma 3 (Weyl inequality [11]) *If* X, Y *and* Z *are* $n \times n$ *Hermitian matrices, with* $X = Y + Z$, *where* X, Y, Z *have eigenvalues, respectively,* $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$, $\lambda_1(Y) \geq \cdots \geq \lambda_n(Y)$, $\lambda_1(Z) \geq \cdots \geq \lambda_n(Z)$, *then the following inequalities hold:*

$$\lambda_i(Y) + \lambda_n(Z) \leq \lambda_i(X) \leq \lambda_i(Y) + \lambda_1(Z).$$

It is not difficult to verify that the matrix $\overline{L}/\sqrt{p(1-p)}$ satisfies the conditions of Lemma 2. Consequently,

$$\lim_{n \to \infty} \frac{\|\overline{L}\|}{\sqrt{p(1-p)n}} = 0 \quad a.s.$$

This means

$$\|\overline{L}\| = o(1)n \quad a.s. \tag{1}$$

Now, let $R := (n-1)I_n - p(J_n - I_n)$. By the Weyl inequality, Lemma 3, it follows that $\lambda_i(R) + \lambda_n(\overline{L}) \leq \lambda_i(L) \leq \lambda_i(R) + \lambda_1(\overline{L})$. Since Eq. (1), we have $\lambda_i(L) = \lambda_i(R) + o(1)n$, *a.s.* Moreover, it is easy to see that the eigenvalues of R are equal to pn (with multiplicity $n-1$) and 0 (with multiplicity 1). Therefore, L has eigenvalues

$$\lambda_i(L) = (p + o(1))n, \quad a.s., \quad \text{for } 1 \leq i \leq n-1,$$
$$\lambda_n(L) = o(1)n.$$

We now consider the eigenvalues of $\rho_G = L/\mathrm{Tr}(\Delta)$. Note that $\mathrm{Tr}(\Delta) = 2\sum_{i>j} A_{i,j}$. Recall that $[A]_{i,j}$ $(i > j)$ are *i.i.d.* with mean p and variance $\sqrt{p(1-p)}$. Implying the strong law of large numbers, it follows that, with probability 1, $\lim_{n \to \infty} \left(\sum_{i>j} [A]_{i,j}\right) / \frac{n(n-1)}{2} = p$. Thus, we have

$$\sum_{i>j} [A]_{i,j} = (p/2 + o(1))n^2 \quad a.s.$$

Consequently,

$$\mathrm{Tr}(\Delta) = (p + o(1))n^2 \quad a.s.$$

The eigenvalues of ρ_G are then

$$\lambda_i(\rho_G) = \frac{(p + o(1)n}{(p + o(1))n^2} = \frac{(1 + o(1))}{n}, \text{ for } 1 \leq i \leq n-1,$$
$$\lambda_n(\rho_G) = o(1)/n \quad a.s.$$

At this point, we can state that, for almost every graph $G = G_n(p)$,

$$\begin{aligned} S(\rho_G) &= -\sum_{i=1}^{n} \lambda_i(\rho_G) \log_2 \lambda_i(\rho_G) \\ &= -\sum_{i=1}^{n-1} \frac{1 + o(1)}{n} \log_2 \frac{1 + o(1)}{n} - \frac{o(1)}{n} \log_2 \frac{o(1)}{n} \\ &= -\frac{(1 + o(1))(n-1)}{n} \\ &\log_2 \frac{1 + o(1)}{n} - \frac{o(1)}{n} \log_2 \frac{o(1)}{n} \\ &= (1 + o(1)) \log_2 n, \end{aligned}$$

concluding the proof of the theorem.

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