On Three and Four Vicious Walkers

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Abstract

We establish a reflection principle for three lattice walkers and use this principle to reduce the enumeration of configurations of three vicious walkers to that of configurations of two vicious walkers. Precisely, the reflection principle leads to a bijection between three walks (L_1, L_2, L_3) such that L_2 intersects both L_1 and L_3 and three walks (L_1, L_2, L_3) such that L_1 intersects L_3. Hence we find a combinatorial interpretation of the formula for the generating function for the number of configurations of three vicious walkers, originally derived by Bousquet-Mélou by using the kernel method, and independently by Gessel by using tableaux and symmetric functions. This answers a question posed by Gessel and Bousquet-Mélou. We also find a reflection principle for four vicious walks that leads to a combinatorial interpretation of a formula derived from Gessel's theorem.

Keywords: vicious walkers, watermelon, Catalan numbers, Ballot numbers, reflection principle.

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1 Introduction

The vicious walker model was introduced by Fisher [5] in 1984 and has drawn much attention. A walker is said to be vicious if he does not like to meet any other walker at any point. Formally speaking, a configuration of r vicious walkers, called r vicious walks, of length n, is an r-tuple of pairwise nonintersecting lattice walks of length n, consisting of up steps U (i.e., $(1,1)$) and down steps D (i.e., $(1,-1)$), starting from $(0,2i_1),(0,2i_2),\ldots,(0,2i_r)$ and ending at $(n,e_1),(n,e_2),\ldots,(n,e_r)$ where $i_r > \cdots > i_2 > i_1 = 0$ and $e_r > \cdots > e_2 > e_1$. Precisely, two lattice paths are said to be nonintersecting if they do not share any common points. In particular, a watermelon of length n is a configuration consisting of r chains, or paths, of length n which start at the points $(0,0),(0,2),\ldots,(0,2r-2)$ and end at the points

$(n, k), (n, k+2), \ldots, (n, k+2r-2)$ for some k. In other words, a watermelon is a vicious walker configuration starting at adjacent points and ending at adjacent points. Note that two lattice points are said to be adjacent if they are on the same vertical line and their y-coordinates differ by 2. It is known that configurations of vicious walkers can be represented by tableaux. So the theory of symmetric functions can be employed to study vicious walkers, see [10, 11, 12, 13, 15, 16].

The main objective of this paper is to present a combinatorial approach to the enumeration of configurations of three vicious walkers. Let us fix the starting points $(0, 0)$, $(0, 2i)$ and $(0, 2i+2j)$. Let $V(i, j, n)$ be the set of three vicious walks (L_1, L_2, L_3) of length n, where L_1 is the path of the first walker starting from $(0, 0)$, L_2 is the path of the second walker starting from $(0, 2i)$, and L_3 is the path of the third walker starting from $(0, 2i+2j)$. Define the generating function $V_{i,j}(t)$ to be

$$V_{i,j}(t) = \sum_{n=0}^{\infty} |V(i, j, n)| t^n, \tag{1.1}$$

where $|\cdot|$ denotes the cardinality of a set.

The enumeration of configurations of three vicious walkers has been solved independently by Bousquet-Mélou [1] by using the obstinate kernel method, and by Gessel [9] by using tableaux and symmetric functions. They obtained a formula for $V_{i,j}(t)$ in terms of the generating function of the Catalan numbers.

Let $C(t)$ be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, that is,

$$C(t) = \sum_{n=0}^{\infty} C_n t^n.$$

Recall that $C(t)$ satisfies the recurrence relation

$$C(t) = 1 + tC^2(t). \tag{1.2}$$

Let

$$D(t) = tC^2(t) = C(t) - 1 = \sum_{n=0} C_{n+1} t^{n+1}. \tag{1.3}$$

The following elegant formula is due to Bousquet-Mélou [1] and Gessel [9].

Theorem 1.1 (Bousquet-Mélou [1] and Gessel [9])**.**

$$V_{i,j}(t) = \frac{1}{1-8t}(1 - D^i(2t))(1 - D^j(2t)). \tag{1.4}$$

In view of the relation (1.3) and the identity

$$\left(\frac{1+D(t)}{1-D(t)}\right)^2 = \frac{1}{1-4t}, \tag{1.5}$$

Gessel derived the following form of the formula for $V_{i,j}(t)$.

Theorem 1.2 (Gessel [9]). *For any $i, j \geq 1$, we have*

$$V_{i,j}(t) = C^2(2t)\big(1+D(2t)+\cdots+D^{i-1}(2t)\big)\big(1+D(2t)+\cdots+D^{j-1}(2t)\big). \tag{1.6}$$

Both Bousquet-Mélou [1] and Gessel [9] proposed the problem of finding a combinatorial interpretation of the formula for $V_{i,j}(t)$. The question of Bousquet-Mélou is concerned with the formula (1.4), while the question of Gessel is concerned with the formula in the form of (1.6). In this paper, we will present a combinatorial interpretation of (1.4). As will be seen, the algebraic manipulations to transform the formula (1.4) to (1.6) can be explained combinatorially. So we have obtained combinatorial interpretations of both formulas (1.4) and (1.6).

In Section 3, we also take a different approach to the enumeration of configurations of two vicious walkers. By reformulating the problem in terms of pairs of intersecting walks, we give a decomposition of a pair of converging walks, that is, two walks that do not intersect until they reach the same ending point, into two-chain watermelons, or 2-watermelons. Then we can use Pólya's formula for the number of 2-watermelons of length n to derive the formula for the number of two vicious walks of length n. In Section 4, we make a connection between the Labelle merging algorithm, in the form presented by Chen, Pang, Qu and Stanley [3], and the classical ballot numbers. In the last section, we present a reflection principle for the enumeration of configurations of four vicious walkers with prescribed starting points. More precisely, we give a combinatorial proof of a formula on the number of four vicious walks derived from Gessel's theorem [9].

2 The Reflection Principle

In this section, we will establish a reflection principle so that we can reduce the enumeration of three vicious walkers to that of two vicious walkers. This reduction leads to a combinatorial interpretation of the formula for $V_{i,j}(t)$, as defined by (1.1).

Let us recall some basic definitions. Two walks L_1 and L_2 are said to be intersecting, denoted $L_1 \cap L_2 \neq \emptyset$, if L_1 and L_2 share a common point. Let

$U(i,j,n)$ be the set of all 3-walks (L_1, L_2, L_3) of length n, where L_1, L_2 and L_3 start from $(0,0)$, $(0,2i)$ and $(0,2i+2j)$ respectively. Let

$$U_{i,j}(t) = \sum_{n=0}^{\infty} |U(i,j,n)| t^n.$$

It is obvious that

$$U_{i,j}(t) = \frac{1}{1-8t}. \tag{2.1}$$

We use $W_{12}(n)$, or W_{12} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 and L_2 are nonintersecting. Similarly, we use $W_{23}(n)$, or W_{23} for short, to denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_2 and L_3 are nonintersecting. Clearly, the set $V(i,j,n)$ of three vicious walks of length n can be expressed as $W_{12} \cap W_{23}$. By the principle of inclusion and exclusion, we see that

$$|V(i,j,n)| = |W_{12} \cap W_{23}| = |W_{12}| + |W_{23}| - |W_{12} \cup W_{23}|. \tag{2.2}$$

In order to compute $|W_{12} \cup W_{23}|$, we let $M_{12,23}(n)$, or $M_{12,23}$ for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_2 intersects both L_1 and L_3. Clearly, we have

$$|W_{12} \cup W_{23}| = |U(i,j,n)| - |M_{12,23}|. \tag{2.3}$$

We are now in a position to establish a reflection principle to deal with the enumeration of $M_{12,23}(n)$. Let $M_{13}(n)$, or M_{13} for short, denote the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 intersects L_3. Then we have the following correspondence.

Theorem 2.1. *For $n \geq 1$, there exists a bijection between $M_{12,23}(n)$ and $M_{13}(n)$.*

Proof. We construct a map Φ from $M_{12,23}(n)$ to $M_{13}(n)$ as follows. Let (L_1, L_2, L_3) be a 3-walk in $M_{12,23}(n)$. We consider the following two cases. If $L_1 \cap L_3 \neq \emptyset$, then it is clear that $(L_1, L_2, L_3) \in M_{13}(n)$. In this case, we define $\Phi((L_1, L_2, L_3)) = (L_1, L_2, L_3)$.

We may now assume that $L_1 \cap L_3 = \emptyset$. We first consider the case that L_2 meets L_1 before it meets L_3. Suppose that P is the first intersection point of L_2 and L_1. We now conduct the usual reflection operation on L_1 and L_2, and denote the resulting paths by L_1' and L_2'. Namely, L_1' consists of the first segment of L_1 up to the point P followed by the last segment of L_2 starting from the point P, and L_2' consists of the first segment of L_2 up to the point P followed by the last segment of L_1 starting from the point P. Figure 2.1 is an illustration of the reflection.

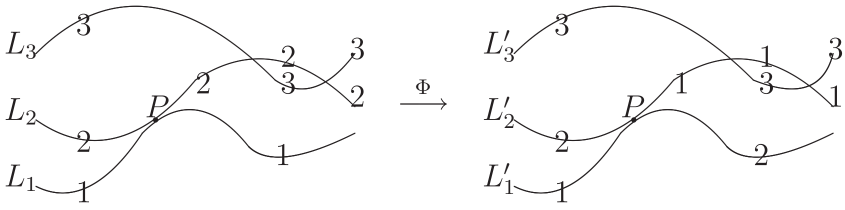

Figure 2.1: The reflection principle.

Let $L'_3 = L_3$ and $\Phi((L_1, L_2, L_3)) = (L'_1, L'_2, L'_3)$. It is clear that L'_1 must meet L'_3. Thus we have $(L'_1, L'_2, L'_3) \in M_{13}(n)$.

It is not difficult to see that the above procedure is reversible. We are still left with the case when L_2 intersects L_3 before meeting L_1. This case is analogous to the case that we have considered. Thus we have reached the conclusion that Φ is a bijection. ∎

Combining (2.2), (2.3) and Theorem 2.1, we obtain the following relation

$$|V(i, j, n)| = |W_{12}| + |W_{23}| + |M_{13}| - |U(i, j, n)|. \tag{2.4}$$

Let W_{13} be the set of three walks (L_1, L_2, L_3) in $U(i, j, n)$ such that L_1 never meets L_3, and define the generating functions for $|W_{12}|$, $|W_{23}|$ and $|W_{13}|$ by $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ respectively. From (2.4) it follows that

$$|V(i, j, n)| = |W_{12}| + |W_{23}| - |W_{13}|. \tag{2.5}$$

Proposition 2.2.

$$V_{i,j}(t) = W_{12}(t) + W_{23}(t) - W_{13}(t). \tag{2.6}$$

The above formula can be viewed as a reduction of the three vicious walkers problem to that of two vicious walkers. Let $N(i, n)$ be the set of two vicious walks (L_1, L_2) of length n starting at $(0, 0)$ and $(0, 2i)$ respectively, and denote the corresponding generating function by

$$N_i(t) = \sum_{n=0}^{\infty} |N(i, n)| t^n.$$

Bousquet-Mélou [1] and Gessel [9] obtained the following formula

$$N_i(t) = \frac{1}{1-4t}(1 - D^i(t)). \tag{2.7}$$

As pointed out by Gessel [9], the above formula for $N_i(2t)$ can be deduced from the formula (1.6) for $V_{i,j}(t)$ by taking the limit $j \to \infty$, and by using the identity (1.5).

Using the above formula for $N_i(t)$, one can derive the following formulas for the generating functions $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$:

$$W_{12}(t) = \frac{1 - D^i(2t)}{1 - 8t}, \; W_{23}(t) = \frac{1 - D^j(2t)}{1 - 8t}, \; W_{13}(t) = \frac{1 - D^{i+j}(2t)}{1 - 8t}. \quad (2.8)$$

Clearly, formula (1.4) in Theorem 1.1 follows from the above formulas and the relation (2.6).

We note that Gessel [9] obtained the following identity

$$V_{i,j}(t) = N_i(2t) + N_j(2t) - N_{i+j}(2t), \quad (2.9)$$

in accordance with the combinatorial statement (2.6) derived from the reflection principle.

As to the question of finding a combinatorial interpretation of the generating function formula (1.4), the reflection principle (Theorem 2.1) along with the combinatorial interpretations of the formulas for $W_{12}(t)$, $W_{23}(t)$ and $W_{13}(t)$ can be considered as an answer because the principle of inclusion and exclusion for two sets can be easily justified combinatorially. In the next section, we will present a combinatorial treatment of the formula (2.7) for two vicious walkers. Moreover, we note that one can give a combinatorial reasoning of the transformation from the formula (1.4) to the formula (1.6).

It is to deduce (1.6) from (1.4) by utilizing the identity (1.5), which can be explained combinatorially in two steps. The first step is to show that

$$4^n = \sum_{k=0}^{2n} \binom{2k}{k} \binom{2n-2k}{n-k}, \quad (2.10)$$

which is equivalent to the identity

$$\sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}. \quad (2.11)$$

There are several combinatorial proofs of (2.10), see, for example, Kleitman [14] and Sved [22]. The second step is to show that

$$\frac{1 + D(t)}{1 - D(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n. \quad (2.12)$$

Note that $\frac{1+D(t)}{1-D(t)}$ can be written as $\frac{C(t)}{1-tC^2(t)}$. A combinatorial interpretation of the identity

$$\frac{C(t)}{1 - tC^2(t)} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n$$

is given by Chen, Li and Shapiro [2] in terms of doubly rooted plane trees and the butterfly decomposition.

3 Converging Walks and 2-Watermelons

In this section, we present a different approach to the two vicious walkers problem by counting pairs of converging walks. A pair of walks is said to be converging if they never meet until they reach a common ending point. We will show that pairs of converging walks can be enumerated by applying Pólya's formula for two-chain watermelons, or 2-watermelons [19]. Precisely, we will give a decomposition of a pair of converging walks into 2-watermelons.

Recall that $M_{13}(n)$ is defined in the previous section. Let $M_{12}(n)$, or M_{12} for short, be the set of 3-walks (L_1, L_2, L_3) in $U(i,j,n)$ such that L_1 intersects L_2. Similarly, we can define $M_{23}(n)$, or M_{23} for short. Clearly, we have

$$|M_{12}| = |U(i,j,n)| - |W_{12}|, \quad |M_{23}| = |U(i,j,n)| - |W_{23}|.$$

From (2.4) it follows that

$$|V(i,j,n)| = |U(i,j,n)| + |M_{13}| - |M_{12}| - |M_{23}|.$$

Let $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ denote the generating functions for $|M_{12}(n)|$, $|M_{23}(n)|$ and $|M_{13}(n)|$, respectively.

Proposition 3.1. *We have*

$$V_{i,j}(t) = U_{i,j}(t) - M_{12}(t) - M_{23}(t) + M_{13}(t). \tag{3.1}$$

We will show that $M_{12}(t)$, $M_{13}(t)$ and $M_{23}(t)$ can be computed by using the following formula for the number of 2-watermelons as derived by Levine [18] and Pólya [19], see also, Fürlinger and Hofbauer [6], Gessel [7], and Shapiro [21].

Proposition 3.2. *The number of* 2*-watermelons with each walk having* n *steps is* C_{n+1}*.*

Using the above formula, one sees that the generating function of the number of 2-watermelons equals $C^2(t)$. Note that 2-watermelons of length n correspond to pairs of converging walks of length $n+1$ with adjacent starting points. In general, let $T(i,n)$ be the set of pairs of converging walks (L_1, L_2) of length n, where L_1 starts from $(0,0)$ and L_2 starts from $(0,2i)$. Define

$$T_i(t) = \sum_{n \geq 0} |T(i,n)| t^n.$$

Proposition 3.3. *For any* $i \geq 1$, $T_i(t) = D^i(t)$*.*

Proof. Let $L_1 = A_0A_1 \ldots A_n$ and $L_2 = B_0B_1 \ldots B_n$, where a walk is represented by a sequence of points. For $0 \le k \le i$, let j_k be the minimum index such that the difference of the y-coordinates of (A_{j_k}, B_{j_k}) equals to $2i - 2k$. It is clear that $j_0 = 0$ and $j_i = n$. We now decompose (L_1, L_2) into i 2-walks: $(L_1^{(1)}, L_2^{(1)}), \ldots, (L_1^{(i)}, L_2^{(i)})$, where $L_1^{(k)} = A_{j_{k-1}}A_{j_{k-1}+1} \ldots A_{j_k}$ and $L_2^{(k)} = B_{j_{k-1}}B_{j_{k-1}+1} \ldots B_{j_k}$. Figure 3.1 is an illustration of the decomposition.

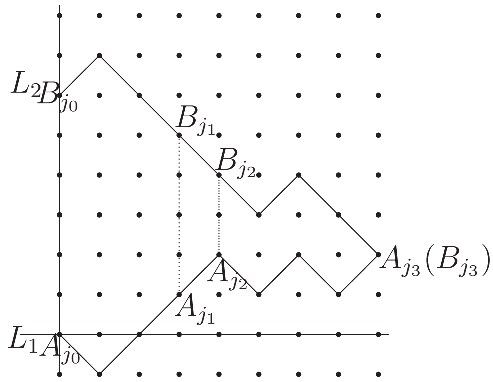

Figure 3.1: The decomposition of a pair of converging walks.

Observe that by the choice of j_k, the rightmost pair of steps in $(L_1^{(k)}, L_2^{(k)})$ must be (U, D). Moreover, if we delete this pair of steps, the resulting upper walk can be lowered $2i - 2k$ units without intersecting the lower walk to form a 2-watermelon. See Figure 3.2 for an example.

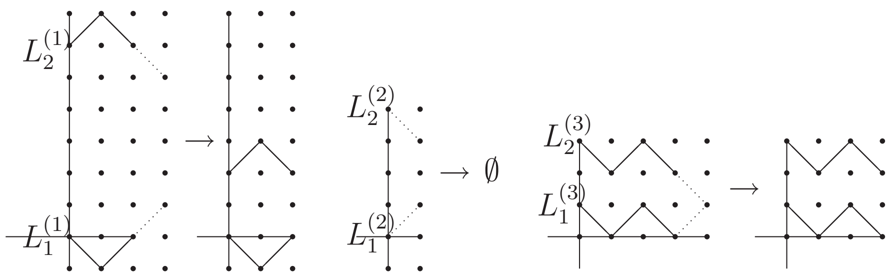

Figure 3.2: From 2-walks to 2-watermelons.

By Proposition 3.2, The generating function for the number of 2-walks $(L_1^{(k)}, L_2^{(k)})$ equals $D(t) = t \cdot C^2(t)$. This completes the proof. ∎

Let $M(i, n)$ be the set of intersecting 2-walks (L_1, L_2) of length n, where L_1 and L_2 start from $(0,0), (0, 2i)$ respectively. Define

$$M_i(t) = \sum_{n \ge 0} |M(i,n)| t^n.$$

Observe that every pair of intersecting paths (L_1, L_2) can be decomposed into a pair of converging paths and a pair of arbitrary paths starting from the same point. Thus we have the following formula.

Corollary 3.4. *For any $i \geq 1$,*

$$M_i(t) = \frac{D^i(t)}{1-4t}.$$

It is obvious that

$$M_i(t) + N_i(t) = \frac{1}{1-4t}. \tag{3.2}$$

So the formula (2.7) for $N_i(t)$ can be deduced from the above formula. It is easy to see that $M_{12}(t)$, $M_{23}(t)$ and $M_{13}(t)$ can be computed by using the above formula for $M_i(t)$. So we get

$$M_{12}(t) = \frac{D^i(2t)}{1-8t}, \quad M_{23}(t) = \frac{D^j(2t)}{1-8t}, \quad M_{13}(t) = \frac{D^{i+j}(2t)}{1-8t}, \tag{3.3}$$

in agreement with (2.8). Substituting (3.3) into (3.1), we obtain Theorem 1.1.

4 Connection to the Ballot Numbers

In this section, we put the Labelle merging algorithm in a more general setting, and show that the direct correspondence formulated by Chen, Pang, Qu and Stanley [3] leads to a connection between pairs of converging walks and the classical ballot numbers.

Let us recall the direct correspondence given in [3]. We will represent a walk as a sequence of steps rather than points. Let (L_1, L_2) be a 2-watermelon of length n, and let $L_1 = p_1p_2\cdots p_n$ and $L_2 = q_1q_2\cdots q_n$, where $p_i, q_i = U$ or D. Set $U' = D$ and $D' = U$. Using the direct correspondence in [3], the watermelon (L_1, L_2) can be represented by a Dyck path of length $2n+2$:

$$Uq_1p_1'q_2p_2'\cdots q_np_n'D.$$

It is not difficult to see that the above correspondence is a bijection. Figure 4.1 gives an illustration.

Using the same idea, we may encode a pair of converging walks (L_1, L_2) in $T(i,n)$ by a partial Dyck path P in the sense that the starting point of P is not necessarily the point $(0,0)$. We should note that the common definition of a partial Dyck path is a lattice path starting from the origin $(0,0)$ with up and down steps not going below the x-axis. Define $P(i,n)$ to be the set of all partial Dyck paths of length $2n$ which start from $(0,2i)$ and never return to the x-axis except for the final destination. The following proposition establishes the connection between converging walks and partial Dyck paths.

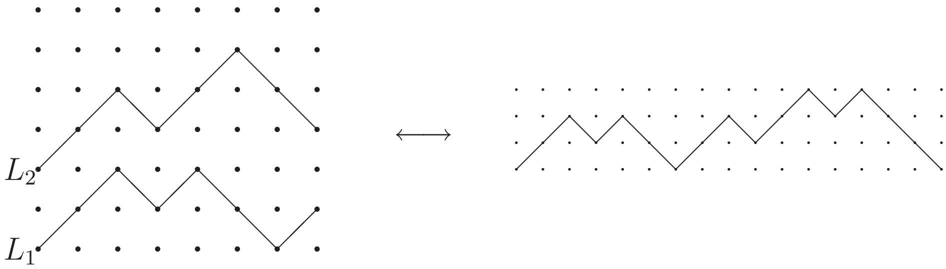

Figure 4.1: From a 2-watermelon to a Dyck path.

Proposition 4.1. *For $n \geq 1$, there exists a bijection between $T(i,n)$ and $P(i,n)$.*

Proof. Given a pair of converging walks (L_1, L_2) in $T(i,n)$, let $L_1 = p_1p_2\cdots p_n$ and $L_2 = q_1q_2\cdots q_n$, where $p_i, q_i = U$ or D. Then (L_1, L_2) can be represented by a partial Dyck path P of length $2n$ starting from $(0, 2i)$:

$$P = q_1p_1'q_2p_2'\cdots q_np_n'.$$

Clearly, P returns to the x-axis at the ending point and never touches the x-axis before the ending point, that is, $P \in P(i,n)$. It is easy to verify that the above correspondence is a bijection. Figure 4.2 is an illustration. ∎

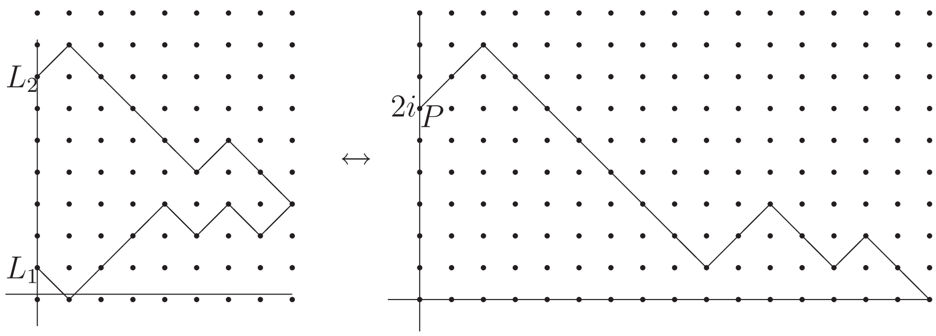

Figure 4.2: From a pair of converging walks to a partial Dyck path.

It is well known that the number of partial Dyck paths in $P(i,n)$ is given by the classical ballot number. Here we give a decomposition of a partial Dyck path into Dyck paths in accordance with the generating function of $|T(i,n)|$ as given in Proposition 3.3.

Given a partial Dyck path P in $P(i,n)$, we can decompose P into i nonempty Dyck paths $P_1, \ldots, P_i$ via the following procedure. Let $P = A_0A_1\cdots A_{2n}$, where P is represented by the sequence of points rather than steps. Let $j_0 = 0$, and for $1 \leq k \leq i$, let j_k be the minimum index such that the y-coordinate of A_{j_k} is two less than that of $A_{j_{k-1}}$. Then we can decompose P into i segments $Q_1, Q_2, \ldots, Q_i$, where Q_k is the segment of P starting at $A_{j_{k-1}}$ and ending at A_{j_k}. Observe that by the choice of j_k, the rightmost two steps of Q_k must be DD. Let P_k denote the Dyck path obtained from Q_k by deleting the last down step and adding an up step before the first step of Q_k. Evidently, P_k is a nonempty Dyck path. This completes the proof. ∎

To conclude this section, we note that $|T(i,n)|$ can be computed by using the Lagrange inversion formula, or by using the formula for the number of Dyck paths of length $2n+2i$ with $2i$ returns to the x-axis, see Deutsch [4]. The explicit formula is as follows:

$$|T(i,n)| = \frac{i}{n}\binom{2n}{n-i}.$$

From formula (2.7), we obtain the explicit formula for $|N(i,n)|$:

$$|N(i,n)| = 4^n - \sum_{k=i}^{n} \frac{i}{k}\binom{2k}{k-i}4^{n-k}. \tag{4.1}$$

We also note that $|T(i,n)|$ can be expressed as the classical ballot number $b(n+i-1, n-i)$, where

$$b(n,i) = \binom{n+i}{i} - \binom{n+i}{i-1} = \frac{n+1-i}{n+1+i}\binom{n+i+1}{i},$$

see, for example, Riordan [20].

5 Four Vicious Walkers

In this section, we present a reflection principle for four vicious walkers that leads to a reduction from four vicious walks to two vicious walks. We first introduce some defintions. Let $U(i,j,k,n)$ be the set of 4-walks (L_1, L_2, L_3, L_4) of length n, where L_1, L_2, L_3 and L_4 start from $(0,0)$, $(0,2i)$, $(0,2i+2j)$ and $(0,2i+2j+2k)$ respectively. Let $V(i,j,k,n)$ be the set of four vicious walks (L_1, L_2, L_3, L_4) in $U(i,j,k,n)$. Define the generating function $V_{i,j,k}(t)$ by

$$V_{i,j,k}(t) = \sum_{n \geq 0} |V(i,j,k,n)| t^n.$$

The following formula for $|V(i,j,k,n)|$ is a consequence of Gessel's theorem [9]. Let $v(i,n)$ denote the number of two vicious walks in $N(i,n)$ as given by the generating function (2.7). Recall that an explicit formula for $v(i,n)$ is given by (4.1).

Theorem 5.1. *For any $i, j, k \geq 1$, we have*

$$|V(i,j,k,n)| = v(i,n)v(k,n) - v(i+j,n)v(j+k,n) + v(i+j+k,n)v(j,n). \quad (5.1)$$

In order to give a combinatorial interpretation of the above formula (5.1), we will establish a reflection principle for certain classes of four vicious walks. For $1 \leq r < s \leq 4$, we use $W_{rs}(n)$, or W_{rs} for short, to denote the set of 4-walks (L_1, L_2, L_3, L_4) in $U(i,j,k,n)$ such that L_r and L_s are nonintersecting. Similarly, for $1 \leq r < s \leq 4$, we use $M_{rs}(n)$, or M_{rs} for short, to denote the set of 4-walks (L_1, L_2, L_3, L_4) in $U(i,j,k,n)$ such that L_r and L_s are intersecting. Clearly, the set $V(i,j,k,n)$ of four vicious walks of length n can be expressed as $W_{12} \cap W_{23} \cap W_{34}$. Clearly, we have

$$|W_{12} \cap W_{23} \cap W_{34}| = |W_{12} \cap W_{34}| - |W_{12} \cap M_{23} \cap W_{34}|. \quad (5.2)$$

Note that it is easy to compute $|W_{12} \cap W_{34}|$ by using the formula for two vicious walks. To compute $|W_{12} \cap M_{23} \cap W_{34}|$, we may rely on the following reflection principle.

Theorem 5.2. *There exists a bijection between the set $W_{13} \cap W_{24}$ and the set $(W_{14} \cap W_{23}) \cup (W_{12} \cap M_{23} \cap W_{34})$.*

Proof. We proceed to construct a map ψ from $W_{13} \cap W_{24}$ to $(W_{14} \cap W_{23}) \cup (W_{12} \cap M_{23} \cap W_{34})$. Let (L_1, L_2, L_3, L_4) be a 4-walk in $W_{13} \cap W_{24}$. We have the following nine cases.

(1) $L_1 \cap L_2 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$ and $L_3 \cap L_4 = \emptyset$, then it is clear that $(L_1, L_2, L_3, L_4) \in W_{12} \cap M_{23} \cap W_{34}$. In this case, we define $\psi((L_1, L_2, L_3, L_4)) = (L_1, L_2, L_3, L_4)$.

(2) $L_1 \cap L_4 = \emptyset$ and $L_2 \cap L_3 = \emptyset$, then it is clear that $(L_1, L_2, L_3, L_4) \in W_{14} \cap W_{23}$. In this case, we define $\psi((L_1, L_2, L_3, L_4)) = (L_1, L_2, L_3, L_4)$.

(3) $L_2 \cap L_3 \neq \emptyset$, either $L_1 \cap L_2 \neq \emptyset$ or $L_3 \cap L_4 \neq \emptyset$, and L_2 meets L_3 before it meets L_1 (when $L_1 \cap L_2 = \emptyset$, we naturally assume that L_2 meets L_3 before it meets L_1), L_3 meets L_2 before it meets L_4 (when $L_3 \cap L_4 = \emptyset$, we naturally assume that L_3 meets L_2 before it meets L_4). In this case, we apply the usual reflection operation on L_2 and L_3, and denote the resulting paths by L_2' and L_3'. Let $L_1' = L_1$, $L_4' = L_4$ and $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. It is easy to verify that $(L_1', L_2', L_3', L_4') \in W_{12} \cap M_{23} \cap W_{34}$.

(4) $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_1 \cap L_2 \neq \emptyset$, and L_2 meets L_1 before it meets L_3.

(5) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$, L_2 meets L_1 before it meets L_3, and L_3 meets L_2 before it meets L_4.

(6) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 = \emptyset$.

In Cases (4), (5), (6), we apply the usual reflection operation on L_1 and L_2, and denote the resulting paths by L_1' and L_2'. Let $L_3' = L_3$, $L_4' = L_4$ and $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. It is easy to verify that $(L_1', L_2', L_3', L_4') \in W_{14} \cap W_{23}$. Figure 5.1 is an illustration of the reflection operation on (L_1, L_2, L_3, L_4) in Case (4).

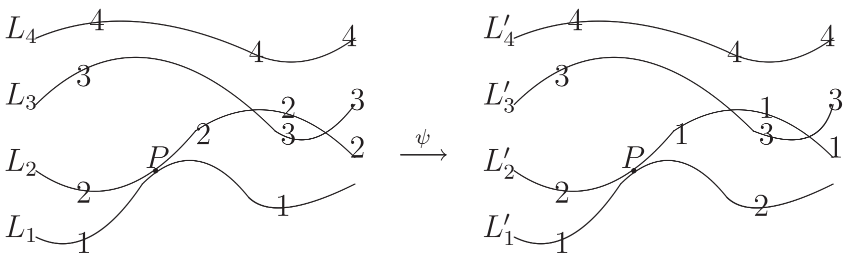

Figure 5.1: The action of the map ψ on a 4-walk (L_1, L_2, L_3, L_4) in Case (4).

(7) $L_1 \cap L_2 = \emptyset$, $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_3 \cap L_4 \neq \emptyset$, L_3 meets L_4 before it meets L_2.

(8) $L_1 \cap L_2 \neq \emptyset$, $L_1 \cap L_4 = \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $L_3 \cap L_4 \neq \emptyset$, L_3 meets L_4 before it meets L_2 and L_2 meets L_3 before it meets L_1.

(9) $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$, L_3 meets L_4 before it meets L_2.

In Cases (7), (8), (9), we use the usual reflection operation on L_3 and L_4, and denote the resulting paths by L_3' and L_4'. Let $L_1' = L_1$, $L_2' = L_2$ and $\psi((L_1, L_2, L_3, L_4)) = (L_1', L_2', L_3', L_4')$. It is easy to verify that $(L_1', L_2', L_3', L_4') \in W_{14} \cap W_{23}$.

It is not difficult to see that the above procedure is reversible. Thus we have reached the conclusion that ψ is a bijection. ∎

Using the above reflection principle, we can give a combinatorial proof of Theorem 5.1. In view of the bijection in Theorem 5.2, we see that

$$|W_{12} \cap M_{23} \cap W_{34}| = |W_{13} \cap W_{24}| - |W_{14} \cap W_{23}|. \tag{5.3}$$

Substituting (5.3) into (5.2), we deduce that

$$|W_{12} \cap W_{23} \cap W_{34}| = |W_{12} \cap W_{34}| - |W_{13} \cap W_{24}| + W_{14} \cap W_{23}|. \tag{5.4}$$

It is clear that a 4-walk in $W_{12} \cap W_{34}$ corresponds to a pair of two vicious walks (V_1, V_2), where V_1 is a two vicious walk (L_1, L_2) of length n, with L_1 and L_2 starting from $(0, 0)$ and $(0, 2i)$ respectively, and V_2 is a two vicious walk (L_3, L_4) of length n, with L_3 and L_4 starting from $(0, 2i+2j)$ and $(0, 2i+2j+2k)$ respectively. So we get

$$|W_{12} \cap W_{34}| = v(i, n) \cdot v(k, n).$$

Similarly, we have

$$|W_{13} \cap W_{24}| = v(i+j,n) \cdot v(j+k,n),$$

and

$$|W_{14} \cap W_{23}| = v(i+j+k,n) \cdot v(j,n).$$

Hence we obtain (5.1). This complete the proof of Theorem 5.1. ∎

The above reflection principle for four vicious walks depends on several cases. It would be interesting to find a simpler reflection principle for 4-vicious walks. It would be also interesting to extend this approach to r-vicious walks for $r \geq 2$ in general.

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