Partition Identities for Ramanujan's Third Order Mock Theta Functions

William Y. C. Chen[1], Kathy Q. Ji[2], and Eric H. Liu[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

[1]chen@nankai.edu.cn, [2]ji@nankai.edu.cn, [3]eric@cfc.nankai.edu.cn

Abstract. We find two involutions on partitions that lead to partition identities for Ramanujan's third order mock theta functions $\phi(-q)$ and $\psi(-q)$. We also give an involution for Fine's partition identity on the mock theta function $f(q)$. The two classical identities of Ramanujan on third order mock theta functions are consequences of these partition identities. Our combinatorial constructions also apply to Andrews' generalizations of Ramanujan's identities.

Keywords: mock theta function, Ramanujan's identities, partition identity, Fine's theorem, involution.

AMS Classifications: 05A17, 11P81

1 Introduction

This paper is concerned with the following three mock theta functions of order 3 defined by Ramanujan,

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.1}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{1.2}$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \tag{1.3}$$

Mock theta functions have been extensively studied, see, for example, Andrews [8], Fine [17, Chapters 2-3], Gordon and McIntosh [18], and Ono [19]. These functions not only have remarkable analytic properties, but also are closely connected to the theory of partitions, see, for example, Agarwal [1], Andrews [4, 9], Andrews and Garvan [7], Andrews, Eriksson, Petrov and Romik [10], and Choi and Kim [14].

In this paper, we find two involutions on partitions that imply two partition identities for Ramanujan's third order mock theta functions $\phi(-q)$ and $\psi(-q)$. We also give an involution for Fine's partition theorem on the mock theta function $f(q)$. These three partition identities lead to the following two identities (1.4) and (1.5) of Ramanujan

$$\phi(-q) - 2\psi(-q) = f(q), \tag{1.4}$$

$$\phi(-q) + 2\psi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty^2}, \tag{1.5}$$

where we have adopted the standard notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The first proofs of (1.4) and (1.5) were given by Watson [20]. Fine [17, p. 60] found another proof by using transformation formulas.

Andrews [3] defined the following functions as generalizations of Ramanujan's mock theta functions and he later found that these generalizations were already in Ramanujan's "lost" notebook [6],

$$f(\alpha;q) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}\alpha^n}{(-q;q)_n(-\alpha;q)_n}, \tag{1.6}$$

$$\phi(\alpha;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\alpha q;q^2)_n}, \tag{1.7}$$

$$\psi(\alpha;q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\alpha;q^2)_n}. \tag{1.8}$$

When $\alpha = q$, the above functions reduce to Ramanujan's mock theta functions. Furthermore, Andrews showed these three functions turn out to be mock theta functions for $\alpha = q^r$, where r is any positive integer. More importantly, Ramanujan's identities (1.4) and (1.5) can be extended to the functions $f(\alpha;q)$, $\phi(\alpha;q)$ and $\psi(\alpha;q)$,

$$\phi(-\alpha;-q) - (1+\alpha q^{-1})\psi(-\alpha;-q) = f(\alpha;q), \tag{1.9}$$

$$\phi(-\alpha;-q) + (1+\alpha q^{-1})\psi(-\alpha;-q) = \frac{(q;q)_\infty}{(-q;q)_\infty(-\alpha;q)_\infty}, \tag{1.10}$$

see Andrews [3, p. 78, Eqs. (3a)–(3b)]. Clearly, the above identities (1.9) and (1.10) specialize to (1.4) and (1.5) by setting $\alpha = q$.

The connection between Ramanujan's third order mock theta function and the theory of partitions was first explored by Fine. In [17, p. 55, Chapter 3], he derived the following identity from his transformation formula:

$$f(q) = 1 + \sum_{k\geq 1} \frac{(-1)^{k-1}q^k}{(-q;q)_k} = 1 + \frac{1}{(-q;q)_\infty} \sum_{k\geq 1} (-1)^{k-1} q^k (-q^{k+1};q)_\infty. \tag{1.11}$$

In fact, (1.11) can be easily established from the combinatorial definition (2.1) of $f(q)$. The following partition identity for $f(q)$ can be deduced from (1.11).

Theorem 1.1 (Fine) *Let $p_{do}(n)$ denote the number of partitions of n into distinct parts with the smallest part being odd. Then*

$$(-q;q)_\infty f(q) = 1 + 2\sum_{n\geq 1} p_{do}(n) q^n. \tag{1.12}$$

The following partition identities can be derived from our involutions.

Theorem 1.2 *We have*

$$(-q;q)_\infty \phi(-q) = 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{1.13}$$

Theorem 1.3 *We have*

$$2(-q;q)_\infty \psi(-q) = -\sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{1.14}$$

It can be seen that the above partition identities lead to Ramanujan's identities (1.4) and (1.5). It follows from Theorem 1.2 and Theorem 1.3 that

$$\begin{aligned} &(-q;q)_\infty \phi(-q) \mp 2(-q;q)_\infty \psi(-q) \\ &= 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\ &\quad \mp \left(-\sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \right). \end{aligned}$$

Using Theorem 1.1 and Gauss' identity

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{1.15}$$

we arrive at Ramanujan's identities (1.4) and (1.5).

In fact, we can deduce two partition identities for $\phi(-q)$ and $\psi(-q)$ analogous to Fine's identity for $f(q)$ by employing the following partition theorem of Bessenrodt and Pak [12] which extends a theorem of Fine in [16, Theorem 5]. It is worth mentioning that there are other involutions which also imply this partition theorem, see, Berndt, Kim and Yee [11], Chen and Liu [13], and Yee [21, 22].

Theorem 1.4 (Bessenrodt and Pak) *Let $p_{do}^e(n)$ ($p_{do}^o(n)$) denote the number of partitions of n into even (odd) distinct parts with the smallest part being odd. Then*

$$\sum_{n=1}^{\infty}[p_{do}^e(n) - p_{do}^o(n)]q^n = \sum_{k=1}^{\infty}(-1)^k q^{k^2}. \tag{1.16}$$

In light of Theorem 1.4, we deduce the following two partition identities from Theorems 1.2 and 1.3 respectively.

Theorem 1.5 *We have*

$$(-q;q)_\infty \phi(-q) = 1 + 2\sum_{n\geq 1} p_{do}^e(n)q^n. \tag{1.17}$$

Theorem 1.6 *We have*

$$(-q;q)_\infty \psi(-q) = -\sum_{n\geq 1} p_{do}^o(n)q^n. \tag{1.18}$$

This paper is organized as follows. In Section 2, we provide an involution for Fine's theorem 1.1. In Sections 3 and 4, we give two involutions that imply the partition identities for $\phi(-q)$ and $\psi(-q)$ stated in Theorems 1.2 and 1.3. Section 5 is devoted to partition identities for $f(\alpha q;q), \phi(-\alpha q;-q)$, and $\psi(-\alpha q;-q)$ based on our involutions, which lead to Andrews' identities (1.9) and (1.10).

2 An involution for Fine's theorem

In this section we give an involution for Fine's partition theorem. Let P denote the set of partitions, and let D denote the set of partitions with distinct parts. For technical reasons, it is necessary to assume that the sets P and D contain the empty partition. The rank of a partition λ, denoted by $r(\lambda)$, is defined as the largest part minus the number of parts, as introduced by Dyson [15]. The empty partition is assumed to have rank zero. The sum of parts of a partition λ is denoted by $|\lambda|$. A pair of partitions (λ,μ) is called a bipartition of n if $|\lambda|+|\mu|=n$. Fine [17, p.49] found the following combinatorial interpretation for $f(q)$,

$$f(q) = \sum_{\lambda\in P}(-1)^{r(\lambda)}q^{|\lambda|}. \tag{2.1}$$

Thus,

$$(-q;q)_\infty f(q) = \sum_{(\lambda,\mu)\in P\times D}(-1)^{r(\lambda)}q^{|\lambda|+|\mu|}. \tag{2.2}$$

Therefore, to prove Theorem 1.1, it suffices to establish the following relation

$$\sum_{(\lambda,\mu)\in P\times D} (-1)^{r(\lambda)} q^{|\lambda|+|\mu|} = 1 + 2\sum_{n\geq 1} p_{do}(n) q^n. \tag{2.3}$$

We shall construct a sign reversing involution to justify (2.3). Recall that an involution τ on a set A is a bijection on A such that τ^2 is the identity map. If any element a in A is given a sign which is either plus or minus, then an involution is called sign reversing provided that $\tau(a)$ and a have opposite signs. For the purpose of constructing sign reversing involutions, throughout this paper, all involutions are assumed to contain no fixed points.

Proof of Theorem 1.1. For a bipartition $(\lambda,\mu)\in P\times D$, let $s(\mu)$ denote the smallest part of μ, $m(\lambda)$ denote the number of occurrences of the largest part of λ and $l(\lambda)$ denote the number of parts of λ. We set $s(\mu)=+\infty$ if $\mu=\emptyset$ and $m(\lambda)=+\infty$ if $\lambda=\emptyset$. Let U be the set of the empty bipartition $(\emptyset,\emptyset)$, bipartitions $(\emptyset,\mu)\in P\times D$ with $s(\mu)$ being odd and bipartitions $(\lambda,\mu)\in P\times D$, where $\lambda=(1,1,\ldots,1)$, $l(\lambda)$ is odd, and $s(\mu)>l(\lambda)$. Let V_n be the set of bipartitions (λ,μ) of n that are in $P\times D$ but not in U.

We claim that the weighted sum of the set U equals the right-hand side of (2.3), namely,

$$\sum_{(\lambda,\mu)\in U} (-1)^{r(\lambda)} q^{|\lambda|+|\mu|} = 1 + 2\sum_{n\geq 1} p_{do}(n) q^n. \tag{2.4}$$

For a bipartition $(\emptyset,\mu)\in U$, by definition, μ is a partition into distinct parts with the smallest part being odd. For a bipartition $(\lambda,\mu)\in U$, where $\lambda=(1,1,\ldots,1)$, since $l(\lambda)$ is odd, we see that $r(\lambda)$ is even. Thus we get partition $\nu\in D$ with $s(\nu)$ being odd by moving all the parts of λ to μ as a single part. Hence we have verified (2.4).

We now proceed to construct a sign reversing involution Υ on V_n so that the weighted sum of V_n vanishes. Hence the theorem is proved. To do so, let $(\lambda,\mu)\in V_n$. We define $\Upsilon(\lambda,\mu)$ by the following procedure. We consider two cases.

(1) If $s(\mu)\leq m(\lambda)$, then set $\Upsilon(\lambda,\mu)$ to be the bipartition obtained from (λ,μ) by deleting the smallest part in μ and adding 1 to each of the first $s(\mu)$ parts $\lambda_1,\lambda_2,\ldots,\lambda_{s(\mu)}$ of λ.

(2) If $s(\mu)>m(\lambda)$, then set $\Upsilon(\lambda,\mu)$ to be the bipartition obtained from (λ,μ) by subtracting 1 from each of the first $m(\lambda)$ parts $\lambda_1,\lambda_2,\ldots,\lambda_{m(\lambda)}$ of λ and adding a part of size $m(\lambda)$ to μ.

It is straightforward to check that the above mapping Υ is an involution on V_n and changes the parity of the rank of λ. This completes the proof. ∎

For example, there are eight bipartitions $(\lambda,\mu)\in P\times D$ of 4 with the rank of λ being even,

$$((3),(1)),((2,1),(1)),((1,1,1),(1)),((1),(2,1)),((2,2),\emptyset),(\emptyset,(4)),((1),(3)),(\emptyset,(3,1)),$$

while there are six bipartitions $(\lambda, \mu) \in P \times D$ of 4 with the rank of λ being odd,

$$((4), \emptyset), ((3,1), \emptyset), ((2,1,1), \emptyset), ((2),(2)), ((1,1),(2)), ((1,1,1,1), \emptyset),$$

and there is only one partition of 4 into distinct parts with the smallest part being odd, i.e., $(3,1)$. Meanwhile, the set U includes bipartitions $((1),(3))$ and $(\emptyset,(3,1))$ which both correspond to $(3,1)$.

The involution Υ yields the following correspondence:

$$\begin{aligned} &((3),(1)) \leftrightarrow ((4),\emptyset), \quad ((2,1),(1)) \leftrightarrow ((3,1),\emptyset), \quad ((1,1,1),(1)) \leftrightarrow ((2,1,1),\emptyset), \\ &((1),(2,1)) \leftrightarrow ((2),(2)), \quad ((2,2),\emptyset) \leftrightarrow ((1,1),(2)), \quad (\emptyset,(4)) \leftrightarrow ((1,1,1,1),\emptyset). \end{aligned}$$

3 A partition identity for $\phi(-q)$

In this section, we shall prove the partition identity for $\phi(-q)$ as stated in Theorem 1.2. Let us begin with an interpretation of $\phi(-q)$ given by Fine [17, p.49]. Let DO denote the set of partitions with distinct odd parts. Like the conventions for P and D in the previous section, we assume that the set DO contains the empty partition and we further set $(\lambda_1+1)/2 = 0$ if $\lambda = \emptyset$. In this notation, Fine's combinatorial interpretation of $\phi(-q)$ asserts that

$$\phi(-q) = \sum_{\lambda \in DO} (-1)^{\frac{\lambda_1+1}{2}} q^{|\lambda|}. \tag{3.1}$$

Note that Choi and Kim found another interpretation of $\phi(q)$ in terms of n-color partitions [14, Theorem 3.1]. From (3.1) we have

$$(-q;q)_\infty \phi(-q) = \sum_{(\lambda,\mu) \in DO \times D} (-1)^{\frac{\lambda_1+1}{2}} q^{|\lambda|+|\mu|}. \tag{3.2}$$

Therefore, Theorem 1.2 can be deduced from the following relation

$$\sum_{(\lambda,\mu) \in DO \times D} (-1)^{\frac{\lambda_1+1}{2}} q^{|\lambda|+|\mu|} = 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{3.3}$$

We shall construct a sign reversing involution Φ for (3.3). In order to deal with the sum $\sum_{k \geq 1} (-1)^k q^{k^2}$ on the right-hand side of (3.3), special attention has to be paid to partitions of the form $Q_k = (2k-1, 2k-3, \ldots, 1)$. Notice that Q_k is a partition of k^2.

Proof of Theorem 1.2. The right-hand side of (3.3) corresponds to the following bipartitions: (1) The empty bipartition $(\emptyset, \emptyset)$. (2) Bipartitions $(\emptyset, \mu) \in DO \times D$ with $s(\mu)$ being odd. (3) Bipartitions $(Q_k, \emptyset)$ for $k \geq 1$. Let W denote the set consisting of all such bipartitions, and let X_n denote the set of bipartititions of n in $DO \times D$ but not in W. We aim to construct an involution Φ on X_n.

To describe the involution Φ, we introduce some notation. Assume that $\lambda \in DO$. We define $c(\lambda)$ to be the maximum number of consecutive odd parts of λ starting with the first part. For example, $c(11, 9, 7, 3, 1) = 3$. For a partition $\mu \in D$, define $s_o(\mu)$ ($s_e(\mu)$) to be the smallest odd (even) part of μ, where we set $s_o(\mu) = +\infty$ ($s_e(\mu) = +\infty$) if μ has no odd (even) part.

The map Φ consists of two parts. Let (λ, μ) be a bipartition in X_n. We shall denote $\Phi(\lambda, \mu)$ by (λ', μ').

Part I of Φ. If $2c(\lambda) \geq s_e(\mu)$, then λ' is obtained from λ by adding 2 to each of the first $s_e(\mu)/2$ parts $\lambda_1, \lambda_2, \cdots, \lambda_{s_e(\mu)/2}$ of λ, and μ' is obtained from μ by removing its smallest even part.

If $2c(\lambda) < s_e(\mu)$ and $\lambda_{c(\lambda)} > 1$, then λ' is obtained from λ by subtracting 2 from each of the first $c(\lambda)$ parts $\lambda_1, \lambda_2, \ldots, \lambda_{c(\lambda)}$ of λ, and μ' is obtained from μ by adding a part of size $2c(\lambda)$.

It is easy to see the above procedure is well defined and the remaining bipartitions that are not covered in this part are those bipartitions (Q_k, μ) with $s_e(\mu) > 2k$ for $k \geq 1$ and $(\emptyset, \mu)$ with $s(\mu)$ being even. This is the task of the second part of the involution. For convenience, we set $Q_0 = \emptyset$.

Part II of Φ. If $s_e(\mu) > s_o(\mu) + (2k - 1)$, then set $\lambda' = Q_{k-1}$ and set μ' to be the partition obtained from μ by deleting its smallest odd part and adding an even part of size $s_o(\mu) + (2k - 1)$ to μ.

If $s_e(\mu) \leq s_o(\mu) + (2k - 1)$, then $\lambda' = Q_{k+1}$ and μ' is obtained from μ by deleting its smallest even part and adding an odd part of size $s_e(\mu) - (2k + 1)$.

It is readily seen that Φ is an involution on X_n and it changes the parity of $(\lambda_1 + 1)/2$. This completes the proof. ∎

For example, when $n = 7$, there are six bipartitions $(\lambda, \mu) \in DO \times D$ such that $(\lambda_1 + 1)/2$ is odd,

$$((5), (2)), ((5, 1), (1)), ((1), (6)), ((1), (4, 2)), ((1), (3, 2, 1)), ((1), (5, 1)),$$

and there are ten bipartitions $(\lambda, \mu) \in DO \times D$ such that $(\lambda_1 + 1)/2$ is even,

$$\begin{array}{lllll} ((7), \emptyset), & ((3, 1), (2, 1)), & ((3, 1), (3)), & ((3), (4)), & ((3), (3, 1)), \\ (\emptyset, (7)), & (\emptyset, (6, 1)), & (\emptyset, (5, 2)), & (\emptyset, (4, 3)), & (\emptyset, (4, 2, 1)). \end{array}$$

Whereas the set W consists of bipartitions $(\emptyset, (7)), (\emptyset, (6, 1)), (\emptyset, (4, 3)), (\emptyset, (4, 2, 1))$.

The involution Φ gives the following pairs of bipartitions:

$$\begin{array}{lll} ((5), (2)) \leftrightarrow ((7), \emptyset), & ((5, 1), (1)) \leftrightarrow ((3, 1), (2, 1)), & ((1), (6)) \leftrightarrow ((3, 1), (3)), \\ ((1), (4, 2)) \leftrightarrow ((3), (4)), & ((1), (3, 2, 1)) \leftrightarrow ((3), (3, 1)) & ((1), (5, 1)) \leftrightarrow (\emptyset, (5, 2)). \end{array}$$

4 A partition identity for $\psi(-q)$

The objective of this section is to prove Theorem 1.3 for $\psi(-q)$. There is also a combinatorial interpretation of $\psi(-q)$ given by Fine [17, p.49]. Let OC denote the set of partitions into odd parts without gaps. For obvious reasons, here we need to assume that the set OC does not contain the empty partition. Fine [17, p.49] showed that

$$\psi(-q)=\sum_{\lambda\in OC}(-1)^{l(\lambda)}q^{|\lambda|}. \tag{4.1}$$

Note that Agarwal [1, 2] found two combinatorial interpretations for $\psi(q)$ by using q-difference equations. Let D^0 denote the set of partitions with distinct parts where the zero part is allowed. So the number of partitions of n in the set D^0 is twice the number of partitions of n in the set D. Hence we have

$$2(-q;q)_\infty\psi(-q)=\sum_{(\lambda,\mu)\in OC\times D^0}(-1)^{l(\lambda)}q^{|\lambda|+|\mu|}. \tag{4.2}$$

Thus Theorem 1.3 can be rewritten as follows

$$\sum_{(\lambda,\mu)\in OC\times D^0}(-1)^{l(\lambda)}q^{|\lambda|+|\mu|}=-\sum_{n=1}^{\infty}p_{do}(n)q^n+\sum_{k=1}^{\infty}(-1)^kq^{k^2}. \tag{4.3}$$

Proof of Theorem 1.3. Let Y denote the set of bipartitions $(Q_k,\emptyset)$ and bipartitions $((1),\mu)\in OC\times D^0$, where $s_e(\mu)+1<s_o(\mu)$. Let Z_n denote the set of bipartitions (λ,μ) of n in the set $OC\times D^0$ but not in the set Y. We claim that the weighted sum of the set Y equals the right-hand side of (4.3), that is,

$$\sum_{(\lambda,\mu)\in Y}(-1)^{l(\lambda)}q^{|\lambda|+|\mu|}=-\sum_{n=1}^{\infty}p_{do}(n)q^n+\sum_{k=1}^{\infty}(-1)^kq^{k^2}, \tag{4.4}$$

since the bipartitions $((1),\mu)\in Y$ correspond to the sum $-\sum_{n=1}^{\infty}p_{do}(n)q^n$ by adding 1 to the smallest part of μ and the bipartitions $(Q_k,\emptyset)$ correspond to the sum $\sum_{k\geq1}(-1)^kq^{k^2}$. Therefore, in order to prove Theorem 1.3, it suffices to construct an involution Ψ on the set Z_n so that the bipartitions in Z_n cancel each other in the corresponding weighted sum.

Let $r_p(\lambda)$ be the smallest part of λ which occurs at least twice in λ, and let $r_p(\lambda)=+\infty$ if λ has no repeated parts. For $(\lambda,\mu)\in Z_n$, let $(\lambda',\mu')=\Psi(\lambda,\mu)$. The involution Ψ can be defined by the following procedure.

Part I of Ψ. If $s_o(\mu)>r_p(\lambda)$, then let λ' be the partition obtained from λ by deleting a part of size $r_p(\lambda)$ and let μ' be the partition derived from μ by adding a part of size $r_p(\lambda)$. On the other hand, if $s_o(\mu)\leq r_p(\lambda)$, then let λ' be the partition obtained from λ by adding a part of size $s_o(\mu)$ and let μ' be the partition derived from μ by deleting a part of size $s_o(\mu)$.

The above process is well defined and the bipartitions not covered by this part are those bipartitions (Q_k, μ) for which $s_o(\mu) > 2k-1$. We continue to describe the second part of Ψ.

Part II of Ψ. Assume that (Q_k, μ) is a bipartition such that $s_o(\mu) > 2k-1$. If μ has a zero part, then set $\lambda' = Q_{k-1}$ and set μ' to be the partition obtained from μ by adding a part of size $2k-1$ and deleting the zero part.

If μ has no zero part and $s_o(\mu) = 2k+1$, then set $\lambda' = Q_{k+1}$ and set μ' to be the partition obtained from μ by removing a part of size $2k+1$ and adding a zero part.

It can be seen that the above mapping is well defined except for those bipartitions (Q_k, μ) such that μ has no zero part and $s_o(\mu) > 2k+1$. Indeed, it is the purpose of the third part of Ψ to deal with these remaining bipartitions.

Part III of Ψ. If $s_o(\mu) > s_e(\mu) + (2k-1)$, then set $\lambda' = Q_{k-1}$ and set μ' to be the partition obtained from μ by deleting the smallest even part of μ and adding an odd part of size $s_e(\mu) + (2k-1)$.

If $s_o(\mu) \leq s_e(\mu) + (2k-1)$, then set $\lambda' = Q_{k+1}$ and set μ' to be the partition obtained from μ by deleting the smallest odd part of μ and adding an even part of size $s_o(\mu) - (2k+1)$.

It can be checked that the map Ψ is an involution on Z_n and it changes the parity of the length of λ. This completes the proof. ∎

For example, when $n = 4$ there are six bipartitions $(\lambda, \mu) \in OC \times D^0$ such that $l(\lambda)$ is odd,

$$((1),(3)),((1,1,1),(1)),((1,1,1),(1,0)),((1),(2,1)),((1),(2,1,0)),((1),(3,0)).$$

and there are six bipartitions $(\lambda, \mu) \in OC \times D^0$ such that $l(\lambda)$ is even,

$$((3,1),\emptyset),((3,1),(0)),((1,1,1,1),\emptyset),((1,1,1,1),(0)),((1,1),(2)),((1,1),(2,0)).$$

Notice that there are two bipartitions $((3,1),\emptyset)$ and $((1),(3,0))$ in Y corresponding to the partition $(3,1)$.

The involution Ψ is illustrated as follows:

$$((1),(3)) \leftrightarrow ((3,1),(0)), \quad ((1,1,1),(1)) \leftrightarrow ((1,1,1,1),\emptyset), \quad ((1),(2,1)) \leftrightarrow ((1,1),(2)),$$
$$((1),(2,1,0)) \leftrightarrow ((1,1),(2,0)), \quad ((1,1,1),(1,0)) \leftrightarrow ((1,1,1,1),(0)).$$

As another example, the involution Ψ gives the following correspondence:

$$((9,7,5,3,1),(16,15,8,6,2)) \leftrightarrow ((7,5,3,1),(16,15,11,8,6)).$$

This example involves the Part III in the construction of Ψ.

5 Andrews' generalizations

This section is devoted to proofs of Andrews' identities (1.9) and (1.10). First, we give combinatorial interpretations for $f(\alpha q; q), \phi(-\alpha q; -q)$, and $\psi(-\alpha q; -q)$ by extending the arguments of Fine. More precisely, we have the following partition theoretic interpretations.

Theorem 5.1 *We have*

$$f(\alpha q; q) = \sum_{\lambda \in P} (-1)^{r(\lambda)} \alpha^{\lambda_1} q^{|\lambda|}, \tag{5.1}$$

$$\phi(-\alpha q; -q) = \sum_{\lambda \in DO} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2} - l(\lambda)} q^{|\lambda|}, \tag{5.2}$$

$$\psi(-\alpha q; -q) = \sum_{\lambda \in OC} (-1)^{l(\lambda)} \alpha^{l(\lambda) - \frac{\lambda_1+1}{2}} q^{|\lambda|}. \tag{5.3}$$

Proof. Recall that

$$f(\alpha q; q) = \sum_{n=0}^{\infty} \frac{q^{n^2} \alpha^n}{(-q; q)_n (-\alpha q; q)_n},$$

it is easy to check that (5.1) follows from the Durfee square dissection of a partition $\lambda \in P$, see Figure 1.

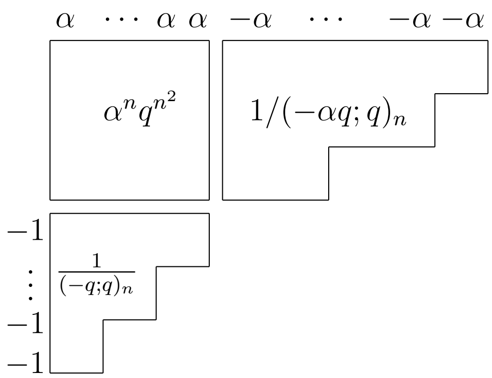

Figure 1: The Durfee square dissection.

From the definition of $\phi(\alpha; q)$, we see that

$$\phi(-\alpha q; -q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-\alpha q^2; q^2)_n}.$$

The term $(-1)^n q^{n^2}$ corresponds to a partition π of the form $Q_n = (2n-1, 2n-3, \ldots, 3, 1)$, which has weight $(-1)^{(\pi_1+1)/2}$. Moreover, $1/(-\alpha q^2; q^2)_n$ is the generating function for partitions σ with at most n even parts and with no odd parts. The weight of such a partition is endowed with weight $(-\alpha)^{\sigma_1/2}$.

Define $\lambda = \pi + \sigma = (\pi_1+\sigma_1, \pi_2+\sigma_2, \ldots)$. We see that $\lambda \in DO$, namely, λ is a partition into distinct odd parts. Now, the weight of λ equals $(-1)^{(\lambda_1+1)/2}\alpha^{(\lambda_1+1)/2-l(\lambda)}$. So (5.2) has been verified.

For the combinatorial interpretation for $\psi(-\alpha q; -q)$, we note that

$$\psi(-\alpha q; -q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(-\alpha q; q^2)_n}.$$

The summand can be expanded as follows

$$\begin{aligned}\frac{(-1)^n q^{n^2}}{(-\alpha q; q^2)_n} &= \frac{-q}{1+\alpha q}\frac{-q^3}{1+\alpha q^3}\cdots\frac{-q^{2n-1}}{1+\alpha q^{2n-1}}\\ &= (-q + \alpha q^{1+1} - \alpha^2 q^{1+1+1}\cdots)(-q^3 + \alpha q^{3+3} - \alpha^2 q^{3+3+3} + \cdots)\\ &\qquad \cdots(-q^{2n-1} + \alpha q^{2(2n-1)} - \alpha^2 q^{3(2n-1)}\cdots).\end{aligned}$$

It follows that the summand $(-1)^n q^{n^2}/(-\alpha q; q^2)_n$ is the generating function of partitions λ in OC with the largest part not exceeding $2n-1$ and with weight $(-1)^{l(\lambda)}\alpha^{l(\lambda)-(\lambda_1+1)/2}$. This proves (5.3). ∎

We can extend Fine's partition theorem to Andrews' function $f(\alpha q; q)$. According to the combinatorial interpretation (5.1) of $f(\alpha q; q)$, we obtain

$$(-\alpha q; q)_\infty f(\alpha q; q) = \sum_{(\lambda,\mu)\in P\times D} (-1)^{r(\lambda)}\alpha^{\lambda_1+l(\mu)}q^{|\lambda|+|\mu|},$$

which can be restated as follows by using the sets U and V_n defined in Section 2,

$$\begin{aligned}(-\alpha q; q)_\infty f(\alpha q; q) = 1 &+ \sum_{(\emptyset,\mu)\in U}\alpha^{l(\mu)}q^{|\mu|} + \sum_{(\lambda,\mu)\in U}(-1)^{r(\lambda)}\alpha^{\lambda_1+l(\mu)}q^{|\lambda|+|\mu|}\\ &+ \sum_{n\geq 1} q^n \sum_{(\lambda,\mu)\in V_n}(-1)^{r(\lambda)}\alpha^{\lambda_1+l(\mu)}.\end{aligned} \tag{5.4}$$

It can be seen that the involution Υ defined on the set V_n in Section 2 preserves the quantity $\lambda_1 + l(\mu)$ but changes the parity of $r(\lambda)$. This yields

$$\sum_{(\lambda,\mu)\in V_n}(-1)^{r(\lambda)}\alpha^{\lambda_1+l(\mu)} = 0. \tag{5.5}$$

As we have seen before, each bipartition $(\lambda, \mu) \in U$ can be viewed as a partition $\nu \in D$ with $s(\nu)$ being odd, and whose weight is $\alpha^{l(\nu)}$. Consequently, using (5.4) and (5.5), we obtain the following partition theorem.

Theorem 5.2 *Let P_{do} denote the set of partitions into distinct parts with the smallest part being odd. Then*

$$(-\alpha q;q)_\infty f(\alpha q;q) = 1 + 2\sum_{\nu\in P_{do}} \alpha^{l(\nu)} q^{|\nu|}. \tag{5.6}$$

Next, we give a generalization of Theorem 1.2 to $\phi(-\alpha q;-q)$. By the combinatorial interpretation (5.2), we find that

$$\begin{aligned}(-\alpha q;q)_\infty \phi(-\alpha q;-q) &= \sum_{(\lambda,\mu)\in DO\times D} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2}-l(\lambda)+l(\mu)} q^{|\lambda|+|\mu|} \\ &= 1 + \sum_{(Q_k,\emptyset)\in W} (-1)^k q^{|Q_k|} + \sum_{(\emptyset,\mu)\in W} \alpha^{l(\mu)} q^{|\mu|} \\ &\quad + \sum_{n\geq 1} q^n \sum_{(\lambda,\mu)\in X_n} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2}-l(\lambda)+l(\mu)}, \end{aligned} \tag{5.7}$$

where the sets W and X_n are defined in Section 3. Moreover, we observe that the involution Φ defined on the set X_n in Section 3 changes the parity of $(\lambda_1+1)/2$ and preserves the quantity

$$l(\mu) - l(\lambda) + \frac{\lambda_1+1}{2}.$$

So we have

$$\sum_{(\lambda,\mu)\in X_n} (-1)^{\frac{\lambda_1+1}{2}} \alpha^{\frac{\lambda_1+1}{2}-l(\lambda)+l(\mu)} = 0. \tag{5.8}$$

In view of (5.7) and (5.8), we obtain the following partition identity for $\phi(-\alpha q;-q)$.

Theorem 5.3 *We have*

$$(-\alpha q;q)_\infty \phi(-\alpha q;-q) = 1 + \sum_{\nu\in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{5.9}$$

We are now in a position to give a generalization of Theorem 1.3 to $\psi(-\alpha q;-q)$. By the combinatorial interpretation (5.3), we find that

$$\begin{aligned}(-\alpha;q)_\infty \psi(-\alpha q;-q) &= \sum_{(\lambda,\mu)\in OC\times D^0} (-1)^{l(\lambda)} \alpha^{l(\lambda)-\frac{\lambda_1+1}{2}+l(\mu)} q^{|\lambda|+|\mu|} \\ &= \sum_{(Q_k,\emptyset)\in Y} (-1)^k q^{|Q_k|} - \sum_{((1),\mu)\in Y} \alpha^{l(\mu)} q^{|\mu|+1} \\ &\quad + \sum_{n\geq 1} q^n \sum_{(\lambda,\mu)\in Z_n} (-1)^{l(\lambda)} \alpha^{l(\lambda)-\frac{\lambda_1+1}{2}+l(\mu)}, \end{aligned} \tag{5.10}$$

where the sets Y and Z_n are defined in Section 4. It can be verified that the involution Ψ defined on the set Z_n in Section 4 changes the parity of $l(\lambda)$ and keeps the quantity

$$l(\lambda) - \frac{\lambda_1 + 1}{2} + l(\mu)$$

invariant. So we get

$$\sum_{(\lambda,\mu)\in Z_n} (-1)^{l(\lambda)} \alpha^{l(\lambda) - \frac{\lambda_1+1}{2} + l(\mu)} = 0. \tag{5.11}$$

As we have seen in Section 4, each bipartition $((1), \mu) \in Y$ can be viewed as a partition $\nu \in P_{do}$ with weight $\alpha^{l(\nu)}$. Using (5.10) and (5.11), we deduce the following partition theorem.

Theorem 5.4 *We have*

$$(-\alpha; q)_\infty \psi(-\alpha q; -q) = -\sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{5.12}$$

Based on the above partition theorems for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$, we can deduce Andrews' generalizations of Ramanujan's identities. More precisely, it follows from Theorem 5.3 and Theorem 5.4 that

$$\begin{aligned} &(-\alpha q; q)_\infty \phi(-\alpha q; -q) \mp (-\alpha; q)_\infty \psi(-\alpha q; -q) \\ &= 1 + \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\ &\quad \mp \left(-\sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \right). \end{aligned}$$

By Theorem 5.2 and Gauss' identity (1.15), we obtain identities (1.9) and (1.10) by replacing αq with α.

Acknowledgments. We wish to thank the referee for helpful comments. This work was supported by the 973 Project, the PCSIRT Project and the Doctoral Program Fund of the Ministry of Education, the National Science Foundation of China.

References

[1] A.K. Agarwal, n-Color partition theoretic interpretations of some mock theta functions, Electron. J. Combin. 11 (2004) N14.

[2] A.K. Agarwal, New combinatorial interpretations of some mock theta functions, Online J. Analytic Combin. 2 (2007) #5.

[3] G.E. Andrews, On basic hypergeometric series, mock-theta functions, and partitions (I), Quart. J. Math. (2) 17 (1966) 64–80.

[4] G.E. Andrews, On basic hypergeometric series, mock-theta functions, and partitions (II), Quart. J. Math. (2) 17 (1966) 132–143.

[5] G.E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

[6] G.E. Andrews, An introduction to Ramanujan's "Lost" Notebook, Amer. Math. Monthly 86 (1979) 89–108.

[7] G.E. Andrews and F.G. Garvan, Ramanujan's "lost" notebook VI: The mock theta conjectures, Adv. Math. 73 (1989) 242–255.

[8] G.E. Andrews, Mock theta functions, In: Theta Functions, Bowdoin 1987, Part 2, Proc. Symp. Pure Math., vol. 49, American Mathematical Society, Providence, RI, (1989) 283–298.

[9] G.E. Andrews, Partitions with short sequences and mock theta functions, Proc. Natl. Acad. Sci. USA 102 (2005) 4666–4671.

[10] G.E. Andrews, H. Eriksson, F. Petrov and D. Romik, Integrals, partitions and MacMahon's theorem, J. Combin. Theory Ser. A 114 (2007) 545–554.

[11] B.C. Berndt, B. Kim and A.J. Yee, Ramanujan's Lost Notebook: Combinatorial proofs of identities associated with Heine's transformation or partial theta functions, J. Combin. Theory Ser. A., to appear.

[12] B.C. Bessenrodt and I. Pak, Partition congruences by involutions, European J. Combin. 25 (2004) 1139–1149.

[13] W.Y.C. Chen and E.H. Liu, A Franklin type involution for squares, Adv. Appl. Math., to appear.

[14] Y.-S. Choi and B. Kim, Partition identities from third and sixth order mock theta functions, preprint.

[15] F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[16] N.J. Fine, Some new results on partitions, Proc. Nat. Acad. Sci. USA 34 (1948) 616–618.

[17] N.J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys 27, AMS Providence, 1988.

[18] B. Gordon and R. McIntosh, A survey of classsical mock theta functions, preprint.

[19] K. Ono, Unearthing the visions of a master: Harmonic Maass forms and number theory, In: Proceeding of the 2008 Harvard-MIT Current Developments in Mathematics Conference, International Press, Somerville, MA, 2009, pp. 347–454.

[20] G.N. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. 11 (1936) 55–80.

[21] A.J. Yee, Bijective proofs of a theorem of Fine and related partition identities, Inter. J. Number Theory 5 (2009) 209–218.

[22] A.J. Yee, Ramanujan's partial theta series and parity in partitions, Ramanujan J., to appear.