Tenacity and rupture degree of permutation graphs of complete bipartite graphs*

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Abstract

Computer or communication networks are so designed that they don't easily get disrupted under external attack and, moreover, these are easily reconstructed when they do get disrupted. These desirable properties of networks can be measured by various parameters such as connectivity, toughness, tenacity and rupture degree. Among these parameters, tenacity and rupture degree are comparatively better parameters to measure the stability of networks. In this paper, the authors give the exact values for the tenacity and rupture degree of permutation graphs of complete bipartite graphs.

Keywords: rupture degree, tenacity, permutation graph.

AMS subject classification(2000): 05C40, 05C90, 68R10, 94C15

1. Introduction

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set V and edge set E. We use Bondy and Murty

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[1] for terminology and notations not defined here. A set of vertices S of G is called a vertex cut set if $G - S$ is disconnected.

Measures of the vulnerability of graphs are currently of growing interest among graph theorists and network designers. Among vulnerability parameters, much have been done recently on the toughness, binding number of different classes of graphs since these parameters are more sensitive to the structure of the graph than is the connectivity of the graph. In [4], Guichard *et al.* given the integrity, toughness, and binding number for permutation graphs of complete and complete bipartite graphs.

In the following two definitions, $m(G - S)$, and $\omega(G - S)$, respectively, denotes the order of the largest component and number of components in $G - S$.

The *tenacity* of a graph G, $T(G)$, which is defined by Cozzens in [3], is defined as

$$T(G) = min\{\frac{|S| + m(G - S)}{\omega(G - S)} : S \subseteq V(G) \ is\ a\ vertex\ cut\ set\ of\ G\}.$$

The *rupture degree* of a noncomplete graph G, $r(G)$, introduced by Li, Zhang and Li in [6], is defined as

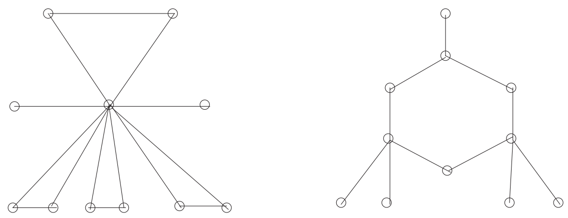
$$r(G) = max\{\omega(G-S)-|S|-m(G-S) : S \subseteq V(G) \ is\ a\ vertex\ cut\ set\ of\ G\}.$$

In particular, the tenacity and rupture degree of a complete graph K_n is defined to be n and $1 - n$ respectively.

Clearly, of all the above parameters, tenacity and rupture degree are comparatively appropriate for measuring the vulnerability of networks. Similarly to the relation between the toughness and scattering number, the rupture degree and tenacity also differ in showing the vulnerability of networks. This can be shown as follows. Consider the graphs G_1 and G_2 in Figure 1, It is not difficult to check that $T(G_1) = T(G_2) = \frac{1}{2}$, but $r(G_1) = 3$ and $r(G_2) = 4$. Clearly $r(G_1) \neq r(G_2)$. On the other hand, we consider graphs $G_3 = K_1 + (K_{n-b-1} \cup E_b)$ and $G_4 = K_2 + (K_{n-b-3} \cup E_{b+1})$, it is obvious that $r(G_3) = r(G_4)$, but $T(G_3) \neq T(G_4)$ unless $n = 2b + 1$, where b is an integer. Hence rupture degree is a better parameter for distinguishing the vulnerability of these two graphs G_1 and G_2, but the tenacity is a better parameter for distinguishing the vulnerability of these two graphs G_3 and G_4.

It is easy to see that the higher the tenacity (the less the rupture degree) of a network the more stable it is considered to be.

In [2], the authors introduced permutation graphs and proceeded to characterize those which are planar.



G_1 G_2

Figure 1.

For a graph G with n vertices labelled $1, 2, \cdots, n$, $n \geq 4$, and a permutation $\alpha \in S_n$, the *symetric group* on the n symbols $\{1, 2, \cdots, n\}$, the *α-permutation graph of* G, $P_\alpha(G)$ consists of two disjoint copies of G, G_x and G_y, along with the n edges obtained by join x_i in G_x with $y_{\alpha(i)}$ in G_y, $i = 1, 2, \cdots, n$.

It is well known that permutation graphs have high connectivity properties, as is shown in [8] and [9]. As special graphs, some vulnerability parameters of permutation graphs of complete bipartite graphs have been determined in [4]. In [5], we give the following decision problem.

Problem 2.1 Not r-Rupture

Instance: An incomplete connected graph G, and an integer r.

Question: Does there exist an $X \subset V(G)$ with $\omega(G - X) \geq 2$ such that $\omega(G - X) > |X| + m(G - X) + r$?

And by this decision problem, we proved that computing the rupture degree of a graph is NP-hard in general and so is the problem of determining the tenacity of a graph [7], so it is an interesting problem to determine these two parameters for some special graphs.

In this paper, Formulas for computing the rupture degree and tenacity for permutation graphs of complete bipartite graphs are determined.

2. Tenacity and rupture degree of the permutation Graphs of complete bipartite graphs

In this section, we fix our attention on permutation graph of complete bipartite graph, $P_\alpha(K_{m,n})$. Assume that $m \leq n$ and that M and N are the sets of the partitions of size m and n respectively. Furthermore, assume that the vertices of M are labelled $1, 2, \cdots, m$ and that vertices of N are labelled $m + 1, m + 2, \cdots, m + n$. For the permutation graph $P_\alpha(K_{m,n})$, let M_x and M_y denote the partitions of the first copy of $K_{m,n}$, N_x and N_y denote the partitions of the second copy of $K_{m,n}$, and let q denote the number of vertices in M_x that are joined by permutation edges to vertices in M_y. It is well known that the connectivity, toughness, integrity and the binding number of $P_\alpha(K_{m,n})$ can be expressed in terms of the parameters m, n, and/or q as follows.

Theorem 2.1([4]) *For α in S_{m+n}, and $m \leq n$*

$$t(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m}{m+n-q} & if\ q < \frac{n^2+m^2}{n+3m} \\ \frac{n+m}{n+q} & if\ q \geq \frac{n^2+m^2}{n+3m}. \end{cases}$$

Theorem 2.2([4]) *For α in S_{m+n}, and $m \leq n$*

$$I(P_\alpha(K_{m,n})) = \begin{cases} 2m+1 & if\ m = n\ and\ q \in \{0, m\} \\ 2m+2 & otherwise. \end{cases}$$

Theorem 2.3 ([2]) *For α in S_{m+n}, and $m \leq n$*

$$b(P_\alpha(K_{m,n})) = \begin{cases} \frac{n+q}{q} & if\ q < \frac{nm}{2m+n-1} \\ \frac{2m+2n-1}{m+2n-1} & if\ \frac{nm}{2n+m-1} \leq q < \frac{m^2+3mn-2m}{4n+2m-2} \\ \frac{3m+n-2q}{n+m} & if\ \frac{m^2+3mn-2m}{4n+2m-2} \leq q. \end{cases}$$

Theorem 2.4([8]) *For α in S_{m+n}, and $m \leq n$, $\kappa(P_\alpha(K_{m,n}) = m+1$.*

In the following, we determine the rupture degree and tenacity of the permutation graph of complete bipartite graphs in terms of the parameters m, n and/or q.

In the proofs of the remaining theorems we will use the following definitions and observations. Let M'_x be the set of vertices in M_x that are joined by permutation edges to vertices in M_y and let M'_y be these vertices in M_y. So $|M'_y| = |M'_x| = q$. Let $M''_x = M_x - M'_x$ and $M''_y = M_y - M'_y$ and thus $|M''_y| = |M''_x| = m - q$. Now the vertices in M''_x are adjacent to vertices in N_y by permutation edges, we call this vertex set N''_y. Similarly define N''_x to be the set of vertices in N_x adjacent to the vertices in M''_y by permutation edges. Thus $|N''_x| = |N''_y| = m - q$. Finally let $N'_x = N_x - N''_x$ and $N'_y = N_y - N''_y$, so $|N'_x| = |N'_y| = n - m + q$. Note that since $0 \leq q \leq m$, some of these sets may be empty. Let $K = \{M'_x, M''_x, M'_y, M''_y, N'_x, N''_x, N'_y, N''_y\}$. The relationship among these sets in K is shown in Figure 2.

Remark: It is easy to see that, when $m = n = 1$, whether $q = 1$ or $q = 0$, the two graphs are isomorphic. So under this condition, we assume that $q = 0$.

To prove our main result we first give a lemma.

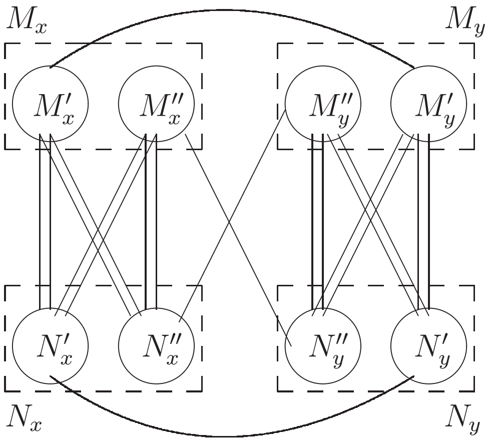


Figure 2. Relationship among the sets in K

Lemma 2.1 *For α in S_{m+n} and $m \leq n$, there exists a vertex cut set S of graph $P_\alpha(K_{m,n})$ with $T(P_\alpha(K_{m,n})) = \frac{|S|+m(P_\alpha(K_{m,n})-S)}{\omega(P_\alpha(K_{m,n})-S)}$, such that for all Z in K, if $Z \cap S$ is not empty, then $Z \subset S$.*

Proof. By the symmetry of the permutation graph of complete bipartite graph, We do the case when $Z = M'_x$. Let S' be the minimum vertex cut set of $P_\alpha(K_{m,n})$ with $T(P_\alpha(K_{m,n})) = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$ and let $A_x = S' \cap M'_x$ and $B_x = M'_x - A_x$. We let A_y be the neighborhood of A_x in M'_y, and B_y be the neighborhood of B_x in M'_y. Suppose that A_x and B_x are both nonempty, i.e., $m \geq 2$. We first note that $T = M_x \cup M_y$ is a vertex cut set of $P_\alpha(K_{m,n})$. So by the definition of tenacity, we have $T(P_\alpha(K_{m,n})) \leq \frac{2m+2}{m+n-q}$. The proof proceeds in four cases.

Case 1. If N_x and N_y are both contained in S', let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + 1$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S'|+m(P_\alpha(K_{m,n})-S')-(|A_x|-1)}{\omega(P_\alpha(K_{m,n})-S')} \leq \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$. Thus, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, which contradicts the minimality of S'.

Case 2. If N_x is contained in S' but N_y is not contained in S', then let x_i be an element of A_x and so $y_{\alpha(i)}$ is in A_y. Let $T = S' - \{x_i\}$, then $|T| = |S'| - 1$. If $y_{\alpha(i)}$ is not contained in S', then $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + 1$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$. Thus, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, but T has one less vertex than that of S', a contradiction. If $y_{\alpha(i)}$ is contained in S', then $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S') + 1$, $m(P_\alpha(K_{m,n}) - T) = m(P_\alpha(K_{m,n}) - S')$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S'|-1+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')+1} < \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, again a contradiction.

Case 3. If N_x is not contained in S' but N_y is contained in S', then let x_i be in A_x and so $y_{\alpha(i)}$ is in A_y. If $y_{\alpha(i)}$ is in S', let $T = S' - \{x_i\}$, then $|T| = |S'| - 1$, $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) = m(P_\alpha(K_{m,n}) - S') + 1$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, but T has one less vertex than that of S', a contradiction. Hence $A_y \cap S'$ is empty. Now let x_i be in B_x, then $y_{\alpha(i)}$ is contained in B_y. If $y_{\alpha(i)}$ is in S', then let $T = S' - \{y_{\alpha(i)}\} \cup \{x_i\}$. Thus, $|T| = |S'|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S') + 1$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - 1$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, a contradiction. So $B_y \cap S$ is empty. Thus $M'_y \cap S'$ is empty. Let $T = S' \cup B_x$, then $|T| = |S'| + |B_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S') + |B_x|$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - |B_x|$. So $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')+|B_x|} < \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, this contradicts the definition of Tenacity.

Case 4. If N_x and N_y are not contained in S', then consider M_y. If M_y is contained in S', let $T = S' \cup B_x$, then $|T| = |S'| + |B_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') - |B_x|$. So, $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$. Thus T is a vertex cut set with $\frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')}$, and $M'_x \subseteq T$. If M_y is not contained in S', then it is easy to see that all of the vertices in $N_x - S'$ are in the same component since B_x is nonempty, and all of the vertices in $N_y - S'$ are in the same component since $M_y \cap S'$ is nonempty. Thus $P_\alpha(K_{m,n}) - S'$ has exactly two components, one in each copy of $K_{m,n}$. If neither x_i nor $y_{\alpha(i)}$ is not in S' then S' is not a cut set. Thus at least one of x_i and $y_{\alpha(i)}$ is in S' for all $i = 1, 2, \cdots, n+m$. Thus we know that $|S'| \geq n+m$, $\omega(P_\alpha(K_{m,n}) - S') = 2$. Let C be the component of $P_\alpha(K_{m,n}) - S'$ containing B_x, then $T(P_\alpha(K_{m,n})) = \frac{|S'|+m(P_\alpha(K_{m,n})-S')}{\omega(P_\alpha(K_{m,n})-S')} \geq \frac{|S'|+|V(C)|}{\omega(P_\alpha(K_{m,n})-S')} \geq \frac{|S'|+|B_x|+|N_x-S'|}{\omega(P_\alpha(K_{m,n})-S')} \geq \frac{n+m+2}{2} \geq \frac{n+m+2}{m+n-q} \geq \frac{2m+2}{m+n-q}$.

On the other hand, by the previous remark we know that $T(P_\alpha(K_{m,n})) \leq \frac{2m+2}{m+n-q}$. Hence, in this case $T(P_\alpha(K_{m,n})) = \frac{2m+2}{m+n-q}$. Let $T = M_x \cup M_y$. Then T is a vertex cut set with $T(P_\alpha(K_{m,n})) = \frac{|T|+m(P_\alpha(K_{m,n})-T)}{\omega(P_\alpha(K_{m,n})-T)}$ and M'_x is contained in T.

From above we know that the lemma is true if $Z = M'_x$. The above proof works for the other cases of $Z \in K$. The details are omitted. ∎

By the above lemma we can obtain the tenacity of the permutation graph of complete bipartite graph.

Theorem 2.4 *For α in S_{m+n} and $m \leq n$*
(1) *if* $1 = m = n$, $T(P_\alpha(K_{m,n})) = \frac{3}{2}$.
(2) *if* $1 = m < n$,
when $q = 0$, $T(P_\alpha(K_{m,n})) = \frac{4}{n+1}$.

when $q = 1$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{n+2}{n+1} & if\ 2 \leq n \leq 3 \\ \frac{4}{n} & if\ n > 3. \end{cases}$$

(3) *if* $2 \leq m \leq n$,
when $q = 0$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+1}{m+n} & if\ m = n \\ \frac{2m+2}{m+n} & if\ m < n. \end{cases}$$

when $q = m$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+2}{n} & if\ m < \frac{m^2+n^2+m-n}{3m+n+3} \\ \frac{n+m+1}{n+m} & if\ m \geq \frac{m^2+n^2+m-n}{3m+n+3}. \end{cases}$$

when $1 \leq q \leq m-1$,

$$T(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m+2}{m+n-q} & if\ q\ < \frac{m^2+2n+n^2}{3m+n+4} \\ \frac{n+m+2}{n+q} & if\ q\ \geq \frac{m^2+2n+n^2}{3m+n+4}. \end{cases}$$

Proof. By Lemma 2.1 we know that the vertex set satisfying the condition must be the union of the elements of K. It is easy to find 55 vertex cut sets of this type. But most of these sets are trivial, all but 4 of these sets may be discarded as giving too larger values for $\frac{|S|+m(P_\alpha(K_{m,n})-S)}{\omega(P_\alpha(K_{m,n})-S)}$. The remaining sets are $S_1 = M_x \cup M''_y$, $S_2 = M_x \cup M_y$, $S_3 = M_x \cup N_y$, $S_4 = M_x \cup N'_y$, and the values for $\frac{|S|+m(P_\alpha(K_{m,n})-S)}{\omega(P_\alpha(K_{m,n})-S)}$ given by these sets are $v_1 = \frac{m+2n+q}{m-q+1}$, $v_2 = \frac{2m+2}{n+m-q}$, $v_3 = \frac{m+n+2}{q+n}$ if $q \neq m$ or $v_3 = \frac{m+n+1}{q+n}$ if $q = m$, $v_4 = \frac{3m+n-q}{n-m+q+1}$. We distinguish three cases.
Case 1. When $m = n = 1$, it is easy to see that
(a) if $q = 0$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_2 = \frac{3}{2}$.
(b) if $q = 1$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_4 = \frac{3}{2}$.
So under this condition $T(P_\alpha(K_{m,n})) = \frac{3}{2}$.
Case 2. When $1 = m < n$,
Subcase 2.1 if $q = 0$, then $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_2 = \frac{4}{n+1}$.
Subcase 2.2 if $q = 1$, then
(a) when $2 \leq n \leq 3$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_4 = \frac{n+2}{n+1}$.
(b) when $n > 3$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_2 = \frac{4}{n}$.
Case 3. When $2 \leq m \leq n$,
Subcase 3.1 if $q = 0$,
(a) when $m = n$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_2 = \frac{m+n+1}{m+n}$.
(b) when $m < n$, $T(P_\alpha(K_{m,n})) = min\{v_1, \cdots, v_4\} = v_2 = \frac{m+n+2}{m+n}$.
Subcase 3.2 if $1 \leq q \leq m$, for fixed m and n, when q increases, the following

occur. The value v_1 increases, so the minimum value of v_1 is $\frac{m+2n+1}{m}$. v_4 decreases, so the minimum value of v_4 is $\frac{2m+n}{n+1}$. When $q \neq m$ increases, The value v_3 decreases, so the maximum value of v_3 is $\frac{m+n+2}{n+1}$. It is easy to check that the minimum value of v_1 is larger than the maximum value of v_2, and the minimum value of v_4 is larger than the maximum value of v_3. And it is also easily checked that when $q = m$, the minimum value of v_4 is larger than the value of $v_3 = \frac{m+n+1}{q+n}$. So S_1 and S_4 should be discarded. Now the value of v_3 decreases as q increases, and the intersection point for v_2 and v_3 occurs where $q = \frac{m^2+2n+n^2}{3m+n+4}$, when $v_3 = \frac{m+n+2}{q+n}$ and where $q = \frac{m^2+n^2+m-n}{3m+n+3}$, when $v_3 = \frac{m+n+1}{q+n}$. Thus the theorem holds. ∎

The following theorem gives us the rupture degree of permutation graph of complete bipartite graphs. Note that $T = M_x \cup M_y$ is a vertex cut set of $P_\alpha(K_{m,n})$, so by the definition of the rupture degree we have $r(P_\alpha(K_{m,n})) \geq \omega(P_\alpha(K_{m,n}) - T) - |T| - m(P_\alpha(K_{m,n}) - T) \geq n - m - q - 2$. In order to prove this theorem we first introduce a lemma.

Lemma 2.2 *For α in S_{m+n} and $m \leq n$, there exists a vertex cut set S of $P_\alpha(K_{m,n})$ with $r(P_\alpha(K_{m,n})) = \omega(P_\alpha(K_{m,n}) - S) - |S| - m(P_\alpha(K_{m,n}) - S)$ such that for all Z in K, if $Z \cap S$ is not empty, then $Z \subset S$.*

Proof. By the symmetry of the permutation graph of complete bipartite graph, We do the case when $Z = M'_x$. Let S' be the minimum vertex cut set of $P_\alpha(K_{m,n})$ with $r(P_\alpha(K_{m,n})) = \omega(P_\alpha(K_{m,n}) - S') - |S'| - m(P_\alpha(K_{m,n}) - S')$, and let $A_x = S' \cap M'_x$ and $B_x = M'_x - A_x$. Suppose that A_x and B_x are both nonempty, i.e., $m \geq 2$. We distinguish four cases.

Case 1. If N_x and N_y are both contained in S', let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + 1$. So,

$$\begin{aligned}
&\omega(P_\alpha(K_{m,n}) - T) - |T| - m(P_\alpha(K_{m,n}) - T) \\
&\geq \omega(P_\alpha(K_{m,n}) - S') - |S'| - m(P_\alpha(K_{m,n}) - S') + |A_x| - 1 \\
&\geq \omega(P_\alpha(K_{m,n}) - S') - |S'| - m(P_\alpha(K_{m,n}) - S'),
\end{aligned}$$

a contradiction to the minimality of S'.

Case 2. If N_x is contained in S' but N_y is not contained in S', let $T = S' - A_x$, then $|T| = |S'| - |A_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S')$, $m(P_\alpha(K_{m,n}) - T) \leq m(P_\alpha(K_{m,n}) - S') + |A_x|$. So,

$$\begin{aligned}
&\omega(P_\alpha(K_{m,n}) - T) - |T| - m(P_\alpha(K_{m,n}) - T) \\
&\geq \omega(P_\alpha(K_{m,n}) - S' - |S'| - m(P_\alpha(K_{m,n}) - S'),
\end{aligned}$$

which contradicts the minimality of S'.

Case 3. If N_x is not contained in S' but N_y is contained in S', then $M'_y \cap S'$ is empty, the proof is similar to that of Case 3 in Lemma 2.1. Let $T = S' \cup B_x$. Thus $|T| = |S'| + |B_x|$, $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S') + |B_x|$,

$m(P_\alpha(K_{m,n})-T)\leq m(P_\alpha(K_{m,n})-S')-|B_x|$. So,

$$\omega(P_\alpha(K_{m,n})-T)-|T|-m(P_\alpha(K_{m,n})-T)$$
$$\geq \omega(P_\alpha(K_{m,n})-S')-|S^{'}|-m(P_\alpha(K_{m,n})-S')+|B_x|$$
$$>\omega(P_\alpha(K_{m,n})-S')-|S'|-m(P_\alpha(K_{m,n})-S'),$$

which contradicts the definition of *rupture degree*.

Case 4. If neither N_x nor N_y is contained in $S^{'}$, then consider M_y. If M_y is contained in S', let $T=S'\cup B_x$, then $|T|=|S'|+|B_x|$, $\omega(P_\alpha(K_{m,n})-T)\geq\omega(P_\alpha(K_{m,n})-S')$, $m(P_\alpha(K_{m,n})-T)\leq m(P_\alpha(K_{m,n})-S')-|B_x|$. So, $\omega(P_\alpha(K_{m,n})-T)-|T|-m(P_\alpha(K_{m,n})-T)\geq\omega(P_\alpha(K_{m,n})-S')-|S'|-m(P_\alpha(K_{m,n})-S')$. Thus T is a vertex cut set with $\omega(P_\alpha(K_{m,n})-T)-|T|-m(P_\alpha(K_{m,n})-T)=\omega(P_\alpha(K_{m,n})-S')-|S'|-m(P_\alpha(K_{m,n})-S')$, and $M_x'\subseteq T$. If M_y is not contained in S', then it is easy to see that all of the vertices in N_x-S' are in the same component since B_x is nonempty, and all of the vertices in N_y-S' are in the same component since $M_y\cap S'$ is nonempty. Thus $P_\alpha(K_{m,n})-S'$ has exactly two components, one in each copy of $K_{m,n}$. If neither x_i nor $y_{\alpha(i)}$ is not in S', then S' is not a cut set. Thus at least one of x_i and $y_{\alpha(i)}$ is in S' for all $i=1,2,\cdots,n+m$. Thus we know that $|S'|\geq n+m$, $\omega(P_\alpha(K_{m,n})-S')=2$. Let C be the component of $P_\alpha(K_{m,n})-S'$ containing B_x, then $r(P_\alpha(K_{m,n}))=\omega(P_\alpha(K_{m,n})-S')-|S'|-m(P_\alpha(K_{m,n})-S')\leq\omega(P_\alpha(K_{m,n})-S')-|S'|-|V(C)|\leq\omega(P_\alpha(K_{m,n})-S')-|S'|-|B_x|-|N_x-S'|\leq 2-n-m-2\leq n-m-q-2$.

On the other hand, by the previous remark we know that $r(P_\alpha(K_{m,n}))\geq n-m-q-2$. Hence, in this case $r(P_\alpha(K_{m,n}))=n-m-q-2$. Let $T=M_x\cup M_y$. Then T is a cut set with $r(P_\alpha(K_{m,n}))=\omega(P_\alpha(K_{m,n})-T)-|T|-m(P_\alpha(K_{m,n})-T)$ and $M_x'\subset T$.

From above we know that the lemma is true if $Z=M_x'$. The above proof works for the other cases of $Z\in K$. The details are omitted. ∎

Theorem 2.5 *For α in S_{m+n} and $m\leq n$*
(1) *if* $m=n=1$, $r(P_\alpha(K_{m,n}))=-1$.
(2) *if* $1=m<n$,
when $q=0$, $r(P_\alpha(K_{m,n}))=n-3$.
when $q=1$,

$$r(P_\alpha(K_{m,n}))=\begin{cases}-1 & if\ 2\leq n\leq 3\\ n-4 & if\ n>3.\end{cases}$$

(3) *if* $2\leq m\leq n$,
when $q=0$,

$$r(P_\alpha(K_{m,n}))=\begin{cases}n-m-2 & if\ n-m\geq 2\\ -1 & if\ n-m<2.\end{cases}$$

when $q = m$,

$$r(P_\alpha(K_{m,n})) = \begin{cases} n-2m-2 & if\ q\ \le \frac{n-1}{2} \\ -1 & if\ q\ > \frac{n-1}{2}. \end{cases}$$

when $1 \le q \le m-1$,

$$r(P_\alpha(K_{m,n})) = \begin{cases} n-m-q-2 & if\ q\ \le \frac{n}{2} \\ q-m-2 & if\ q\ > \frac{n}{2}. \end{cases}$$

Proof. By Lemma 2.2 we know that the vertex set satisfying the condition must be the union of the elements of K. It is easy to find 55 vertex cut sets of this type. But most of these sets are trivial, all but 5 of these sets may be discarded as giving too less values for $\omega(P_\alpha(K_{m,n})-S)-|S|-m(P_\alpha(K_{m,n})-S)$. The remaining sets are $S_1 = M_x \cup M_y$, $S_2 = M_x \cup N_y$, $S_3 = M_x \cup M_y \cup N'_x$, $S_4 = M_x \cup N'_y$, $S_5 = M_x \cup M''_y \cup N'_y$ and the values for $\omega(P_\alpha(K_{m,n})-S)-|S|-m(P_\alpha(K_{m,n})-S)$ given by these sets are $v_1 = n-m-q-2$, $v_2 = q-m-2$ if $q \neq m$ or $v_2 = -1$ if $q = m$, $v_3 = -2q-1$, $v_4 = -4m+2q+1$, $v_5 = -2m+1$. So we distinguish three cases.
Case 1. When $m = n = 1$, it is easy to see that
$r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\} = v_5 = -1$.
Case 2. When $1 = m < n$,
Subcase 2.1 If $q = 0$, it is easy to see that
$r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\} = v_1 = n-3$.
Subcase 2.2 If $q = 1$, it is easy to see that
(a) when $n = 2$, then $r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\} = v_5 = -1$.
(b) when $n > 2$, then $r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\} = v_1 = n-4$.
Case 3. When $2 \le m \le n$,
Subcase 3.1 If $q = 0$, it is easy to see that
(a) when $n-m \ge 2$, then $r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\}$
$= v_1 = n-m-2$.
(b) when $n-m < 2$, then $r(P_\alpha(K_{m,n})) = max\{v_1, v_2, \cdots, v_5\} = v_3 = -1$.
Subcase 3.2 If $q = m$, then, when $q \le \frac{n-1}{2}$, $max\{v_1, v_2, \cdots, v_5\} = v_1 = n-2m-2$; when $q \ge \frac{n-1}{2}$, $max\{v_1, v_2, \cdots, v_5\} = v_2 = -1$.
Subcase 3.3 If $1 \le q \le m-1$, under this condition, for fixed m and n, when $q \le \frac{n}{2}$, then $max\{v_1, v_2, \cdots, v_5\} = v_1 = n-m-q-2$; when $q \ge \frac{n}{2}$, $max\{v_1, v_2, \cdots, v_5\} = v_2 = q-m-2$.
The proof is thus completed. ∎

3. Conclusion

The rupture degree and tenacity of a graph, to some extent, represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. Hence, the rupture degree and tenacity can be used to measure the vulnerability of networks. So clearly, it is of prime importance to determine this parameter for a graph. In this paper, we have obtained the exact values for the rupture degree and tenacity of permutation graphs of complete bipartite graphs. To make further progress in this direction, one could try to characterize the graphs with given rupture degree or tenacity.

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New york, 1976.

[2] G. Chartrand and F. Harary, Planar permutation graphs, *Ann. Inst. H. Poincare Sec. B (N.S.)* 3(1967), 433-438.

[3] M. Cozzens, D. Moazzami and S. Stueckle, The tenacity of a graph, *Proc. 7th International Conference on the Theory and Applications of Graphs*, Wiley, New York, 1995, 1111-1122.

[4] D. Guichard, B. Piazza and S. Stueckle, On the vulnerability of permutation graphs of complete and complete bipartite graphs, *Ars Combinatoria* 31(1991), 149-157.

[5] F.W. Li and X.L. Li, Computing the rupture degrees of graphs, *Proc. 7th international symposium on parallel architectures, algorithms and networks*, IEEE computer society, Los Alamitos, California, 2004, 368-373.

[6] Y. K. Li, S. G. Zhang and X. L. Li, Rupture degree of graphs, *Int. J. Comput. Math.*,82(7)(2005),793-803.

[7] D. E. Mann, *The tenacity of trees*, Ph.D.Thesis, Northeastern University, 1993.

[8] B. Piazza, Connectivity and hamilitonian properties of permutation graphs, Ph.D.Dissertation, Clemsom University, August 1985.

[9] B. Piazza and S. Stueckle, On the cut frequency vector of permutation graphs, *Congr.Numer.* 49(1985), 143-145.

[10] A. Kirlangic, The Rupture Degree and Gear Graphs, *Bull. Malays.Math. Sci. Soc.* (2) 32(1) (2009), 31-36.