SOME SEMISYMMETRIC GRAPHS ARISING FROM FINITE VECTOR SPACES

HU YE CHEN, HUA HAN, AND ZAI PING LU

Abstract. A graph is worthy if no two vertices have the same neighborhood. In this paper, we characterize the automorphism groups of unworthy edge-transitive bipartite graphs, and present some worthy semisymmetric graphs arising from vector spaces over finite fields. We also determine the automorphism groups of these graphs.

Keywords. Automorphism group, semisymmetric graph, vector space, flag.

1. introduction

All graphs in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. Denote by $\mathsf{Aut}\Gamma$ the automorphism group of Γ, i.e., the subgroup of the symmetric group $\mathrm{Sym}(V)$ preserving the adjacency of Γ. Then $\mathsf{Aut}\Gamma$ acts naturally on the edge set E of Γ by

$$\{u, w\}^g = \{u^g, w^g\};\ \forall \{u, w\} \in E,\ g \in \mathsf{Aut}\Gamma.$$

The graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on V or E, respectively. If Γ is regular, edge-transitive but not vertex-transitive, then Γ is called a *semisymmetric* ([10]) graph. It is well-know that a semisymmetric graph is bipartite with two parts the orbits of its automorphism group on the vertices.

In 1972, Folkman [7] constructed some examples of semisymmetric graphs and posed eight problems on the existence of semisymmetric graphs with restricted order or valency. Folkman's problems stimulated a wide interest in the study of semisymmetric graphs. As a result, various constructions and also classification results of semisymmetric graphs have been published, see [1, 2, 3, 4, 6, 5, 8, 9, 11, 12, 14] for example.

A graph is worthy if no two vertices have the same neighborhood. It is easy to see that every unworthy semisymmetric graph can be reconstructed from some worthy edge-transitive bipartite graph by replacing each edge with a suitable complete bipartite graph, see [14]. Thus, in the field of semisymmetric graphs, worthy graphs

2000 Mathematics Subject Classification. 05C25, 20B25.

This work was supported by National Natural Science Foundation of China (11971248, 11731002) and the Fundamental Research Funds for the Central Universities.

play an important role. In this paper, we construct several families of semisymmetric graphs arising from finite vector spaces, most of which are worthy.

2. Automorphisms of graphs with given repeatednesses

Let $\Gamma = (V, E)$ be a connected bipartite graph with bipartition $V = U \cup W$. Let $\mathsf{Aut}^+\Gamma$ be the subgroup of $\mathsf{Aut}\Gamma$ which fixes the bipartition of Γ. Assume that $\mathsf{Aut}^+\Gamma$ acts transitively on E. Then U and W are orbits of $\mathsf{Aut}^+\Gamma$ on V. For $v \in V$, denote by $\Gamma(v)$ the set of neighbors of v in Γ.

For $u \in U$ and $w \in W$, set

$$u^* = \{u' \in U \mid \Gamma(u') = \Gamma(u)\},\ w^* = \{w' \in W \mid \Gamma(w') = \Gamma(w)\}.$$

Let $U^* = \{u^* \mid u \in U\}$ and $W^* = \{w^* \mid w \in W\}$. Then U^* and W^* are $\mathsf{Aut}^+\Gamma$-invariant partitions of U and W, respectively. The group $\mathsf{Aut}^+\Gamma$ induces two transitive actions on U^* and W^* by

$$(u^*)^g = (u^g)^*,\ (w^*)^g = (w^g)^*;\ u \in U,\ w \in W,\ g \in \mathsf{Aut}^+\Gamma.$$

Moreover, the sizes $r_U := |u^*|$ and $r_W := |w^*|$ are independent of the choices of $u \in U$ and $w \in W$, called the *repeatednesses* of Γ (see [14]). Let

$$r = r_U,\ s = r_W,\ k = |\Gamma(u)|,\ l = |\Gamma(w)|, m = |U^*|,\ n = |W^*|.$$

Then

$$s \,\big|\, k,\ r \,\big|\, l,\ mrk = nsl.$$

Note that $U^* \cup W^*$ is an $\mathsf{Aut}\Gamma$-invariant partition of V. We define a bipartite graph $\Gamma^* = (V^*, E^*)$ with $V^* = U^* \cup W^*$ and $\{u^*, w^*\} \in E^*$ if and only if the subgraph $[u^*, w^*]$ of Γ induced by $u^* \cup w^*$ is (isomorphic to) the complete bipartite graph $\mathsf{K}_{r,s}$. Then $\mathsf{Aut}^+\Gamma$ induces a subgroup of $\mathsf{Aut}^+\Gamma^*$, which acts transitively on E^*. Moreover, the vertices in U^* have valency $k^* := \frac{k}{s}$ in Γ^*, and the vertices in W^* have valency $l^* := \frac{l}{r}$ in Γ^*. Clearly, no two vertices in Γ^* have the same neighborhood. Note that each $\sigma \in \mathsf{Aut}\Gamma \setminus \mathsf{Aut}^+\Gamma$ if exists induces an automorphism of Γ^* interchanging U^* and W^*. Then the following lemma holds, see also [14].

Lemma 2.1. *If $\mathsf{Aut}^+\Gamma^* = \mathsf{Aut}\Gamma^*$ then $\mathsf{Aut}^+\Gamma = \mathsf{Aut}\Gamma$. In particular, if $k = l$ but $r \neq s$ than Γ is semisymmetric.*

Set

$$U^* = \{u_i^* \mid 1 \leq i \leq m\},\ W^* = \{w_j^* \mid 1 \leq j \leq n\}.$$

Let M be the subgroup of the symmetric group $\mathrm{Sym}(U)$ fixing every u_i^* set-wise, and let N be the subgroup of $\mathrm{Sym}(W)$ fixing every w_j^* set-wise. Then $M \leq \mathsf{Aut}^+\Gamma$, where M acts naturally on U and trivially on W. Thus M is contained in the kernel of $\mathsf{Aut}^+\Gamma$ acting on W. On the other hand, if g lies in the kernel of $\mathsf{Aut}^+\Gamma$ acting on W, then u and u^g have the same neighborhood. It follows that M is the kernel of $\mathsf{Aut}^+\Gamma$

acting on W. Similarly, N is the kernel of $\mathsf{Aut}^+\Gamma$ acting on U. Then the following lemma holds.

Lemma 2.2. *Let K_U and K_W be the kernels of $\mathsf{Aut}^+\Gamma$ acting on U and W, respectively. Then*

$$K_U = \mathrm{Sym}(w_1^*) \times \cdots \times \mathrm{Sym}(w_n^*),\ K_W = \mathrm{Sym}(u_1^*) \times \cdots \times \mathrm{Sym}(u_m^*),$$

and $K_UK_W = K_U \times K_W$ is normal in $\mathsf{Aut}\Gamma$.

In particular, we have the following corollary.

Lemma 2.3. (i) *$\mathsf{Aut}^+\Gamma$ is faithful on U if and only if $s = 1$;*
(ii) *$\mathsf{Aut}^+\Gamma$ is faithful on W if and only if $r = 1$.*

Write $u_i^* = \{u_{1i}, \ldots, u_{ri}\}$ and $w_j^* = \{w_{1j}, \ldots, w_{sj}\}$, where $1 \le i \le m$ and $1 \le j \le n$. Then we may assume that the actions of K_U and K_W are given by

$$u_{e\,i}^z = \begin{cases} u_{e^z\,i} & \text{if } z \in \mathrm{Sym}(u_i^*), \\ u_{e\,i} & \text{otherwise;} \end{cases}$$

$$w_{f\,j}^z = \begin{cases} w_{f^z\,j} & \text{if } z \in \mathrm{Sym}(w_j^*); \\ w_{f\,j} & \text{otherwise.} \end{cases}$$

Consider the semidirect product $G := (K_U \times K_W){:}\mathsf{Aut}^+\Gamma^*$, where $\sigma \in \mathsf{Aut}^+\Gamma^*$ acts on $K_U \times K_W$ by

$$(y_1, \ldots, y_n; x_1, \ldots, x_m)^\sigma = (y_{1^{\sigma^{-1}}}, \ldots, y_{n^{\sigma^{-1}}}; x_{1^{\sigma^{-1}}}, \ldots, x_{m^{\sigma^{-1}}}).$$

Then G has an action on $V = U \cup W$ defined as follows: for $g = (y_1, \ldots, y_n; x_1, \ldots, x_m; \sigma) \in G$, $1 \le i \le m$, $1 \le j \le n$, $1 \le e \le r$ and $1 \le f \le s$,

$$u_{e\,i}^g = u_{e^{x_i}\,i^\sigma},\ w_{f\,j}^g = w_{f^{y_j}\,j^\sigma}.$$

It is easily shown that this action is faithful, and each $g \in G$ gives an automorphism of Γ which fixes the bipartition of Γ. Thus $\mathsf{Aut}^+\Gamma \ge (K_U \times K_W){:}\mathsf{Aut}^+\Gamma^*$. (Note that the action of G on U induces a group isomorphic to the wreath product $\mathrm{S}_r \wr \mathsf{Aut}^+\Gamma^*$, and the action of G on W induces a group isomorphic to the wreath product $\mathrm{S}_s \wr \mathsf{Aut}^+\Gamma^*$.)

Theorem 2.4. $\mathsf{Aut}^+\Gamma = (K_U \times K_W){:}\mathsf{Aut}^+\Gamma^*$.

Proof. By Lemma 2.2, $\mathsf{Aut}^+\Gamma^*$ is faithful on both U^* and W^*. Then $K_U \times K_W$ is the kernel of $\mathsf{Aut}\Gamma$ acting on V^*. In particular, $\mathsf{Aut}^+\Gamma/(K_U \times K_W)$ is isomorphic to a subgroup of $\mathsf{Aut}^+\Gamma^*$. Recalling that $\mathsf{Aut}^+\Gamma \ge (K_U \times K_W){:}\mathsf{Aut}^+\Gamma^*$, our theorem follows. $\square$

Suppose that $\mathsf{Aut}\Gamma^* \ne \mathsf{Aut}^+\Gamma^*$. Take $\sigma \in \mathsf{Aut}\Gamma^* \setminus \mathsf{Aut}^+\Gamma^*$. Then σ interchanges U^* and W^*, and hence $m = n$ and $k^* = l^*$. In particular, Γ is regular, that is $k = l$, if and only if $r = s$. Assume that $r = s$. We define

$$\widetilde{\sigma} : u_{e\,i} \mapsto w_{e\,i'},\ w_{f\,j} \mapsto u_{f\,j'};\ 1 \le e, f \le r, 1 \le i, j \le m,$$

where i' and j' are such that $(u_i^*)^\sigma = w_{i'}^*$ and $(w_j^*)^\sigma = u_{j'}^*$. Then it is easy to check that $\widetilde{\sigma}$ is an automorphism of Γ, which interchanges U and W. Thus, by Lemma 2.1 and Theorem 2.4, we have the following result.

Corollary 2.5. *If $k = l$ then Γ is semisymmetric if and only if either $r \neq s$ or Γ^* is semisymmetric.*

3. A construction of edge-transitive bipartite graphs

Let n be an integer no less than 4. For a positive power q of some prime p, let $\mathbb{F}_q$ be the finite field of order q, $\mathbb{F}_q^n$ the n-dimensional column vector space over $\mathbb{F}_q$, and $\mathrm{PG}(n-1,q)$ the $(n-1)$-dimensional projective geometry over $\mathbb{F}_q$. Denote by $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{H}$ the sets of 1-dimensional (i.e., points in $\mathrm{PG}(n-1,q)$), 2-dimensional (i.e., lines in $\mathrm{PG}(n-1,q)$) and $(n-1)$-dimensional subspaces (i.e., hyperplanes in $\mathrm{PG}(n-1,q)$) of $\mathbb{F}_q^n$, respectively. Recall that an (n_1,n_2)-flag of $\mathbb{F}_q^n$ is an ordered pair $(\mathbf{u},\mathbf{w})$ of an n_1-dimensional subspace $\mathbf{u}$ and an n_2-dimensional subspace $\mathbf{w}$ with $1 \leq n_1 < n_2 \leq n-1$ and $\mathbf{u} \subset \mathbf{w}$. Let U and W be the sets of $(1,2)$-flags and $(1,n-1)$-flags of $\mathbb{F}_q^n$, respectively. Then

$$U = \{(\mathbf{p},\mathbf{l}) \mid \mathbf{p} \in \mathcal{P},\ \mathbf{p} \subset \mathbf{l} \in \mathcal{L}\},\ W = \{(\mathbf{p},\mathbf{h}) \mid \mathbf{p} \in \mathcal{P},\ \mathbf{p} \subset \mathbf{h} \in \mathcal{H}\},$$

and

$$|U| = |W| = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q-1)}.$$

Consider the actions of the projective semilinear group $\mathrm{P\Gamma L}(n,q)$ on U and W. Then $\mathrm{P\Gamma L}(n,q)$ is transitive (and faithful) on both U and W. For $(\mathbf{p},\mathbf{l}) \in U$, the stabilizer $\mathrm{P\Gamma L}(n,q)_{(\mathbf{p},\mathbf{l})}$ has exactly 7 orbits on W:

$O_1 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} = \mathbf{p} \oplus \mathbf{p}', \mathbf{l} \cap \mathbf{h} = \mathbf{p}'\}$, which has length q^{n-1};
$O_2 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} \cap \mathbf{h} = \mathbf{p} \neq \mathbf{p}'\}$, which has length $\frac{q^{n-1}(q^{n-2}-1)}{q-1}$;
$O_3 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} \cap \mathbf{h} \in \mathcal{P} \setminus \{\mathbf{p},\mathbf{p}'\}\}$, which has length $\frac{q^n(q^{n-2}-1)}{q-1}$;
$O_4 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} = \mathbf{p} \oplus \mathbf{p}' \subset \mathbf{h}\}$, which has length $\frac{q(q^{n-2}-1)}{q-1}$;
$O_5 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} \subset \mathbf{h}, \mathbf{l} \cap \mathbf{p}' = 0\}$, which has length $\frac{q^2(q^{n-2}-1)(q^{n-3}-1)}{(q-1)(q-1)}$;
$O_6 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} \cap \mathbf{h} = \mathbf{p} = \mathbf{p}'\}$, which has length q^{n-2};
$O_7 = \{(\mathbf{p}',\mathbf{h}) \in W \mid \mathbf{l} \subset \mathbf{h}, \mathbf{p} = \mathbf{p}'\}$, which has length $\frac{q^{n-2}-1}{q-1}$.

For each i define a bipartite graph $\mathcal{F}_i(n,q;1,2;1,n-1)$ on $U \cup W$ with edge set

$$E_i = \{\{(\mathbf{p}^g,\mathbf{l}^g),((\mathbf{p}')^g,\mathbf{h}^g)\} \mid g \in \mathrm{GL}(n,q),\ (\mathbf{p}',\mathbf{h}) \in O_i\}.$$

Then every $\mathcal{F}_i(n,q;1,2;1,n-1)$ admits $\mathrm{P\Gamma L}(n,q)$ acting transitively on the edge set but not on the vertex set.

It is easy to check that both $\mathcal{F}_6(n,q;1,2;1,n-1)$ and $\mathcal{F}_7(n,q;1,2;1,n-1)$ are not connected. In the following section, we shall prove that, for $1 \leq i \leq 5$, the graphs $\mathcal{F}_i(n,q;1,2;1,n-1)$ are connected and semisymmetric.

4. The numbers of pathes in $\mathcal{F}_i$ with length 2

For $1 \le i \le 5$, we let $\Gamma_i = \mathcal{F}_i(n, q; 1, 2; 1, n-1)$. For distinct vertices v_1 and v_2 of Γ_i, let $\theta_{\Gamma_i}(v_1, v_2)$ be the number of pathes with length 2 joining v_1 and v_2, that is, $\theta_{\Gamma_i}(v_1, v_2) = |\Gamma_i(v_1) \cap \Gamma_i(v_2)|$. Set

$$\begin{aligned} \Theta_i(U) &= \{\theta_{\Gamma_i}(u_1, u_2) \mid u_1, u_2 \in U,\ u_1 \neq u_2\}, \\ \Theta_i(W) &= \{\theta_{\Gamma_i}(w_1, w_2)| \mid w_1, w_2 \in W,\ w_1 \neq w_2\}. \end{aligned}$$

Note that every Γ_i is regular and bipartite. Then Γ_i is connected if and only if every pair of vertices in U are joined by some path in Γ_i, or equivalently, every pair of vertices in W are joined by some path in Γ_i. Further, if Γ_i is connected and $\Theta_i(U) \neq \Theta_i(W)$ then there is no automorphism of Γ_i interchanging U and W, and so Γ_i is semisymmetric in this case.

Lemma 4.1. *Let $\mathbf{l}_1, \mathbf{l}_2 \in \mathcal{L}$ and write $\mathbf{l}_1 = \mathbf{p}_1 + \mathbf{p}_1'$ and $\mathbf{l}_2 = \mathbf{p}_2 + \mathbf{p}_2'$, where $\mathbf{p}_1, \mathbf{p}_1', \mathbf{p}_2, \mathbf{p}_2' \in \mathcal{P}$ with $\mathbf{p}_1' \neq \mathbf{p}_2'$. Then there are $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}$ such that $\mathbf{l}_i \cap \mathbf{h}_i = \mathbf{p}_i'$, $\mathbf{h}_i = \mathbf{p}_i' + \mathbf{h}_1 \cap \mathbf{h}_2$, where $i = 1, 2$.*

Proof. Let $\mathcal{H}_i = \{\mathbf{h} \in \mathcal{H} \mid \mathbf{p}_i' \subset \mathbf{h}\}$, and $\mathcal{H}_i' = \{\mathbf{h} \in \mathcal{H} \mid \mathbf{l}_i \subset \mathbf{h}\}$, where $i = 1, 2$. Then $|\mathcal{H}_i \setminus \mathcal{H}_i'| = q^{n-2}$, and each $\mathbf{h} \in \mathcal{H}_i \setminus \mathcal{H}_i'$ intersects $\mathbf{l}_i$ at $\mathbf{p}_i'$. Let $\mathcal{H}_{12} = \mathcal{H}_1 \cap \mathcal{H}_2$, the set of $(n-1)$-dimensional subspaces containing both $\mathbf{p}_1'$ and $\mathbf{p}_2'$. Then $|\mathcal{H}_{12}| = \frac{q^{n-2}-1}{q-1}$, and so $\mathcal{H}_i \setminus \mathcal{H}_i' \setminus \mathcal{H}_{12} \neq \emptyset$. This yields our lemma. □

Theorem 4.2. (i) $\Theta_1(U) = \{0, q^{n-3}(q-1), q^{n-2}(q-1)\}$.
(ii) $\Theta_1(W) = \{0, q-1, q^{n-2}(q-1)\}$.
(iii) *Γ_1 is connected and semisymmetric.*

Proof. (1) Let $u_1 = (\mathbf{p}_1, \mathbf{l}_1)$ and $u_2 = (\mathbf{p}_2, \mathbf{l}_2)$ be distinct $(1,2)$-flags. Then, by the construction of Γ_1, it is easily shown that $\Gamma_1(u_1) \cap \Gamma_1(u_2) = \emptyset$ if and only if either $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$, or $\mathbf{l}_1 \cap \mathbf{l}_2 \in \{\mathbf{p}_1, \mathbf{p}_2\}$.

Suppose that $\Gamma_1(u_1) \cap \Gamma_1(u_2) \neq \emptyset$. Then either $\mathbf{l}_1 = \mathbf{l}_2$, or $\mathbf{l}_1$ and $\mathbf{l}_2$ intersect at some $\mathbf{p} \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$. Note that a $(1, n-1)$-flag $(\mathbf{q}, \mathbf{h})$ is connected $\Gamma(u_1) \cap \Gamma(u_2)$ if and only if $\mathbf{q} \notin \{\mathbf{p}_1, \mathbf{p}_2\}$ and $\mathbf{q} = \mathbf{l}_1 \cap \mathbf{h} = \mathbf{l}_2 \cap \mathbf{h}$. Let $\mathcal{H}_1 = \{\mathbf{h} \in \mathcal{H} \mid \mathbf{l}_1 \cap \mathbf{h} = \mathbf{l}_2 \cap \mathbf{h} \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}\}$. Then $\theta_{\Gamma_1}(u_1, u_2) = |\mathcal{H}_1|$. If $\mathbf{l}_1 = \mathbf{l}_2$ then

$$|\mathcal{H}_1| = \frac{q^n - 1}{q-1} - 2\frac{q^{n-1} - 1}{q-1} + \frac{q^{n-2} - 1}{q-1} = q^{n-2}(q-1).$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{p} \notin \{\mathbf{p}_1, \mathbf{p}_2\}$ then

$$|\mathcal{H}_1| = \frac{q^{n-1} - 1}{q-1} - 2\frac{q^{n-2} - 1}{q-1} + \frac{q^{n-3} - 1}{q-1} = q^{n-3}(q-1).$$

Thus $\Theta_1(U)$ is known as in (i).

Suppose that $\Gamma_1(u_1) \cap \Gamma_1(u_2) = \emptyset$. We write $\mathbf{l}_1 = \mathbf{q}_1 \oplus \mathbf{q}_1'$ and $\mathbf{l}_2 = \mathbf{q}_2 \oplus \mathbf{q}_2'$, where $\mathbf{q}_1 = \mathbf{p}_1$ and $\mathbf{q}_2 = \mathbf{p}_2$ if $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$, or $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{p}_1$ if $\mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{p}_1$, or $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{p}_2$ if $\mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{p}_2$. By Lemma 4.1, we take $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}$ such that $\mathbf{l}_i \cap \mathbf{h}_i = \mathbf{q}_i'$, $\mathbf{h}_i = \mathbf{q}_i' + \mathbf{h}_1 \cap \mathbf{h}_2$, where $i = 1, 2$. Then, choosing a suitable 1-dimensional subspace $\mathbf{q}$ of $\mathbf{q}_1' + \mathbf{q}_2'$, we get a path: $(\mathbf{p}_1, \mathbf{l}_1)$, $(\mathbf{q}_1', \mathbf{h}_1)$, $(\mathbf{q}, \mathbf{q}_1' + \mathbf{q}_2')$, $(\mathbf{q}_2', \mathbf{h}_2)$, $(\mathbf{p}_2, \mathbf{l}_2)$. It follows that every pair of vertices in U are joined by some path, and so Γ_1 is connected.

(2) Take distinct $(1, n-1)$-flags $w_1 = (\mathbf{p}_1, \mathbf{h}_1) \in W$ and $w_2 = (\mathbf{p}_2, \mathbf{h}_2) \in W$. Then $\Gamma_1(w_1) \cap \Gamma_1(w_2) \neq \emptyset$ if and only if either $\mathbf{p}_1 = \mathbf{p}_2$ or $\mathbf{h}_i = \mathbf{p}_i \oplus (\mathbf{h}_1 \cap \mathbf{h}_2)$ for $i = 1, 2$. For the latter case, $(\mathbf{q}, \mathbf{l}) \in \Gamma_1(w_1) \cap \Gamma_1(w_2)$ if and only if $\mathbf{l} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{q} \not\in \{\mathbf{p}_1, \mathbf{p}_2\}$, yielding $\theta_{\Gamma_1}(w_1, w_2) = q - 1$. Let $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$. Then $(\mathbf{q}, \mathbf{l}) \in \Gamma_1(w_1) \cap \Gamma_1(w_2)$ if and only if $\mathbf{l} = \mathbf{p} + \mathbf{q}$. Let $\mathcal{L}_1 = \{\mathbf{l} \in \mathcal{L} \mid \mathbf{l} \cap \mathbf{h}_1 = \mathbf{l} \cap \mathbf{h}_2 = \mathbf{p}\}$. Then

$$|\mathcal{L}_1| = \frac{q^{n-1} - 1}{q - 1} - 2\frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-3} - 1}{q - 1} = q^{n-3}(q - 1),$$

and each $\mathbf{l} \in \mathcal{L}_1$ contributes q common neighbors $(\mathbf{q}, \mathbf{l})$ of w_1 and w_2. Thus $\theta_{\Gamma_1}(w_1, w_2) = q^{n-2}(q - 1)$. Then part (ii) of this theorem follows.

Finally, noting that Γ_1 is connected and $\Theta_1(U) \neq \Theta_1(W)$, there is no automorphism of Γ_1 interchanging U and W. Thus Γ is semisymmetric. $\square$

By the argument in the above proof, we have the following fact.

Corollary 4.3. (i) *If $u_1, u_2 \in U$, then $\theta_{\Gamma_1}(u_1, u_2) = q^{n-2}(q - 1)$ if and only if $u_1 = (\mathbf{p}_1, \mathbf{l})$ and $u_2 = (\mathbf{p}_2, \mathbf{l})$ for some $\mathbf{l} \in \mathcal{L}$.*

(ii) *If $w_1, w_2 \in W$, then $\theta_{\Gamma_1}(w_1, w_2) = q^{n-2}(q - 1)$ if and only if $w_1 = (\mathbf{p}, \mathbf{h}_1)$ and $w_2 = (\mathbf{p}, \mathbf{h}_2)$ for some $\mathbf{p} \in \mathcal{P}$.*

Theorem 4.4. (i) $\Theta_2(U) = \{0, q^{n-4}(q^{n-1} - 2q + 1), \frac{q^{n-3}(q^{n-1} - 2q + 1)}{q - 1}, q^{n-2}(q^{n-2} - 1)\}$.

(ii) $\Theta_2(W) = \{q^{n-3}(q^{n-2} - 2q + 1), q^{n-2}(q^{n-3} - 1), q^{n-3}(q^{n-2} - 1), \frac{q^{n-2}(q^{n-1} - 2q + 1)}{q - 1}\}$.

(iii) *Γ_2 is connected and semisymmetric.*

Proof. (1) Take distinct vertices $u_1 = (\mathbf{p}_1, \mathbf{l}_1)$ and $u_2 = (\mathbf{p}_2, \mathbf{l}_2)$ in U. Then $\Gamma_2(u_1) \cap \Gamma_2(u_2) \neq \emptyset$ if and only if $\mathbf{p}_1 = \mathbf{p}_2$, or $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$, or $\mathbf{l}_1 \cap \mathbf{l}_2 \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$.

If $\mathbf{p}_1 = \mathbf{p}_2$ then $(\mathbf{q}, \mathbf{h}) \in \Gamma_2(u_1) \cap \Gamma_2(u_2)$ if and only if $\mathbf{l}_1 \cap \mathbf{h} = \mathbf{l}_2 \cap \mathbf{h} = \mathbf{p}_1 \neq \mathbf{q}$, and hence

$$\theta_{\Gamma_2}(u_1, u_2) = \frac{q^{n-1} - q}{q - 1}\left(\frac{q^{n-1} - 1}{q - 1} - 2\frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-3} - 1}{q - 1}\right) = q^{n-2}(q^{n-2} - 1).$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$ then

$$\theta_{\Gamma_2}(u_1, u_2) = \left(\frac{q^{n-1} - 1}{q - 1} - 2\right)\left(\frac{q^{n-2} - 1}{q - 1} - 2\frac{q^{n-3} - 1}{q - 1} + \frac{q^{n-4} - 1}{q - 1}\right) = q^{n-4}(q^{n-1} - 2q + 1).$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$ then

$$\theta_{\Gamma_2}(u_1, u_2) = \left(\frac{q^{n-1} - 1}{q - 1} - 2\right)\left(\frac{q^{n-2} - 1}{q - 1} - \frac{q^{n-3} - 1}{q - 1}\right) = \frac{q^{n-3}(q^{n-1} - 2q + 1)}{q - 1}.$$

Thus $\Theta_2(U)$ is known as in part (i).

(2) Take distinct $(1, n-1)$-flags $w_1 = (\mathbf{p}_1, \mathbf{h}_1) \in W$ and $w_2 = (\mathbf{p}_2, \mathbf{h}_2) \in W$. Then $(\mathbf{q}, \mathbf{l}) \in \Gamma_2(w_1) \cap \Gamma_2(w_2)$ if and only if $\mathbf{l} \cap \mathbf{h}_1 = \mathbf{l} \cap \mathbf{h}_2 = \mathbf{q} \not\in \{\mathbf{p}_1, \mathbf{p}_2\}$. Let $\mathcal{L}_1 = \{\mathbf{l} \in \mathcal{L} \mid \mathbf{l} \cap \mathbf{h}_1 = \mathbf{l} \cap \mathbf{h}_2 \not\in \{\mathbf{p}_1, \mathbf{p}_2\}\}$. Then each $\mathbf{l} \in \mathcal{L}_1$ contributes a unique common neighbor $(\mathbf{l} \cap \mathbf{h}_1, \mathbf{l})$ of w_1 and w_2, and then $\theta_{\Gamma_2}(w_1, w_2) = |\mathcal{L}_1|$.

If $\mathbf{h}_1 = \mathbf{h}_2$ then

$$\theta_{\Gamma_2}(w_1, w_2) = |\mathcal{L}_1| = \frac{(q^{n-1} - 2q + 1)(q^n - q^{n-1})}{q(q-1)^2} = q^{n-2}(q^{n-1} - 2q + 1).$$

Thus we assume that $\mathbf{h}_1 \cap \mathbf{h}_2$ has dimension $n-2$. If $\mathbf{h}_i = \mathbf{p}_i \oplus (\mathbf{h}_1 \cap \mathbf{h}_2)$ for $i = 1, 2$, then

$$\theta_{\Gamma_2}(w_1, w_2) = |\mathcal{L}_1| = \frac{(q^{n-2} - 1)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-3}(q^{n-2} - 1).$$

If $\mathbf{p}_1 \neq \mathbf{p}_2$ and $\mathbf{p}_1 + \mathbf{p}_2 \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$ then

$$\theta_{\Gamma_2}(w_1, w_2) = |\mathcal{L}_1| = \frac{(q^{n-2} - 2q + 1)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-3}(q^{n-2} - 2q + 1).$$

If $\mathbf{p}_1 = \mathbf{p}_2$ or only one of $\mathbf{p}_1$ and $\mathbf{p}_2$ is contained in $\mathbf{h}_1 \cap \mathbf{h}_2$, then

$$\theta_{\Gamma_2}(w_1, w_2) = |\mathcal{L}_1| = \frac{(q^{n-2} - q)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-2}(q^{n-3} - 1).$$

By the above argument, we have $\Theta_2(W)$ as in part (ii). In particular, since $n \geq 4$, every pair of distinct vertices in W have common neighbors, and hence Γ_2 is connected. Noting that $\Theta_2(U) \neq \Theta_2(W)$, part (iii) follows. □

By the argument in the above proof, we have the following fact.

Corollary 4.5. (i) *If $u_1, u_2 \in U$ then $\theta_{\Gamma_2}(u_1, u_2) = q^{n-2}(q^{n-2} - 1)$ if and only if $u_1 = (\mathbf{p}, \mathbf{l}_1)$ and $u_2 = (\mathbf{p}, \mathbf{l}_2)$ for some $\mathbf{p} \in \mathcal{P}$.*

(ii) *If $w_1, w_2 \in W$ then $\theta_{\Gamma_2}(w_1, w_2) = q^{n-2}(q^{n-1} - 2q + 1)$ if and only if $w_1 = (\mathbf{p}_1, \mathbf{h})$ and $w_2 = (\mathbf{p}_2, \mathbf{h})$ for some $\mathbf{h} \in \mathcal{H}$.*

Theorem 4.6. (i) $\Theta_3(U) = \{q^{n-3}(q^n - 2q^2 + 2q - 1), q^{n-2}(q^{n-1} - 2q + 1), q^{n-1}(q^{n-2} - 1), \frac{q^{n-1}(q^{n-1} - 2q + 1)}{q-1}\}$.

(ii) $\Theta_3(W) = \{(q^{n-2} - 1)(q^{n-1} - q + 1), q^{n-2}(q^{n-1} - 2q + 1), q^{n-1}(q^{n-2} - 1), \frac{q^{n-1}(q^{n-1} - 2q + 1)}{q-1}\}$.

(iii) *Γ_3 is connected and semisymmetric.*

Proof. (1) Take distinct $(1, 2)$-flags $u_1 = (\mathbf{p}_1, \mathbf{l}_1)$ and $u_2 = (\mathbf{p}_2, \mathbf{l}_2)$. Note that $(\mathbf{q}, \mathbf{h}) \in \Gamma_3(u_1) \cap \Gamma_3(u_2)$ if and only if $\mathbf{l}_1 \cap \mathbf{h} \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{q}\}$ and $\mathbf{l}_2 \cap \mathbf{h} \in \mathcal{P} \setminus \{\mathbf{p}_2, \mathbf{q}\}$. Let $\mathcal{H}_1 = \{\mathbf{h} \in \mathcal{H} \mid \mathbf{p}_1 \cap \mathbf{h} = \mathbf{p}_2 \cap \mathbf{h} = 0\}$.

If $\mathbf{p}_1 = \mathbf{p}_2$ then

$$|\mathcal{H}_1| = \frac{q^n - 1}{q - 1} - \frac{q^{n-1} - 1}{q - 1} = q^{n-1},$$

and each $\mathbf{h} \in \mathcal{H}_1$ contributes $\frac{(q^{n-1}-1)}{q-1} - 2$ to $\theta_{\Gamma_3}(u_1, u_2)$, and thus

$$\theta_{\Gamma_3}(u_1, u_2) = \frac{q^{n-1}(q^{n-1} - 2q + 1)}{q-1}.$$

Assume that $\mathbf{p}_1 \neq \mathbf{p}_2$. Then

$$|\mathcal{H}_1| = \frac{q^n - 1}{q-1} - 2\frac{q^{n-1} - 1}{q-1} + \frac{q^{n-2} - 1}{q-1} = q^{n-2}(q-1).$$

If $\mathbf{l}_1 = \mathbf{l}_2$ then each $\mathbf{h} \in \mathcal{H}_1$ contributes $\frac{(q^{n-1}-1)}{q-1} - 1$ to $\theta_{\Gamma_3}(u_1, u_2)$, and thus

$$\theta_{\Gamma_3}(u_1, u_2) = q^{n-1}(q^{n-2} - 1).$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$ or $\mathbf{l}_1 \cap \mathbf{l}_2 \in \{\mathbf{p}_1, \mathbf{p}_2\}$, then each $\mathbf{h} \in \mathcal{H}_1$ contributes $\frac{(q^{n-1}-1)}{q-1} - 2$ to $\theta_{\Gamma_3}(u_1, u_2)$, yielding

$$\theta_{\Gamma_3}(u_1, u_2) = q^{n-2}(q^{n-1} - 2q + 1).$$

The remain case is that $\mathbf{l}_1 \cap \mathbf{l}_2 \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$. Let $\mathcal{H}_2 = \{\mathbf{h} \in \mathcal{H}_1 \mid \mathbf{l}_1 \cap \mathbf{l}_2 \subset \mathbf{h}\}$. Then $|\mathcal{H}_2| = q^{n-3}(q-1)$, each $\mathbf{h} \in \mathcal{H}_2$ contributes $\frac{(q^{n-1}-1)}{q-1} - 1$ to $\theta_{\Gamma_3}(u_1, u_2)$, and each $\mathbf{h} \in \mathcal{H}_1 \setminus \mathcal{H}_2$ contributes $\frac{(q^{n-1}-1)}{q-1} - 2$ to $\theta_{\Gamma_3}(u_1, u_2)$. It follows that

$$\theta_{\Gamma_3}(u_1, u_2) = q^{n-3}(q^n - 2q^2 + 2q - 1).$$

Thus $\Theta_3(U)$ is known as in part (i). In particular, any two vertices in U have common neighbors, and so Γ is connected.

(2) Take distinct $(1, n-1)$-flags $w_1 = (\mathbf{p}_1, \mathbf{h}_1)$ and $w_2 = (\mathbf{p}_2, \mathbf{h}_2)$. Then $(\mathbf{q}, \mathbf{l}) \in \Gamma_3(w_1) \cap \Gamma_3(w_2)$ if and only if $\mathbf{l} \cap \mathbf{h}_i \in \mathcal{P} \setminus \{\mathbf{p}_i, \mathbf{q}\}$ for $i = 1, 2$.

Let $\mathcal{L}_1 = \{\mathbf{l} \in \mathcal{L} \mid \mathbf{l} \cap \mathbf{h}_i \in \mathcal{P} \setminus \{\mathbf{p}_i\}, i = 1, 2\}$. If $\mathbf{h}_1 = \mathbf{h}_2$ then

$$|\mathcal{L}_1| = \frac{(q^{n-1} - 2q + 1)(q^n - q^{n-1})}{q(q-1)^2},$$

and each $\mathbf{l} \in \mathcal{L}_1$ contributes q to $\theta_{\Gamma_3}(w_1, w_2)$, and thus

$$\theta_{\Gamma_3}(w_1, w_2) = \frac{q^{n-1}(q^{n-1} - 2q + 1)}{q-1}.$$

Assume next that $\mathbf{h}_1 \cap \mathbf{h}_2$ have dimension $n-2$. Let $\mathcal{L}_2 = \{\mathbf{l} \in \mathcal{L}_1 \mid \mathbf{l} \cap \mathbf{h}_1 = \mathbf{l} \cap \mathbf{h}_2\}$. Then each $\mathbf{l} \in \mathcal{L}_2$ contributes q to $\theta_{\Gamma_3}(w_1, w_2)$, and each $\mathbf{l} \in \mathcal{L}_1 \setminus \mathcal{L}_2$ contributes $q - 1$ to $\theta_{\Gamma_3}(w_1, w_2)$.

Let $\mathbf{p}_1 = \mathbf{p}_2$. Then

$$|\mathcal{L}_2| = \frac{(q^{n-2} - q)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-2}(q^{n-3} - 1)$$

and

$$|\mathcal{L}_1 \setminus \mathcal{L}_2| = \frac{(q^{n-1} - q^{n-2})^2}{(q-1)^2} = q^{2(n-2)}.$$

Thus

$$\theta_{\Gamma_3}(w_1, w_2) = qq^{n-2}(q^{n-3} - 1) + (q-1)q^{2(n-2)} = q^{n-1}(q^{n-2} - 1).$$

Let $\mathbf{p}_1 \neq \mathbf{p}_2$. If $\mathbf{h}_i = \mathbf{p}_i \oplus \mathbf{h}_1 \cap \mathbf{h}_2$ for $i = 1, 2$, then

$$|\mathcal{L}_2| = \frac{(q^{n-2}-1)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-3}(q^{n-2}-1)$$

and

$$|\mathcal{L}_1 \setminus \mathcal{L}_2| = \frac{(q^{n-1} - q^{n-2} - q + 1)^2}{(q-1)^2} = (q^{n-2}-1)^2,$$

and hence

$$\theta_{\Gamma_3}(w_1, w_2) = (q-1)(q^{n-2}-1)^2 + qq^{n-3}(q^{n-2}-1) = (q^{n-2}-1)(q^{n-1}-q+1).$$

If $\mathbf{p}_1 + \mathbf{p}_2 \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$, then

$$|\mathcal{L}_2| = \frac{(q^{n-2}-2q+1)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-3}(q^{n-2}-2q+1)$$

and

$$|\mathcal{L}_1 \setminus \mathcal{L}_2| = \frac{(q^{n-1} - q^{n-2})^2}{(q-1)^2} = q^{2(n-2)},$$

and so

$$\theta_{\Gamma_3}(w_1, w_2) = (q-1)q^{2(n-2)} + qq^{n-3}(q^{n-2}-2q+1) = q^{n-2}(q^{n-1}-2q+1).$$

If only one of $\mathbf{p}_1$ and $\mathbf{p}_2$ is contained in $\mathbf{h}_1 \cap \mathbf{h}_2$, then

$$|\mathcal{L}_2| = \frac{(q^{n-2}-q)(q^n - 2q^{n-1} + q^{n-2})}{q(q-1)^2} = q^{n-2}(q^{n-3}-1)$$

and

$$|\mathcal{L}_1 \setminus \mathcal{L}_2| = \frac{(q^{n-1} - q^{n-2})(q^{n-1} - q^{n-2} - q + 1)}{(q-1)^2} = q^{n-2}(q^{n-2}-1),$$

and so

$$\theta_{\Gamma_3}(w_1, w_2) = (q-1)q^{n-2}(q^{n-2}-1) + qq^{n-2}(q^{n-3}-1) = q^{n-2}(q^{n-1}-2q+1).$$

Thus $\Theta_3(W)$ is known as in part (ii).

Clearly, $\Theta_3(U) \neq \Theta_3(W)$. Recalling that Γ_3 is connected, Γ_3 is semisymmetric. □

By the argument in the above proof, we have the following fact.

Corollary 4.7. (i) *If $u_1, u_2 \in U$ then $\theta_{\Gamma_3}(u_1, u_2) = q^{n-1}(q^{n-2}-1)$ if and only if $u_1 = (\mathbf{p}_1, \mathbf{l})$ and $u_2 = (\mathbf{p}_2, \mathbf{l})$ for $\mathbf{l} \in \mathcal{L}$.*

(ii) *If $w_1, w_2 \in W$ then $\theta_{\Gamma_3}(w_1, w_2) = \frac{q^{n-1}(q^{n-1}-2q+1)}{q-1}$ if and only if $w_1 = (\mathbf{p}_1, \mathbf{h})$ and $w_2 = (\mathbf{p}_2, \mathbf{h})$ for $\mathbf{h} \in \mathcal{H}$.*

Theorem 4.8. (i) $\Theta_4(U) = \{0, \frac{q^{n-3}-1}{q-1}, q^{n-2}-1\}$.

(ii) $\Theta_4(W) = \{0, q-1, \frac{q(q^{n-3}-1)}{q-1}\}$.

(iii) *Γ_4 is connected and semisymmetric.*

Proof. (1) Let $u_1 = (\mathbf{p}_1, \mathbf{l}_1) \in U$ and $u_2 = (\mathbf{p}_2, \mathbf{l}_2) \in U$ with $u_1 \neq u_2$. Then $(\mathbf{q}, \mathbf{h}) \in \Gamma_4(u_1) \cap \Gamma_4(u_2)$ if and only if $\mathbf{l}_i = \mathbf{p}_i \oplus \mathbf{q} \subset \mathbf{h}$, $i = 1, 2$. In particular, if $\Gamma_4(u_1) \cap \Gamma_4(u_2) \neq \emptyset$ then either $\mathbf{l}_1 = \mathbf{l}_2$, or $\mathbf{l}_1 \cap \mathbf{l}_2 \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$.

If $\mathbf{l}_1 = \mathbf{l}_2 = \mathbf{l}$ then $(\mathbf{q}, \mathbf{h}) \in \Gamma_4(u_1) \cap \Gamma_4(u_2)$ if and only if $\mathbf{l} = \mathbf{p}_1 \oplus \mathbf{q} = \mathbf{p}_2 \oplus \mathbf{q} = \mathbf{p}_1 \oplus \mathbf{p}_2 \subset \mathbf{h}$, thus

$$\theta_{\Gamma_4}(u_1, u_2) = (q-1)\frac{q^{n-2}-1}{q-1} = q^{n-2} - 1.$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 \in \mathcal{P} \setminus \{\mathbf{p}_1, \mathbf{p}_2\}$, then $(\mathbf{q}, \mathbf{h}) \in \Gamma_4(u_1) \cap \Gamma_4(u_2)$ if and only if $\mathbf{l}_1 + \mathbf{l}_2 \subset \mathbf{h}$ and $\mathbf{q} = \mathbf{l}_1 \cap \mathbf{l}_2$, and so $\theta_{\Gamma_4}(u_1, u_2) = \frac{q^{n-3}-1}{q-1}$. Thus

$$\Theta_4(U) = \{0, \frac{q^{n-3}-1}{q-1}, q^{n-2} - 1\},$$

as in part (i).

Suppose that $\Gamma_4(u_1) \cap \Gamma_4(u_2) = \emptyset$. Then $\mathbf{p}_1 = \mathbf{p}_2$, or $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$, or $\mathbf{l}_1 \cap \mathbf{l}_2 \in \{\mathbf{p}_1, \mathbf{p}_2\}$. If $\mathbf{p}_1 = \mathbf{p}_2$, writing $\mathbf{l}_i = \mathbf{p}_1 \oplus \mathbf{q}_i$ for $i = 1, 2$, choosing $\mathbf{h} \in \mathcal{H}$ with $\mathbf{p}_1 + \mathbf{q}_1 + \mathbf{q}_2 \subset \mathbf{h}$, and taking $\mathbf{q} \in \mathcal{P}$ with $\mathbf{q}_1 + \mathbf{q} = \mathbf{q}_2 + \mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$, then we get a path:

$$u_1 = (\mathbf{p}_1, \mathbf{l}_1),\ (\mathbf{q}_1, \mathbf{h})\,(\mathbf{q}, \mathbf{q}_1 + \mathbf{q}_2),\ (\mathbf{q}_2, \mathbf{h}),\ (\mathbf{p}_2, \mathbf{l}_2) = u_2.$$

If $\mathbf{l}_1 \cap \mathbf{l}_2 = 0$ then there are a path between $(\mathbf{p}_1, \mathbf{l}_1)$ and $(\mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}_2)$ and a path between $(\mathbf{p}_2, \mathbf{l}_2)$ and $(\mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_2)$, and so there is a path between u_1 and u_2 as $(\mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}_2)$ and $(\mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_2)$ have common neighbors. Assume that $\mathbf{l}_1 \cap \mathbf{l}_2 \in \{\mathbf{p}_1, \mathbf{p}_2\}$. Without loss of generality, we let $\mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{p}_1$, and write $\mathbf{l}_1 = \mathbf{p}_1 \oplus \mathbf{q}_1$. Choose $\mathbf{q}_2 \in \mathcal{P}$ with $\mathbf{q}_2 \cap (\mathbf{l}_1 + \mathbf{l}_2) = 0$. Let $\mathbf{l} = \mathbf{q}_1 + \mathbf{q}_2$. Then $\mathbf{l} \cap \mathbf{l}_2 = 0$, and so $(\mathbf{q}_2, \mathbf{l})$ and u_2 are joined by a path. Noting that $\mathbf{l}_1 \cap \mathbf{l} = \mathbf{q}_1 \notin \{\mathbf{p}_1, \mathbf{q}_2\}$, we know that u_1 and $(\mathbf{q}_2, \mathbf{l})$ have common neighbors. It follows that u_1 and u_2 are joined by a path. Then Γ_4 is connected.

(2) Take distinct $(1, n-1)$-flags $w_1 = (\mathbf{p}_1, \mathbf{h}_1)$ and $w_2 = (\mathbf{p}_2, \mathbf{h}_2)$. Then $(\mathbf{q}, \mathbf{l}) \in \Gamma_4(w_1) \cap \Gamma_4(w_2)$ if and only if $\mathbf{l} = \mathbf{p}_i \oplus \mathbf{q} \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$ for $i = 1, 2$. In particular, if $\Gamma_4(w_1) \cap \Gamma_4(w_2) \neq \emptyset$ then $\mathbf{p}_1 + \mathbf{p}_2 \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$.

Let $\mathbf{p}_1 + \mathbf{p}_2 \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$. If $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ then $(\mathbf{q}, \mathbf{l}) \in \Gamma_4(w_1) \cap \Gamma_4(w_2)$ if and only if $\mathbf{l} = \mathbf{p} \oplus \mathbf{q} \subseteq \mathbf{h}_1 \cap \mathbf{h}_2$, and hence $\theta_{\Gamma_4}(w_1, w_2) = q\frac{q^{n-3}-1}{q-1}$. If $\mathbf{p}_1 \neq \mathbf{p}_2$ then $(\mathbf{q}, \mathbf{l}) \in \Gamma_4(w_1) \cap \Gamma_4(w_2)$ if and only if $\mathbf{l} = \mathbf{p}_1 \oplus \mathbf{p}_2 = \mathbf{p}_1 \oplus \mathbf{q} = \mathbf{p}_2 \oplus \mathbf{q} \subset \mathbf{h}_1 \cap \mathbf{h}_2$, and then $\theta_{\Gamma_4}(w_1, w_2) = q - 1$. Thus $\Theta_4(W)$ is known as in part (ii).

Since $\Theta_4(U) \neq \Theta_4(W)$, recalling that Γ_4 is connected, we conclude that Γ_4 is semisymmetric. $\square$

By the argument in the above proof, we have the following fact.

Corollary 4.9. (i) *If $u_1, u_2 \in U$ then $\theta_{\Gamma_4}(u_1, u_2) = q^{n-2} - 1$ if and only if $u_1 = (\mathbf{p}_1, \mathbf{l})$ and $u_2 = (\mathbf{p}_2, \mathbf{l})$ for $\mathbf{l} \in \mathcal{L}$.*

(ii) *If $w_1, w_2 \in W$ then $\theta_{\Gamma_4}(w_1, w_2) = \frac{q(q^{n-3}-1)}{q-1}$ if and only if $w_1 = (\mathbf{p}, \mathbf{h}_1)$ and $w_2 = (\mathbf{p}, \mathbf{h}_2)$ for $\mathbf{p} \in \mathcal{P}$.*

Theorem 4.10. *Let $t = q^{n-4} - 1$.*

(i) $\Theta_5(U)=\{\frac{q^{n-1}-2q^2+1}{(q-1)^2}t, \frac{q(q^{n-3}-1)(q^{n-2}-2q+1)}{(q-1)^2}, \frac{q^2(q^{n-2}-1)(q^{n-3}-1)}{(q-1)^2}\}$.
(ii) $\Theta_5(W)=\{\frac{q(q^{n-2}-q^2-q+1)}{(q-1)^2}t, \frac{q^2(q^{n-3}-1)}{(q-1)^2}t, \frac{(q^{n-2}-1)(q^{n-3}-1)}{(q-1)^2}, \frac{q(q^{n-3}-1)(q^{n-1}-q^2-q+1)}{(q-1)^2}\}$.
(iii) *Γ_5 is connected and semisymmetric.*

Proof. (1) Let $u_1=(\mathbf{p}_1,\mathbf{l}_1)\in U$ and $u_2=(\mathbf{p}_2,\mathbf{l}_2)\in U$ with $u_1\neq u_2$. Then $(\mathbf{q},\mathbf{h})\in\Gamma_5(u_1)\cap\Gamma_5(u_2)$ if and only if $\mathbf{l}_1+\mathbf{l}_2\subseteq\mathbf{h}$ and $\mathbf{l}_1\cap\mathbf{q}=\mathbf{l}_2\cap\mathbf{q}=0$. In particular, if $\Gamma_5(u_1)\cap\Gamma_5(u_2)=\emptyset$ then $n=4$ and $\mathbf{l}_1\cap\mathbf{l}_2=0$. Thus, if $\mathbf{l}_1\cap\mathbf{l}_2\neq 0$ or $n\geq 4$ then there is a path joining u_1 and u_2. Suppose that $\mathbf{l}_1\cap\mathbf{l}_2=0$, and write $\mathbf{l}_i=\mathbf{p}_i\oplus\mathbf{q}_i$ for $i=1,2$. Then $(\mathbf{q}_1,\mathbf{q}_1+\mathbf{q}_2)$ and each of u_1 and u_2 have common neighbors, and so u_1 and u_2 are joined by a path. Therefore Γ is connected.

If $\mathbf{l}_1=\mathbf{l}_2$ then

$$\theta_{\Gamma_5}(u_1,u_2)=\frac{q^{n-1}-q^2}{q-1}\frac{q^{n-2}-1}{q-1}=\frac{q^2(q^{n-2}-1)(q^{n-3}-1)}{(q-1)^2}.$$

If $\mathbf{l}_1\cap\mathbf{l}_2\in\mathcal{P}$ then

$$\theta_{\Gamma_5}(u_1,u_2)=\frac{q(q^{n-3}-1)(q^{n-2}-2q+1)}{(q-1)^2}.$$

If $\mathbf{l}_1\cap\mathbf{l}_2=0$ then

$$\theta_{\Gamma_5}(u_1,u_2)=\frac{(q^{n-4}-1)(q^{n-1}-2q^2+1)}{(q-1)^2}.$$

Thus $\Theta_5(U)$ is known as in par (i).

(2) Let $w_1=(\mathbf{p}_1,\mathbf{h}_1)\in W$ and $w_2=(\mathbf{p}_2,\mathbf{h}_2)\in W$ with $w_1\neq w_2$. Then $(\mathbf{q},\mathbf{l})\in\Gamma_5(u_1)\cap\Gamma_5(u_2)$ if and only if $\mathbf{l}\subseteq\mathbf{h}_1\cap\mathbf{h}_2$ and $\mathbf{p}_1\cap\mathbf{l}=\mathbf{p}_2\cap\mathbf{l}=0$. Let $\mathcal{L}_1=\{\mathbf{l}\in\mathcal{L}\mid \mathbf{l}\subseteq\mathbf{h}_1\cap\mathbf{h}_2,\mathbf{p}_1\cap\mathbf{l}=\mathbf{p}_2\cap\mathbf{l}=0\}$. Then $\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|$.

If $\mathbf{h}_1=\mathbf{h}_2$ then

$$|\mathcal{L}_1|=\frac{(q^{n-1}-1)(q^{n-2}-1)}{(q^2-1)(q-1)}-2\frac{q^{n-2}-1}{q-1}+1,$$

and so

$$\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|=\frac{q(q^{n-3}-1)(q^{n-1}-q^2-q+1)}{(q-1)^2}.$$

If $\mathbf{p}_1=\mathbf{p}_2$ then

$$|\mathcal{L}_1|=\frac{(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)(q-1)}-\frac{q^{n-3}-1}{q-1},$$

and then

$$\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|=\frac{q^2(q^{n-4}-1)(q^{n-3}-1)}{(q-1)^2}.$$

Let $\mathbf{h}_1\neq\mathbf{h}_2$ and $\mathbf{p}_1\neq\mathbf{p}_2$. If $\mathbf{p}_1\cap\mathbf{h}_1\cap\mathbf{h}_2=0=\mathbf{p}_2\cap\mathbf{h}_1\cap\mathbf{h}_2$ then $|\mathcal{L}_1|=\frac{(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)(q-1)}$, and so

$$\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|=\frac{(q^{n-2}-1)(q^{n-3}-1)}{(q-1)^2}.$$

If $\mathbf{p}_1+\mathbf{p}_2\subseteq \mathbf{h}_1\cap\mathbf{h}_2$ then

$$|\mathcal{L}_1|=\frac{(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)(q-1)}-2\frac{q^{n-3}-1}{q-1}+1,$$

and so

$$\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|=\frac{q(q^{n-4}-1))(q^{n-2}-q^2-q+1)}{(q-1)^2}.$$

If only one of $\mathbf{p}_1$ and $\mathbf{p}_2$ is contained in $\mathbf{h}_1\cap\mathbf{h}_2$, then

$$|\mathcal{L}_1|=\frac{(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)(q-1)}-\frac{q^{n-3}-1}{q-1},$$

and so

$$\theta_{\Gamma_5}(w_1,w_2)=(q+1)|\mathcal{L}_1|=\frac{q^2(q^{n-4}-1))(q^{n-3}-1)}{(q-1)^2}.$$

Thus $\Theta_5(W)$ is known as in part (ii).

Noting that $\Theta_5(U)\neq\Theta_5(W)$, we conclude that Γ_5 is semisymmetric. □

By the argument in the above proof and Theorems 4.2, 4.4, 4.6 and 4.8, we have the following fact.

Corollary 4.11. (i) *If $w_1,w_2\in W$ then $\theta_{\Gamma_5}(w_1,w_2)=\frac{q(q^{n-3}-1)(q^{n-1}-q^2-q+1)}{(q-1)^2}$ if and only if $w_1=(\mathbf{p}_1,\mathbf{h})$ and $w_2=(\mathbf{p}_2,\mathbf{h})$ for $\mathbf{h}\in\mathcal{H}$.*

(ii) *Two vertices $v_1,v_2\in U\cup W$ have the same neighborhood in Γ_i if and only if $i=5$ and $v_1=(\mathbf{p}_1,\mathbf{l})$, $v_1=(\mathbf{p}_2,\mathbf{l})$ for $\mathbf{l}\in\mathcal{L}$.*

5. The automorphism groups of graphs $\mathcal{F}_i$

For $1\leq i\leq 5$, let A_i be the automorphism group of $\Gamma_i=\mathcal{F}_i(n,q;1,2;1,n-1)$. Clearly, every A_i contains the projective semilinear group $\mathrm{P\Gamma L}(n,q)$ as a subgroup. Then, by Lemma 2.3 and Corollary 4.11, the following lemma holds.

Lemma 5.1. (i) *$\mathrm{P\Gamma L}(n,q)\leq A_i$ for $1\leq i\leq 5$.*

(ii) *A_i is faithful on both U and W, where $1\leq i\leq 4$.*

(iii) *A_5 is faithful on U.*

Theorem 5.2. *$A_i=\mathrm{P\Gamma L}(n,q)$ for $1\leq i\leq 4$.*

Proof. (1) Let $i=1$ or 4. For $\mathbf{q}\in\mathcal{P}$ and $\mathbf{l}\in\mathcal{L}$, set $U_{\mathbf{l}}=\{(\mathbf{p},\mathbf{l})\mid(\mathbf{p},\mathbf{l})\in U\}$ and $W_{\mathbf{q}}=\{(\mathbf{q},\mathbf{h})\mid(\mathbf{q},\mathbf{h})\in W\}$. Then, by Corollaries 4.3 and 4.9, we conclude that $\overline{U}:=\{U_{\mathbf{l}}\mid\mathbf{l}\in\mathcal{L}\}$ and $\overline{W}:=\{W_{\mathbf{q}}\mid\mathbf{q}\in\mathcal{P}\}$ are A_i-invariant partitions of U and W, respectively.

Define a bipartite graph $\overline{\Gamma_i}$ on $\overline{U}\cup\overline{W}$ such that $\{U_{\mathbf{l}},W_{\mathbf{q}}\}$ is an edge if and only if there are some $u\in U_{\mathbf{l}}$ and $w\in W_{\mathbf{q}}$ adjacent in Γ_i. Then $\overline{\Gamma_i}$ is isomorphic to the point-line incidence graph of the projective geometry $\mathrm{PG}(n-1,q)$. Let K be the kernel of A_i acting on $\overline{U}\cup\overline{W}$. Then A_i/K is isomorphic to a subgroup of $\mathsf{Aut}\overline{\Gamma_i}\cong\mathrm{P\Gamma L}(n,q)$.

Take an edge $\{U_{\mathbf{l}}, W_{\mathbf{q}}\}$ of $\overline{\Gamma_i}$, and consider the subgraph $[U_{\mathbf{l}}, W_{\mathbf{q}}]$ of Γ_i induced by $U_{\mathbf{l}} \cup W_{\mathbf{q}}$. Then $U_{\mathbf{l}}$ contains only one isolated vertex of $[U_{\mathbf{l}}, W_{\mathbf{q}}]$, say $(\mathbf{q}, \mathbf{l})$. Noting that K fixes both $U_{\mathbf{l}}$ and $W_{\mathbf{q}}$ set-wise, K fixes the vertex $(\mathbf{q}, \mathbf{l})$. Since K is normal in A_i, all K-orbits on U have the same length. It follows that K fixes U point-wise. Then $K = 1$ as A_i is faithful on U by Lemma 5.1. Then we have $A_i = \mathrm{P\Gamma L}(n, q)$ as $\mathrm{P\Gamma L}(n, q) \le \mathrm{A}_i$.

(2) For $\mathbf{p} \in \mathcal{P}$ and $\mathbf{h} \in \mathcal{H}$, set $U_{\mathbf{p}} = \{(\mathbf{p}, \mathbf{l}) \mid (\mathbf{p}, \mathbf{l}) \in U\}$ and $W_{\mathbf{h}} = \{(\mathbf{q}, \mathbf{h}) \mid (\mathbf{q}, \mathbf{h}) \in W\}$. Then, by Corollary 4.5, we conclude that $\overline{U} := \{U_{\mathbf{p}} \mid \mathbf{p} \in \mathcal{P}\}$ and $\overline{W} := \{W_{\mathbf{h}} \mid \mathbf{h} \in \mathcal{H}\}$ are A_2-invariant partitions of U and W, respectively.

Define a bipartite graph $\overline{\Gamma_2}$ on $\overline{U} \cup \overline{W}$ such that $\{U_{\mathbf{p}}, W_{\mathbf{h}}\}$ is an edge if and only if there are some $u \in U_{\mathbf{p}}$ and $w \in W_{\mathbf{h}}$ adjacent in Γ_2. Then $\overline{\Gamma_2}$ is isomorphic to the point-hyperplane incidence graph of $\mathrm{PG}(n-1, q)$, and so $\mathsf{Aut}\overline{\Gamma_2} \cong \mathrm{P\Gamma L}(n, q).\mathbb{Z}_2$. Let K be the kernel of A_2 acting on $\overline{U} \cup \overline{W}$. Then A_2/K is isomorphic to a subgroup of $\mathsf{Aut}\overline{\Gamma_2}$. Since A_2 fixes U and W set-wise, A_2/K isomorphic to a subgroup of $\mathrm{P\Gamma L}(n, q)$.

Take an edge $\{U_{\mathbf{p}}, W_{\mathbf{h}}\}$ of $\overline{\Gamma_2}$, and consider the subgraph $[U_{\mathbf{p}}, W_{\mathbf{h}}]$. Then $W_{\mathbf{h}}$ contains a unique isolated vertex of $[U_{\mathbf{p}}, W_{\mathbf{h}}]$, say $(\mathbf{p}, \mathbf{h})$. Since K fixes both $U_{\mathbf{p}}$ and $W_{\mathbf{h}}$ set-wise, K fixes the vertex $(\mathbf{p}, \mathbf{h})$. It follows that K acts trivially on W, and so $K = 1$. Then we have $A_2 = \mathrm{P\Gamma L}(n, q)$.

(3) For $\mathbf{h} \in \mathcal{H}$ and $\mathbf{l} \in \mathcal{L}$, set $U_{\mathbf{l}} = \{(\mathbf{p}, \mathbf{l}) \mid (\mathbf{p}, \mathbf{l}) \in U\}$ and $W_{\mathbf{h}} = \{(\mathbf{q}, \mathbf{h}) \mid (\mathbf{q}, \mathbf{h}) \in W\}$. Then, by Corollary 4.7, we conclude that $\overline{U} := \{U_{\mathbf{l}} \mid \mathbf{l} \in \mathcal{L}\}$ and $\overline{W} := \{W_{\mathbf{h}} \mid \mathbf{h} \in \mathcal{H}\}$ are A_3-invariant partitions of U and W, respectively.

Define a bipartite graph $\overline{\Gamma_3}$ on $\overline{U} \cup \overline{W}$ such that $\{U_{\mathbf{l}}, W_{\mathbf{h}}\}$ is an edge if and only if there are some $u \in U_{\mathbf{l}}$ and $w \in W_{\mathbf{h}}$ adjacent in Γ_3. Then $\overline{\Gamma_3}$ is isomorphic to the graph on $\mathcal{L} \cup \mathcal{H}$ such that $\{\mathbf{l}, \mathbf{h}\}$ is an edge if and only if $\mathbf{l} \cap \mathbf{h}$ has dimension 1. Note that $\mathsf{Aut}\overline{\Gamma_3}$ contains a subgroup isomorphic to $\mathrm{P\Gamma L}(n, q)$ which acts 2-transitively on $\overline{W}$. It follows from [13, Proposition 6.1] that $\mathsf{Aut}\overline{\Gamma_3}$ and $\mathrm{P\Gamma L}(n, q)$ have isomorphic socle. Thus we have $\mathsf{Aut}\overline{\Gamma_3} \cong \mathrm{P\Gamma L}(n, q)$.

Let K be the kernel of A_3 acting on $\overline{U} \cup \overline{W}$. Then A_3/K is isomorphic to a subgroup of $\mathsf{Aut}\overline{\Gamma_3} \cong \mathrm{P\Gamma L}(n, q)$. If $\{U_{\mathbf{l}}, W_{\mathbf{h}}\}$ is an edge of $\overline{\Gamma_3}$ then $W_{\mathbf{h}}$ contains a unique isolated vertex of $[U_{\mathbf{l}}, W_{\mathbf{h}}]$, say $(\mathbf{l} \cap \mathbf{h}, \mathbf{h})$. Then a similar argument as in (2) implies that $K = 1$, and hence $A_3 = \mathrm{P\Gamma L}(n, q)$. $\square$

Theorem 5.3. *$A_5 \cong (\underbrace{\mathrm{S}_{q+1} \times \cdots \times \mathrm{S}_{q+1}}_{m \text{ factors}}){:}\mathrm{P\Gamma L}(n, q)$, where $m = |\mathcal{L}| = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$.*

Proof. By Corollary 4.11, Γ_5 has repeatednesses $q + 1$ and 1. For $\mathbf{l} \in \mathcal{L}$, let $\mathbf{l}^*$ be the set of $(1, 2)$ flags with form of $(\mathbf{p}, \mathbf{l})$. Set $U^* = \{\mathbf{l}^* \mid \mathbf{l} \in \mathcal{L}\}$, and consider the graph Γ^* on $U^* \cup W$ with edge set $\{\{\mathbf{l}^*, (\mathbf{p}, \mathbf{h})\} \mid \mathbf{l} \in \mathcal{L}, (\mathbf{p}, \mathbf{h}) \in W, \mathbf{l} \subset \mathbf{h}, \mathbf{l} \cap \mathbf{p} = 0\}$. By Lemma 2.1, Theorem 2.4 and Corollary 4.11, we have

$$A_5 \cong (\underbrace{\mathrm{S}_{q+1} \times \cdots \times \mathrm{S}_{q+1}}_{m \text{ factors}}){:}\mathsf{Aut}\Gamma^*,$$

where $m = |\mathcal{L}|$. It is easily shown that $\mathsf{Aut}\Gamma^*$ contains a subgroup isomorphic to $\mathrm{P\Gamma L}(n, q)$. Next we show $\mathsf{Aut}\Gamma^* \cong \mathrm{P\Gamma L}(n, q)$, and then the result follows.

By Lemma 2.2, $\mathsf{Aut}\Gamma^*$ is faithful on both U^* and W. For $\mathbf{h} \in \mathcal{H}$, denote by $W_{\mathbf{h}}$ the set of $(1, n-1)$-flags with form of $(\mathbf{p}, \mathbf{h})$. Then, by Corollary 4.11, $\overline{W} = \{W_{\mathbf{h}} \mid \mathbf{h} \in \mathcal{H}\}$ is a A_5-invariant partition and hence an $\mathsf{Aut}\Gamma^*$-invariant partition of W. Consider the graph $\overline{\Gamma}$ on $U^* \cup \overline{W}$ with edge set $\{\{\mathbf{l}^*, W_h\} \mid \mathbf{l} \subset \mathbf{h}, \mathbf{l} \in \mathcal{L}, \mathbf{h} \in \mathcal{H}\}$. Then $\overline{\Gamma}$ is (isomorphic to) the line-hyperplane incidence graph of $\mathrm{PG}(n-1, q)$, and $\mathsf{Aut}\Gamma^*$ induces a subgroup of $\mathsf{Aut}\overline{\Gamma}$. Noting that $\mathsf{Aut}\overline{\Gamma}$ contains a subgroup isomorphic to $\mathrm{P\Gamma L}(n, q)$, by [13, Proposition 6.1], we conclude that $\mathsf{Aut}\overline{\Gamma} \cong \mathrm{P\Gamma L}(n, q)$. Let K be the kernel of $\mathsf{Aut}\Gamma^*$ acting on $\overline{W}$. Note that no two vertices in $\overline{\Gamma}$ has the same neighborhood. It follows that K acts trivially on U^*, and so $K = 1$ as $\mathsf{Aut}\Gamma^*$ is faithful on U^*. Then $\mathsf{Aut}\Gamma^*$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(n, q)$. This implies that $\mathsf{Aut}\Gamma^* \cong \mathrm{P\Gamma L}(n, q)$. Then our theorem follows. $\square$

References

[1] I. Z. Bouwer, On edge but not vertex transitive graphs, *J. Combin Theory Ser. B* **12** (1972), 32-40.

[2] P. Cara, S. Rottey, G. Van de Voorde, A construction for infinite families of semisymmetric graphs revealing their full automorphism group, *J. Algebraic Combin.* **39** (2014), 967-988.

[3] M. Conder, A. Malnič, D. Marušič and P. Potočnik, A census of semisymmetric cubic graphs on up to 768 vertices. *J. Algebr. Comb.* **23** (2006), 255–294.

[4] S. F. Du and D. Marušič, An infinite family of biprimitive semisymmetric graphs, *J. Graph Theory* **32** (1999), 217-228.

[5] S. F. Du and M. Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Comm. Algebra* **28** (2000), 2685-2714.

[6] S. F. Du, F. R. Wang and L. Zhang, An infinite family of semisymmetric graphs constructed from affine geometries, *European J. Combin.* **24** (2003), 897-902.

[7] J. Folkman, Regular line-symmetric graphs, *J. Combin. Theory Ser. B* **3** (1967), 215-232.

[8] M. E. Iofinova and A. A. Ivanov, Biprimitive cubic graphs (Russian), *Investigation in Algebraic Theory of Combinatorial Objects*, Proc. of the Seminar, Institute for System Studies, Moscow, 1985, 124-134.

[9] A. V. Ivanov, On edge but not vertex transitive regular graphs, *Ann. Discrete Math.* **34** (1987), 273–286.

[10] M.H. Klin, On edge but not vertex transitive graphs, *Coll. Math. Soc. J. Bolyai* (Algebraic Methods in Graph Theory, Vol. 25, Szeged, 1978), Budapest, 1981, pp. 399C403.

[11] G. X. Liu and Z. P. Lu, On edge-transitive cubic graphs of square-free order, *European J. Combin.* **45** (2015), 41-46.

[12] C. W. Parker, Semisymmetric cubic graphs of twice odd order, *European J. Combin.* **28** (2007), 572-591.

[13] C. E. Praeger, The inclusion problem for finite primitive permutation groups, *Proc. Lond. Math. Soc.* **60** (1990), 68-88.

[14] S. Wilson, A worthy family of semisymmetric graphs, *Discrete Math.* **271** (2003), 283-294.

H. Y. Chen, Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, 443002, P. R. China

E-mail address: chenhy280@163.com

H. Han, College of Science, Tianjin University of Technology, Tianjin 300384, P. R. China

E-mail address: hh1204@mail.nankai.edu.cn

Z. P. Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lu@nankai.edu.cn