On Classification of Extendability of Cayley Graphs on Dicyclic Groups *

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Abstract

Let G be a group and S a subset of G such that the identity element $1 \notin S$ and $x^{-1} \in S$ for each $x \in S$. The *Cayley graph* $X(G; S)$ on a group G has the elements of G as its vertices and edges joining g and gs for all $g \in G$ and $s \in S$. A graph is said to be *k-extendable* if it contains k independent edges and any k independent edges can be extended to a perfect matching. In this paper, we prove that every connected Cayley graph on dicyclic groups is 2-extendable and also investigate the 3-extendability in $X(G; S)$.

Keywords: dicyclic group, Cayley graph, k-extendable graphs.

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1 Introduction

For a simple graph X, we use $V(X)$ and $E(X)$ to denote vertex-set and edge-set of X, respectively. For any set $S \subseteq V(X)$, we use $X[S]$ to denote the subgraph of X induced by S.

Let G be a group and S a subset of G such that the identity element $1 \notin S$ and $x^{-1} \in S$ for each $x \in S$. *Cayley graph* $X(G; S)$ on a group G has the elements of G as its vertices and edges joining g and gs for all $g \in G$ and $s \in S$. An edge xy in $X(G; S)$ is called *type a* (or an *a-edge*) if $x^{-1}y = a$ or a^{-1}. Hence, if xy is of

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type a, then either $y = xa$ or $x = ya$. It is well-known that every Cayley graph is vertex-transitive. For $S \subseteq G$, we denote by $\langle S \rangle$ the subgroup of G induced by S. When G is a cyclic group, Cayley graphs $X(G; S)$ are also referred as *circulants*.

The *dicyclic group* Q_{2n} is a group which is generated by two elements a and x, where $a^{2n} = 1$, $x^2 = a^n$ and $x^{-1}ax = a^{-1}$. We denote $\{mx \mid m \in \langle a \rangle\}$ by $\langle a \rangle x$. From the relations $a^{2n} = 1$, $x^2 = a^n$ and $x^{-1}ax = a^{-1}$, we can easily verify $(a^i x)^{-1} = a^{i+n}x$, $xa^i x = a^{n-i}$ and $a^i x = xa^{2n-i}$, which are useful later. It is not hard to see that Q_{2n} has a cyclic subgroup $\langle a \rangle$ of index $2n$, which is isomorphic to Z_{2n}. Moreover, $Q_{2n} = \langle a \rangle \cup \langle a \rangle x$ and $|Q_{2n}| = 4n$.

A *perfect matching* of a graph X is a set of independent edges which together cover all the vertices of X. For a positive integer k, if M is a set of k independent edges of X (i.e., k-matching) and M^* is a perfect matching of X such that $M \subseteq M^*$, we call M^* a *perfect matching extension* of M, or *M can be extended to M^**. A graph X is said to be *k-extendable* if it contains a k-matching and any k-matching of X can be extended to a perfect matching of X. We use $c_0(G)$ to denote the number of odd components in G.

The concept of k-extendability was introduced by Plummer [5] in 1980. Stong [6] showed that 1-factorization exists for all generating sets of abelian groups of even order, dihedral groups, dicyclic groups, all minimal generating sets of nilpotent groups of even order and $D_m \times Z_n$. Chan, Chen and Yu [1] classified the 2-extendable Cayley graphs on abelian groups. Later, Chen, Liu and Yu [3] classified the 2-extendable Cayley graphs on dihedral groups. These classifications will be useful in our proof of the main theorem. In this paper, we show that any connected Cayley graph $X = X(Q_{2n}; S)$ is 2-extendable. In Section 3, we study 3-extendability of X and classify 3-extendability of Cayley graph with regularity at most 5.

From the generator and relation definition of Q_{2n}, it follows that the maps defined on generators by

$$x \mapsto a^i x, \ a \mapsto a.$$

and

$$x \mapsto x, \ a \mapsto a^d, \ gcd(d, 2n) = 1$$

are group automorphisms. We will exploit this symmetry to simplify many types of dicyclic Cayley graphs.

2 2-extendability and connectivity

In this section, we study 2-extendability of Cayley graph with a given set S.

Theorem 2.1 *Let $X = X(Q_{2n}; S)$ be a connected Cayley graph on the dicyclic group Q_{2n} ($n \geq 1$). Then X is 2-extendable.*

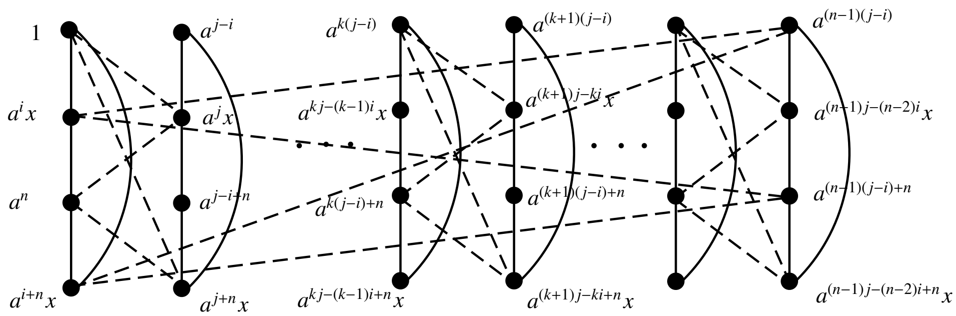

Figure 1: Lemma 2.3

To show Theorem 2.1, we consider several cases based on the given subset S. For $S = \{a^i x\}$, from the definition of Cayley graph, it is easy to see that $X(Q_{2n}; S)$ is the union of n 4-cycles,

$$\bigcup_{j=0}^{n-1} C_j = \bigcup_{j=0}^{n-1} \{(a^j)(a^{i+j}x)(a^{j+n})(a^{j+i+n}x)(a^j)\},$$

where the superscripts are taken in modula $2n$. So we have the following lemma.

Lemma 2.2 *$X(Q_{2n}; \{a^i x\})$ is a disconnected graph for any i. Furthermore, it is the union of n 4-cycles.*

We call 4-cycles in $X(Q_{2n}; S)$ generated by $a^i x$-edges and $a^{i+n}x$-edges *basic cycles*.

Lemma 2.3 *$X(Q_{2n}; \{a^i x, a^j x\})$ is connected if and only if $gcd(n, j-i) = 1$. Furthermore, if $X(Q_{2n}; \{a^i x, a^j x\})$ is connected, then it is 2-extendable.*

Proof. It is easy to see that $X(Q_{2n}; \{a^i x, a^j x\})$ is connected if and only if $\{a^i x, a^j x\}$ is a generating set of Q_{2n}. By exploiting symmetry, we can reduce $\{a^i x, a^j x\}$ to $\{x, a^{j-i}\}$, clearly, $\{x, a^{j-i}\}$ is a generating set of Q_{2n} if and only if $gcd(n,\ j-i) = 1$. We can arrange the vertices of each basic cycle of $X(Q_{2n}; \{a^i x\})$ in a column and connect them by all $a^j x$-edges and $a^{j+n}x$-edges. The resulting graph, shown in Figure 1, is connected. Let $X = X(Q_{2n}; \{a^i x, a^j x\})$. Without loss of generality, assume $0 \le i < j < n$. Since X is connected, we arrange the vertices of each basic cycle of $X(Q_{2n};\{a^i x\})$ and their adjacency as in Figure 1. Let $M = \{e_1, e_2\}$ be a set of any two independent edges. Consider the following two cases.

Case 1. e_1 and e_2 are the same type, say $a^i x$.

They lie in either the same basic cycle or two distinct basic cycles. Clearly, M can be extended to a perfect matching of X.

Case 2. e_1 is of type $a^i x$ and e_2 is of type $a^j x$.

Since X is vertex-transitive, we may assume $e_1 = (1)(a^i x), e_2 = (a^{k(j-i)})(a^{(k+1)j-ki}x)$. First, we show that regardless n is odd or even, the edge $(a^{(n-1)(j-i)})(a^{i+n}x)$ is in X. By the definition of Cayley graph, $(a^{(n-1)(j-i)})^{-1}(a^{i+n}x) = a^{j-n(j-i)}x$. If n is even and $j-i$ is odd, then $j-n(j-i) \equiv j+n \pmod{2n}$; if n is odd, then $j-n(j-i) \equiv j \pmod{2n}$. Thus, in either case, $(a^{(n-1)(j-i)})(a^{i+n}x)$ is an edge of X.

Let

$$\begin{aligned} M^* = \{&e_1,\ (a^n)(a^{j+n}x),\ (a^j x)(a^{j-i+n}),\ (a^{(n-1)(j-i)})(a^{i+n}x)\} \\ &\cup \{(a^{j-i})(a^{2j-i}x),\ \dots,\ e_2,\ \dots,\ (a^{(n-2)(j-i)})(a^{(n-1)j-(n-2)i}x)\} \\ &\cup \{(a^{2(j-i)+n})(a^{2j-i+n}x),\ \dots,\ (a^{(k+1)(j-i)+n})(a^{(k+1)j-ki+n}x),\ \dots, \\ &\quad (a^{(n-1)(j-i)+n})(a^{(n-1)j-(n-2)i+n}x)\}. \end{aligned}$$

Then M^* is a perfect matching and thus M can be extended to a perfect matching of X. ∎

Since $a^i x$ and a^k generate the same subgroup of Q_{2n} as $a^i x$ and $a^{i+k}x$, by Lemma2.3 we have the following consequence.

Corollary 2.4 *$X(Q_{2n};\{a^i x, a^k\})$ is connected if and only if one of the following holds:*

(i) n is odd and $gcd(k,\ 2n) = 2$;

(ii) $gcd(k, 2n) = 1$.

The following classic result of Chen and Quimpo [2] is the first study of extendability of Cayley graphs.

Lemma 2.5 *(Chen and Quimpo [2]) Every Cayley graph of even order over an abelian group is* 1*-extendable.*

From now on, we assume that $X(Q_{2n};S)$ is connected. For convenience, let $S' = S \cap \langle a\rangle$ and $S'' = S \cap (\langle a\rangle x)$. Clearly, $S = S' \cup S''$ and $S'' \neq \emptyset$ as $X(Q_{2n};S)$ is connected. Without loss of generality, assume $x \in S''$. Let E_s be the set of edges of type s for $s \in S''$. Then E_s is a perfect matching of $X(Q_{2n};S)$. We denote $E_1 = E(X[\langle a\rangle]), E_2 = E(X[\langle a\rangle x])$ and $E_3 = E(X(Q_{2n};S''))$. Then $E(X(Q_{2n};S)) = E_1 \cup E_2 \cup E_3$.

Proof of Theorem 2.1. If $n = 1$, then $X = X(Q_2; S)$ is complete graph K_4. In this case, X is 2-extendable. So we may assume that $n \geqslant 2$. Let e_1 and e_2 be any two independent edges of X and $M = \{e_1, e_2\}$.

Case 1. $M \subseteq E_1$ or E_2.

Since $X[\langle a\rangle] \cong X[\langle a\rangle x]$, we may assume that $M \subseteq E_1$. Suppose $e_1 = (a^i)(a^j)$ and $e_2 = (a^k)(a^h)$, then i, j, k and h are all distinct integers in modulus $2n$. Let

$$M^* = (E_x \cup \{e_1, e_2, (a^i x)(a^j x), (a^k x)(a^h x)\}) - \{(a^i)(a^i x), (a^j)(a^j x), (a^k)(a^k x), (a^h)(a^h x)\}.$$

So M can be extended to M^*.

Case 2. $M \cap E_3 \neq \emptyset$ and $M \cap (E_1 \cup E_2) \neq \emptyset$.

Without loss of generality, we assume that $e_1 = (a^i)(a^j) \in E_1$ and $e_2 = (a^k)(a^{k+h}x) \in E_3$, where i, j and k are all distinct in modulus $2n$ and $a^h x \in S''$. Then

$$(E_{a^h x} \cup \{e_1, (a^{i+h}x)(a^{j+h}x)\}) - \{(a^i)(a^{i+h}x), (a^j)(a^{j+h}x)\}$$

is a perfect matching containing M.

Case 3. $e_1 \in E_1, e_2 \in E_2$.

Let $G_1, G_2, \cdots, G_r$ be the components of $X[\langle a\rangle]$, then $G_i \cong G_j$ for $1 \leqslant i, j \leqslant r$. Let G'_i be the subgraph of $X[\langle a\rangle x]$ induced by $\{ mx \mid m \in V(G_i)\}$. Then $G'_i \cong G_i$ for $1 \leqslant i \leqslant r$.

We consider the following subcases.

Case 3.1. e_1 and e_2 lie in G_i and G'_j, respectively, where $i \neq j$.

Let $e_1 = (a^i)(a^j)$, $e_2 = (a^k x)(a^h x)$, then

$$(E_x \cup \{e_1, e_2, (a^i x)(a^j x), (a^k)(a^h)\}) - \{(a^i)(a^i x), (a^j)(a^j x), (a^k)(a^k x), (a^h)(a^h x)\}$$

is a perfect matching containing e_1 and e_2.

Case 3.2. e_1 and e_2 lie in G_i and G'_i, respectively.

Let $e_1 = (a^i)(a^j)$ and $e_2 = (a^k x)(a^h x)$.

If $\mathrm{X}(\langle a\rangle; S')$ is connected, since $\mathrm{X}(\langle a\rangle; S')$ and $\mathrm{X}(\langle a\rangle x; S')$ are connected graphs of order $2n$, then, by Lemma 2.5, both of them are 1-extendable. Hence e_1 can be extended to a perfect matching M_1 in $\mathrm{X}(\langle a\rangle; S')$ and e_2 can be extended to a perfect matching M_2 in $\mathrm{X}(\langle a\rangle x; S')$. Thus $M_1 \cup M_2$ is a perfect matching of X as required. If $\mathrm{X}(\langle a\rangle; S')$ is disconnected, so is $X(\langle a\rangle x; S')$. Since X is connected, there exists an $a^m x \in S''$ such that $a^{m+i}x \notin V(G'_i)$. In this case,

$$\begin{aligned}&(E_{a^m x} \cup \{e_1, e_2, (a^{i+m}x)(a^{j+m}x), (a^{k-m})(a^{h-m})\}) - \\ &\{(a^i)(a^{i+m}x), (a^j)(a^{j+m}x), (a^{k-m})(a^k x), (a^{h-m})(a^h x)\}\end{aligned}$$

is a perfect matching containing e_1 and e_2.

Case 4. $M \subseteq E_3$.

We consider the following two subcases.

Case 4.1. e_1 and e_2 are of same type $a^i x$.

Since $X(Q_{2n};\{a^ix\})$ is a union of basic cycles by Lemma 2.2, then e_1 and e_2 lie in either the same basic cycle or two distinct basic cycles. In either case, M can be extended to a perfect matching of X.

Case 4.2. e_1 and e_2 are of different types a^ix and a^jx, respectively, where $i \neq j$.

Without loss of generality, assume $0 \leq i < j < n$. Then e_1 and e_2 lie in a spanning subgraph of X generated by $\{a^ix, a^jx\}$. If X is connected, by Lemma 2.3, M can be extended to a perfect matching of X. Thus, we only need to consider X is disconnected. From Lemma 2.3, there exists a greatest common divisor of n and $j-i$, say d, satisfying $\frac{j-i}{d} \cdot n = \frac{n}{d}(j-i)$, this implies $X(Q_{2n};\{a^ix, a^jx\})$ has d components. If e_1 and e_2 lie in the same component, we can find a perfect matching by the similar way as in Lemma 2.3, otherwise, e_1 and e_2 lie in two distinct basic cycles, they can be extended to a perfect matching of X as well. ∎

3 3-extendability of Cayley graphs on dicyclic groups

In this section, we discuss 3-extendability of Cayley graphs on dicyclic groups with low regularities. For Cayley graphs with regularity at most 5, we classify 3-extendability of $X(Q_{2n};S)$.

If the regularity of $X(Q_{2n};S)$ is less than 4, none of Cayley graphs $X(Q_{2n};S)$ is connected. So we only discuss the graphs of the regularity at least 4.

If $X(Q_{2n};S)$ is 4-regular, by Lemma 2.3 and Corollary2.4, only two families of Cayley graphs are connected, namely, $X(Q_{2n};\{a^ix, a^jx\})$ for $gcd(n, j-i) = 1$ and $X(Q_{2n};\{a^k, a^ix\})$ for either $gcd(k, 2n) = 2$ and n is odd or $gcd(k, 2n) = 1$.

Proposition 3.1 *The following* 4*-regular connected Cayley graphs on dicyclic groups are not* 3*-extendable.*

(i) $X(Q_{2n};\{a^ix, a^jx\})$ *for* $n \geq 3$*;*

(ii) $X(Q_{2n};\{a^k, a^ix\})$ *for* $gcd(k, 2n) = 2$ *and n is odd;*

(iii) $X(Q_{2n};\{a^k, a^ix\})$ *for* $gcd(k, 2n) = 1$ *and n is odd.*

Proof. For (i), we see that any perfect matching which contains the edges $(1)(a^ix)$ and $(a^n)(a^{i+n}x)$ must contain only edges generated by a^ix. In fact, choose any a^jx-edge $(a^{k(j-i)})(a^{(k+1)j-ki}x)$, we only need to prove that $\hat{X} = X - \{1, a^ix, a^n, a^{i+n}x, a^{k(j-i)}, a^{(k+1)j-ki}x\}$ contains no perfect matching (see Figure 1). Let

$$S = \bigcup_{m=k+2}^{n-1} \{a^{mj-(m-1)i}x, a^{mj-(m-1)i+n}x\} \cup \{a^{(k+1)j-ki+n}x\}.$$

Then $c_0(\hat{X} - S) = |S| + 2 > |S|$, by Tutte's 1-Factor Theorem, $\hat{X}$ has no perfect matching. Therefore, X is not 3-extendable.

The proofs of the other two classes are similar, we can choose three independent edges, deleting them and their end-vertices, leaving a bipartite graph with different number of vertices in the two classes, so we only present the detailed proof of (*iii*) here.

Assume that $gcd(k, 2n) = 1$ and n is odd. Choose $e_1 = (1)(a^k)$, $e_2 = (a^{i+k}x)(a^{i+2k}x)$ and $e_3 = (a^n)(a^{i+n}x)$.

Let

$$T = \bigcup_{m=2}^{\frac{2n-k}{2k}} \{a^{(2m-1)k}, a^{n+(2m-1)k}, a^{i+2mk}x, a^{i+n+2mk}x\} \cup \{a^{n+k}, a^{i+n+2k}x\}.$$

Then T is the set of circled vertices in Figure 2. Set $G_1 = G - \cup_{i=1}^{3} V(e_i)$, then all components of $G_1 - T$ are isolated vertices, $|T| = 2n - 4$ and the number of isolated vertices of $G_1 - T$ is $2n - 2$. Thus G_1 is a bipartite graph with bipartition T and $G_1 - T$. Therefore, G_1 has no perfect matching or X is not 3-extendable. ∎

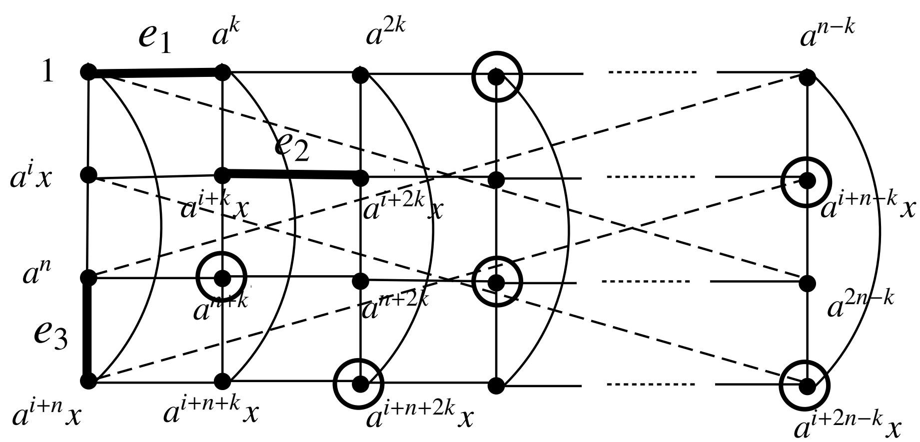

Figure 2: Illustration of condition (iii) in Proposition 3.1

We consider $X(Q_{2n}; S)$ as two subgraphs $G' = X[\langle a\rangle]$ and $G'' = X[\langle a\rangle x]$, joined by two perfect matchings consisting of all $a^i x$-edges and $a^{i+n}x$-edges. Recall the notions of E_1, E_2, E_3, we need them in the proof of next theorem and also call the edges in E_1 and E_2 *parallel edges*.

For any edge $e = (a^m)(a^{m+k}) \in E(G')$, there exists a bijection $\theta : E(G') \longrightarrow E(G'')$ such that $\theta(e) = (a^{m+i}x)(a^{m+k+i}x)$ and a bijection $\delta : E(G') \longrightarrow E(G'')$ such that $\delta(e) = (a^{m+i+n}x)(a^{m+k+i+n}x)$. The *shadows* of e in G' onto G'' are $\theta(e)$ and $\delta(e)$ under $a^i x$-edges and $a^{i+n}x$-edges, respectively. Similarly, we define the *shadows* of an edge e in G'' onto G'.

Theorem 3.2 *A 4-regular connected Cayley graph $\hat{X}$ on dicyclic group is 3-extendable if and only if $\hat{X} \cong X(Q_{2n}; \{a^k, a^i x\})$, $gcd(k, 2n) = 1$ and n is even, $n \geq 4$.*

Proof. From Proposition 3.1, we only need to show that if $gcd(k, 2n) = 1$ and n is even, $n \geq 4$, then $\hat{X}$ is 3-extendable. By exploiting symmetry, generating set $\{a^k, a^i x\}$ of $\hat{X}$ could reduce to $\{x, a\}$.

Let $M = \{e_1, e_2, e_3\}$ be a set of any three independent edges of $\hat{X}$, $\mathbb{C}$ be the union of all basic cycles. In Table 1, we list all possible cases according to the locations of edges in M.

For Case 1 and Case 2.1, let C^* be the union of basic cycles containing e_1, e_2 and e_3. As each basic cycle in C^* has a perfect matching containing e_j and $\hat{X} - V(C^*)$ has a perfect matching, then M can be extended to a perfect matching of $\hat{X}$ in Case 1. For Case 2.1, no matter whether $C_1 = C_2$ or not, we take all the $a^i x$-edges in $\hat{X} - V(C^*)$ except two which are contained in the same 4-cycle with e_3 and its shadow under $a^i x$-edge, and replace the above two edges with e_3 and its corresponding shadow, then this yields a perfect matching of $\hat{X}$.

For Case 2.2, $C_1 \neq C_2$. Let e_3 join C_1 and another basic cycle C_5. If $C_5 \neq C_2$, $X[C_1 \cup C_2 \cup C_5]$ has a perfect matching M_1 containing M, $X[G \setminus C_1 \cup C_2 \cup C_5]$ has a perfect matching M_2, generated by the union of perfect matchings in each remaining basic cycle. Then $M_1 \cup M_2$ is the required perfect matching of $\hat{X}$. If $C_5 = C_2$, for the case that e_1, e_2 are contained in an $a - x$ alternating 4-cycle, $X[C_1 \cup C_5]$ has a perfect matching M_1 containing M, $X[G \setminus (C_1 \cup C_5)]$ has a perfect matching M_2. Then $M_1 \cup M_2$ is the required perfect matching of $\hat{X}$. Otherwise, without loss of generality, let $e_1 = (1)(x), e_2 = (ax)(a^{n+1}), e_3 = (a^n x)(a^{n+1} x)$, let

$$\overline{M} = \left(\bigcup_{j=0}^{\frac{n}{2}-2} (a^{(2j+1)})(a^{(2j+2)}) \right) \cup \left(\bigcup_{j=2}^{n-2} \{(a^j x)(a^{j+1} x), (a^{n+j})(a^{n+j+1}), (a^{n+j} x)(a^{n+j+1} x)\} \right)$$
$$\cup \{e_1, e_2, e_3, (a^{n-1})(a^n)\}.$$

Then $\overline{M}$ is a perfect matching of $\hat{X}$ containing M.

For all the subcases of Case 3.1 and Case 3.2.1, we could always find a perfect matching of $X[C_{1,1} \cup C_{1,2} \cup C_{2,1} \cup C_{2,2} \cup C_3]$, which containing M. The rest is similar to the discussions above.

For Case 3.2.2, we just consider the following location of M, to find a perfect matching for other locations of M are similar as in Case 2.2.

Without loss of generality, let $e_1 = (1)(a), e_2 = (a^{n+1} x)(a^{n+2} x), e_3 = (x)(a^n)$

Table 1: Summary of the locations of edges in M

cases	subcases	subsubcases
1. $\lvert M \cap E(\mathbb{C})\rvert = 3$;	Nil.	Nil.
2. $\lvert M \cap E(\mathbb{C})\rvert = 2$, say, $e_1 \in C_1$, $e_2 \in C_2$, $e_3 \notin E(\mathbb{C})$;	1. $\lvert V(e_3) \cap V(C_1, C_2)\rvert = 0$;	Nil.
	2. $\lvert V(e_3) \cap V(C_1, C_2)\rvert \geq 1$.	Nil.
3. $\lvert M \cap E(\mathbb{C})\rvert = 1$, say $e_1, e_2 \notin E(\mathbb{C})$, $e_3 \in C_3 \subseteq \mathbb{C}$;	1. $\lvert V(e_1, e_2) \cap V(C_3)\rvert = 0$;	[a]1. $\lvert \cap_{i_1,i_2=1,2} V(C_{i_1,i_2})\rvert = 0$;
		2. either $C_{1,1} = C_{2,1}$ $(C_{2,2})$ or $C_{1,2} = C_{2,1}$ $(C_{2,2})$;
		3. $C_{1,1} = C_{2,1}, C_{1,2} = C_{2,2}$.
	2. $\lvert V(e_1, e_2) \cap V(C_3)\rvert = 1$, say, $\lvert V(e_1) \cap V(C_3)\rvert = 1$ and $C_{1,1} = C_3$;	1. $C_{2,j} \neq C_{1,2}$ for $j = 1, 2$;
		2. $C_{2,j} = C_{1,2}$ for some j, say $j = 1$.
	3. $\lvert V(e_1, e_2) \cap C_3\rvert = 2$.	
4. $\lvert M \cap E(\mathbb{C})\rvert = 0$.	1. $\lvert M \cap G'\rvert = 3$ (or G'');	
	[b]2. $\lvert M \cap G'\rvert = 2$, say $e_1, e_2 \in G', e_3 \in G''$.	1. $\lvert V(e_3) \cap V(\sigma(e_1, e_2))\rvert = 0$;
		2. $e_3 = \sigma(e_1)$ or $\sigma(e_2)$;
		3. $\lvert V(e_3) \cap V(\sigma(e_1 \cup e_2))\rvert = 1$;
		4. $\lvert V(e_3) \cap V(\sigma(e_1 \cup e_2))\rvert = 2$.

[a] Let $C_{1,1}$ and $C_{1,2}$ be two basic cycles that e_1 connects, $C_{2,1}$ and $C_{2,2}$ be two basic cycles that e_2 connects.
[b] σ means the θ or δ shadow of e_j, $j = 1, 2$.

and let

$$M^* = \left(\bigcup_{j=1}^{\frac{n}{2}-1}(a^{2j})(a^{(2j+1)})\right) \cup \left(\bigcup_{j=1}^{n-2}(a^j x)(a^{(n+j)})\right) \cup$$
$$\left(\bigcup_{j=2}^{\frac{n}{2}-1}(a^{n+2j-1}x)(a^{n+2j}x)\right) \cup \{(a^{n-1}x)(a^n x), (a^{2n-1})(a^{2n-1}x), e_1, e_2, e_3\}.$$

Then $M \subset M^*$ and M^* is a perfect matching of $\hat{X}$.

For Case 3.3, if e_1, e_2 join another common basic cycle different from C_3, denoted it by C_6, it can be dealt with similarly as in Case 2.2. If not, without loss of generality, let $e_1 = (1)(a), e_2 = (a^{n+1}x)(a^{n+2}x), e_3 = (ax)(a^{n+1})$, then the above M^* is also the required perfect matching.

For Case 4.1, let $E^* = E_x \bigcup (\cup_{j=1}^{3}\{e_j$ and its corresponding shadow under x-edge$\})$. Then E^* contains three vertex-disjoint $a - x$ alternating cycles. In each alternating cycle, we replace x-edges with e_j and its corresponding shadow, then there exists a perfect matching in $\hat{X}$ containing M.

For Case 4.2, if e_1, e_2 have four distinct shadows in G''.

Case 4.2.1-Case 4.2.3 are similar as in Case 4.1. Next we deal with Case 4.2.4.

If e_3 joins two shadows of the same e_j ($j = 1$ or 2), it contradicts to the definition of Cayley graphs except in the case $n = 2$. Without loss of generality, let $e_1 = (1)(a), e_2 = (a^j)(a^{(j+1)}), e_3 = (ax)(a^2x)$, l_1 be the shadow of e_1 under x-edge and l_2, l_3 be shadows of e_2 under x-edge and $a^n x$-edge, respectively. We only consider the case $gcd(j, 2n) = 1$, for the case $gcd(j, 2n) = 2$, there exists a perfect matching in G' containing e_1, e_2 and a perfect matching in G'' containing e_3. Let $j = 2m - 1$. If e_3 joins one end-vertex of l_2, then there exist two perfect matchings in G' and G'', respectively. Suppose e_3 joins one end-vertex of l_3. Consider the end-vertex a^2x of e_3, the shadow of a^2x under $a^n x$-edge in G' is a^{2-n}, which is also the end-vertex of e_2, then $a^{2-n} = a^{(2m-1)}$, that is, $(2m - 3) + n \equiv 0 \pmod 2$, contradicting the fact that $gcd(k, 2n) = 1$ and n is even.

If e_1, e_2 have two common shadows in G''. We just consider that e_3 joins one of these two common shadows, otherwise it is similar to Case 4.2.1-Case 4.2.3. By the hypothesis, k is odd and n is even, then there is a perfect matching M_1 in G' containing e_1, e_2 and a perfect matching M_2 in G'' containing e_3, so $M_1 \cup M_2$ is the required perfect matching.

The proof is complete. ∎

For 5-regular Cayley graphs $X(Q_{2n}; S)$, there are only three types of connected graphs, namely, $X(Q_{2n}; \{a^i x, a^j x, a^n\})$ with $gcd(j - i, n) = 1$, $X(Q_{2n}; \{a^i x, a^k, a^n\})$ with $gcd(k, 2n) = 1$ and graph $X(Q_{2n}; \{a^i x, a^k, a^n\})$ with $gcd(k, 2n) = 2$ and n is odd.

Note $X(Q_{2n}; \{a^i x, a^j x, a^n\})$ is not 3-extendable since $e_1 = (1)(a^n), e_2 = (a^i x)(a^{(n-1)(j-i)+n})$ and $e_3 = (a^{(n-1)(j-i)})(a^{(n-1)j-(n-2)i}x)$ can not be extended to a perfect matching.

For other two types, $E_{a^i x}, E_{a^{i+n} x}$ and E_{a^n} generate a disjoint union of the complete graph K_4. From the discussion of 4-regular Cayley graphs, using the similar arguments as in Theorem 3.2, we have the following theorem but omit the proof.

Theorem 3.3 *Every connected 5-regular Cayley graph on a dicyclic group is 3-extendable except one family $X(Q_{2n}; \{a^i x, a^j x, a^n\})$.*

For Cayley graphs with regularity more than 5, it becomes too tedious to manage with case by case analysis and a new technique is required to classify 3-extendability. However, experiments suggest the following conjecture.

Conjecture 3.4 *The connected Cayley graphs on dicyclic groups of regularity more than 5 are 3-extendable.*

For general $k \geq 4$, the current technique is powerless to deal with k-extendability of Cayley graphs on dicyclic groups. However, as a conclusion of the paper, we provide the following two families of non-k-extendable Cayley graphs.

Proposition 3.5 *Let k be an odd integer. Then $(k+1)$-regular connected Cayley graphs $X(Q_{2n}; S)$ are not k-extendable, if*

(i) $S = \{a^{i_1} x, a^{i_2} x, \cdots, a^{i_{\frac{k+1}{2}}} x\}$;

(ii) $S = \{a^h, a^{i_1} x, \cdots, a^{i_{\frac{k-1}{2}}} x\}$, *where* $h = i_2 - i_1$ *or* $i_2 - i_1 - n$.

Proof. (i) From Lemma 2.3, without loss of generality, assume $gcd(i_2 - i_1, n) = 1$. Then the vertices $a^{i_2} x$ and $a^{i_2+n} x$ have exactly the same neighbors. Choose k edges as follows:

$$e_1 = (1)(a^{i_1} x), e_2 = (a^n)(a^{i_1+n} x), e_3 = (a^{i_2-i_1})(a^{2i_2-i_1} x), \ldots,$$

$$e_{2(m-3)+4} = (a^{n+i_2-i_m})(a^{n+2i_2-i_m} x), e_{2(m-3)+5} = (a^{i_2-i_m})(a^{2i_2-i_m} x) \text{ for } 3 \leq m \leq \frac{k+1}{2}.$$

After deleting these k edges, then the neighbor set of $a^{i_2} x$ and $a^{i_2+n} x$ turns out to be $\{a^{i_2-i_1+n}\}$ and thus $X(Q_{2n}; S)$ are not k-extendable.

For (ii), it can be verified similarly. ∎

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