ON THE INDEX OF SEQUENCES OVER CYCLIC GROUPS

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ABSTRACT. Let G be a finite cyclic group of order $n \ge 2$. Every sequence S over G can be written in the form $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ where $g \in G$ and $n_1, \ldots, n_l \in [1, \operatorname{ord}(g)]$, and the index $\operatorname{ind}(S)$ of S is defined as the minimum of $(n_1 + \ldots + n_l)/\operatorname{ord}(g)$ over all $g \in G$ with $\operatorname{ord}(g) = n$. In this paper we prove that a sequence S over G of length $|S| = n$ having an element with multiplicity at least $\frac{n}{2}$ has a subsequence T with $\operatorname{ind}(T) = 1$, and if the group order n is a prime, then the assumption on the multiplicity can be relaxed to $\frac{n-2}{10}$. On the other hand, if $n = 4k + 2$ with $k \ge 5$, we provide an example of a sequence S having length $|S| > n$ and an element with multiplicity $\frac{n}{2} - 1$ which has no subsequence T with $\operatorname{ind}(T) = 1$. This disproves a conjecture given twenty years ago by Lemke and Kleitman.

1. Introduction and Main Results

Let G be an additively written, finite cyclic group and $g \in G$ with $\operatorname{ord}(g) = |G|$. For a sequence

$$S = (n_1 g) \cdot \ldots \cdot (n_l g) \quad \text{over } G, \quad \text{where } l \in \mathbb{N}_0 \quad \text{and} \quad n_1, \ldots, n_l \in [1, n],$$

we set

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n},$$

and then

$$\operatorname{ind}(S) = \min\{\|S\|_h \mid h \in G \text{ with } \operatorname{ord}(h) = |G|\} \in \mathbb{Q}_{\ge 0}$$

denotes the *index* of S. The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Lemke and Kleitman ([11]), used as key tool by Geroldinger ([6, page 736]), and then investigated by Gao [3] in a systematical way. Since then it has attracted a lot of attention in recent years (see [1, 2, 5, 8, 12, 13, 14, 15, 16]). We briefly discuss some key results.

If S is a minimal zero-sum sequence, then $|S| \le 3$, as well as $|S| \ge \lfloor \frac{n}{2} \rfloor + 2$, implies that $\operatorname{ind}(S) = 1$ (see [1], [14], [16]). In contrast to that, it was shown that for every $k \in [5, \lfloor \frac{n}{2} \rfloor + 1]$, there is a minimal zero-sum subsequence T of length $|T| = k$ and with $\operatorname{ind}(T) \ge 2$, and that the same is true for $k = 4$ and $\gcd(n, 6) \ne 1$. This leads to the conjecture that, in case $\gcd(n, 6) = 1$, every minimal zero-sum sequence S over G of length $|S| = 4$ has $\operatorname{ind}(S) = 1$. Li, Plyley, Yuan and Zeng [12] recently proved that this holds true if n is a prime power, but the general case is still open.

In 1989, Lemke and Kleitman stated the following conjecture ([11, page 344]), which we formulate in the present language.

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Conjecture 1.1. *Let G be a cyclic group of order n, d a divisor of n, and let S be a sequence over G of length $|S| = n$. Then there exists a subsequence T of S and element $g \in G$ with* $\mathrm{ord}(g) = n$ *such that*

$$d \,\big|\, n\|T\|_g \,\big|\, n\,.$$

In the special case $d = n$, this is equivalent to the existence of a subsequence T with $\mathrm{ind}(T) = 1$.

Indeed the above is the third of three interesting conjectures stated by Lemke and Kleitman in [11]. Their first conjecture has turned out to be true for all finite abelian groups (see [7]), and the second one is still open. In this paper we demonstrate that the above conjecture fails in general (see Theorem 1.2), but that it holds true under an additional assumption on the highest multiplicity of an element occurring in the sequence. Here are the main results of the present paper (for any undefined terminology or notation the reader is referred to the beginning of Section 2).

Theorem 1.2. *Let G be a cyclic group of order $n \ge 2$, where $n = 4k+2$ for some $k \ge 5$, and let $g \in G$ with* $\mathrm{ord}(g) = n$. *Then the sequence*

$$S = g^{\frac{n}{2}-3}\left(\frac{n}{2}g\right)\left((\frac{n}{2}+1)g\right)^{\frac{n}{2}-1}\left((\frac{n}{2}+2)g\right)^{\lfloor\frac{n}{4}\rfloor-2}$$

has no subsequence T with $\mathrm{ind}(T) = 1$.

Theorem 1.3. *Let G be a cyclic group of order $n \ge 2$ and S be a sequence over G of length $|S| = n$. If $\mathsf{h}(S) < 4$ or $\mathsf{h}(S) \ge n/2$, then S has a subsequence T with* $\mathrm{ind}(T) = 1$ *and length* $|T| \le \mathsf{h}(S)$.

Theorem 1.4. *Let G be a cyclic group of prime order $p > 24318$ and S be a sequence over G of length $|S| = p$. If $\mathsf{h}(S) \ge \frac{p-2}{10}$, then S has a subsequence T with* $\mathrm{ind}(T) = 1$.

In Section 2 we summarize our notations and give the proof of Theorem 1.2. In the following two sections we provide the proofs of Theorem 1.3 and of Theorem 1.4. We end the paper with a further conjecture and some open problems (see Section 5).

2. Notations and Proof of Theorem 1.2

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers, and for rational numbers $a, b \in \mathbb{Q}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let G be an additively written abelian group and $G_0 \subset G$ a subset. We fix the notation concerning sequences over G_0 (which is consistent with [4] and [9]). Let $\mathcal{F}(G_0)$ be the free abelian monoid with basis G_0. The elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. We write sequences $S \in \mathcal{F}(G_0)$ in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,,$$

where $l \in \mathbb{N}_0$, $g_1, \ldots, g_l \in G_0$, $\mathsf{v}_g(S) \in \mathbb{N}_0$ and $\mathsf{v}_g(S) = 0$ for almost all $g \in G_0$. We call $|S| = l$ the *length* of S, $\sigma(S) = g_1 + \ldots + g_l$ the *sum* of S, $\mathsf{v}_g(S)$ the *multiplicity* of g in S, $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$ the *support* of S, and we denote by

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|] \qquad \text{the } \textit{maximum of the multiplicities} \text{ of } S\,.$$

For every group homomorphism $\varphi\colon G \to H$, we set $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l) \in \mathcal{F}(H)$, and if φ is the multiplication by some $m \in \mathbb{N}$, then we set $mS = \varphi(S)$. We say that S is a *zero-sum* *sequence* if $\sigma(S)=0$, and it is called a *minimal zero-sum sequence* if $\sigma(S)=0$ but $\sum_{i\in I} g_i \neq 0$ for all $\emptyset \neq I \subsetneq [1,l]$. Suppose that G is finite cyclic. Then a simple calculation (see [8, Lemma 5.1.2]) shows that

$$\begin{aligned}\operatorname{ind}(S) &= \min\{\|S\|_h \mid h \in G \text{ with } \operatorname{supp}(S) \subset \langle h\rangle\} \\ &= \min\{\|S\|_h \mid h \in G \text{ with } \langle \operatorname{supp}(S)\rangle = \langle h\rangle\}\,.\end{aligned}$$

Proof of Theorem 1.2. Assume to the contrary that S has a subsequence T with $\operatorname{ind}(T)=1$. Then there exists an element $h \in G$ with $\operatorname{ord}(h)=n$ such that $\|T\|_h = 1$. We set

$$g = jh \qquad \text{and} \qquad T = g^x \Big(\frac{n}{2}g\Big)^y \Big((\frac{n}{2}+1)g\Big)^z \Big((\frac{n}{2}+2)g\Big)^w$$

where $j \in [1, n-1]$ with $\gcd(j,n)=1$, $x \in [0, n/2-3]$, $y \in [0,1]$, $z \in [0, n/2-1]$ and $w \in [0, n/4-2]$. Then

$$n\|T\|_g = (x+z+2w) + \frac{n}{2}(y+z+w) \equiv 0 \pmod n. \tag{1}$$

Case 1. $j < \frac{n}{4}$.

Then

$$T = (jh)^x \Big(\frac{n}{2}h\Big)^y \Big((\frac{n}{2}+j)h\Big)^z \Big((\frac{n}{2}+2j)h\Big)^w\,.$$

Since $\|T\|_h = 1$, we infer that $y+z+w \le 1$ which implies that $n\|T\|_g \le x + (\frac{n}{2}+2) \le \frac{n}{2} - 3 + \frac{n}{2} + 2 < n$, a contradiction.

Case 2. $\frac{n}{4} < j < \frac{n}{2}$.

Then

$$T = (jh)^x \Big(\frac{n}{2}h\Big)^y \Big((\frac{n}{2}+j)h\Big)^z \Big((2j-\frac{n}{2})h\Big)^w\,.$$

Since $\|T\|_h = 1$, we infer that $x \le 3$ and $z \le 1$ which implies that $n\|T\|_g \le x+z+2w \le 3+1+2\,(\lfloor\frac{n}{4}\rfloor - 2) < \frac{n}{2}$. Since $x+z+2w > 0$ and again by $\|T\|_h = 1$, we derive that $x+z+2w \equiv 0 \pmod{\frac{n}{2}}$, a contradiction.

Case 3. $\frac{n}{2} < j < \frac{3n}{4}$.

Then

$$T = (jh)^x \Big(\frac{n}{2}h\Big)^y \Big((j-\frac{n}{2})h\Big)^z \Big((2j-\frac{n}{2})h\Big)^w\,.$$

Since $\|T\|_h = 1$, we infer that $x+y+w \le 1$. We assert that

$$x+y+w = 1. \tag{2}$$

Otherwise, $x=y=w=0$ and $n\|T\|_g = z + \frac{n}{2}z \not\equiv 0 \pmod{\frac{n}{2}}$, a contradiction to $n\|T\|_g \equiv 0 \pmod n$. Note that $0 < x+z+2w < n$. By (1), we have that

$$x+z+2w = \frac{n}{2} \tag{3}$$

and

$$y+z+w \equiv 1 \pmod 2. \tag{4}$$

By (2) and (3), we have $y+z+w \equiv z+w-y = \frac{n}{2} - 1 \equiv 0 \pmod 2$, a contradiction to (4).

Case 4. $\frac{3n}{4} < j < n$.

Then

$$T = (jh)^x \left(\frac{n}{2}h\right)^y \left((j-\frac{n}{2})h\right)^z \left((2j-\frac{3n}{2})h\right)^w .$$

Since $\|T\|_h = 1$, we infer that $x \le 1$ and $z \le 3$ which implies that $n\|T\|_g \le x + z + 2w \le 1 + 3 + 2\left(\lfloor \frac{n}{4} \rfloor - 2\right) < \frac{n}{2}$. Clearly, $x + z + 2w > 0$. From (1), we derive a contradiction. □

3. Proof of Theorem 1.3

We need the following two results. A simple proof of the first one can be found in [8, Proposition 4.2.6] (for historical comments see [10]), and a proof of Lemma 3.2 is given in [13].

Lemma 3.1. *Let G be a finite cyclic group and S be a sequence over G of length $|S| \ge |G|$. Then S has a zero-sum subsequence T of length $|T| \in [1, \mathsf{h}(S)]$.*

Lemma 3.2. *Let G be a finite cyclic group and S be a minimal zero-sum sequence over G of length $|S| \in [1, 3]$. Then* $\operatorname{ind}(S) = 1$.

Proof of Theorem 1.3. We set $n = |G|$ and $h = \mathsf{h}(S)$. If $h < 4$, then the assertion follows from Lemmas 3.1 and 3.2. Suppose that $h \ge n/2$. Let $g \in G$ with $\mathsf{v}_g(S) = h$. If $\operatorname{ord}(g) < n$, then $\operatorname{ord}(g) \le n/2 \le h$, and $T = g^{\operatorname{ord}(g)}$ has the required properties. If $0 \mid S$, then $T = 0$ has the required properties.

Suppose that $\operatorname{ord}(g) = n$ and that $0 \nmid S$. Then we can write S in the form

$$S = g^h (b_1 g) \cdot \ldots \cdot (b_{n-h} g) \quad \text{where} \quad b_1, \ldots, b_{n-h} \in [2, n-1] .$$

Assume to the contrary that S has no subsequence T with the required properties. We continue with the following assertion.

A. For every subset $I \subset [1, n-h]$ we have $\sum_{i \in I} b_i \le n - h + |I| - 1 .$

If **A** holds, then we apply it with $I = [1, n-h]$ and obtain that

$$\sum_{i=1}^{n-h} b_i \le 2(n-h) - 1 ,$$

a contradiction to $b_1, \ldots, b_{n-h} \in [2, n-1]$. We prove **A** by induction on $|I|$. If there were an $i \in [1, n-h]$ such that $b_i \ge n - h + 1$, then $T = g^{n-b_i}(b_i g)$ were a subsequence of S with $\operatorname{ind}(T) = 1$ and length $|T| = n - b_i + 1 \le h$, a contradiction. Let $I \subset [1, n-h]$ with $|I| = k + 1 \ge 2$, say $I = [1, k+1]$, and suppose that **A** holds for all proper subsets of I. We set $\beta = b_1 + \ldots + b_{k+1}$. By induction hypothesis we get $\beta - b_i \le n - h + k - 1$ for every $i \in [1, k+1]$, which implies that

$$\beta = \frac{1}{k}(k\beta) = \frac{1}{k}\sum_{i=1}^{k+1}(\beta - b_i) \le \frac{(k+1)(n-h+k-1)}{k} \le n$$

(to get the last inequality, use that $h \ge n/2$ and $k \le n - h - 1$). Thus, if $\beta \ge n - h + k + 1$, then $T = g^{n-\beta}(b_1 g) \cdot \ldots \cdot (b_{k+1} g)$ is a subsequence of S with $\operatorname{ind}(T) = 1$ and length $|T| = n - \beta + k + 1 \le h$. This is a contradiction, and thus **A** is proved. □

Note that the sequence S given in Theorem 1.2 satisfies $\mathsf{h}(S) = \frac{n}{2} - 1$. Thus the assumption in Theorem 1.3, that $\mathsf{h}(S) \ge \frac{n}{2}$, cannot be weakened for $n \equiv 2 \pmod 4$.

4. Proof of Theorem 1.4

We fix our notations which remain valid throughout the whole section. Let G be a prime cyclic group of order $|G| = p > 24318$, $G^\bullet = G \setminus \{0\}$, and let S be a sequence over $G^\bullet$ of length $|S| = p$. If $g \in G^\bullet$, $A \subset \mathbb{Z}$ and $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ with $n_1, \ldots, n_l \in [1, p-1]$, then we set

$$S(A, g) = \prod_{i \in [1,l], n_i \in A} (n_i g) \,.$$

For an element $g \in G^\bullet$, we set

$$\Sigma_g(S) = \{p \, \|T\|_g \ \mid \ T \text{ is a subsequence of } S \text{ with } \|T\|_g \le 1\} \,,$$

and we denote by $\mathsf{m}_g(S)$ the maximal $t \in [1, p]$ such that $\Sigma_g(T) = [1, t]$ for some subsequence T of S. We define

$$\mathsf{m}(S) = \max\{\mathsf{m}_g(S) \mid g \in G^\bullet\} \,.$$

From now on we fix an element $g \in G^\bullet$ such that $\mathsf{m}_g(S) = \mathsf{m}(S)$.

Lemma 4.1. *Let T be a subsequence of S such that $\Sigma_g(T) = [1, \mathsf{m}(S)]$. Then $|T| \le \mathsf{m}(S)$, and if $x \in [1, p-1]$ such that $(xg) \mid ST^{-1}$, then $x \ge \mathsf{m}(S) + 2$. Furthermore, if $\mathsf{m}(S) = p$, or if there exists an $x \in [1, p-1]$ such that $(xg) \mid ST^{-1}$ and $x \ge p - \mathsf{m}(S)$, then S has a subsequence with index* 1.

Proof. By definition, we have $|T| \le p \, \|T\|_g = \mathsf{m}(S)$. If there is some $x \in [1, p-1]$ with $(xg) \mid ST^{-1}$ and $x \le \mathsf{m}(S) + 1$, then $\Sigma_g((xg)T) = [1, \min\{p, \mathsf{m}(S) + x\}]$, a contradiction to the maximality of $\mathsf{m}(S)$. The second part of this lemma is clear. □

From now on we suppose that S has no subsequence with index 1.

Let $k \ge 2$ be a positive integer, and let $F[\frac{1}{k}, \frac{k-1}{k}]$ be all irreducible fractions between $\frac{1}{k}$ and $\frac{k-1}{k}$ and with denominators in $[2, k]$, i.e.,

$$F\left[\frac{1}{k}, \frac{k-1}{k}\right] = \left\{ \frac{a}{b} \ \Big| \ a \in \mathbb{N}, \, b \in [2, k] \text{ with } \gcd(a, b) = 1 \text{ and } \frac{1}{k} \le \frac{a}{b} \le \frac{k-1}{k} \right\}.$$

Lemma 4.2. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two adjacent fractions in $F[\frac{1}{k}, \frac{k-1}{k}]$ with $\frac{a}{b} < \frac{c}{d}$. Then we have*

1. $b + d \ge k + 1$.
2. $bc - ad = 1$.

Proof. 1. Note that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Since $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent, it follows that the irreducible fraction with value $\frac{a+c}{b+d}$ is not in $F[\frac{1}{k}, \frac{k-1}{k}]$. This forces that $b + d \ge k + 1$.

2. Since $\gcd(a, b) = 1$, there are two integers u and v such that $bu + av = 1$. Note that $b(u + ma) + a(v - mb) = 1$ holds for any integer m. Let $x = u + ma$ and $y = mb - v$. Then,

$bx - ay = 1$. By choosing m suitably we may assume that $y \le k$ and $y + b \ge k + 1$. It follows that $y \ge k + 1 - b > 0$ and $x > 0$. From $bx - ay = 1$ we get

$$\frac{x}{y} - \frac{a}{b} = \frac{1}{by}.$$

If $y > 1$, then $\frac{x}{y}$ is a fraction in $F[\frac{1}{k}, \frac{k-1}{k}]$. So, either $\frac{c}{d} = \frac{x}{y}$ and we are done, or $\frac{c}{d} < \frac{x}{y}$. For the latter case we have $\frac{1}{by} = \frac{x}{y} - \frac{a}{b} = (\frac{x}{y} - \frac{c}{d}) + (\frac{c}{d} - \frac{a}{b}) = \frac{b(dx-cy)+y(cb-ad)}{byd} \ge \frac{b+y}{byd}$. This implies that $d \ge b + y \ge k + 1$, a contradiction.

Now assume that $y = 1$ and we must have $b = k$. It follows from $bx - ay = 1$ that $a = kx - 1$. Therefore, $x = 1$ and $a = k - 1$. So, $\frac{a}{b} = \frac{k-1}{k}$ is the biggest fraction in $F[\frac{1}{k}, \frac{k-1}{k}]$, a contradiction. □

We set

$$k = \left\lfloor \frac{p}{\mathsf{m}(S)} \right\rfloor, \quad f = \left| F\left[\frac{1}{k}, \frac{k-1}{k}\right] \right|,$$

and we arrange all fractions in $F[\frac{1}{k}, \frac{k-1}{k}]$ increasingly; so let

$$\frac{a_1}{b_1} < \ldots < \frac{a_f}{b_f}$$

denote the elements of $F[\frac{1}{k}, \frac{k-1}{k}]$. Furthermore, we set

$$S_1 = S([1, \mathsf{m}(S)], g) \quad S_2 = S([\mathsf{m}(S) + 2, \frac{p-1}{b_1}], g)$$

and, for every $i \in [1, f]$, we set

$$S_{2i+1} = S\Big(\Big[\frac{a_i p + 1}{b_i}, \frac{a_i p + \mathsf{m}(S)}{b_i}\Big], g\Big) \quad \text{and} \quad S_{2i+2} = S\Big(\Big[\frac{a_i p + \mathsf{m}(S) + 1}{b_i}, \frac{a_{i+1} p - 1}{b_{i+1}}\Big], g\Big).$$

Furthermore, for every $i \in [2, k]$, we define

$$R_i = S(\{x \in [1, p] \mid \text{If } x_i \in [1, p] \text{ with } p \,|\, (x_i - ix), \text{ then } x_i \in [1, \mathsf{m}(S)] \text{ and } \gcd(x_i, i) = 1\}, g).$$

Lemma 4.3. *We have* $S = \prod_{j=1}^{2f+1} S_j$.

Proof. This is clear by construction. □

Lemma 4.4. *Suppose that*

$$4 \le \mathsf{m}(S) \le \frac{p-3}{2} \quad \textit{and} \quad \max\left\{\frac{p - \mathsf{m}(S) - 2}{\mathsf{m}(S)}, \frac{p - \mathsf{m}(S)}{\mathsf{m}(S) + 1}\right\} \le k \le \frac{p+1}{\mathsf{m}(S)}.$$

1. $|S_{2i+2}| \le b_{i+1} - 1$ *for every* $i \in [0, f-1]$.
2. $p = |S| \le \mathsf{m}(S) + \sum_{i=2}^{k} \sum_{j \in [1, i-1] \text{ with } \gcd(i,j)=1} (i-1) + \sum_{i=2}^{k} |R_i|$.

Proof. 1. Suppose that $i = 0$. Then $S_2 = S([\mathsf{m}(S)+2, \frac{p-1}{b_1}], g)$ and $b_1 = k$. If $|S_2| \ge b_1 = k$, then we can take a k-term subsequence U of S_2. Note that $p - 1 \ge p\|U\|_g \ge k(\mathsf{m}(S)+2) \ge p - \mathsf{m}(S)$ and one can find a subsequence V of S_1 such that UV has index 1, a contradiction.

Now suppose that $i \in [1, f-1]$, and assume to the contrary that $|S_{2i+2}| \ge b_{i+1}$. We choose an arbitrary b_{i+1}-term subsequence X of S_{2i+2}, and write $b_i S$ in the form

$$b_i S = (x_1 g) \cdot \ldots \cdot (x_p g) \quad \text{with} \quad x_1, \ldots, x_p \in [1, p-1]\,.$$

It follows from Lemma 4.2 that $a_{i+1} b_i - a_i b_{i+1} = 1$, and so $b_i(\frac{a_{i+1} p - 1}{b_{i+1}}) - a_i p = \frac{p - b_i}{b_{i+1}}$. Thus for every $\nu \in [1, p]$ with $(x_\nu g) \mid S_{2i+2}$, we infer that $x_\nu \in [\mathsf m(S)+1, \frac{p-b_i}{b_{i+1}}]$ and $x_\nu \equiv -a_i p \pmod{b_i}$. Therefore we get, since by Lemma 4.2 , $b_i + b_{i+1} \ge k+1$,

$$p - b_i \ge p \|b_i X\|_g \ge b_{i+1}(\mathsf m(S)+1) \ge p - b_i \mathsf m(S)$$

and

$$p\|b_i X\|_g \equiv -b_{i+1} a_i p = (1 - a_{i+1} b_i) p \equiv p \pmod{b_i}\,.$$

Therefore there exists a subsequence Y of S_1 such that $p\|b_i(XY)\|_g = p$, a contradiction.

2. For every $\ell \in [2, k]$, we have $R_\ell = \prod_{b_i = \ell} S_{2i+1}$, and hence

$$S = S_1 \prod_{i=0}^{f-1} S_{2i+2} \prod_{\ell=2}^{k} R_\ell\,.$$

Now 2. follows from 1. □

Lemma 4.5. *Let $\ell \in \mathbb N_{\ge 2}$ and $S \in \mathcal F(\mathbb Z)$ be a sequence of length $|S| = \ell$. Suppose that every element from S is co-prime to ℓ. Then for every $m \in \mathbb Z$ there exists a subsequence S_m such that $\sigma(S_m) \equiv m \pmod{\ell}$. Moreover, if $m \notin \ell\mathbb Z$, then we get $S_m \ne S$.*

Proof. Let $\varphi \colon \mathbb Z \to \mathbb Z/\ell\mathbb Z$ be the canonical epimorphism and $\varphi(S) = a_1 \cdot \ldots \cdot a_l$. We denote by $A = \{a_1, 0\} + \ldots + \{a_{\ell-1}, 0\} \subset \mathbb Z/\ell\mathbb Z$ the sumset, and by $H = \operatorname{Stab}(A)$ the stabilizer of A. Clearly, it suffices to verify that $A = \mathbb Z/\ell\mathbb Z$. If H would be a proper subgroup of $\mathbb Z/\ell\mathbb Z$, then Kneser's Theorem would imply that

$$|A| \ge \sum_{i=1}^{\ell-1} |\{a_i, 0\} + H| - (\ell-2)|H| = (\ell-1)2|H| - (\ell-2)|H| \ge \ell\,,$$

whence $A = H = Z/\ell\mathbb Z$. Thus $H = \mathbb Z/\ell\mathbb Z$, which implies that $A = \mathbb Z/\ell\mathbb Z$, and we are done. □

Lemma 4.6. *Let $t, \ell \in [2, k-1]$ with $t < \ell$ and $d = \gcd(t, \ell) < t$, and let $u \in [2, \mathsf m(S)]$. If*

$$\frac{(t-d)p - \ell}{t\ell} \le \mathsf m(S) \le \frac{dp}{\ell} - t(u-1)\,, \quad \text{then} \quad |R_t| = 0 \quad \text{or} \quad |R_\ell| \le \frac{p - \ell\mathsf m(S) - 2\ell + 1}{u} + 2\ell - 1\,.$$

Proof. Suppose that $|R_t| > 0$. Let $x \in [1, p-1]$ such that $(xg) \mid R_t$, and let $x_\ell \in [1, p-1]$ such that $p \mid (\ell x - x_\ell)$. By the definition of R_t, we get

$$x_\ell \in \bigcup_{i \in [1, t-1] \text{ with } \gcd(i,t)=1} \Big[\frac{\ell i p + \ell}{t}, \frac{\ell i p + \ell \mathsf m(S)}{t}\Big]\,,$$

and thus,

$$x_\ell \in \bigcup_{i \in [1, t-1] \text{ with } d \mid i} \Big[\frac{ip + \ell}{t}, \frac{ip + \ell\mathsf m(S)}{t}\Big] \subset \Big[\frac{dp + \ell}{t}, \frac{(t-d)p + \ell \mathsf m(S)}{t}\Big] \subset \Big[p - \ell\mathsf m(S), p - \ell(u-1)\Big]\,.$$

If $|(\ell R_\ell)([1, u-1], g)| \ge \ell$, then, by Lemma 4.5 and the definition of R_t, we may choose a subsequence W of R_ℓ of length at most ℓ such that $(\ell W)([1, u-1], g) = \ell W$ and $x_\ell + p\|\ell W\|_g \equiv p$

(mod ℓ). Since $p\|\ell W\|_g \le \ell(u-1)$, we have $x_\ell+p\|\ell W\|_g \in [p-\ell\mathsf{m}(S), p]$. Thus, we can construct a subsequence of $(xg)WS_1$ of index 1, a contradiction. Therefore,

$$|(\ell R_\ell)([1, u-1], g)| \le \ell - 1. \tag{5}$$

If $|R_\ell| < \ell$ then we are done. Otherwise, by Lemma 4.5, we get a subsequence R_0 of R_ℓ with $p\|\ell R_0\|_p \equiv p \pmod{\ell}$ and

$$|R_0| \ge |R_\ell| - \ell. \tag{6}$$

We assert that

$$p\|\ell R_0\|_p \le p - \ell\mathsf{m}(S) - \ell. \tag{7}$$

Assume to the contrary that $p\|\ell R_0\|_p \ge p - \ell\mathsf{m}(S)$, choose T to be the minimal subsequence of R_0 such that $p\|\ell T\|_g \ge p - \ell\mathsf{m}(S)$ and $p\|\ell T\|_g \equiv p \pmod{\ell}$. If $p\|\ell T\|_g \le p$, then we can construct a subsequence of TS_1 with index 1, a contradiction. Now suppose that $p\|\ell T\|_g > p$. If $y \in [1, p-1]$ such that $(yg) \mid R_\ell$ and $y_\ell \in [1, p-1]$ such that $p \mid (\ell y - y_\ell)$, then $y_\ell \in [1, \mathsf{m}(S)]$ and $\gcd(y_\ell, \ell) = 1$. By Lemma 4.5, by dropping at most ℓ terms from T, we get a proper subsequence $\tilde{T}$ such that $p\|\ell\tilde{T}\|_g \ge p - \ell\mathsf{m}(S)$ and $p\|\ell\tilde{T}\|_g \equiv p \pmod{\ell}$, a contradiction to the minimality of T. Therefore, (7) holds.

By (5), we have that $p\|\ell R_0\|_g \ge (\ell-1) + u\big(|R_0| - \ell + 1\big)$. This together with (7) gives that $|R_0| \le \frac{p-\ell\mathsf{m}(S)-2\ell+1}{u} + \ell - 1$. Now the lemma follows from (6). □

Lemma 4.7. *Let $t \in [2, k]$, and let $1 = \alpha_1 < \alpha_2 < \ldots$ denote all positive integers coprime to t. If*

$$\mathsf{m}(S) \le \frac{p - 2t + w\alpha_{u+1} + 2}{t + \sum_{i=2}^{u} \alpha_i} \quad \textit{for some} \quad w, u \in \mathbb{N}_0,$$

then

$$|R_t| \le \frac{p - (t + \sum_{i=2}^{u}\alpha_i)\mathsf{m}(S) - 2t + 2}{\alpha_{u+1}} + \delta_u(u-1)\mathsf{m}(S) + 2t + w \quad \textit{where} \quad \delta_u = \begin{cases} 0 & \textit{for } u = 0 \\ 1 & \textit{for } u \ge 1 \end{cases}$$

Proof. Assume to the contrary that $|R_t|$ is strictly larger than the above bound. Since

$$\mathsf{m}(S) \le \frac{p - 2t + w\alpha_{u+1} + 2}{t + \sum_{i=2}^{u}\alpha_i}, \quad \text{it follows that} \quad |R_t| \ge 2t + 1.$$

By Lemma 4.5, there exists a nonempty subsequence R_0 of R_t with

$$p\|tR_0\|_g \equiv p \pmod{t} \quad \text{and} \quad |R_0| \ge |R_t| - t. \tag{8}$$

Similarly to Lemma 4.6, we can prove that

$$p\|tR_0\|_g \le p - t\mathsf{m}(S) - t. \tag{9}$$

Note that tR_0 contains $\alpha_1 g = g$ at most $t-2$ times, because otherwise we would get

$$\mathsf{m}(S) \ge \mathsf{m}_g(tS) \ge t\mathsf{m}_g(S) + t - 1 > \mathsf{m}_g(S) = \mathsf{m}(S),$$

a contradiction. Since $\mathsf{v}_{\alpha_i g}(S) \le \mathsf{h}(S) \le \mathsf{m}(S)$ for all $i \ge 2$, it follows that

$$p\|tR_0\|_g \ge \alpha_1(t-2) + \Big(\sum_{i=2}^{u}\alpha_i\Big)\mathsf{m}(S) + \alpha_{u+1}\big(|R_0| - (u-1)\mathsf{m}(S) - (t-2)\big).$$

By (9), we have $|R_0| \le \frac{p-(t+\sum\limits_{i=2}^{u} a_i)\mathsf{m}(S)-2t+2}{\alpha_{u+1}} + \delta(u-1)\mathsf{m}(S) + t - 2$. By (8), we derive a contradiction. $\square$

Proof of Theorem 1.4. We use all the notations which have been fixed at the beginning of this section. In particular, we assume to the contrary that there exists a sequence $S \in \mathcal{F}(G^\bullet)$ of length $|S| = p$ which has no subsequence with index 1. We have to derive a contradiction.

Clearly, we have $\mathsf{h}(S) \le \mathsf{m}(S) \le p-1$. Lemma 4.1 implies that, for every $x \in [1, p-1]$ with $(xg) \mid ST^{-1}$, we have $\mathsf{m}(S) + 2 \le x \le p - \mathsf{m}(S) - 1$. Thus it follows that

$$\frac{p-2}{10} \le \mathsf{h}(S) \le \mathsf{m}(S) \le \frac{p-3}{2}\,.$$

We distinguish several cases.

Case 1. $\frac{p-2}{3} \le \mathsf{m}(S) \le \frac{p-3}{2}$.

With $k = 2$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 1 + |R_2|.$$

Applying Lemma 4.7 with $u = 0$ and $w = 6$, we infer that

$$|R_2| \le p - 2\mathsf{m}(S) + 8.$$

It follows that $p \le \mathsf{m}(S) + 1 + |R_2| = \mathsf{m}(S) + 1 + p - 2\mathsf{m}(S) + 8 < p$, a contradiction.

Case 2. $\frac{p+3}{4} \le \mathsf{m}(S) \le \frac{p-4}{3}$.

With $k = 3$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 1 + 2 + 2 + |R_2| + |R_3|\,.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we infer that

$$|R_2| \le \frac{p - 2\mathsf{m}(S) + 28}{3} \quad \text{and} \quad |R_3| \le \frac{p - 3\mathsf{m}(S) + 20}{2}\,.$$

It follows that

$$p \le \mathsf{m}(S) + 5 + \sum_{i=2}^{3} |R_i| = \mathsf{m}(S) + 5 + \frac{p - 2\mathsf{m}(S) + 28}{3} + \frac{p - 3\mathsf{m}(S) + 20}{2} < p\,,$$

a contradiction.

Case 3. $\frac{p-2}{5} \le \mathsf{m}(S) \le \frac{p+1}{4}$.

With $k = 4$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 1 + 2 \cdot 2 + 3 \cdot 2 + |R_2| + |R_3| + |R_4|\,.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we infer that

$$|R_2| \le \frac{p - 2\mathsf{m}(S) + 28}{3}\,, \quad |R_3| \le \frac{p - 3\mathsf{m}(S) + 20}{2} \quad \text{and} \quad |R_4| \le \frac{p - 4\mathsf{m}(S) + 36}{3}\,.$$

It follows that

$$p \le \mathsf{m}(S) + 11 + \frac{p - 2\mathsf{m}(S) + 28}{3} + \frac{p - 3\mathsf{m}(S) + 20}{2} + \frac{p - 4\mathsf{m}(S) + 36}{3} < p\,,$$

a contradiction.

Case 4. $\frac{p-1}{6} \le \mathsf{m}(S) \le \frac{p-3}{5}$.

With $k=5$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 27 + \sum_{i=2}^{5} |R_i| \,.$$

Applying Lemma 4.7 with $u=1$ and $w=6$, we infer that

$$|R_2| \le \frac{p-2\mathsf{m}(S)+28}{3}\,, \quad |R_3| \le \frac{p-3\mathsf{m}(S)+20}{2}\,,$$
$$|R_4| \le \frac{p-4\mathsf{m}(S)+36}{3}\,, \quad |R_5| \le \frac{p-5\mathsf{m}(S)+24}{2}\,.$$

Applying Lemma 4.6 with $t=2, \ell=3$ and $u=12$, we obtain that either

$$|R_2| = 0 \quad \text{or} \quad |R_3| \le \frac{p-3\mathsf{m}(S)+55}{12}\,,$$

and therefore

$$|R_2|+|R_3| \le \max\{\frac{p-2\mathsf{m}(S)+28}{3} + \frac{p-3\mathsf{m}(S)+55}{12}, \frac{p-3\mathsf{m}(S)+20}{2}\} = \frac{5p-11\mathsf{m}(S)+167}{12}\,.$$

Summing up we obtain that

$$\begin{aligned} p &\le \mathsf{m}(S) + 27 + \sum_{i=2}^{5} |R_i| = \mathsf{m}(S) + 27 + (|R_2|+|R_3|) + |R_4| + |R_5| \\ &\le \frac{5p-11\mathsf{m}(S)+167}{12} + \frac{p-4\mathsf{m}(S)+36}{3} + \frac{p-5\mathsf{m}(S)+24}{2} + 27 < p\,, \end{aligned}$$

a contradiction.

Case 5. $\frac{p-5}{7} \le \mathsf{m}(S) \le \frac{p-5}{6}$.

With $k=6$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 37 + \sum_{i=2}^{6} |R_i| \,.$$

Applying Lemma 4.7 with $u=2$ and $w=0$, we infer that

$$|R_2| \le \frac{p+18}{5} \quad \text{and} \quad |R_3| \le \frac{p-\mathsf{m}(S)+20}{4}\,.$$

Applying Lemma 4.7 with $u=1$ and $w=6$, we infer that

$$|R_4| \le \frac{p-4\mathsf{m}(S)+36}{3}\,, \quad |R_5| \le \frac{p-5\mathsf{m}(S)+24}{2}\,, \quad \text{and} \quad |R_6| \le \frac{p-6\mathsf{m}(S)+80}{5}\,.$$

Summing up we obtain that

$$\begin{aligned} p &\le \mathsf{m}(S) + 37 + \sum_{i=2}^{6} |R_i| \\ &= \mathsf{m}(S) + 37 + \frac{p+18}{5} + \frac{p-\mathsf{m}(S)+20}{4} + \frac{p-4\mathsf{m}(S)+36}{3} + \frac{p-5\mathsf{m}(S)+24}{2} + \frac{p-6\mathsf{m}(S)+80}{5} \\ &< p\,, \end{aligned}$$

a contradiction.

Case 6. $\frac{p-2}{8} \le \mathsf{m}(S) \le \frac{p-3}{7}$.

With $k = 7$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 73 + \sum_{i=2}^{7} |R_i| \,.$$

Applying Lemma 4.7 with $u = 2$ and $w = 0$, we infer that

$$|R_2| \le \frac{p+18}{5} \quad \text{and} \quad |R_3| \le \frac{p - \mathsf{m}(S) + 20}{4} \,.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we infer that

$$|R_4| \le \frac{p - 4\mathsf{m}(S) + 36}{3} \;, \quad |R_5| \le \frac{p - 5\mathsf{m}(S) + 24}{2} \;,$$
$$|R_6| \le \frac{p - 6\mathsf{m}(S) + 80}{5} \;, \quad |R_7| \le \frac{p - 7\mathsf{m}(S) + 28}{2} \,.$$

Applying Lemma 4.6, with $t = 2$, $\ell = 5$ and $u = 10$, we infer that

$$|R_2| + |R_5| \le \max\{\frac{p - 5\mathsf{m}(S) + 4}{2}, \frac{p+18}{5} + \frac{p - 5\mathsf{m}(S) - 9}{10} + 9\} = \frac{3p - 5\mathsf{m}(S) + 117}{10} \,.$$

Summing up we obtain that

$$\begin{aligned} p &\le \mathsf{m}(S) + 73 + \sum_{i=2}^{7} |R_i| = \mathsf{m}(S) + 73 + (|R_2| + |R_5|) + |R_3| + |R_4| + |R_6| + |R_7| \\ &\le \mathsf{m}(S) + 73 + \frac{3p - 5\mathsf{m}(S) + 117}{10} + \frac{p - \mathsf{m}(S) + 20}{4} + \frac{p - 4\mathsf{m}(S) + 36}{3} \\ &\quad + \frac{p - 6\mathsf{m}(S) + 80}{5} + \frac{p - 7\mathsf{m}(S) + 28}{2} < p \,, \end{aligned}$$

a contradiction.

Case 7. $\frac{p-2}{9} \le \mathsf{m}(S) \le \frac{p-3}{8}$.

With $k = 8$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S) + 111 + \sum_{i=2}^{8} |R_i| \,.$$

Applying Lemma 4.7 with $u = 2$ and $w = 0$, we infer that

$$|R_2| \le \frac{p+18}{5} \;, \quad |R_3| \le \frac{p - \mathsf{m}(S) + 20}{4} \;,$$
$$|R_4| \le \frac{p - 2\mathsf{m}(S) + 34}{5} \;, \quad |R_5| \le \frac{p - 4\mathsf{m}(S) + 22}{3} \,.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we infer that

$$|R_6| \le \frac{p - 6\mathsf{m}(S) + 80}{5} \;, \quad |R_7| \le \frac{p - 7\mathsf{m}(S) + 28}{2} \quad \text{and} \quad |R_8| \le \frac{p - 8\mathsf{m}(S) + 52}{3} \,.$$

Applying Lemma 4.6 with $t = 2$, $\ell \in \{5, 7\}$ and $u = 20$, we can prove that either

$$|R_2| = 0 \quad \text{or} \quad |R_i| \le \frac{p - i\mathsf{m}(S) - 2i + 1}{20} + 2i - 1 \quad \text{for} \quad i \in \{5, 7\} \,,$$

and therefore

$$\begin{aligned}|R_2|+|R_5|+|R_7| \le \max\Big\{&\frac{p-4\mathsf{m}(S)+22}{3}+\frac{p-7\mathsf{m}(S)+28}{2},\\ &\frac{p-\mathsf{m}(S)+20}{4}+\frac{p-5\mathsf{m}(S)-9}{20}+9+\frac{p-7\mathsf{m}(S)-13}{20}+13\Big\}\\ =&\ \frac{5p-29\mathsf{m}(S)+128}{6}\,.\end{aligned}$$

Applying Lemma 4.6 with $t=4, \ell=6$ and $u=10$, we obtain that either

$$|R_4|=0 \quad \text{or} \quad |R_6| \le \frac{p-6\mathsf{m}(S)-11}{10}+11$$

and therefore

$$|R_4|+|R_6| \le \max\Big\{\frac{p-2\mathsf{m}(S)+34}{5}+\frac{p-6\mathsf{m}(S)-11}{10}+11, \frac{p-6\mathsf{m}(S)+80}{5}\Big\} = \frac{3p-10\mathsf{m}(S)+167}{10}\,.$$

Summing up we obtain that

$$\begin{aligned}p &\le \mathsf{m}(S)+111+\sum_{i=2}^{8}|R_i| = \mathsf{m}(S)+111+(|R_2|+|R_5|+|R_7|)+(|R_4|+|R_6|)+|R_3|+|R_8|\\ &\le \mathsf{m}(S)+111+\frac{5p-29\mathsf{m}(S)+128}{6}+\frac{3p-10\mathsf{m}(S)+167}{10}+\frac{p-\mathsf{m}(S)+20}{4}+\frac{p-8\mathsf{m}(S)+52}{3}\\ &< p\,,\end{aligned}$$

a contradiction.

Case 8. $\frac{p-2}{10} \le \mathsf{m}(S) \le \frac{p-4}{9}$.

With $k=9$ in Lemma 4.4, we have

$$p \le \mathsf{m}(S)+159+\sum_{i=2}^{9}|R_i|\,.$$

Applying Lemma 4.7 with $u=2$ and $w=0$, we infer that

$$|R_2| \le \frac{p+18}{5}\,, \quad |R_3| \le \frac{p-\mathsf{m}(S)+20}{4}\,,$$
$$|R_4| \le \frac{p-2\mathsf{m}(S)+34}{5}\,, \quad |R_5| \le \frac{p-4\mathsf{m}(S)+22}{3}\,.$$

Applying 4.7 with $u=1$ and $w=6$, we infer that

$$|R_6| \le \frac{p-6\mathsf{m}(S)+80}{5}\,, \quad |R_7| \le \frac{p-7\mathsf{m}(S)+28}{2}\,,$$
$$|R_8| \le \frac{p-8\mathsf{m}(S)+52}{3}\,, \quad |R_9| \le \frac{p-9\mathsf{m}(S)+32}{2}\,.$$

Applying Lemma 4.6 with $t=2, \ell \in \{5,7\}$ and $u=10$, we obtain that either

$$|R_2|=0 \quad \text{or} \quad |R_i| \le \frac{p-i\mathsf{m}(S)-2i+1}{10}+2i-1 \quad \text{for} \quad i \in \{5,7\}\,,$$

and therefore

$$\begin{aligned}|R_2|+|R_5|+|R_7| \le \max\Big\{&\frac{p-4\mathsf{m}(S)+22}{3}+\frac{p-7\mathsf{m}(S)+28}{2},\\ &\frac{p+18}{5}+\frac{p-5\mathsf{m}(S)-9}{10}+9+\frac{p-7\mathsf{m}(S)-13}{10}+13\Big\} = \frac{5p-29\mathsf{m}(S)+128}{6}\,.\end{aligned}$$

Applying Lemma 4.6 with $t=3, \ell=8$ and $u=5$, we obtain that either

$$|R_3|=0 \quad \text{or} \quad |R_8| \le \frac{p-8\mathsf m(S)-15}{8}+15\,,$$

and therefore

$$|R_3|+|R_8| \le \max\Big\{\frac{p-\mathsf m(S)+20}{4}+\frac{p-8\mathsf m(S)-15}{8}+15, \frac{p-8\mathsf m(S)+52}{3}\Big\} = \frac{3p-10\mathsf m(S)}{8}+20\,.$$

Summing up we obtain that

$$\begin{aligned} p \le \mathsf m(S)+159+\sum_{i=2}^{9}|R_i| &= M+159+(|R_2|+|R_5|+|R_7|)+(|R_3|+|R_8|)+|R_4|+|R_6|+|R_9| \\ \le \mathsf m(S)+159+\frac{5p-29\mathsf m(S)+128}{6}+(\frac{3p-10\mathsf m(S)}{8}+20) & \\ +\frac{p-2\mathsf m(S)+34}{5}+\frac{p-6\mathsf m(S)+80}{5}+\frac{p-9\mathsf m(S)+32}{2} < p\,, & \end{aligned}$$

a contradiction. □

5. A Conjecture and Open Problems

In spite of Theorem 1.2 and in view of Lemma 3.1, we formulate a conjecture which sharpens the original Lemke-Kleitman Conjecture for prime cyclic groups.

Conjecture 5.1. *Let G be a cyclic group of prime order and S be a sequence over G of length $|S|=|G|$. Then S has a subsequence T with* $\operatorname{ind}(T)=1$ *and length $|T|\in[1,\mathsf h(S)]$.*

Let G be a cyclic group of order $n\ge 2$. We denote by

- $\mathsf t(n)$ the smallest integer $\ell\in\mathbb N$ such that every sequence S over G of length $|S|\ge\ell$ has a subsequence T with $\operatorname{ind}(T)=1$.
- $\mathsf T(n)$ the smallest integer $\ell\in\mathbb N$ such that every squarefree sequence S over G of length $|S|\ge\ell$ has a subsequence T with $\operatorname{ind}(T)=1$.

By Theorem 1.2, it follows that $\mathsf t(n)\ge n+\lfloor\frac{n}{4}\rfloor-4$ for $n=4k+2\ge 22$.

Open Problem. *Determine* $\mathsf t(n)$ *and* $\mathsf T(n)$ *for all* $n\ge 2$.

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