# Note on the Rainbow $k$-Connectivity of Regular Complete Bipartite Graphs*

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**Abstract**

A path in an edge-colored graph $G$, where adjacent edges may be colored the same, is called a rainbow path if no two edges of the path are colored the same. For a $\kappa$-connected graph $G$ and an integer $k$ with $1 \leq k \leq \kappa$, the rainbow $k$-connectivity $rc_k(G)$ of $G$ is defined as the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that any two distinct vertices of $G$ are connected by $k$ internally disjoint rainbow paths. Denote by $K_{r,r}$ an $r$-regular complete bipartite graph. Chartrand et al. in "G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, The rainbow connectivity of a graph, Networks 54(2009), 75-81" left an open question of determining an integer $g(k)$ for which the rainbow $k$-connectivity of $K_{r,r}$ is 3 for every integer $r \geq g(k)$. This short note is to solve this question by showing that $rc_k(K_{r,r}) = 3$ for every integer $r \geq 2k\lceil \frac{k}{2} \rceil$, where $k \geq 2$ is a positive integer.

**Keywords:** edge-colored graph, rainbow path, rainbow $k$-connectivity, regular complete bipartite graph

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All graphs considered in this paper are simple, finite and undirected. Let $G$ be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is called *rainbow* if no two edges of it are colored the same. A well-known result shows that in every $\kappa$-connected graph $G$ with $\kappa \geq 1$, there are $k$ internally disjoint $u - v$ paths connecting any two distinct vertices $u$ and $v$ for every integer $k$ with $1 \leq k \leq \kappa$. Chartrand et al. [2] defined the *rainbow k-connectivity* $rc_k(G)$ of $G$, which is the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that for any two distinct vertices $u$ and $v$ of $G$, there exist at least $k$ internally disjoint $u-v$ rainbow paths.

The concept of rainbow $k$-connectivity has applications in transferring information of high security in communication networks. For details we refer to [2] and [3].

In [2], Chartrand et al. studied the rainbow $k$-connectivity of the complete graph $K_n$ for various pairs $k, n$ of integers. It was shown in [2] that for every integer $k \geq 2$, there exists an integer $f(k)$ such that $rc_k(K_n) = 2$ for every integer $n \geq f(k)$. In [4], we improved the upper bound of $f(k)$ from $(k+1)^2$ to $ck^{\frac{3}{2}} + C$ (here $0 < c < 1$ and $C = o(k^{\frac{3}{2}})$), i.e., from $O(k^2)$ to $O(k^{\frac{3}{2}})$. Chartrand et al. in [2] also investigated the rainbow $k$-connectivity of $r$-regular complete bipartite graphs for some pairs $k, r$ of integers with $2 \leq k \leq r$, and they obtained the following results.

**Proposition 1.** For each integer $r \geq 2$,

$$rc_2(K_{r,r}) = \begin{cases} 4 & \text{if } r = 2 \\ 3 & \text{if } r \geq 3. \end{cases}$$

.

**Proposition 2.** For each integer $r \geq 3$, $rc_3(K_{r,r}) = 3$.

**Theorem 3.** For every integer $k \geq 2$, there exists an integer $r$ such that $rc_k(K_{r,r}) = 3$.

Moreover, they showed that $r = 2k\lceil \frac{k}{2} \rceil$ is a desired integer for Theorem 3. However, they could not show a similar result as for complete graphs, and therefore they left an open question: For every integer $k \geq 2$, determine an integer (function) $g(k)$, for which $rc_k(K_{r,r}) = 3$ for every integer $r \geq g(k)$, that is, the rainbow $k$-connectivity of the complete bipartite graph $K_{r,r}$ is essentially 3. This short note is to solve this question by showing that $rc_k(K_{r,r}) = 3$ for every integer $r \geq 2k\lceil \frac{k}{2} \rceil$. We use a method similar to but more complicated than the proof of Theorem 3 in [2]. For notation and terminology not defined here, we refer to [1].

**Theorem 4.** For every integer $k \geq 2$, there exists an integer $g(k)$ such that $rc_k(K_{r,r}) = 3$ for any $r \geq g(k)$.

*Proof.* Let $g(k) = 2k\lceil \frac{k}{2} \rceil$. We will show that $rc_k(K_{r,r}) = 3$ for every $k \geq 2$, where $r \geq 2k\lceil \frac{k}{2} \rceil$ is an integer. By Propositions 1 and 2, we know that the conclusion holds for $k = 2, 3$. So we assume $k \geq 4$.

We first assume that $k$ is even. Then, $g(k) = 2k \cdot \frac{k}{2}$. Since $r \geq g(k)$, then $r = k_1 \cdot (2k) + r_1$, where $k_1 \geq \frac{k}{2}, 1 \leq r_1 \leq 2k - 1$. Let the bipartite sets of $G = K_{r,r} = K_{k_1 \cdot (2k)+r_1, k_1 \cdot (2k)+r_1}$ be $U$ and $W$. Let $U'$, $W'$ be the set of first $k_1 \cdot (2k)$ vertices of $U$, $W$, respectively. $U \setminus U' = \{u_1, \ldots, u_{r_1}\}$ and $W \setminus W' = \{w_1, \ldots, w_{r_1}\}$. Suppose that

$$U' = U'_1 \cup \ldots \cup U'_{2k}, W' = W'_1 \cup \ldots \cup W'_{2k},$$

where $U'_i = \{u_{i,1}, \ldots, u_{i,k_1}\}$ and $W'_i = \{w_{j,1}, \ldots, w_{j,k_1}\}$ for $1 \leq i, j \leq 2k$. Let $G'$ be an induced subgraph of $G$ with bipartite sets $U'$ and $W'$. Suppose that

$$U = U_1 \cup \ldots \cup U_{2k}, W = W_1 \cup \ldots \cup W_{2k},$$

where $U_i = U'_i \cup \{u_i\}$, $W_j = W'_j \cup \{w_j\}$ for $1 \leq i, j \leq r_1$ and $U_i = U'_i$, $W_j = W'_j$ for $r_1 + 1 \leq i, j \leq 2k$.

We now give $G$ a 3-edge coloring as follows: Let $G'_1$ be the spanning subgraph of $G'$ such that $E(G'_1) = \{u_{i,p}w_{j,p} : 1 \leq i, j \leq 2k, 1 \leq p \leq k_1, i$ and $j$ are of the same parity$\}$. Let $G_1$ be the spanning subgraph of $G$ such that $E(G_1) = E(G'_1) \cup \{u_iw_j : 1 \leq i, j \leq r_1, i$ and $j$ are of the same parity$\}$. Let $G_2$ be the spanning of subgraph of $G$ such that

$$G_2 = H_1 \cup \ldots \cup H_{2k},$$

where $H_1$ has bipartite sets $U_1$ and $W_{2k}$, $H_i$ $(2 \leq i \leq 2k)$ has bipartite sets $U_i$ and $W_{i-1}$. So, $H_i = K_{m,n}(\{m, n\} = \{k_1, k_1 + 1\})$. See Figure 0.1 for the case $r = 18$, $k = 4$, $r_1 = 2$. Finally, let

$$G_3 = G - (E(G_1) \cup E(G_2)).$$

Assign each edge of $G_i (1 \leq i \leq 3)$ the color $i$.

Next we will show that the above edge-coloring is a $k$-rainbow coloring, that is, there are at least $k$ internally disjoint rainbow paths connecting any two distinct vertices $u, v$ of $G$. We will consider the following two cases:

**Case 1.** $u \in V(G')$. Without loss of generality, let $u = u_{1,1}$.

**Subcase 1.1.** $u$ and $v$ belong to the same bipartite set of $G$.

**Subsubcase 1.1.1.** $v \in U_1$. Then $G$ contains the $k$ internally disjoint $u_{1,1} - v$ rainbow paths $u_{1,1}, w_{i,1}, v$ where $1 \leq i \leq 2k - 1$ and $i$ is odd.

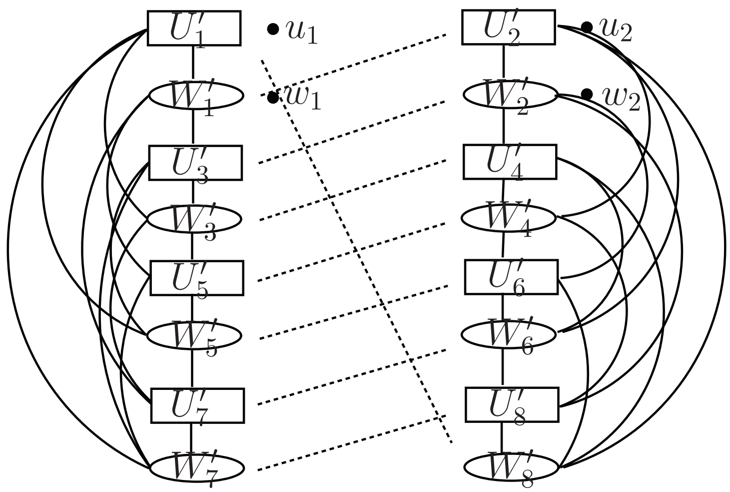


Figure 0.1 The figure for the case $r = 18$, $k = 4$, $r_1 = 2$.

**Subsubcase 1.1.2.** $v \in U_i$, $3 \le i \le 2k - 1$, and $i$ is odd, say $v \in U_3$. Then $G$ contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{2,j}, v$ and $u_{1,1}, w_{2k,j}, v$, where $1 \le j \le k_1$.

**Subsubcase 1.1.3.** $v \in U_i$, $2 \le i \le 2k$, and $i$ is even, say $v \in U_2$. Then $G$ contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{1,j}, v$ and $u_{1,1}, w_{2k,j}, v$, where $1 \le j \le k_1$.

**Subcase 1.2.** $u$ and $v$ belong to different bipartite sets, and so $v \in W$.

**Subsubcase 1.2.1.** $v \in W_i$, where $1 \le i \le 2k - 1$ and $i$ is odd, say $v \in W_1$. Then $G$ contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{2,j}, u_{2,j}, v$ and $u_{1,1}, w_{2k,j}, u_{2k,j}, v$, where $1 \le j \le k_1$.

**Subsubcase 1.2.2.** $v \in W_i$, where $2 \le i \le 2k$ and $i$ is even, say $v \in W_2$. If $v \in W'_2$, without loss of generality, let $v = w_{2,1}$, then $G$ contains the $u_{1,1} - v$ path $u_{1,1}, v$ together with the $u_{1,1} - v$ rainbow paths $u_{1,1}, w_{3,j}, u_{3,j}, v$; $u_{1,1}, w_{3,1}, u_{4,j}, v$ and $u_{1,1}, w_{2k,j}, u_{2k,j}, v$, where $2 \le j \le k_1$. The cases for $v = w_2$ and $v \in W_{2k}$ are similar.

**Case 2.** $u \in V(G) \setminus V(G')$, that is, $u \in \{u_1, \ldots, u_{r_1}; w_1, \ldots, w_{r_1}\}$. Without loss of generality, let $u = u_1$. By Case 1, we only need to show that there are at least $k$ internally disjoint rainbow paths connecting $u$ and $v$ for every $v \in V(G) \setminus V(G')$.

**Subcase 2.1.** $u$ and $v$ belong to the same bipartite set of $G$.

**Subsubcase 2.1.1.** $v = u_i$, $3 \leq i \leq 2k-1$ and $i$ is odd, say $v = u_3$. Then $G$ contains the $2k_1 \geq k$ internally disjoint $u-v$ rainbow paths $u_1, w_{2,j}, u_3$ and $u_1, w_{2k,j}, u_3$, where $1 \leq j \leq k_1$.

**Subsubcase 2.1.2.** $v = u_i$, $2 \leq i \leq 2k$ and $i$ is even, say $v = u_2$. Then $G$ contains the $2k_1 \geq k$ internally disjoint $u-v$ rainbow paths $u_1, w_{1,j}, u_2$ and $u_1, w_{2k,j}, u_2$, where $1 \leq j \leq k_1$.

**Subcase 2.2.** $u$ and $v$ belong to different bipartite sets of $G$.

**Subsubcase 2.2.1.** $v = w_i$, $1 \leq i \leq 2k-1$ and $i$ is odd, say $v = w_1$. Then $G$ contains the $2k_1 \geq k$ internally disjoint $u-v$ rainbow paths $u_1, w_{2,j}, u_{2,j}, w_1$ and $u_1, w_{2k,j}, u_{2k,j}, w_1$ where $1 \leq j \leq k_1$.

**Subsubcase 2.2.2.** $v = w_i$, $2 \leq i \leq 2k$ and $i$ is even, say $v = w_2$. Then $G$ contains the $2k_1 \geq k$ internally disjoint $u-v$ rainbow paths $u_1, w_{1,j}, u_{3,j}, w_2$ and $u_1, w_{2k,j}, u_{2k,j}, w_2$ where $1 \leq j \leq k_1$.

So the conclusion holds for the case that $k$ is even.

Next we assume that $k$ is odd. Then $g(k) = 2k \cdot \frac{k+1}{2}$. Since $r \geq g(k)$, then $r = k_2 \cdot (2k) + r_2$, where $k_2 \geq \frac{k+1}{2}$, $1 \leq r_2 \leq 2k-1$. Then with a similar argument to the case that $k$ is even, we can show that the conclusion also holds when $k$ is odd. ∎

**Remark 2.5.** In [4] we showed that for every pair of integers $k \geq 2$ and $r \geq 1$, there is an integer $f(k,r)$ such that if $\ell \geq f(k,r)$, then the rainbow $k$-connectivity of an $r$-regular complete $\ell$-partite graph is 2, where $r$-regular means that every partite set has the same number $r$ of elements. That is, for sufficiently many number $\ell$ of partite sets, the rainbow $k$-connectivity of an $r$-regular complete $\ell$-partite graph is 2. Theorem 4 of this note implies that for sufficiently large size $r$ of every partite set, the rainbow $k$-connectivity of an $r$-regular complete $\ell$-partite graph is at most 3. So, an interesting question is to think about the question of determining some bounds on $k, r, \ell$ that tell us the rainbow $k$-connectivity of an $r$-regular complete $\ell$-partite graph is 2 or 3.

# References


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