

Extremal H -free planar graphs

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Abstract

Given a graph H , a graph is H -free if it does not contain H as a subgraph. We continue to study the topic of “extremal” planar graphs initiated by Dowden [J. Graph Theory 83 (2016) 213–230], that is, how many edges can an H -free planar graph on n vertices have? We define $ex_p(n, H)$ to be the maximum number of edges in an H -free planar graph on n vertices. We first obtain several sufficient conditions on H which yield $ex_p(n, H) = 3n - 6$ for all $n \geq |V(H)|$. We discover that the chromatic number of H does not play a role, as in the celebrated Erdős-Stone Theorem. We then completely determine $ex_p(n, H)$ when H is a wheel or a star. Finally, we examine the case when H is a (t, r) -fan, that is, H is isomorphic to $K_1 + tK_{r-1}$, where $t \geq 2$ and $r \geq 3$ are integers. However, determining $ex_p(n, H)$, when H is a planar subcubic graph, remains wide open.

Mathematics Subject Classifications: 05C10, 05C35

Key words. Turán number, extremal planar graph, planar triangulation

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1 Introduction

All graphs considered in this paper are finite and simple. We use K_t , C_t and P_t to denote the complete graph, cycle, and path on t vertices, respectively. Given a graph G , we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree. For a vertex $v \in V(G)$, we will use $N_G(v)$ to denote the set of vertices in G which are adjacent to v . Let $d_G(v) = |N_G(v)|$ denote the degree of the vertex v in G and $N_G[v] = N_G(v) \cup \{v\}$. A vertex is a k -vertex in G if it has degree k . We use $n_k(G)$ to denote the number of k -vertices in G . For any set $S \subset V(G)$, the subgraph of G induced on S , denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. We denote by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$. If $S = \{v\}$, then we simply write $G \setminus v$. For any two disjoint sets A and B of $V(G)$, we use $e_G(A, B)$ to denote the number of edges between A and B in G . The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex-disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). For a positive integer t , we use tG to denote disjoint union of t copies of a graph G . Given two isomorphic graphs G and H , we may (with a slight but common abuse of notation) write $G = H$. For any positive integer k , let $[k] := \{1, 2, \dots, k\}$.

Given a graph H , a graph is H -free if it does not contain H as a subgraph. One of the best known results in extremal graph theory is Turán's Theorem [15], which gives the maximum number of edges that a K_t -free graph on n vertices can have. The celebrated Erdős-Stone Theorem [7] then extends this to the case when K_t is replaced by an arbitrary graph H with at least one edge, showing that the maximum number of edges possible is $(1 + o(1)) \binom{n}{2} \left(\frac{\chi(H) - 2}{\chi(H) - 1} \right)$, where $\chi(H)$ denotes the chromatic number of H . This latter result has been called the "fundamental theorem of extremal graph theory" [1]. Turán-type problems when host graphs are hypergraphs are notoriously difficult. A large quantity of work in this area has been carried out in determining the maximum number of edges in a k -uniform hypergraph on n vertices without containing k -uniform linear paths and cycles (see, for example, [9, 10, 12]). Surveys on Turán-type problems of graphs and hypergraphs can be found in [8, 11].

In this paper, we continue to study the topic of "extremal" planar graphs, that is, how many edges can an H -free planar graph on n vertices have? We define $ex_p(n, H)$ to be the maximum number of edges in an H -free planar graph on n vertices. Dowden [3] initiated the study of $ex_p(n, H)$ and proved the following results, where each bound is tight.

Theorem 1 ([3]). *Let n be a positive integer.*

- (a) $ex_p(n, C_4) \leq 15(n - 2)/7$ for all $n \geq 4$.
- (b) $ex_p(n, C_5) \leq (12n - 33)/5$ for all $n \geq 11$.

Let Θ_4 and Θ_5 be obtained from C_4 and C_5 , respectively, by adding an additional edge joining two non-consecutive vertices. The present authors [13] studied $ex_{\mathcal{P}}(n, H)$ when $H \in \{\Theta_4, \Theta_5, C_6\}$ and when H is a path on at most 9 vertices. Results from [13] are summarized below.

Theorem 2 ([13]). *Let n be a positive integer.*

- (a) $ex_{\mathcal{P}}(n, \Theta_4) \leq 12(n - 2)/5$ for all $n \geq 4$, with equality when $n \equiv 12 \pmod{20}$.
- (b) $ex_{\mathcal{P}}(n, \Theta_5) \leq 5(n - 2)/2$ for all $n \geq 5$, with equality when $n \equiv 50 \pmod{120}$.
- (c) $ex_{\mathcal{P}}(n, C_6) \leq 18(n - 2)/7$ for all $n \geq 6$.
- (d) $ex_{\mathcal{P}}(n, P_9) \leq \max\{9n/4, (5n - 8)/2\}$.

It seems quite non-trivial to determine $ex_{\mathcal{P}}(n, C_t)$ for all $t \geq 7$. In this paper, we first investigate planar graphs H satisfying $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$. This partially answers a question of Dowden [3]. As observed in [3], for all $n \geq 6$, the planar triangulation $2K_1 + C_{n-2}$ is K_4 -free. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all graphs H which contains K_4 as a subgraph and $n \geq \max\{|H|, 6\}$. In particular, $ex_{\mathcal{P}}(n, K_5^-) = 3n - 6$ for all $n \geq 6$, where K_p^- denotes the graph obtained from K_p by deleting one edge. Proposition 3 below describes several sufficient conditions on K_4 -free planar graphs H such that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$.

Proposition 3. *Let H be a K_4 -free planar graph and let $n \geq |H|$ be an integer. Then $ex_{\mathcal{P}}(n, H) = 3n - 6$ if one of the following holds.*

- (a) $\chi(H) = 4$ and $n \geq |H| + 2$.
- (b) $\Delta(H) \geq 7$.
- (c) $\Delta(H) = 6$ and either $n_6(H) + n_5(H) \geq 2$ or $n_6(H) + n_5(H) = 1$ and $n_4(H) \geq 5$.
- (d) $\Delta(H) = 5$ and either H has at least three 5-vertices or H has exactly two adjacent 5-vertices.
- (e) $\Delta(H) = 4$ and $n_4(H) \geq 7$.
- (f) H is 3-regular with $|H| \geq 9$ or H has at least three vertex-disjoint cycles or H has exactly one vertex u of degree $\Delta(H) \in \{4, 5, 6\}$ such that $\Delta(H[N(u)]) \geq 3$.
- (g) $\delta(H) \geq 4$ or H has exactly one vertex of degree at most 3.

Proposition 3 implies that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all H with $n \geq |H| + 2$ and either $\chi(H) = 4$ or $\chi(H) = 3$ and $\Delta(H) \geq 7$. Note that $\Theta_4 = K_4^-$, and both K_4^- and $K_1 + 2K_2$ have chromatic number 3. Theorem 2(a) and Theorem 5 (see below) then demonstrate that the chromatic number of H does not play a role, as in the Erdős-Stone Theorem.

By Proposition 3, $ex_{\mathcal{P}}(n, H)$ remains unknown for K_4 -free planar graphs H with $\Delta(H) = 6$, $n_6(H) + n_5(H) = 1$ and $n_4(H) \leq 4$; or $\Delta(H) = 5$ and $n_5(H) \leq 2$ (and

the two 5-vertices are not adjacent when $n_5(H) = 2$); or $\Delta(H) = 4$ and $n_4(H) \leq 6$; or $\Delta(H) \leq 3$. In particular, by Proposition 3(f), $ex_p(n, H)$ remains unknown for K_4 -free planar graphs H with exactly one vertex, say u , of degree $\Delta(H) \leq 6$ and $\Delta(H[N(u)]) \leq 2$. It seems non-trivial to determine $ex_p(n, H)$ for all such H . We next study $ex_p(n, W_k)$, where $W_k := K_1 + C_k$ is a wheel on $k + 1 \geq 5$ vertices. Unlike the classic Turán number of W_k (see [4, 5] for more information), the planar Turán number of W_k can be completely determined. We establish this in Theorem 4.

Theorem 4. *Let n, k be integers with $n \geq k + 1 \geq 5$. Then*

$$ex_p(n, W_k) = \begin{cases} 3n - 6 & \text{if } k \geq 6, \text{ or } k = 5 \text{ and } n \neq 7, \text{ or } k = 4 \text{ and } n \geq 12 \\ 3n - 7 & \text{if } k = 4 \text{ and } n \in \{5, 6\}, \text{ or } k = 5 \text{ and } n = 7 \\ 3n - 8 & \text{if } k = 4 \text{ and } 7 \leq n \leq 11. \end{cases}$$

A graph is a (t, r) -fan if it is isomorphic to $K_1 + tK_{r-1}$, where $t \geq 2$ and $r \geq 2$ are integers. The classical Turán number of (t, r) -fan, namely, $ex(n, K_1 + tK_{r-1})$, has also been studied when n is sufficiently large (see [2, 6] for more information). We next study $ex_p(n, H)$ when H is a K_4 -free (t, r) -fan, in particular, when $H \in \{K_1 + 2K_2, K_{1,t}, K_1 + 3K_2\}$. Theorem 5 below establishes a sharp upper bound for $ex_p(n, K_1 + 2K_2)$, and Theorem 6 completely determines the value of $ex_p(n, K_{1,t})$ for all $t \geq 3$. However, the upper bound for $ex_p(n, K_1 + 3K_2)$ when $n \geq 15$ in Theorem 7 is not tight.

Theorem 5. *Let $n \geq 5$ be an integer. Then*

$$2n - 3 \leq ex_p(n, K_1 + 2K_2) \leq \frac{19n}{8} - 4.$$

Furthermore, $ex_p(n, K_1 + 2K_2) = \frac{19n}{8} - 4$ if and only if n is divisible by 8.

Theorem 6. *Let n, t be integers with $n \geq t + 1 \geq 4$. Then*

$$ex_p(n, K_{1,t}) = \begin{cases} 3n - 6 & \text{if } t \geq 7, \text{ or } t = 6 \text{ and } n \in \{7, 8, 9, 10, 12\} \\ 3n - 7 & \text{if } t = 6 \text{ and } n = 11 \\ 3n - 8 & \text{if } t = 6 \text{ and } n \in \{13, 14\}, \text{ or } t = 5 \text{ and } n = 7 \\ \lfloor \frac{(t-1)n}{2} \rfloor & \text{if } t \in \{3, 4\}, \text{ or } t = 5 \text{ and } n \neq 7, \text{ or } t = 6 \text{ and } n \geq 15. \end{cases}$$

Theorem 7. *Let $n \geq 7$ be an integer. Then*

$$\left\lfloor \frac{5n}{2} \right\rfloor \leq ex_p(n, K_1 + 3K_2) < \frac{17n}{6} - 4$$

for all $n \geq 15$ and

$$ex_p(n, K_1 + 3K_2) = \begin{cases} 3n - 6 & \text{if } n \in \{7, 8, 9, 10, 12\} \\ 3n - 7 & \text{if } n = 11 \\ 3n - 8 & \text{if } n \in \{13, 14\}. \end{cases}$$

We need to introduce more notation. Given a plane graph G and an integer $i \geq 3$, an i -face in G is a face of order i . Let f_i denote the number of i -faces in G . Let T_n denote a plane triangulation on $n \geq 3$ vertices, and let T_n^- be obtained from T_n with one edge removed. For every integer $n \geq 5$, let O_n denote the unique outerplane graph with $2n - 3$ edges, maximum degree 4, and the outer face of order n ; let O'_n be a different drawing of O_n with one unique inner face of order n ; and finally, let O_n^* be the planar triangulation obtained from O_n and O'_n by identifying the outer face of O_n with the unique n -face of O'_n in such a way that O_n^* is a simple graph. The proof of Proposition 3 relies heavily on the fact that O_n^* , $K_1 + O_{n-1}$ and $2K_1 + C_{n-2}$ are planar triangulations.

Finally, we shall make use of the following lemma in the proof of Theorem 6 and Theorem 7.

Lemma 8 ([14]). *There does not exist a 4-regular planar graph on 7 vertices, or a 5-regular planar graph on 14 vertices, or a planar graph on $n \in \{11, 13\}$ vertices with exactly one vertex of degree 4 and $n - 1$ vertices of degree 5.*

2 Proof of Proposition 3

Let H and n be given as in the statement. To prove (a), assume $\chi(H) = 4$ and $n \geq |H| + 2$. Since the planar triangulation $2K_1 + C_{n-2}$ has no subgraph on $|H|$ vertices with chromatic number 4, we see that $2K_1 + C_{n-2}$ is H -free. Hence, $ex_p(n, H) = 3n - 6$ when $\chi(H) = 4$ and $n \geq |H| + 2$.

To prove (b), assume that $\Delta(H) \geq 7$. Then $n \geq |H| \geq 8$, and the planar triangulation O_n^* is H -free because $\Delta(O_n^*) = 6$. Hence, $ex_p(n, H) = 3n - 6$ for all $n \geq |H|$.

To prove (c), assume $\Delta(H) = 6$. Then $n \geq |H| \geq 7$. Assume first $n_6(H) + n_5(H) \geq 2$. Let $x, y \in V(H)$ be such that $d_H(x) = 6$ and $d_H(y) \geq 5$. Then the planar triangulation $K_1 + O_{n-1}$ is H -free when $xy \notin E(H)$, and the planar triangulation $2K_1 + C_{n-2}$ is H -free when $xy \in E(H)$. Hence, $ex_p(n, H) = 3n - 6$ when $n_6(H) + n_5(H) \geq 2$. Next assume that $n_6(H) + n_5(H) = 1$ and $n_4(H) \geq 5$. Then $n_5(H) = 0$. Note that the planar triangulation $2K_1 + C_{n-2}$ is H -free when $n_4(H) \geq 6$. We may further assume that $n_4(H) = 5$. Let $u, v_1, \dots, v_5 \in V(H)$ with $d_H(u) = 6$ and $d_H(v_i) = 4$ for all $i \in [5]$. Let $M := H[\{u, v_1, \dots, v_5\}]$. We may assume that $uv_1 \notin E(M)$ when $d_M(u) = 4$. Then the

planar triangulation $2K_1 + C_{n-2}$ is H -free when $d_M(u) = 5$, or $d_M(u) \leq 3$, or $d_M(u) = 4$ and $d_M(v_1) \leq 3$; and the planar triangulation O_n^* is H -free when $N_M(u) = N_M(v_1) = \{v_2, v_3, v_4, v_5\}$. It follows that $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$. This proves (c).

To prove (d), assume $\Delta(H) = 5$ and $n_5(H) \geq 2$. Then $n \geq |H| \geq 6$. Let u, v be two distinct 5-vertices in H . Then either $n_5(H) \geq 3$ or $n_5(H) = 2$ with $uv \in E(H)$. Note that the planar triangulation $2K_1 + C_{n-2}$ has exactly two non-adjacent vertices of degree at least 5 when $n \geq 7$ and has maximum degree 4 when $n = 6$. Hence, $2K_1 + C_{n-2}$ is H -free, and so $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$. This proves (d).

To prove (e) and (f). Assume $\Delta(H) = 4$ and $n_4(H) \geq 7$, or H is 3-regular with $|H| \geq 9$, or H has at least three vertex-disjoint cycles, or H has exactly one vertex u of degree $\Delta(H) \in \{4, 5, 6\}$ such that $\Delta(H[N(u)]) \geq 3$. Then the planar triangulation $2K_1 + C_{n-2}$ is H -free. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$.

It remains to prove (g). Assume $\delta(H) \geq 4$ or H has exactly one vertex of degree at most 3. Then $n \geq |H| \geq 5$. Note that the planar triangulation $K_1 + O_{n-1}$ is 3-degenerate and every subgraph of $K_1 + O_{n-1}$ has at least two vertices of degree at most 3, because every subgraph of O_{n-1} has at least two vertices of degree at most 2. Hence, $K_1 + O_{n-1}$ is H -free, and so $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geq |H|$. This completes the proof of Proposition 3. ■

3 Proof of Theorem 4

Let n, k be given as in the statement. Assume $k \geq 7$. By Proposition 3(b), $ex_{\mathcal{P}}(n, W_k) = 3n - 6$ for all $n \geq k + 1$. Assume next $k \in \{5, 6\}$. Since the planar triangulation $2K_1 + C_{n-2}$ is W_k -free when $n \geq k + 3$ or $n = k + 1$, we see that $ex_{\mathcal{P}}(n, W_k) = 3n - 6$ when $n \geq k + 3$ or $n = k + 1$. We next determine $ex_{\mathcal{P}}(n, W_k)$ when $n = k + 2$. For $k = 6$ and $n = 8$, the plane triangulation on 8 vertices depicted in Figure 1(a) is W_6 -free and so $ex_{\mathcal{P}}(n, W_6) = 3n - 6$ when $n = 8$. For $k = 5$ and $n = 7$, note that the plane graph with 7 vertices and 14 edges

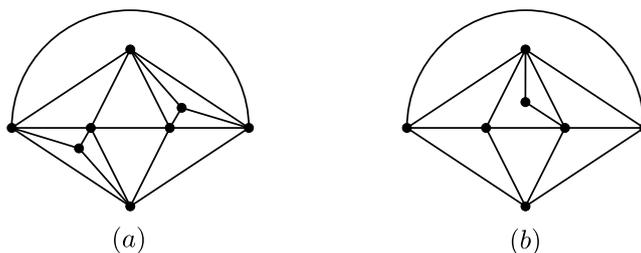


Figure 1: W_k -free plane graphs with $n = k + 2$ vertices and $3n - 12 + k$ edges, where $k \in \{5, 6\}$.

given in Figure 1(b) is W_5 -free. Thus, $ex_{\mathcal{P}}(7, W_5) \geq 3 \cdot 7 - 6 - 1 = 14$. On the other

hand, all plane triangulations on seven vertices¹ are depicted in Figure 2, each containing a copy of W_5 . Hence, $ex_{\mathcal{P}}(n, W_5) = 3n - 7$ when $n = 7$.

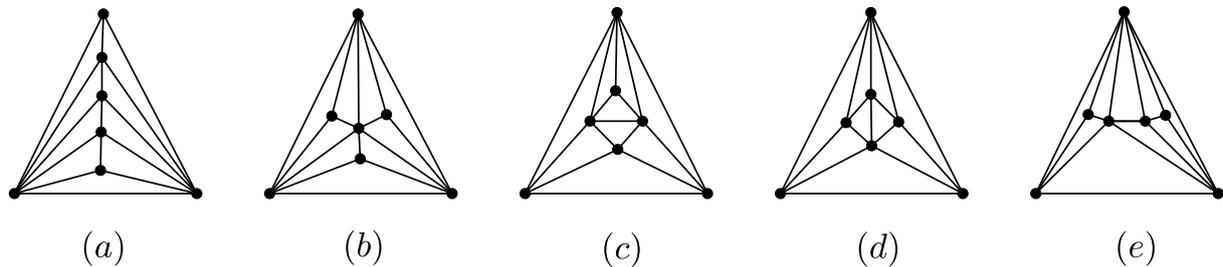


Figure 2: All plane triangulations on 7 vertices. Furthermore, each T_7 has a copy of W_4 and W_5 , and each T_7^- has a copy of W_4 .

It remains to consider the case when $k = 4$. To show $ex_{\mathcal{P}}(n, W_4) = 3n - 6$ for all $n \geq 12$, assume first that $n = 5t + 2$ for some integer $t \geq 2$. Let L_t be a plane triangulation on $n = 5t + 2$ vertices constructed as follows: for each $i \in [t]$, let C^i be a cycle with vertices $u_{i,1}, u_{i,2}, \dots, u_{i,5}$ in order. Let L_t be the plane triangulation obtained from disjoint union of C^1, \dots, C^t by adding edges $u_{i,j}u_{i+1,j}$ and $u_{i,j}u_{i+1,j+1}$ for all $i \in [t-1]$ and $j \in [5]$, where all arithmetic on the index $j + 1$ here is done modulo 5, and finally adding two new non-adjacent vertices u and v such that u is adjacent to all vertices of C^1 and v is adjacent to all vertices of C^t . The graph L_t when $t = 3$ is depicted in Figure 3. It is worth noting that L_t is K_4 -free, $d_{L_t}(u) = d_{L_t}(v) = 5$, $d_{L_t}(u_{1,j}) = d_{L_t}(u_{t,j}) = 5$, $d_{L_t}(u_{i,j}) = 6$ for $2 \leq i \leq t-1$ and $j \in [5]$. Furthermore, the subgraph induced by the neighborhood of each vertex in L_t is isomorphic to either C_5 or C_6 . Hence, L_t is W_4 -free and so $ex_{\mathcal{P}}(n, W_4) = 3n - 6$ when $n = 5t + 2$ for some integer $t \geq 2$.

Next assume that $n = 5t + 2 + i$ for some $i \in [4]$, where $t \geq 2$ is an integer. Note that the plane triangulation L_t on $5t + 2$ vertices constructed above contains at least four pairwise vertex-disjoint faces. Let F_1, \dots, F_i be any i pairwise vertex-disjoint faces of L_t , and let L_t^i be the plane triangulation obtained from L_t by adding one vertex, say x_j , of degree 3 to each F_j for all $j \in [i]$. Clearly, L_t^i is a plane triangulation on $n = 5t + 2 + i$ vertices. By the choice of F_1, \dots, F_i , we see that x_1, \dots, x_i are pairwise non-adjacent in L_t^i , and no two of x_1, \dots, x_i have common neighbors in L_t^i . We next show that L_t^i is W_4 -free for all $i \in [4]$. Suppose that L_t^i contains a copy of W_4 for some $i \in [4]$. Let

¹To find all plane triangulations on 7 vertices, let H be a plane triangulation on 7 vertices. Then $e(H) = 15$ and H must be 3-connected with maximum degree 5 or 6. Let $u \in V(H)$ be a vertex of maximum degree. If $d(u) = 6$, then H has at most two vertices of degree 6 and $H - u$ has neither K_4 nor $K_{2,3}$ minor. Thus $H - u$ must be outer planar and so H is isomorphic to one of the plane triangulations in Figure 2(a,b,e). If $d(u) = 5$, then the degree sequence of H is either $(5, 5, 5, 4, 4, 4, 3)$ or $(5, 5, 4, 4, 4, 4, 4)$. Since H has no $K_{3,3}$ minor, one can then check that H is isomorphic to one of the plane triangulations in Figure 2(c,d).

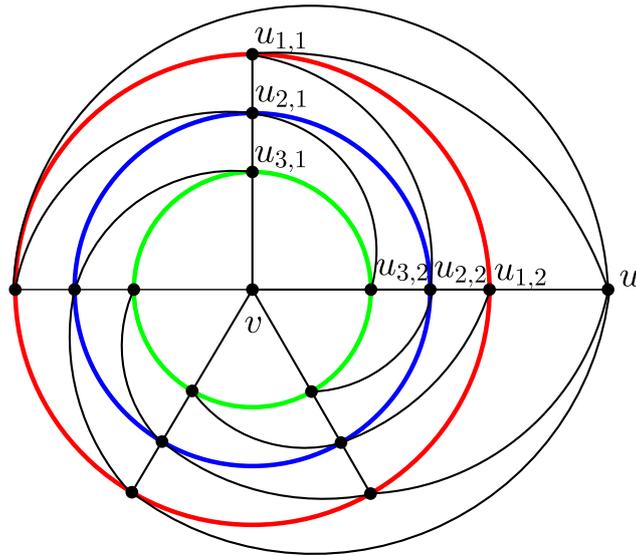


Figure 3: The plane triangulation L_t when $t = 3$, where C^1, C^2, C^3 are in red, blue and green.

H be a W_4 in L_t^i . Then H must contain exactly one, say x_1 , of x_1, \dots, x_i , because L_t is W_4 -free, and no two of x_1, \dots, x_i are adjacent or have common neighbors in L_t^i . Let $y, z \in V(H)$ be the two neighbors of x_1 such that $yz \notin E(H)$. By the choice of x_1 , we see that $yz \in E(L_t)$. But then $L_t[V(H \setminus x_1)] = K_4$ and so L_t contains K_4 as a subgraph, a contradiction. Therefore, $ex_p(n, W_4) = 3n - 6$ for all $n \geq 12$.

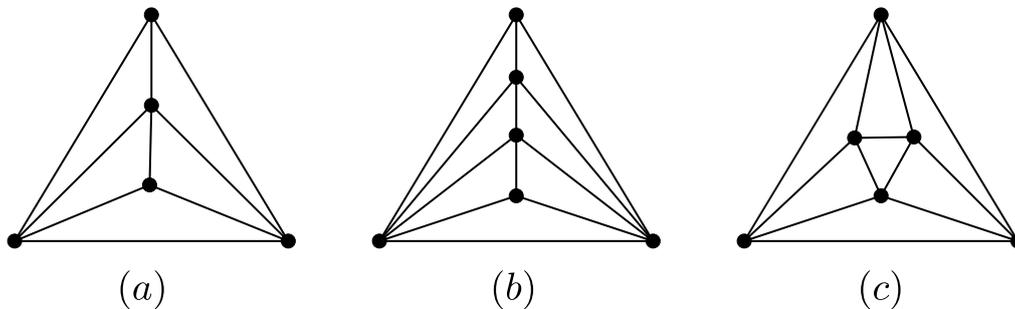


Figure 4: All plane triangulations on 5 and 6 vertices.

We next show that $ex_p(n, W_4) = 3n - 7$ when $n \in \{5, 6\}$. Note that all plane triangulations on $n \in \{5, 6\}$ vertices are depicted in Figure 4, each containing a copy of W_4 . Thus, $ex_p(n, W_4) \leq 3n - 7$. On the other hand, for all $n \in \{5, 6\}$, the planar graph $K_2 + (K_2 \cup K_{n-4})$ has $3n - 7$ edges and is W_4 -free. Hence, $ex_p(n, W_4) = 3n - 7$ when $n \in \{5, 6\}$.

Finally, we show that $ex_p(n, W_4) = 3n - 8$ for all $n \in \{7, 8, 9, 10, 11\}$. The plane graph

J , given in Figure 5, is W_4 -free with $n = 11$ vertices and $3n - 8$ edges. Let B be the set of

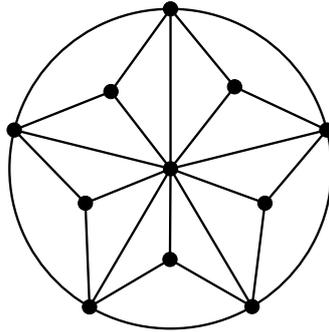


Figure 5: Graph J .

all vertices of degree 3 in J . Then $|B| = 5$. For each $n \in \{7, 8, 9, 10\}$, let J_n be the plane graph obtained from J by deleting $11 - n$ vertices in B . Then J_n is an induced subgraph of J . Clearly, J_n is W_4 -free with n vertices and $3n - 8$ edges. Hence, $ex_p(n, W_4) \geq 3n - 8$ for all $n \in \{7, 8, 9, 10, 11\}$. We next show that $ex_p(n, W_4) \leq 3n - 8$ for all $n \in \{7, 8, 9, 10, 11\}$. Suppose this is not true. Let G be a W_4 -free planar graph on $n \in \{7, 8, 9, 10, 11\}$ vertices with $e(G) \geq 3n - 7$. We choose such a G with n minimum. Then $G = T_n$ or $G = T_n^-$. Since each T_7 , depicted in Figure 2, contains a copy of W_4 , and each T_7^- also contains a copy of W_4 , it follows that $n \in \{8, 9, 10, 11\}$. Let $u \in V(G)$ with $d_G(u) = \delta(G)$. Then $\delta(G) \leq 4$, else $e(G) \geq \frac{5n}{2} > 3n - 6$ because $n \leq 11$, a contradiction. Next, if $\delta(G) \leq 3$, then $e(G \setminus u) \leq 3(n - 1) - 8$ by minimality of n and the fact that $ex_p(n, W_4) \leq 3n - 8$ when $n = 7$. Thus, $e(G) = e(G \setminus u) + d_G(u) \leq 3(n - 1) - 8 + 3 = 3n - 8$, a contradiction. This proves that $\delta(G) = 4$. Since $N_G[u]$ does not contain a copy of W_4 in G , we see that $G \neq T_n$. Thus $G = T_n^-$. We may assume that G is a plane drawing of T_n^- such that the outer face is a 3-face. Let $x_1, y_1 \in V(G)$ be such that $G + x_1y_1 = T_n$. Then x_1 and y_1 must lie on the boundary of the unique 4-face, say F , in G . Let x_1, x_2, y_1, y_2 be the vertices on the boundary of F in order. Then $d_G(v) \geq 5$ for all $v \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$, because $G = T_n^-$ and $N_G[u]$ does not contain a copy of W_4 in G for any $u \in V(G)$ with $d_G(u) = 4$. Thus $2(3n - 7) = 2e(G) \geq 4 \cdot 4 + 5 \cdot (n - 4)$, which implies that $n \in \{10, 11\}$. Suppose each vertex in $\{x_1, x_2, y_1, y_2\}$ has degree 4 in G . Since $G = T_n^-$, there must exist four distinct vertices $z_1, z_2, z_3, z_4 \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$ such that $G[A]$ is isomorphic to the graph given in Figure 6(a), where $A = \{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}$. But then G contains $K_{3,3}$ as a minor, because $n \in \{10, 11\}$ and $d_G(v) \geq 5$ for any $v \in V(G) \setminus A$. Thus, we may assume that $d_G(x_2) \geq 5$. Then $2(3n - 7) = 2e(G) \geq 4 \cdot 3 + 5 \cdot (n - 3)$, which implies that $n = 11$, $d_G(v) = 4$ for all $v \in \{x_1, y_1, y_2\}$ and $\Delta(G) = 5$. Thus there exist five distinct vertices $z_1, z_2, z_3, z_4, z_5 \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$ such that $G[A]$ is isomorphic to the graph given in Figure 6(b), where $A = \{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}$. But then $e_G(A, V(G) \setminus A) = 6$,

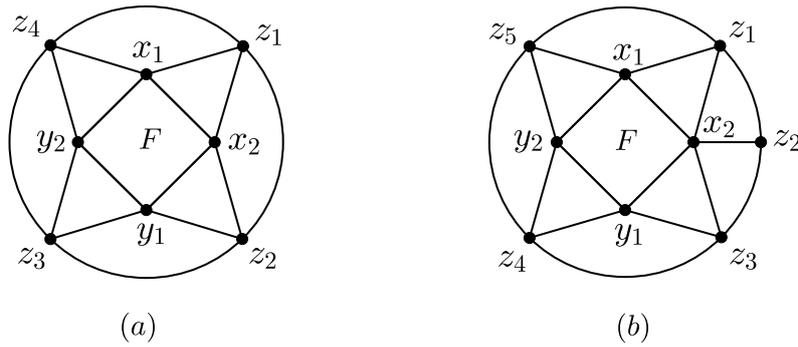


Figure 6: The graph $G[A]$.

contrary to $e_G(V(G) \setminus A, A) \geq 8$ because $n = 11$ and $d_G(v) = 5$ for any $v \in V(G) \setminus A$.

This completes the proof of Theorem 4. ■

4 Proof of Theorem 5

To establish the desired lower bound, note that the planar graph $K_2 + (n - 2)K_1$ is $(K_1 + 2K_2)$ -free for all $n \geq 5$. Hence, $ex_p(n, K_1 + 2K_2) \geq 2n - 3$ for all $n \geq 5$. In particular, $ex_p(5, K_1 + 2K_2) \geq 7$. We next show that every $(K_1 + 2K_2)$ -free planar graph on $n \geq 5$ vertices has at most $19n/8 - 4$ edges. We proceed the proof by induction on n . Assume first $n = 5$. Then $ex_p(5, K_1 + 2K_2) = 7$, because the only plane triangulation on five vertices, given in Figure 4(a), is not $(K_1 + 2K_2)$ -free, and any T_5^- is not $(K_1 + 2K_2)$ -free. Hence, $ex_p(n, K_1 + 2K_2) = 7 < 19n/8 - 4$ when $n = 5$. So we may assume that $n \geq 6$. Let G be a $(K_1 + 2K_2)$ -free plane graph on $n \geq 6$ vertices. Assume there exists a vertex $u \in V(G)$ with $d_G(u) \leq 2$. By the induction hypothesis, $e(G \setminus u) \leq 19(n - 1)/8 - 4$ and so $e(G) = e(G \setminus u) + d_G(u) \leq 19(n - 1)/8 - 4 + 2 < 19n/8 - 4$, as desired. So we may assume that $\delta(G) \geq 3$. Next, assume G is disconnected. Then each component of G either is isomorphic to K_4 or has at least six vertices because $\delta(G) \geq 3$. Let $G_1, \dots, G_p, G_{p+1}, \dots, G_{p+q}$ be all components of G such that $|G_1| = \dots = |G_p| = 4$ and $6 \leq |G_{p+1}| \leq \dots \leq |G_{p+q}|$, where $p \geq 0$ and $q \geq 0$ are integers with $p + q \geq 2$ and $|G_{p+1}| + \dots + |G_{p+q}| = n - 4p$. Then $e(G_i) = 6$ for all $i \in [p]$, and $e(G_j) \leq \frac{19|G_j|}{8} - 4$ for all $j \in \{p + 1, \dots, p + q\}$ by the induction hypothesis. Therefore,

$$\begin{aligned}
 e(G) &\leq 6p + \frac{19(|G_{p+1}| + \dots + |G_{p+q}|)}{8} - 4q \\
 &= \frac{19n}{8} - \frac{7p}{2} - 4q \\
 &\leq \frac{19n}{8} - \frac{7(p+q)}{2} < \frac{19n}{8} - 4,
 \end{aligned}$$

as desired. So we may further assume that G is connected. Then G has no faces of size at most two. Hence,

$$2e(G) = 3f_3 + \sum_{i \geq 4} i f_i \geq 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that

$$f \leq e(G)/2 + f_3/4. \tag{1}$$

Note that each 3-vertex is incident with at most three distinct 3-faces in G . Furthermore, since G is $(K_1 + 2K_2)$ -free, we see that for all $j \geq 4$, each j -vertex is incident with at most two distinct 3-faces in G . Let $U \subseteq V(G)$ denote the set of 3-vertices each incident with exactly three distinct 3-faces in G . Then U must be an independent set in G because G is connected. Furthermore, no two vertices in U have a common neighbor in G , because G is $(K_1 + 2K_2)$ -free. Thus, $4|U| \leq n$ and so $|U| \leq n/4$. It follows that

$$3f_3 \leq 3|U| + 2(n - |U|) = 2n + |U| \leq 9n/4, \tag{2}$$

which implies that $f_3 \leq 3n/4$. This, together with (1), further implies that $f \leq e(G)/2 + 3n/16$. By Euler's formula, $n - 2 = e(G) - f \geq e(G)/2 - 3n/16$. Hence, $e(G) \leq 19n/8 - 4$, as desired.

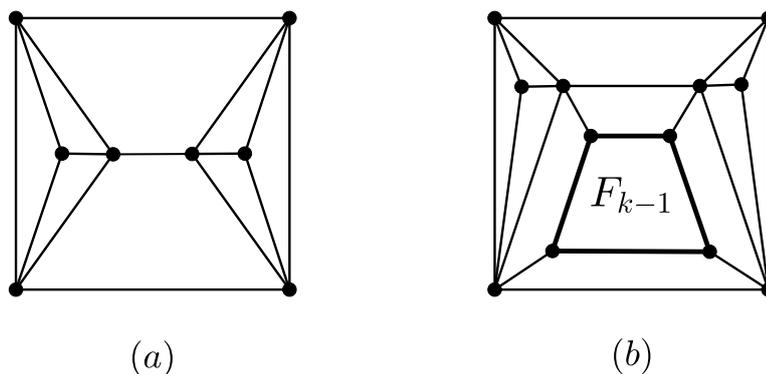


Figure 7: The construction of F_k .

From the proof above, we see that equality in $e(G) \leq 19n/8 - 4$ is achieved for n if and only if equalities hold in both (1) and (2), and in $4|U| \leq n$. This implies that $e(G) = 19n/8 - 4$ for n if and only if G is a connected, $(K_1 + 2K_2)$ -free plane graph on n vertices satisfying: $\delta(G) \geq 3$; each 3-vertex in G is incident with exactly three distinct 3-faces; each vertex of degree at least 4 in G is incident with exactly two distinct 3-faces; each face is either a 3-face or a 4-face. We next construct such an extremal plane graph for n and $K_1 + 2K_2$. Let $n = 8(k + 1)$ for some integer $k \geq 0$. Let F_0 be the graph depicted in Figure 7(a), we then construct F_k of order n recursively for all $k \geq 1$

via the illustration given in Figure 7(b): the entire graph F_{k-1} is placed into the center quadrangle of Figure 7(b) (in such a way that the center bold quadrangle of Figure 7(b) is identified with the outer quadrangle of F_{k-1}). One can check that F_k is $(K_1 + 2K_2)$ -free with $n = 8(k + 1)$ vertices and $19n/8 - 4$ edges for all $k \geq 0$. ■

5 Proof of Theorem 6

By Proposition 3(b), $ex_p(n, K_{1,t}) = 3n - 6$ for all $n \geq t + 1 \geq 8$. So we may assume that $t \leq 6$. We next show that $ex_p(n, K_{1,6}) = 3n - 6$ for all $n \in \{7, 8, 9, 10, 12\}$. Let J_a, J_b, J_c be the plane graphs given in Figure 8. Let J'_a be the plane triangulation obtained from J_a by adding a new vertex adjacent to x_1, x_2, x_3 , J'_b be the plane triangulation obtained from J_b by first deleting the edge x_1x_3 and then adding a new vertex adjacent to x_1, x_2, x_3, x_4 , and J'_c be the plane triangulation obtained from J_c by first deleting the edge x_1x_3 and then adding one new vertex adjacent to x_1, x_2, x_3, x_4, x_5 . Then the plane triangulations J_a, J'_a, J_b, J'_b and J'_c are $K_{1,6}$ -free because each of them has maximum degree 5. Hence, $ex_p(n, K_{1,6}) = 3n - 6$ for all $n \in \{7, 8, 9, 10, 12\}$. By Lemma 8, no

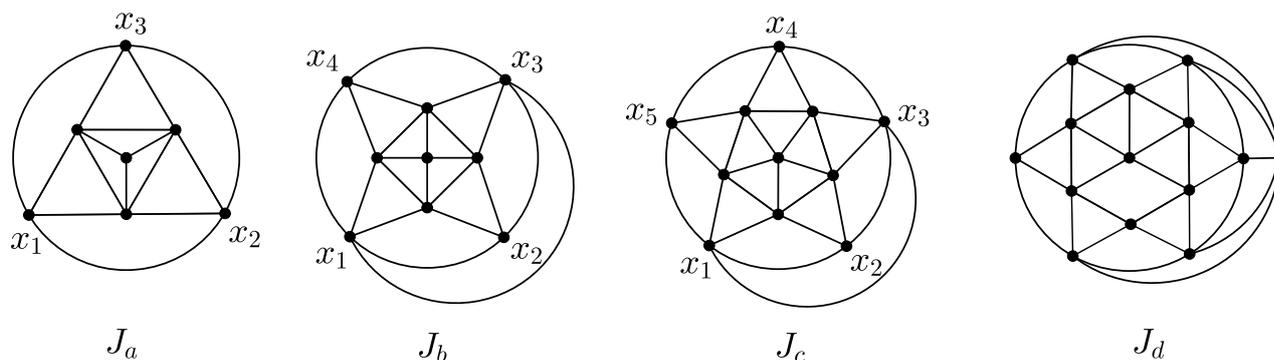


Figure 8: The graphs J_a, J_b, J_c and J_d .

plane triangulation on 11 vertices has maximum degree at most 5. Hence, every plane triangulation on $n \in \{11, 13, 14\}$ vertices has maximum degree at least 6. This implies that $ex_p(n, K_{1,6}) \leq 3n - 7$ for all $n \in \{11, 13, 14\}$. Since J_c given in Figure 8 is a $K_{1,6}$ -free plane graph with $n = 11$ vertices and $3n - 7$ edges, we have $ex_p(n, K_{1,6}) = 3n - 7$ when $n = 11$. By Lemma 8, there does not exist any planar graphs on $n \in \{13, 14\}$ vertices with $3n - 7$ edges and maximum degree at most 5. It follows that $ex_p(n, K_{1,6}) \leq 3n - 8$ when $n \in \{13, 14\}$. Let J''_c be the plane graph obtained from J_c by first deleting the edge x_1x_3 and then adding two new adjacent vertices y_1, y_2 such that y_1 is adjacent to x_1, x_2, x_3 and y_2 is adjacent to x_4, x_5 . Then J''_c and the graph J_d given in Figure 8 are $K_{1,6}$ -free plane graph with $n \in \{13, 14\}$ vertices and $3n - 8$ edges. Hence, $ex_p(n, K_{1,6}) = 3n - 8$ when $n \in \{13, 14\}$.

It is easy to see that $ex_p(n, K_{1,3}) = n$ for all $n \geq 4$, because every $K_{1,3}$ -free planar graph has maximum degree at most 2 and the planar graph C_n is $K_{1,3}$ -free with n edges. We next show that $ex_p(n, K_{1,4}) = \lfloor 3n/2 \rfloor$ for all $n \geq 5$. Clearly, $ex_p(n, K_{1,4}) \leq \lfloor 3n/2 \rfloor$ for all $n \geq 5$, because every $K_{1,4}$ -free planar graph has maximum degree at most 3. Next, for all $n \geq 5$, the planar graph obtained from C_n by adding a matching of size $\lfloor n/2 \rfloor$ is $K_{1,4}$ -free with $\lfloor 3n/2 \rfloor$ edges. Hence, $ex_p(n, K_{1,4}) = \lfloor 3n/2 \rfloor$ for all $n \geq 5$. To determine $ex_p(n, K_{1,5})$ for all $n \geq 6$, since every $K_{1,5}$ -free planar graph on $n \geq 6$ vertices has maximum degree at most 4, we have $ex_p(n, K_{1,5}) \leq 2n$ for all $n \geq 6$. Let J''_a be the plane triangulation obtained from J_a by deleting the unique 3-vertex. Since J''_a is $K_{1,5}$ -free plane graph on $n = 6$ vertices with $2n$ edges, we have $ex_p(n, K_{1,5}) = 2n$ when $n = 6$. By Lemma 8, no planar graph with $n = 7$ vertices and $2n$ edges has maximum degree at most 4. Hence, $ex_p(n, K_{1,5}) \leq 2n - 1$ when $n = 7$. Let J'''_a be the plane graph obtained from J''_a by first deleting the edge x_1x_2 and then adding a new vertex adjacent to x_1, x_2 only. Note that J'''_a is a $K_{1,5}$ -free plane graph on $n = 7$ vertices with $2n - 1$ edges, we see that $ex_p(n, K_{1,5}) = 2n - 1$ when $n = 7$. Next, for all $n \geq 8$, let C be a cycle on $2\lfloor n/2 \rfloor$ vertices with vertices $u_1, \dots, u_{\lfloor n/2 \rfloor}, w_{\lfloor n/2 \rfloor}, \dots, w_1$ in order. Let H be the plane graph obtained from C by adding the path with vertices $w_1, u_2, w_2, u_3, \dots, w_{\lfloor n/2 \rfloor - 1}, u_{\lfloor n/2 \rfloor}$ in order. When n is even, the planar graph $H + u_1u_{\lfloor n/2 \rfloor} + u_1w_{\lfloor n/2 \rfloor} + w_1w_{\lfloor n/2 \rfloor}$ is $K_{1,5}$ -free with $2n$ edges. When n is odd, let H' be obtained from H by first deleting the edge u_2u_3 and then adding a new vertex u adjacent to u_2 and u_3 . Then the planar graph $H' + uu_1 + uu_{\lfloor n/2 \rfloor} + w_1w_{\lfloor n/2 \rfloor} + u_1w_{\lfloor n/2 \rfloor}$ is $K_{1,5}$ -free with $2n$ edges. Hence, $ex_p(n, K_{1,5}) = 2n$ for all $n \geq 8$.

It remains to show that $ex_p(n, K_{1,6}) = \lfloor 5n/2 \rfloor$ for all $n \geq 15$. Clearly, $ex_p(n, K_{1,6}) \leq \lfloor 5n/2 \rfloor$ for all $n \geq 15$, because every $K_{1,6}$ -free planar graph on $n \geq 15$ vertices has maximum degree at most 5. We next show that $ex_p(n, K_{1,6}) \geq \lfloor 5n/2 \rfloor$ for all $n \geq 15$. Let $n := 4q + r \geq 15$, where $q \geq 3$ and $r \in \{0, 1, 2, 3\}$. Let $p \in \{q, q + 1\}$. Let C^1 and C^2 be two vertex-disjoint cycles with vertices x_1, x_2, \dots, x_q in order and y_1, y_2, \dots, y_p in order, respectively. Let C^3 be a cycle of length $q + p$ with vertices $b_1, a_1, b_2, a_2, \dots, b_q, a_q$ in order when $p = q$, and $b_1, a_1, b_2, a_2, \dots, b_q, a_q, b_{q+1}$ in order when $p = q + 1$. Let R_p be the plane graph on $2q + 2p$ vertices obtained from disjoint copies of C^1, C^2 and C^3 by making x_i adjacent to $\{a_i, b_i, b_{i+1}\}$, and y_j adjacent to $\{b_j, a_{j-1}, a_j\}$ for all $1 \leq i \leq q$ and $1 \leq j \leq p$, where all arithmetic on the indices $i + 1$ and $j - 1$ here are done modulo p . Then R_p is $K_{1,6}$ -free planar graph with $10q$ edges when $p = q$ and $10q + 3$ edges when $p = q + 1$. The construction of R_p on $4q$ vertices when $q = 4$ is depicted in Figure 9(a). When $n = 4q$ for $q \geq 4$, the planar graph R_p with $p = q$ is $K_{1,6}$ -free with $10q$ edges and so $ex_p(n, K_{1,6}) = 10q = 5n/2$. When $n = 4q + 1$ for $q \geq 4$, let R^1 be obtained from $R_p - y_2y_3 - y_1y_p$ with $p = q$ by adding a new vertex u adjacent to y_2 and y_3 . Then the planar graph $R^1 + uy_1 + uy_p$ is $K_{1,6}$ -free with $10q + 2$ edges. Hence, $ex_p(n, K_{1,6}) = 10q + 2 = \lfloor 5n/2 \rfloor$ when $n = 4q + 1$ for $q \geq 4$. When $n = 4q + 2$ for $q \geq 4$, let R^2 be obtained from $R_p - y_2y_3 - y_1y_p - x_2x_3 - x_1x_p - b_1a_p$ with $p = q$ by adding

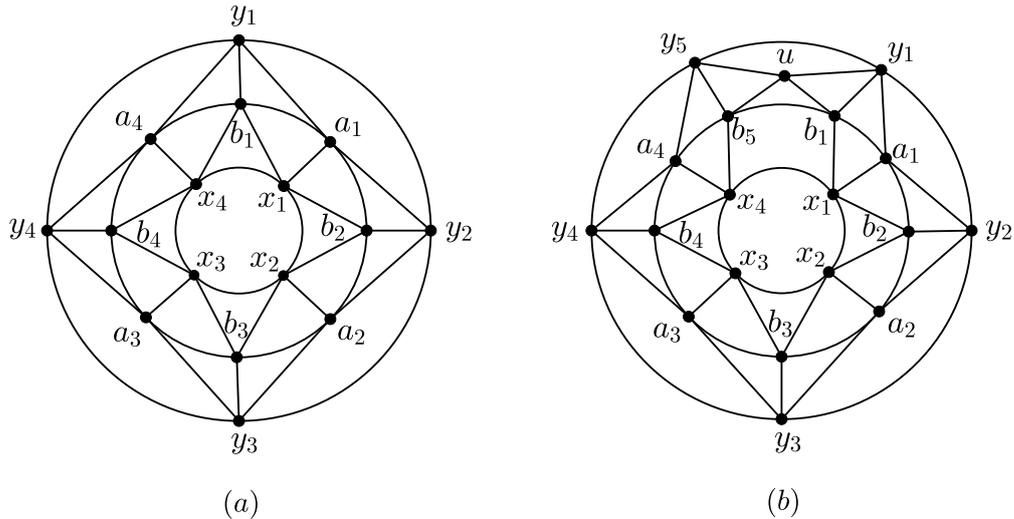


Figure 9: Almost 5-regular plane graphs on $4q + r$ vertices when $q = 4$ and $r \in \{0, 3\}$.

two new vertices u adjacent to y_2 and y_3 and v adjacent to x_2 and x_3 . Then the planar graph $R^2 + uy_1 + uy_p + ua_q + vx_1 + vx_q + vb_1$ is $K_{1,6}$ -free with $10q + 5$ edges. Hence, $ex_p(n, K_{1,6}) = 10q + 5 = 5n/2$ when $n = 4q + 2$ for $q \geq 4$. When $n = 4q + 3$ for $n \geq 3$, the planar graph obtained from R_p with $p = q + 1$ by adding a new vertex u adjacent to y_1, b_1, y_p, b_p , given in Figure 9(b) when $q = 4$, is $K_{1,6}$ -free with $10q + 7$ edges. Hence, $ex_p(n, K_{1,6}) = 10q + 7 = \lfloor 5n/2 \rfloor$ when $n = 4q + 3$ for $q \geq 3$.

6 Proof of Theorem 7

Since the plane triangulations J_a, J'_a, J_b and J'_b constructed in the proof of Theorem 6 is $(K_1 + 3K_2)$ -free, we see that $ex_p(n, K_1 + 3K_2) = 3n - 6$ for all $n \in \{7, 8, 9, 10\}$. To determine $ex_p(11, K_1 + 3K_2)$, note that the plane graph J_c given in Figure 8 with $n = 11$ vertices and $3n - 7$ edges is $(K_1 + 3K_2)$ -free. Thus $ex_p(n, K_1 + 3K_2) \geq 3n - 7$ when $n = 11$. By Lemma 8, no plane triangulation on 11 vertices has maximum degree at most 5. Hence, every plane triangulation on 11 vertices must contain a vertex of degree at least 6 (and so contains a copy of $K_1 + 3K_2$), which implies that $ex_p(n, K_1 + 3K_2) = 3n - 7$ when $n = 11$. Since every $K_{1,6}$ -free graph is certainly $(K_1 + 3K_2)$ -free, by Theorem 6, $ex_p(n, K_1 + 3K_2) = ex_p(n, K_{1,6}) = 3n - 6$ when $n = 12$, $ex_p(n, K_1 + 3K_2) \geq ex_p(n, K_{1,6}) = 3n - 8$ when $n \in \{13, 14\}$, and $ex_p(n, K_1 + 3K_2) \geq ex_p(n, K_{1,6}) = \lfloor 5n/2 \rfloor$ when $n \geq 15$. Since every plane triangulation on $n \in \{13, 14\}$ vertices has maximum degree at least 6, we see that $ex_p(n, K_1 + 3K_2) \leq 3n - 7$ when $n \in \{13, 14\}$. By Lemma 8, every T_n^- with $n \in \{13, 14\}$ has maximum degree at least 6 and so contains a copy of $K_1 + 3K_2$. It follows that $ex_p(n, K_1 + 3K_2) = 3n - 8 < 17n/6 - 4$ when $n \in \{13, 14\}$.

We next show that every $(K_1 + 3K_2)$ -free planar graph G on $n \geq 13$ vertices has at most $17n/6 - 4$ edges. We proceed the proof by induction on n . This is trivially true when $n \in \{13, 14\}$. So we may assume that $n \geq 15$. Assume there exists a vertex $u \in V(G)$ with $d_G(u) \leq 2$. By the induction hypothesis, $e(G \setminus u) \leq 17(n-1)/6 - 4$ and so $e(G) = e(G \setminus u) + d_G(u) \leq 17(n-1)/6 - 4 + 2 < 17n/6 - 4$, as desired. So we may assume that $\delta(G) \geq 3$. Assume next that G is disconnected. Let $G_1, \dots, G_p, G_{p+1}, \dots, G_{p+q}$ be all components of G such that $|G_1| \leq \dots \leq |G_p| \leq 12$ and $13 \leq |G_{p+1}| \leq \dots \leq |G_{p+q}|$, where $p \geq 0$ and $q \geq 0$ are integers with $p + q \geq 2$ and $|G_1| + \dots + |G_{p+q}| = n$. Then $e(G_i) \leq 3|G_i| - 6$ for all $i \in [p]$, and $e(G_j) \leq 17|G_j|/6 - 4$ for all $j \in \{p+1, \dots, p+q\}$ by the induction hypothesis. Therefore,

$$\begin{aligned} e(G) &\leq 3(|G_1| + \dots + |G_p|) - 6p + \frac{17(|G_{p+1}| + \dots + |G_{p+q}|)}{6} - 4q \\ &= \frac{17n}{6} - (6p + 4q) + \frac{|G_1| + \dots + |G_p|}{6} \\ &\leq \frac{17n}{6} - (6p + 4q) + 2p = \frac{17n}{6} - 4(p + q) < \frac{17n}{6} - 4, \end{aligned}$$

as desired. So we may further assume that G is connected. Then G has no faces of size at most two. Hence,

$$2e(G) = 3f_3 + \sum_{i \geq 4} i f_i \geq 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that $f \leq e(G)/2 + f_3/4$. Note that $n_3(G) \geq 0$ and $n_5(G) < n$; and for all $i \in \{3, 4, 5\}$, each i -vertex is incident with at most i 3-faces. Furthermore, for all $j \geq 6$, each j -vertex is incident with at most four 3-faces because G is $(K_1 + 3K_2)$ -free and $n \geq 15$. It follows that

$$3f_3 \leq 3n_3(G) + 4n_4(G) + 5n_5(G) + 4(n - n_3(G) - n_4(G) - n_5(G)) = 4n - n_3(G) + n_5(G) < 5n,$$

which implies that $f_3 < 5n/3$. This, together with the fact that $f \leq e(G)/2 + f_3/4$, further implies that $f < e(G)/2 + 5n/12$. By Euler's formula, $n - 2 = e(G) - f > e(G)/2 - 5n/12$. Hence, $e(G) < 17n/6 - 4$. ■

7 Concluding remarks

The lower bound in Theorem 7 can be further improved when n is divisible by 24. To see this, let $n = 24(k + 1)$ for some integer $k \geq 0$. Let R_5 be the 5-regular plane graph on twelve vertices given in Figure 10(a), and let G_0 be the plane graph obtained from two disjoint copies of R_5 by adding three independent edges between their outer faces, as depicted in Figure 10(b). We construct G_k of order n recursively for all $k \geq 1$ via the illustration given in Figure 10(c): the entire graph G_{k-1} is placed into the center

quadrangle of Figure 10(c) (in such a way that the center bold quadrangle of Figure 10(c) is identified with the outer quadrangle of G_{k-1}). One can check that G_k is $(K_1 + 3K_2)$ -free with $n = 24(k + 1)$ vertices and $67n/24 - 4$ edges for all $k \geq 0$.

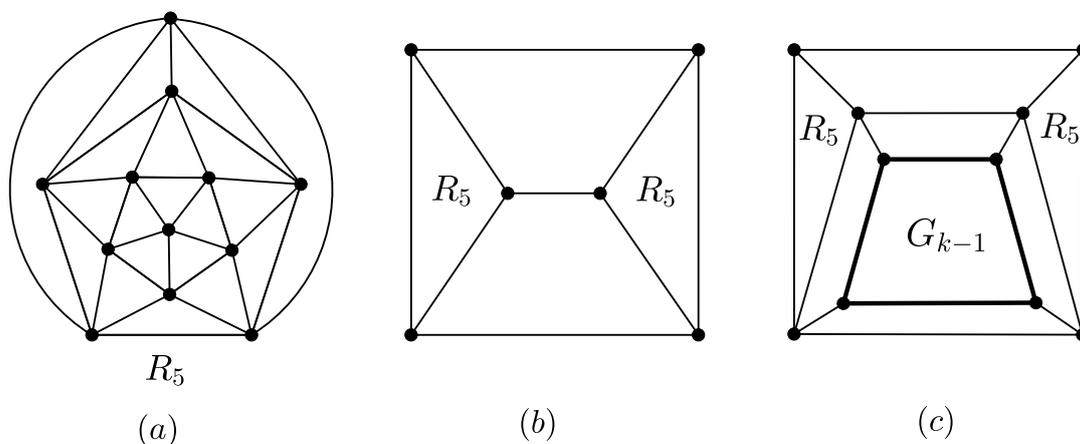


Figure 10: The construction of G_k .

As mentioned earlier, it seems non-trivial to determine $ex_p(n, H)$ for all K_4 -free planar graphs H with exactly one vertex, say u , satisfying $d_H(u) = \Delta(H) \leq 6$ and $\Delta(H[N(u)]) \leq 2$. We conclude this section by giving an upper bound (but not tight) for $ex_p(n, K_1 + H)$, where H is a disjoint union of paths.

Theorem 9. *Let $4 \leq t \leq 6$ be an integer and let H be a graph on t vertices such that H is a disjoint union of paths. Then $ex_p(n, K_1 + H) \leq \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ for all $n \geq t + 1$.*

Proof. Let t and H be given as in the statement. Since H is a subgraph of $K_1 + P_t$, it suffices to show that $ex_p(n, K_1 + P_t) \leq \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ for all $n \geq t + 1$. Let G be a $(K_1 + P_t)$ -free planar graph on $n \geq t + 1$ vertices. We next show that $e(G) \leq \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ by induction on n . This is trivially true when $n = t + 1$ because $e(G) \leq 3(t+1) - 6 \leq \frac{13(t-1)(t+1)}{4t-2} - \frac{12(t-1)}{2t-1}$ for $5 \leq t \leq 6$ and $e(G) \leq 3(t+1) - 7 \leq \frac{13(t-1)(t+1)}{4t-2} - \frac{12(t-1)}{2t-1}$ for $t = 4$. So we may assume that $n \geq t + 2$. We may further assume that $\delta(G) \geq 3$ and G is connected. Hence,

$$2e(G) = 3f_3 + \sum_{i \geq 4} if_i \geq 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that $f \leq e(G)/2 + f_3/4$. Note that for all $3 \leq i \leq t - 1$, each i -vertex is incident with at most i many 3-faces, and for all $j \geq t$, each j -vertex is incident with at

most $(t-2)\lfloor \frac{j}{t-1} \rfloor$ many 3-faces, because G is $(K_1 + P_t)$ -free. It follows that

$$\begin{aligned}
3f_3 &\leq \sum_{i=3}^{t-1} i \cdot n_i(G) + \sum_{j \geq t} (t-2) \left\lfloor \frac{j}{t-1} \right\rfloor \cdot n_j(G) \\
&\leq \sum_{i=3}^{t-1} \left(\frac{i \cdot n_i(G)}{t-1} + \frac{(t-2)i \cdot n_i(G)}{t-1} \right) + \frac{t-2}{t-1} \sum_{j \geq t} j \cdot n_j(G) \\
&= \sum_{i=3}^{t-1} \frac{i \cdot n_i(G)}{t-1} + \frac{t-2}{t-1} \sum_{\ell \geq 3} \ell \cdot n_\ell(G) \\
&= \sum_{i=3}^{t-1} \frac{i \cdot n_i(G)}{t-1} + \left(\frac{t-2}{t-1} \cdot 2e(G) \right) \\
&= \sum_{i=3}^{t-1} n_i(G) - \sum_{i=3}^{t-1} \frac{(t-1-i) \cdot n_i(G)}{t-1} + \left(\frac{t-2}{t-1} \cdot 2e(G) \right) \\
&< n + \frac{t-2}{t-1} \cdot 2e(G),
\end{aligned}$$

which implies that $f_3 < \frac{n}{3} + \frac{2(t-2)}{3(t-1)} \cdot e(G)$. This, together with the fact that $f \leq e(G)/2 + f_3/4$, further implies that $f \leq \frac{e(G)}{2} + \frac{n}{12} + \frac{(t-2)}{6(t-1)} \cdot e(G) = \frac{(4t-5)}{6(t-1)} \cdot e(G) + \frac{n}{12}$. By Euler's formula, $n-2 = e(G) - f \geq \frac{(2t-1)}{6(t-1)} \cdot e(G) - \frac{n}{12}$. Hence, $e(G) \leq \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$.

■

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