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Strong conflict-free connection of graphs[☆]

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ABSTRACT

A path P in an edge-colored graph is called *a conflict-free path* if there exists a color used on only one of the edges of P. An edge-colored graph G is called *conflict-free connected* if for each pair of distinct vertices of G there is a conflict-free path in G connecting them. The graph G is called *strongly conflict-free connected* if for every pair of vertices u and v of G there exists a conflict-free path of length $d_G(u, v)$ in G connecting them. For a connected graph G, the *strong conflict-free connection number* of G, denoted by $scfc(G)$, is defined as the smallest number of colors that are required in order to make G strongly conflict-free connected. In this paper, we first show that if G_t is a connected graph with m (≥ 2) edges and t edge-disjoint triangles, then $scfc(G_t) \leq m - 2t$, and the equality holds if and only if $G_t \cong S_{m-t,t}$. Then we characterize the graphs G with $scfc(G) = k$ for $k \in \{1, m-3, m-2, m-1, m\}$. In the end, we present a complete characterization for the cubic graphs G with $scfc(G) = 2$.

1. Introduction

All graphs mentioned in this paper are simple, undirected and finite. We follow book [2] for undefined notation and terminology. For a graph G, let c: $E(G) \mapsto [r]$ be an edge-coloring of G. For an edge e of G, we denote the color of e by $c(e)$. And we denote the number of vertices, edges in G by n, m, respectively. We denote $[t]$ the set $\{1, 2, \cdots, t\}$ and we define C_s as a cycle of length s. We denote by $d_G(v)$ the degree v in G.

Coloring problems are important topics in graph theory. In recent years, there have appeared a number of colorings raising great concern due to their wide applications in real world. We list a few well-known colorings here. The first of such would be the rainbow connection coloring, which is stated as follows. A path in an edge-colored graph is called a *rainbow path* if all the edges of the path have distinct colors. An edge-colored graph is called (*strongly*) rainbow connected if there is a (*shortest* and) rainbow path between every pair of distinct vertices in the graph. For a connected graph G, the (*strong*) rainbow connection number of G is defined as the smallest number of colors needed to make G (*strongly*) rainbow connected, denoted by ($src(G)$) $rc(G)$. These concepts were first introduced by Chartrand et al. in [6].

Inspired by the rainbow connection coloring, the concept of proper connection coloring was independently posed by Andrews et al. in [1] and Borozan et al. in [3], the only difference from (*strong*) rainbow connection coloring is that distinct colors are only required for adjacent edges instead of all edges on the (*shortest*) path. For an edge-colored connected graph G,

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the smallest number of colors required to give G a (*strong*) proper connection coloring is called the *(strong) proper connection number* of G, denoted by ($spc(G)$) $pc(G)$.

The hypergraph version of conflict-free coloring was first introduced by Even et al. in [9]. A hypergraph H is a pair $H=(X,E)$ where X is the set of vertices, and E is the set of nonempty subsets of X, called hyperedges. The coloring was motivated to solve the problem of assigning frequencies to different base stations in cellular networks, which is defined as a vertex-coloring of H such that every hyperedge contains a vertex with a unique color.

Later on, Czap et al. in [7] introduced the concept of *conflict-free connection coloring* of graphs, motivated by the earlier hypergraph version. A path in an edge-colored graph G is called a *conflict-free path* if there is a color appearing only once on the path. The graph G is called *conflict-free connected* if there is a conflict-free path between each pair of distinct vertices of G. For a connected graph G, the minimum number of colors required to make G conflict-free connected is defined as the *conflict-free connection number* of G, denoted by $cfc(G)$. For more results, the reader can be referred to [4,6,5,8,12].

In this paper, we focus on studying the strong conflict-free connection coloring which was introduced by Ji et al. in [11], where only computational complexity was studied. An edge-colored graph is called *strongly conflict-free connected* if there exists a conflict-free path of length $d_G(u,v)$ for every pair of vertices u and v of G. For a connected graph G, the *strong conflict-free connection number* of G, denoted $scfc(G)$, is the smallest number of colors that are required to make G strongly conflict-free connected.

The paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we show that if G_t is a connected graph with m ($m\geq 2$) edges and t edge-disjoint triangles, then $scfc(G_t)\leq m-2t$, and the equality holds if and only if $G_t\cong S_{m-t,t}$. In Section 4, we characterize the graphs G with $scfc(G)=k$ for $k\in\{1,m-3,m-2,m-1,m\}$. In the last section, we completely characterize the cubic graphs G with $scfc(G)=2$.

2. Basic results and lemmas

In this section, we present some results which will be used in the sequel. In [11], the authors obtained the following computational complexity result.

Theorem 2.1 [7]**.** *If P_n is a path on n vertices, then $cfc(P_n)=\lceil\log_2 n\rceil$.*

Theorem 2.2 [4]**.** *Let G be a connected graph of order n ($n\geq 2$). Then $cfc(G)=n-1$ if and only if $G=K_{1,n-1}$.*

From Theorem 2.1 and 2.2 and the definitions of (strong) conflict-free connection number, we immediately have the following theorem.

Theorem 2.3. *For a tree T, $scfc(T)=cfc(T)$. Therefore, for a path P_n on n vertices, $scfc(P_n)=\lceil\log_2 n\rceil$; for a star S_m with m edges, $scfc(S_m)=m$.*

The authors in [6] obtained the strong rainbow connection number for a wheel graph W_n, where n is the degree of the central vertex, and the complete bipartite graph $K_{s,t}$.

Theorem 2.4 [6]**.** *For $n\geq 3$, let W_n be a wheel. Then $src(W_n)=\lceil\frac{n}{3}\rceil$.*

Theorem 2.5 [6]**.** *For integers s and t with $1\leq s\leq t$, $src(K_{s,t})=\lceil\sqrt[s]{t}\rceil$.*

Theorem 2.6. *For the integers n, s and t with $1\leq s\leq t$, $scfc(W_n)=\lceil\frac{n}{3}\rceil$ and $scfc(K_{s,t})=\lceil\sqrt[s]{t}\rceil$.*

Proof. Note that for a graph G with diameter 2, a strong rainbow path (of length 2) of G is a strong conflict-free path of G, and vice versa. Since $diam(W_n)=2$, then $scfc(W_n)=src(W_n)$. So, $scfc(W_n)=\lceil\frac{n}{3}\rceil$ from Theorem 2.4. Since $diam(K_{s,t})=2$, from Theorem 2.5 we have that $scfc(K_{s,t})=\lceil\sqrt[s]{t}\rceil$. □

Lemma 2.7. *Let C_n be a cycle of order n and let P_n be a spanning subgraph of C_n. Then $scfc(C_n)\leq scfc(P_n)$.*

Proof. Let $P_n=v_1(=u)v_2\cdots v_{n-1}v_n(=v)$ be a path with n vertices. We know that $scfc(P_n)=\lceil\log_2 n\rceil$ by Theorem 2.3. Now we first give a coloring for P_n: color the edge e_i with color $x+1$, where 2^x is the largest power of 2 that divides i. One can see that $\lceil\log_2 n\rceil$ is the largest number in the coloring by Theorem 2.3. Clearly, the color $\lceil\log_2 n\rceil$ only occurs once. Thus, we color the edge uv with $\lceil\log_2 n\rceil$ in C_n if there is only one color occurring once; otherwise, we color the edge uv with $\lceil\log_2 n\rceil-1$. Consequently, the coloring is a strong conflict-free connection coloring of C_n. □

Remark. The proposition does not hold for general graphs. Here is a counterexample. Let $G=C_6$ with the edge set $\{v_1v_2,v_2v_3,v_3v_4,v_4v_5,v_5v_6,v_6v_1\}$. So $scfc(G)=2$. Let $G'=C_6+v_1v_3$. Then $scfc(G')=3$.

Lemma 2.8. *If C_n is a cycle with n ($n\geq 3$) vertices, then*

$$scfc(C_n)=\lceil\log_2 n\rceil-1 or \lceil\log_2 n\rceil.$$

Proof. By Lemma 2.7 and Theorem 2.3, one can see that $scfc(C_n)\leq\lceil\log_2 n\rceil$. It remains to handle with the lower bound. We first consider the case that $diam(C_n)=\frac{n}{2}$ for $n=2k$ ($k\in\mathbb{Z}^+$). Hence, $scfc(C_n)\geq\lceil\log_2(\frac{n}{2}+1)\rceil=\lceil\log_2(n+2)\rceil-1\geq$

$\lceil \log_2 n \rceil - 1$. We then consider the case that $diam(C_n) = \frac{n-1}{2}$ for $n = 2k+1$ $(k \in \mathbb{Z}^+)$. Thus, $scfc(C_n) \geq \lceil \log_2(\frac{n-1}{2}+1) \rceil = \lceil \log_2(n+1) \rceil - 1 \geq \lceil \log_2 n \rceil - 1$. Consequently, $scfc(C_n) = \lceil \log_2 n \rceil - 1$ or $\lceil \log_2 n \rceil$. □

Lemma 2.8 implies the following corollary.

Corollary 2.9. *Let G be a connected graph with m edges and let C be a cycle in G. Then* $scfc(G) \leq m - |C| + \lceil \log_2 |C| \rceil$.

We end this section with an observation and a lemma.

Observation 2.10. Let G be a connected graph with $scfc(G) = |E(G)| - k$ and let H be a connected graph with $scfc(H) \leq |E(H)| - k - 1$. Then there is not a copy of H in G.

Lemma 2.11. *Let G be a connected graph with size m and* $scfc(G) = m - k$. *Then* $diam(G) - \lceil \log_2(diam(G)+1) \rceil \leq k$.

Proof. Let P be the path of length $diam(G)$. Now we define a coloring with $m + \lceil \log_2 diam(G) + 1 \rceil - diam(G)$ colors: assign the edges of P with $\lceil \log_2 diam(G) + 1 \rceil$ colors to make P strongly conflict-free connected; assign each of the remaining $m - diam(G)$ edges a fresh color. Clearly, G is strongly conflict-free connected. Since $scfc(G) = m - k$, then we have that $m - k \leq m + \lceil \log_2 (diam(G)+1) - diam(G) \rceil$. Consequently, $diam(G) - \lceil \log_2(diam(G)+1) \rceil \leq k$. □

3. Upper and lower bounds

At first, let us look at trees. We have one trivial result.

Theorem 3.1. *Let T be a tree of order n. Then we have*

$$\max\{\lceil \log_2(diam(T)+1) \rceil, \Delta(T)\} \leq scfc(T) \leq n - 1.$$

Next, we show a simple lower bound. Let G be a connected graph and let u, v be two vertices of G. If there are t paths between u and v in G, where the degree of internal vertices of the paths in G is 2, then we call the paths *t-parallel paths*.

Theorem 3.2. *Let G be a connected graph and let v, u be two vertices of G with* $d(u,v) \geq 2$. *If one of the following conditions holds, then* $scfc(G) \geq 3$.

1. *There exist a cut-vertex w which splits G into at least three components by deleting w.*
2. *There exists a path P of length at least 4 between u and v, where the edges of the path are bridges.*
3. *There exist 2-parallel paths between u and v, where the length of one path is 2 and the length of the other one is 3.*
4. *There exist 5-parallel paths between u and v.*

Proof. 1. Let $C_1, C_2, \cdots, C_m$ $(m \geq 3)$ be the components when deleting w from G. We choose a vertex u_i which is adjacent to w in each component C_i. Clearly, each pair of u_i and u_j contains the only path, and it contains w. Consequently, we have that $scfc(G) \geq scfc(S_m) = m \geq 3$.

2. Let P be a path of length at least 4. Since every edge of P is a bridge. Hence, we have $scfc(G) \geq scfc(P) \geq 3$.

3. Since the lengths of the two paths are 2 and 3, there is a 5-cycle in G. Clearly, $scfc(G) \geq 3$.

4. Since $d(u,v) \geq 2$, every path between u and v has a length at least 2. If we assign a coloring with 2 colors for the paths, then there always exist at least two internal vertices of the paths which do not contain a strong conflict-free path. Consequently, $scfc(G) \geq 3$. □

We now define a graph class. Let S_k be a star with k edges $uv_1, uv_2, \cdots, uv_k$. We denote by $S_{m-t,t}$ the graph $S_{m-t} + \{v_1v_2, v_3v_4, \cdots, v_{t-1}v_t\}$ $(2 \leq t \leq m)$.

Theorem 3.3. *If* G_t *is a connected graph with* m $(m \geq 2)$ *edges and t edge-disjoint triangles, then* $scfc(G_t) \leq m - 2t$, *and the equality holds if and only if* $G_t \cong S_{m-t,t}$.

Proof. Clearly, $scfc(K_3) = 1$. Now we first give a coloring of G_t: Color each triangle with a distinct color, that is, the three edges of each triangle receive a same color, and color each of the remaining $m - 3t$ edges with a distinct color. Let P be a strong conflict-free path for any pair of vertices u and v in G. Clearly, P contains at most one edge from each triangle. Otherwise, it will produce a contradiction. Thus, G_t is strongly conflict-free connected. So $scfc(G_t) \leq m - 2t$.

We now show that $scfc(G_t) = m - 2t$ if and only if $G_t \cong S_{m-t,t}$.

Sufficiency. Suppose that $G_t \cong S_{m-t,t}$. Clearly, $scfc(S_{m-t,t}) \leq m - 2t$. Note that every pendant edge needs a distinct color and every triangle needs a fresh color. Suppose that there is a coloring of $S_{m-t,t}$ in which on some triangle there is used the same color as on some pendant edge. Then the shortest path is not a conflict-free path between the leaf incident with the pendant edge and one vertex of degree two. Also, if we provide the t triangles with $t-1$ colors, there exist two triangle with the same color. There would also not exist a strong conflict-free path between the vertices of the two triangles. Consequently, $scfc(S_{m-t,t}) \geq m - 2t$.

Necessity. We now show that it holds for the necessity by the following 3 claims.

Claim 1. If $scfc(G_t) = m - 2t$, then every edge of G_t, except of the edges of the triangles, is a cut edge.

Proof of Claim 1. Assume that there is a cycle C $(|C| \geq 3)$ except the t triangles. We know that $scfc(C) \leq \lceil \log_2 |C| \rceil$ by Lemma 2.8. Now we define a coloring with $m - 2t + \lceil \log_2 |C| \rceil - |C| \leq m - 2t - 1$ colors: assign every triangle with a distinct color and assign C with $\lceil \log_2 |C| \rceil$ fresh colors, and the remaining edges are assigned by $m - |E(C)| - 3t$ fresh colors. Clearly, G_t is strongly conflict-free connected. So, $scfc(G_t) \leq m - 2t + \lceil \log_2 |C| \rceil - |C| \leq m - 2t - 1,$ a contradiction.

Claim 2. If $scfc(G_t) = m - 2t,$ then each triangle in G_t contains at least two vertices of degree two.

Proof of Claim 2. Assume that there is at most one vertex of degree two in a triangle $v_1v_2v_3v_1$. Without loss of generality, let u_1v_1 and u_2v_2 be two edges. We will consider the following three cases.

Case 1. Both u_1v_1 and u_2v_2 are not contained in triangles. We define a coloring c of G_t: assign each triangle with a distinct color; assign both u_1v_1 and u_2v_2 with a fresh same color; the remaining $m-2-3t$ edges are colored by $m-2-3t$ fresh colors. We only need to check $u_1 - u_2$ paths. By Claim 1, $u_1v_1v_2u_2$ is the unique strong conflict-free path between u_1 and u_2. Clearly, G_t is strongly conflict-free connected. Hence, $scfc(G_t) \leq (m-2-3t) + 1 + t = m - 2t - 1,$ a contradiction.

Case 2. u_1v_1 and u_2v_2 are contained in different triangles. Let X_1 contain u_1v_1 and let X_2 contain u_2v_2. We now define a coloring of G_t: assign X_1 and X_2 with the same color; assign the other triangles with $t-2$ fresh colors; each of the remaining edges is colored by a fresh color. Clearly, G_t is strongly conflict-free connected. Hence, $scfc(G_t) \leq m - 2t - 1$,a contradiction.

Case 3. One of u_1v_1 and u_2v_2 is contained in a triangle. Similarly, there is a strong conflict-free connection coloring with $m - 2t - 1$ colors, a contradiction. Completing the proof of Claim 2.

Claim 3. Let $C(G_t)$ be the graph induced by all the cut-edges of G_t. Then $C(G_t)$ is a tree with $diam(C(G_t)) \leq 2$.

Proof of Claim 3. Assume $C(G_t)$ is not connected. Let H_1 and H_2 be two connected components of $C(G_t)$. Clearly, the path in G_t which is connected to two vertices $h_1(\in V(H_1))$ and $h_2(\in V(H_2))$ goes through at least one triangle. Thus, the triangle contains at least two vertices of degree at least 3, which contradicts to Claim 2. Assume that $diam(C(G_t)) = k \geq 3$. Let $P = v_0v_1 \cdots v_k$ be a path of length k. Then we define a coloring of G_t with $m - 2t - k + \lceil \log_2(k+1) \rceil$ colors: assign the edges of P with $\lceil \log_2 k \rceil$ colors to make P strongly conflict-free connected from Theorem 2.3; assign each of the t triangles with a fresh color; assign each of the remaining $m - 3t - k$ edges with a fresh color. Clearly, G_t is strongly conflict-free connected, a contradiction. Completing the proof of Claim 3.

From the above claims, we can deduce that $G_t \cong S_{m-t,t}$. □

4. Graphs with large or small *scfc* numbers

In this section, we characterize the connected graphs G of size m with $scfc(G) = k$ for $k \in \{1, m-3, m-2, m-1, m\}$.

For the connected graph G with $scfc(G) = 1,$ we have the trivial result.

Theorem 4.1. *For a nontrivial connected graph G, $scfc(G) = 1$ if and only if G is a complete graph.*

From here on, we start to characterize the graph with large strong conflict-free connection number.

Theorem 4.2. *Let G be a nontrivial connected graph of size m. Then $scfc(G) = m$ if and only if $G \cong S_m$.*

Proof. *Necessity.* Suppose that $G \cong S_m$. we have $scfc(G) = m$ by Theorem 2.3.

Sufficiency. Suppose that $scfc(G) = m$. Assume there is a cycle C in G. Then $scfc(G) \leq m - |C| + \lceil \log_2 |C| \rceil \leq m - 1$ by Corollary 2.9, a contradiction. Hence, G is a tree. Let u and v be two vertices with $d_G(u, v) \geq 3$ in G. Similarly, $scfc(G) \leq m - d_G(u, v) + \lceil \log_2(d_G(u, v) + 1) \rceil \leq m - 1,$ a contradiction. Thus, $G \cong S_m$. □

For convenience, we define some graph-classes before proving the theorem below. Let S_m be a star with $m(\geq 2)$ edges and let u be a leaf of S_m. We define a graph by $\Gamma_{m+1} = (V(S) \cup \{v\}, E(S) \cup \{uv\})$.

Theorem 4.3. *Let G be a connected graph of size m. Then $scfc(G) = m - 1$ if and only if $G \in \{P_4, P_5, \Gamma_m\}$.*

Proof. *Necessity.* We have $scfc(G) = scfc(P_4) = 2 = m - 1$ and $scfc(G) = scfc(P_5) = 3 = m - 1$ by Theorem 2.3. On one hand, we have $scfc(\Gamma_m) \geq \Delta(\Gamma_m) = m - 1$ by Theorem 3.1. On the other hand, we define a coloring of Γ_m by assigning each of the $m-1$ edges of $S_{m-1}(\subset \Gamma_m)$ with a fresh color and choosing one color from the used colors except for the color assigned to the edge incident with u to assign the unique remaining edge. Clearly, G is strongly conflict-free connected. Hence, $scfc(\Gamma_m) = m - 1$.

Sufficiency. Suppose that $scfc(G) = m - 1$. We first show that G is a tree. Assume, to the contrary, that there is a cycle C in G. We have that $scfc(C) \leq |E(C)| - 2$ by Lemma 2.8, and thus $C \not\subseteq G$ by Observation 2.10.

When $diam(G) = 2,$ we have $G \cong S_n$ with $scfc(G) = m$ since G is a tree. But it is a contradiction.

When $diam(G) = 3,$ we show $G \in \{P_4, \Gamma_m\}$. Let $P_4 = v_1v_2v_3v_4$ of G. If $G = P_4,$ then $scfc(G) = m - 1$ by Theorem 2.3. Assume $M_1 = P_4 \cup \{xv_2, yv_3\}$ is a copy of the subgraph of G. It is easy to check that $scfc(M_1) \leq 3 = |E(M_1)| - 2$. So $M_1 \not\subseteq G$ by Observation 2.10. Thus, there is at most one vertex $v_i \in V(P_4)$ with $d_G(v_i) \geq 3$. Let $M_2 = P_4 \cup \{x_1v_2, \cdots, x_{t-2}v_2,\}$ for $t \geq 3$. Obversely, $scfc(M_2) \geq t = |E(M_2)| - 1$ by Theorem 3.1. On the other hand, there is a strong conflict-free connection coloring with t colors for G with $c(e) = 1$ for each $e \in \{v_1v_2, v_3v_4\}$, $c(v_2v_3) = 2$ and $c(x_iv_2) = i$ for $i \in [t-2]$. So, $G \in \{P_4, \Gamma_m\}$.

When $diam(G) = 4,$ we show $G = P_5$. Let $P_5 = v_1v_2v_3v_4v_5$ be a path of G. If $G = P_5,$ then $scfc(G) = scfc(P_5) = m - 1$ by Theorem 2.3. Assume that $M_3 = P_5 \cup \{wv_i\}$ for $i \in [5]$ is a copy of the subgraph of G. By symmetry, $M_3 = P_5 \cup \{wv_2\}$ or $M_3 = P_5 \cup \{wv_3\}$. If $c(v_1v_2) = c(v_3v_4) = 1,$ $c(wv_2) = 3$ $(c(wv_3) = 3)$ and $c(v_2v_3) = 2,$ then we can check $scfc(M_3) \leq |E(M_3)| - 2$. Hence, $M_3 \not\subseteq G$ by Observation 2.10.

For $diam(G) \geq 5$, clearly, we have $diam(G) - \lceil \log_2(diam(G) + 1) \rceil > 1,$ then $scfc(G) \neq m - 1$ by Lemma 2.11, a contradiction. □

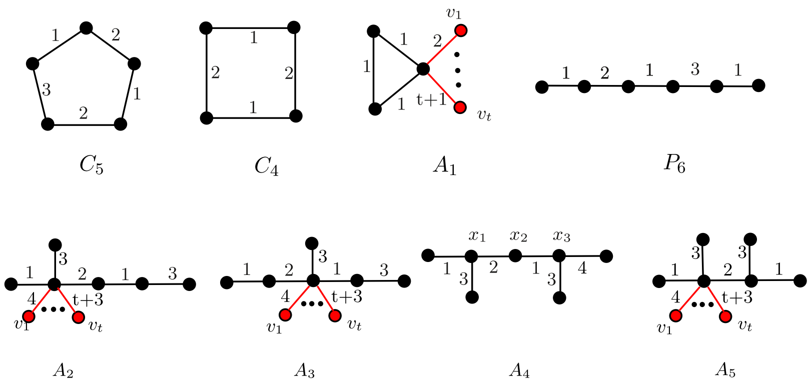

Fig. 1. Graphs with $scfc(G) = m - 2$. (Remark: The graphs A_1, A_2, A_3 and A_5 contain t leaves of the star S_t with $t \geq 0$ in Fig. 1. if they occur in the latter figures, it also means that they are the t leaves of the star S_t with $t \geq 0$).

Theorem 4.4. *Let G be a connected graph with $m\,(m \geq 3)$ edges. Then $scfc(G) = m - 2$ if and only if $G \in \{C_4, C_5, P_6, A_1, A_2, \cdots, A_5\}$ which are demonstrated in Fig. 1.*

Proof. *Necessity.* For $G = P_6$ we have $scfc(G) = scfc(P_6) = 3 = m - 2$ by Theorem 2.3. For $G \in \{C_4, C_5\}$, clearly, we have $scfc(C_4) \geq 2$ and $scfc(C_5) \geq 3$, on the other hand, from the coloring in Fig. 1 we know that $scfc(G) = scfc(C_5) = 3 = m - 2$, $scfc(G) = scfc(C_4) = 2 = m - 2$. For $G = A_i$ with $i \in \{2, 3, 5\}$, we have $scfc(G) = scfc(A_i) \geq t + 3 = m - 2$ by Theorem 3.1. On the other hand, we know that $scfc(G) = scfc(A_i) \leq t + 3 = m - 2$ by the coloring in Fig. 1. Clearly, for $G = A_1$ we have $scfc(G) = scfc(A_1) \geq \Delta(G) - 1 = t + 1 = m - 2$, meanwhile, we have $scfc(G) = scfc(A_1) \leq t + 1 = m - 2$ by the coloring in Fig. 1. For $G = A_4$, the edges incident with x_1 need to be assigned by three distinct colors, say 1,2 and 3. If $c(x_1x_2) = 2$, then $c(x_2x_3) = 1$ or 3. Thus, one of the remaining two edges must be colored by a fresh color. So, $scfc(G) = A_4 \geq 4 = m - 4$. Conversely, we have $scfc(G) = scfc(A_4) \leq 4 = m - 4$ by coloring in Fig. 1.

Sufficiency. Suppose that G contains one cycle with $scfc(G) = m - 2$. Let C be a cycle of length at least 6 in G. We have $scfc(C) \leq |E(C)| - 3$ by Lemma 2.8. It follows that $C \not\subseteq G$ by Observation 2.10. A contradiction. Hence, $|C| \leq 5$.

When $|C| = 3$, we show $G \cong A_1$. Let $C = v_1v_2v_3v_1$. Suppose there are two vertices $v_i, v_j \in V(C)$ with $d_G(v_i) \geq 3$ and $d_G(v_j) \geq 3$. Let $H_1 = C \cup \{v_1u_1, v_2u_2\}$ be a copy of a subgraph of G. We have $scfc(H_1) \leq 2 = |E(H_1)| - 3$ according to the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = 1$ and $c(v_1u_1) = c(v_2u_2) = 2$. Thus, there is not a copy of H_1 in G by Observation 2.10. A contradiction. Then there is at most one vertex $v_i \in V(C)$ with $d_G(v_i) \geq 3$ in G. Thus, let $H_2 = C \cup \{v_1u_1, u_1u_2\}$ be a copy of subgraph of G. Obviously, $scfc(H_2) \leq 2 = |E(H_2)| - 3$. There is not a copy of H_2 in G by Observation 2.10. Hence, we have $diam(G) = 2$. It means that $G \cong A_1$.

When $|C| = 4$, we show $G \cong C_4$. Let $C = v_1v_2v_3v_4v_1$. Suppose there is one vertex $v_i \in V(C)$ with $d_G(v_i) \geq 3$ in G. Let $H_3 = C \cup \{v_1u_1\}$ or $C \cup \{v_1v_3\}$ be a copy of the subgraph of G. Clearly, we have $scfc(H_3) \leq 2 = |E(H_3)| - 3$ by the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_4) = 1$ and $c(v_3v_4) = c(v_1u_1) = 2$ (or $c(v_3v_4) = c(v_1v_3) = 2$). Thus, there is not a copy of H_3 in G by Observation 2.10. Hence, $G \cong C_4$.

When $|C| = 5$, we show $G \cong C_5$. Let $C = v_1v_2v_3v_4v_5v_1$. Suppose there is one vertex $v_i \in V(C)$ with $d_G(v_i) \geq 3$ in G. By the same way, the graph $H_4 = C \cup \{v_1u_1\}$ (or $H_4' = C \cup \{v_1v_3\}$) is not a copy of the subgraph in G by Observation 2.10 since $scfc(H_4) \leq |E(H_4)| - 4$ (or $scfc(H_4') \leq |E(H_4')| - 4$) by the coloring with $c(v_1v_2) = c(v_4v_5) = 1$, $c(v_1v_5) = c(v_2v_3) = 2$ and $c(v_3v_4) = c(v_1u_1) = 3$ (or $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = c(v_4v_5) = 1$ and $c(v_3v_4) = c(v_1u_1) = 2$). Hence, there is not a vertex $v_i \in V(C)$ with $d_G(v_i) \geq 3$ in G. Hence, every vertex $v_i \in V(C)$ have degree 2, then we can deduce that $G \cong C_5$.

Suppose that G is a tree with $scfc(G) = m - 2$. Assume that $diam(G) \geq 6$. Clearly, we have $diam(G) - \lceil \log_2(diam(G) + 1) \rceil > 2$, then $scfc(G) \neq m - 2$ by Lemma 2.11, a contradiction. Thus, $diam(G) \leq 5$.

When $diam(G) = 2$. Clearly, we have $G = S_m$ with $scfc(S_m) = m$, which is a contradiction.

When $diam(G) = 3$, we show $G \cong A_5$. Let $P_4 = v_1v_2v_3v_4$ be a path of G. Assume that the degrees of both v_2 and v_3 are at least 4. Let $H_5 = P_4 \cup \{w_1v_2, w_2v_2, w_3v_3, w_4v_3\}$ be a copy of the subgraph of G. We have $scfc(H_5) \leq 4 = |E(H_5)| - 3$ by the coloring with $c(v_1v_2) = c(v_3v_4) = 1$, $c(w_2v_2) = c(w_4v_3) = 2$, $c(w_1v_2) = c(w_3v_3) = 3$ and $c(v_2v_3) = 4$. Thus, there is not a copy of H_5 in G by Observation 2.10. Hence, there is at most one vertex $v_i \in \{v_2, v_3\}$ with $d_G(v_i) \geq 4$. Together with $scfc(P_4) = 2 = m - 1$ and $scfc(\Gamma_m) = m - 1$ for $G \in \{P_4, \Gamma_m\}$ by Theorem 4.3, we can deduce that $G \cong A_5$.

When $diam(G) = 4$, we show $G \in \{A_2, A_3, A_4\}$. Let $P_5 = v_1v_2v_3v_4v_5$ be a path of G. Assume that there are two adjacent vertices with degree 3, say v_2 and v_3. Let $H_6 = P_5 \cup \{w_1v_2, w_2v_3\}$ be a copy of the subgraph of G. We have $scfc(H_6) \leq 3 = |E(H_6)| - 3$ by the coloring with $c(v_1v_2) = c(v_3v_4) = 1$, $c(w_1v_2) = c(w_2v_3) = c(v_4v_5) = 2$ and $c(v_2v_3) = 3$. Thus, there is not a copy of H_6 in G by Observation 2.10. Furthermore, assume that $H_7 = P_5 \cup \{w_1v_2, w_2v_4, w_3v_4\}$ is a copy of the subgraph of G. We have $scfc(H_7) \leq 4 = |E(H_7)| - 3$ by the coloring with $c(v_3v_4) = 1$, $c(v_2v_3) = c(w_3v_4) = 2$, $c(v_2w_1) = c(v_4v_5) = 3$ and $c(v_1v_2) = c(w_2v_4) = 4$. Thus, there is not a copy of H_7 in G by Observation 2.10. Together with $scfc(G) = m - 1$ for $G \cong P_5$ from Theorem 4.3, we could deduce that $G \in \{A_2, A_3, A_4\}$.

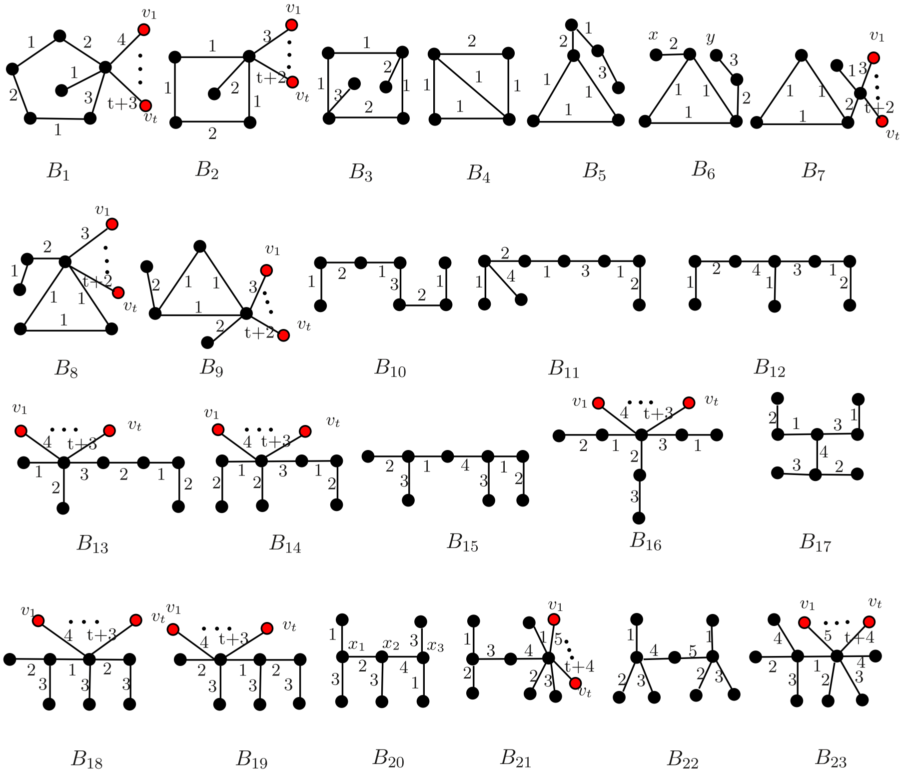

Fig. 2. Graphs with $scfc(G) = m - 3$.

When $diam(G) = 5$, we show $G \cong P_6$. Let $P_6 = v_1v_2v_3v_4v_5v_6$ be a path of G. If $G = P_6$, then $scfc(G) = 3 = m - 2$ by Theorem 2.3. By symmetry, Assume that $H_8 = P_6 \cup \{v_2x\}$ or $H_8 = P_6 \cup \{v_3x\}$ is a copy of the subgraph of G. Clearly, $scfc(H_8) \leq 3 = m - 3$. Thus, there is not a copy of H_8 in G by Observation 2.10. We can deduce that $G \cong P_6$. □

Theorem 4.5. *Let G be a connected graph with $m\,(m \geq 4)$ edges. Then $scfc(G) = m - 3$ if and only if $G \in \{B_1, B_2, \cdots, B_{23}\}$ which are demonstrated in Fig. 2.*

Proof. *Sufficiency.* Clearly, we have $scfc(G) \geq \Delta(G)$ for $G \in \{B_1, B_3, B_7, B_{13}, B_{14}, B_{16}, B_{18}, B_{19}, B_{21}, B_{23}\}$. On the other hand, by the coloring of $G \in \{B_1, B_3, B_7, B_{13}, B_{14}, B_{16}, B_{18}, B_{19}, B_{21}, B_{23}\}$ in Fig. 2, we have $scfc(G) = scfc(B_1) = scfc(B_{13}) = scfc(B_{14}) = scfc(B_{16}) = scfc(B_{18}) = scfc(B_{19}) = t + 3 = m - 3$, $scfc(G) = scfc(B_3) = 3 = m - 3$, $scfc(G) = scfc(B_7) = t + 2 = m - 3$ and $scfc(G) = scfc(B_{21}) = scfc(B_{21}) = t + 4 = m - 3$. Obviously, for $G \in \{B_2, B_4, B_8, B_9\}$ we have $scfc(G) \geq \Delta(G) - 1$. On the other hand, by the coloring of $G \in \{B_2, B_4, B_8, B_9\}$ in Fig. 2, we have $scfc(G) = scfc(B_2) = scfc(B_8) = scfc(B_9) = t + 2 = m - 3$ or $scfc(G) = scfc(B_4) = 2 = m - 3$. For $G = B_{10}$ we have $scfc(G) = scfc(B_{10}) = 3 = m - 3$ by Theorem 2.3. For $G = B_6$, since there is exactly one path of length $d(x, y)$ $(d(x, y) = 4$ between x and y, then we have $scfc(B_6) \geq 3$. By the coloring in Fig. 2, we have $scfc(B_6) = 3 = m - 3$. Similarly, $scfc(B_5) = 3 = m - 3$. For $G = B_{20}$, the edges incident with x_1 need to be assigned by three distinct colors, say 1,2 and 3. Without loss of generality, if $c(x_1x_2) = 1$, then the remaining edges incident with x_2 must be assigned by 2 and 3. Thus, one of the edges incident with x_3, except the edge x_2x_3, must be assigned by a fresh color. Hence, $scfc(G) = scfc(B_{20}) = 4 = m - 3$ in Fig. 2. Clearly, for $G \in \{B_{11}, B_{12}, B_{15}, B_{17}, B_{22}\}$, easily, we have $scfc(B_{11}) = scfc(B_{12}) = scfc(B_{15}) = scfc(B_{17}) = 4 = m - 3$; $scfc(B_{22}) = 5 = m - 3$.

Necessity. Suppose that G contains one cycle with $scfc(G) = m - 3$. Let C be a cycle of length at least 6 in G. We have $scfc(C) \leq |E(C)| - 4$ by Lemma 2.8. We know that there is not a copy of C in G by Observation 2.10. Thus, $|C| \leq 5$.

When $|C| = 5$, we show that $G \cong B_1$. Let $C = v_1v_2v_3v_4v_5v_1$. Suppose that there is a chord in C. Let $W_0 = C \cup v_1v_3$ be a copy of the subgraph of G. We have $scfc(W_0) = scfc(H_4') \leq 2 = |E(H_4')| - 4 = |E(W_0)| - 4$. There is not a copy of W_0 in G by Observation 2.10. A contradiction. Without loss of generality, assume that $W_1 = C \cup \{v_1u_1, v_2u_2\}$ or $W_1 = C \cup \{v_1u_1, v_3u_2\}$ is a copy of the subgraph of G. Clearly, we have $scfc(W_1) \leq |E(W_1)| - 4$ according to the coloring with $c(v_1v_2) = c(v_3v_4) = 1$, $c(v_1v_5) = c(v_2v_3) = 2$ and $c(v_4v_5) = c(v_1u_1) = c(v_2u_2) = 3$ (or $c(v_4v_5) = c(v_1u_1) = c(u_2v_3) = 3$). By Observation 2.10 we know there is not a copy of W_1 in G. By the same way, the graph $W_2 = C \cup \{v_1u_1, u_1u_2\}$ is not a copy of the subgraph of G by Observation 2.10 since $scfc(W_2) \leq 3 = m - 4$ by the coloring with $c(v_1v_2) = c(v_4v_5) = c(u_1u_2) = 1$, $c(v_2v_3) = c(v_1v_5) = 2$ and $c(v_1u_1) = c(v_3v_4) = 3$. Let $W_3 = C \cup \{v_1v_3\}$. Since $scfc(W_3) = scfc(H_4') \leq |W_3| - 4$, we know there is not a copy of W_3 in G by Observation 2.10. In addition, we have $scfc(G) = m - 2$ for $G = C$ by Theorem 4.4. Hence, we deduce that $G \cong B_1$.

When $|C| = 4$, *we show* $G \in \{B_2, B_3, B_4\}$. Let $C = v_1v_2v_3v_4v_1$. We claim that if there is not a chord in C, then $G \in \{B_2, B_3\}$. Now assume that $W_4 = C \cup \{v_1u_1, u_1u_2\}$ is a copy of the subgraph of G. Then we have $scfc(W_4) \leq 2 = |E(W_4)| - 4$ by the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_4) = c(u_1u_2) = 1$ and $c(v_1u_1) = c(v_3v_4) = 2$. By Observation 2.10, W_4 is not a copy of the subgraph of G. Furthermore, we show that there is not two adjacent vertices $v_i, v_j \in V(C)$ with degree at least three in G. Thus, let $W_5 = C \cup \{v_1x_1, v_2x_2\}$, $W_6 = C \cup \{v_1w_1, v_1w_2, v_3w_3\}$. The graphs W_5 and W_6 are not the copies of the subgraphs of G by Observation 2.10 since $scfc(W_5) \leq 2 = m - 4$ and $scfc(W_6) \leq 2 = |E(W_6)| - 4$. Meanwhile, we have $G \not\cong C$ since $scfc(C) = 2 = |E(C)| - 2$ by Theorem 4.4. Hence, we deduce that $G \cong B_2$ or $G \cong B_3$. Next, we claim that if there is a chord in C, then $G = B_4$. We first show there are exactly two vertices of $V(C)$ with degree three. Let $W_7 = C \cup \{v_1v_3, v_1y\}$ and $W_8 = C \cup \{v_2v_4, v_1z\}$. Let K_4 be a complete graph of order 4. The graphs K_4, W_7 and W_8 are not the copies of the subgraphs of G by Observation 2.10 since $scfc(K_4) = 1 = |E(K_4)| - 5$, $scfc(W_7) = scfc(W_8) \leq 2 = |E(W_7)| - 4 = |E(W_8)| - 4$. Clearly, we deduce that $G \cong B_4$.

When $|C| = 3$, *we show* $G \cong B_5$, B_7 *or* B_8. Let $C = v_1v_2v_3v_1$. *We first show that not all the vertices of* $V(C)$ *have degree at least 3.* Assume, to the contrary, that $W_9 = C \cup \{v_1u_1, v_2u_2, v_3u_3\}$ is a copy of the subgraph of G. We have $scfc(W_9) \leq 2 = |E(W_9)| - 4$ by the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = 1$ and $c(v_1u_1) = c(v_2u_2) = c(v_3u_3) = 2$. A contradiction by Observation 2.10. Thus, there are at most two vertices in $V(C)$ with degree at least three. *Suppose that there are exactly two vertices* $v_1, v_2 \in V(C)$ *with* $d_G(v_1) \geq 3$ *and* $d_G(v_2) \geq 3$. Next, let $W_{10} = C \cup \{v_1u_2, u_1u_2, v_2u_3, u_3u_4\}$. Clearly, $scfc(W_{10}) \leq 3 = |E(W_{10})| - 4$ by the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = 1$, $c(v_1u_1) = c(v_2u_3) = 2$ and $c(u_1u_2) = c(u_3u_4) = 3$. Thus W_{10} is not a copy of the subgraph of G by Observation 2.10. Similarly, in the same way the graphs $W_{11} = C \cup \{v_1w_1, v_1w_2, v_2w_3, v_2w_4\}$, $W_{12} = C \cup \{v_1x_1, v_1x_2, v_2x_3, x_3x_4\}$, $W_{13} = C \cup \cup\{v_1y_1, v_2y_2, v_2y_3, y_3y_4\}$ are not the copies of the subgraphs in G since $scfc(W_{11}) \leq 3 = |E(W_{11})| - 4$, $scfc(W_{12}) \leq 3 = |E(W_{12})| - 4$, $scfc(W_{13}) \leq 3 = |E(W_{13})| - 4$. Hence, we have $G \cong B_6$ or $G \cong B_9$ for two vertices v_1, v_2 with $d_G(v_1) \geq 3$ and $d_G(v_2) \geq 3$. *Suppose that there is exactly one vertex* $v_1 \in V(C)$ *with* $d_G(v_1) \geq 3$. Let $W_{14} = C \cup \{v_1w_1, w_1w_2, w_2w_3, w_3w_4\}$, $W_{15} = C \cup \{v_1x_1, x_1x_2, x_1x_3, x_3x_4\}$ and $W_{16} = C \cup \{v_1y_1, y_1y_2, y_2y_3, y_2y_4\}$. Then we have $scfc(W_{14}) \leq 3 = |E(W_{14})| - 4$ according to the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = c(w_1w_2) = c(w_3w_4) = 1$, $c(v_1w_1) = 2$ and $c(w_2w_3)$ and $scfc(W_{15}) \leq 3 = |E(W_{15})| - 4$ (or $scfc(W_{16}) \leq 3 = |E(W_{16})| - 4$) according to the coloring with $c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = c(x_1x_2) = c(x_3x_4) = 1$, $c(v_1x_1) = 2$ and $c(x_1x_3) = 3$ ($c(v_1v_2) = c(v_2v_3) = c(v_1v_3) = c(y_2y_3) = 1$, $c(v_1y_2) = c(y_2y_4) = 2$ and $c(y_1y_2) = 3$). So W_{14}, W_{15} and W_{16} are not the copies of the subgraphs of G by Observation 2.10. In addition, for $G = A_1$, we have $scfc(G) = m - 2 > m - 3$ by Theorem 4.4. Hence, $G \cong B_5$, B_7 or B_8.

Suppose that G is a tree. Assume that $diam(G) \geq 7$. Clearly, $diam(G) - \lceil \log_2(diam(G) + 1) \rceil \geq 4$. From Lemma 2.11, we have $scfc(G) \neq m - 3$. A contradiction. Thus, $diam(G) \leq 6$.

When $diam(G) = 6$, we show $G \in \{B_{10}, B_{11}, B_{12}\}$. Let $P_7 = v_1v_2v_3v_4v_5v_6v_7$ be a path of G. *Suppose* $d_G(v_i) \leq 2$ *for* $(i \in [7])$. Then, clearly, we have $G \cong P_7 = B_{10}$. *Suppose there is at least one vertex* v_i *with* $d_G(v_i) = 3$. Assume that $U_1 = P_7 \cup \{u_1v_3\}$ or $U_2 = P_7 \cup \{v_4u_1, u_1u_2\}$ is a copy of a subgraph of G. Clearly, $scfc(U_1) \leq 3 = |E(U_1)| - 4$ according to the coloring with $c(v_2v_3) = c(v_4v_5) = c(v_6v_7) = 1$, $c(v_1v_2) = c(u_1v_3) = c(v_5v_6) = 2$ and $c(v_3v_4) = 3$ and $scfc(U_2) \leq 4 = |E(U_2)| - 4$ according to the coloring with $c(v_1v_2) = c(u_1u_2) = c(v_3v_4) = c(v_5v_6) = 1$, $c(v_2v_3) = c(v_6v_7) = 2$, $c(v_4v_5) = 3$ and $c(v_4u_1) = 4$. Hence, we can deduce that G must be B_{11} or B_{12}. *Suppose there is a vertex* $v_i \in V(P_7)$ *with* $d_G(v_i) \geq 4$. Then let $U_3 = P_7 \cup \{v_2x_1, v_2x_2\}$, $U_4 = P_7 \cup \{v_3y_1, v_3y_2\}$ and $U_5 = P_7 \cup \{v_4z_1, v_4z_2\}$. Clearly, $scfc(U_3) \leq 4 = |E(U_3)| - 4$ by the coloring with $c(v_1v_2) = c(v_5v_6) = c(v_3v_4) = 1$, $c(v_2x_2) = c(v_4v_5) = 2$, $c(v_2x_1) = c(v_6v_7) = 3$ and $c(v_2v_3) = 4$; $scfc(U_4) \leq 4 = |E(U_4)| - 4$ by the coloring with $c(v_1v_2) = c(v_4v_5) = c(v_6v_7) = c(v_3y_1) = 1$, $c(v_2v_3) = c(v_5v_6) = 2$, $c(v_3y_2) = 3$ and $c(v_3v_4) = 4$; $scfc(U_5) \leq 4 = |E(U_5)| - 4$ by the coloring with $c(v_1v_2) = c(z_2v_4) = c(v_5v_6) = 1$, $c(v_2v_3) = c(v_4z_1) = c(v_6v_7) = 2$, $c(v_4v_5) = 3$ and $c(v_3v_4) = 4$. Hence, G does not contain one copy of one of $\{U_3, U_4, U_5\}$ by Observation 2.10.

When $diam(G) = 5$, we show $G \in \{B_{13}, B_{14}, B_{15}\}$. Let $P_6 = v_1v_2v_3v_4v_5v_6$ be a path of G. Suppose $d_G(v_i) \leq 2$ for $i \in [6]$, then we have $G = P_6$. But, $scfc(G) = scfc(P_6) = 3 = m - 2$ from Theorem 4.4. A contradiction. Suppose there is exactly one vertex $v \in V(P_6)$ with $d_G(v) \geq 3$, then we claim that $G = B_{13}$ or B_{14}. By symmetry, assume, to the contrary, that $U_6 = P_6 \cup \{v_3y_1, y_1y_2\}$ is a copy of the subgraph of G. However, we have $scfc(U_6) \leq 3 = |E(U_6) - 4|$ by the coloring with $c(v_2v_3) = c(v_4v_5) = 1$, $c(v_1v_2) = c(v_5v_6) = c(v_3y_1) = 2$ and $c(y_1y_2) = c(v_3v_4) = 3$. Thus, U_6 is not a copy of the subgraph of G by Observation 2.10. Since $diam(G) = 5$, then $G = B_{13}$ or B_{14}. We then claim $G = B_{15}$ if there are at least two vertices $v_i, v_j \in V(P_6)$ with $d_G(v_i) \geq 3$ and $d_G(v_j) \geq 3$. Assume, to the contrary, that $U_7 = P_6 \cup \{x_1v_2, x_2v_3\}$, $U_8 = P_6 \cup \{y_1x_3, y_2x_4\}$ or $U_9 = P_6 \cup \{z_1v_2, z_2v_5\}$ is a copy of the subgraph of G. Since $scfc(U_7) \leq 3 = |E(U_7)| - 4$ by the coloring with $c(v_1v_2) = c(x_2v_3) = c(v_4v_5) = 1$, $c(x_1v_2) = c(v_3v_4) = 2$ and $c(v_2v_3) = c(v_5v_6) = 3$, and $scfc(U_8) \leq 3 = |E(U_8)| - 4$ by the coloring with $c(v_1v_2) = c(y_1v_3) = c(y_2v_4) = c(v_5v_6) = 1$, $c(v_2v_3) = c(v_4v_5) = 2$ and $c(v_3v_4) = 3$, and $scfc(U_9) \leq 3 = |E(U_9)| - 4$ by the coloring with $c(v_1v_2) = c(v_3v_4) = c(v_5v_6) = 1$, $c(z_1v_2) = c(v_4v_5) = 2$ and $c(v_2v_3) = c(z_2v_5) = 3$. Furthermore, since G does not contain a copy of U_6, then G must be B_{15}.

When $diam(G) = 4$, we show $G \in \{B_{16}, B_{17}, B_{18}, B_{19}, B_{20}, B_{21}, B_{22}\}$. Clearly, $G \neq A_2$, A_3 or A_4 by Theorem 4.4 and $G \neq P_5$ by Theorem 4.3. Let $P_5 = v_1v_2v_3v_4v_5$ be a path of G. Suppose that $d_G(v_2) \geq 3$, $d_G(v_3) \geq 3$ and $d_G(v_4) \geq 3$. Let $U_{10} = P_5 \cup \{x_1v_2, x_2v_3, x_3v_4, x_4v_4\}$ or $U_{11} = P_5 \cup \{x_1v_2, x_2v_3, x_3v_4, x_4v_3\}$. Then we show $G \cong B_{20}$. Since $scfc(U_{10}) \leq |E(U_{10})| - 4$ by the coloring with $c(v_2v_3) = c(x_4v_4) = 1$, $c(x_1v_2) = c(v_4v_5) = 2$, $c(v_1v_2) = c(x_2v_3) = c(x_3v_4) = 3$ and $c(v_3v_4) = 4$, and $scfc(U_{11}) \leq |E(U_{11})| - 4$ by the coloring with $c(v_2v_3) = 1$, $c(x_1v_2) = c(v_3x_4) = c(v_4v_5) = 2$, $c(v_1v_2) = c(x_2v_3) = c(x_3v_4) = 3$ and $c(v_3v_4) = 4$. Thus, both U_{10} and U_{11} are not the copies of the subgraphs of G by Observation 2.10. Let $U_{12} = P_5 \cup \{y_1v_2, y_2v_3, y_2y_3, v_4y_4\}$. Clearly, the graph U_{12} is not a copy of the subgraph of G by Observation 2.10. Hence, $G \cong B_{20}$ when $d_G(v_2) \geq 3$, $d_G(v_3) \geq 3$ and $d_G(v_4) \geq 3$. In the similar way, when there are exactly two vertices $v_i, v_j \in V(P_5)$ with

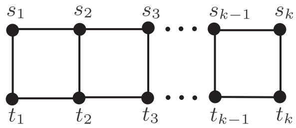

Fig. 3. Ladder L_k.

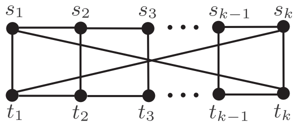

Fig. 4. Möbius M_{2k}.

$d_G(v_i) \geq 3$ and $d_G(v_j) \geq 3$, we have $G \in \{B_{18}, B_{19}, B_{21}, B_{22}\}$. When there is exactly one vertex $v_i \in V(P_5)$ with $d_G(v_i) \geq 3$, we have $G \in \{B_{16}, B_{17}\}$.

When $diam(G) = 3$. Let $P_4 = v_1v_2v_3v_4$ be a path of G. Let $U_{13} = P_5 \cup \{v_2w_1, v_2w_2, v_2w_3, v_3w_4, v_3w_5, v_3w_6\}$. Clearly, U_{13} is not a copy of one subgraph of G by Observation 2.10. Together with $G \neq \Gamma_m$ by Theorem 4.3 and $G \neq A_5$ by Theorem 4.4, we deduce $G \cong B_{23}$. □

5. Cubic graphs with *scfc*-number 2

In this section, we first define some useful definitions and show several lemmas. Next, we will characterize the cubic graphs G with $scfc(G) = 2$ by the lemmas.

We first give a useful definition.

Definition 5.1 [10]**.** A *forced 2-path* in a graph G is a path xyz such that $xz \notin E(G)$ and xyz is the unique 2-path connecting x and z. If each 2-path $u_iu_{i+1}u_{i+2}$ is *forced* for $i = 0, 1, \cdots, k-2$, a k-path $P = u_0u_1 \cdots u_k$ in a graph G is called *forced*. A cycle of a graph G is called a *forced cycle* if any two successive edges of the cycle form a forced 2-path in G. An edge e in a graph G is called a *forced edge* if e is not included in a cycle of length at most 4.

If uv is a forced edge in G and vw is an edge adjacent to uv, then uvw is a forced 2-path in G. The following two results follow directly from the definition.

Lemma 5.2. *Let $P = u_1u_2 \cdots u_k$ be a forced path in G with $scfc(G) = 2$. Then the adjacent edges of P are colored by distinct colors for every strong conflict-free connection coloring with 2 colors.*

Lemma 5.3. *Let $C = u_1u_2 \cdots u_ku_1$ be a forced cycle of length k in G with $scfc(G) = 2$. Then the adjacent edges of C are colored by distinct colors for every strong conflict-free connection coloring with 2 colors and k is even and $k \leq 6$.*

Lemma 5.4. *$scfc(C_k \square K_2) = 2$ if and only if k equals 3, 4 or 6.*

Proof. We have $scfc(C_k \square K_2) \geq 2$ by Theorem 4.1. For $k = 3$, we define a 2-edge-coloring c: for every edge e in the triangles, $c(e) = 1$; Otherwise, $c(e) = 2$. Clearly, the coloring c is a strong conflict-free connection coloring of $C_3 \square K_2$. Hence, $scfc(C_3 \square K_2) = 2$. For $k \geq 4$, Clearly, the graph $C_k \square K_2$ has a forced cycle. Then by Lemma 5.3, since $scfc(C_k \square K_2) = 2$, we have that $k = 4$ or 6. □

Now we define some graph-classes. A k-ladder, denoted by L_k, is defined to be the product graph $P_k \square K_2$, where P_k is the path on k vertices *(see* Fig. 3*)*. The Möbius ladder M_{2k} is the graph obtained from L_k by adding two new edges s_1t_k and t_1s_k *(see* Fig. 4*)*.

Lemma 5.5. *$scfc(M_{2k}) = 2$ if and only if $3 \leq k \leq 7$.*

Proof. Since M_{2k} is not a complete graph, it is clear to see that $scfc(M_{2k}) \geq 2$ for every $k \geq 3$.

First, we show that $scfc(M_{2k}) > 2$ for $k \geq 8$. Clearly, for the pair of vertices s_2 and s_6 there is only one shortest path connecting them, which is $P' = s_2s_3s_4s_5s_6$. For every pair of vertices in P, there is only one shortest path in M_{2k} connecting them. So we have that $scfc(M_{2k}) \geq scfc(P') = 3$.

Second, we show that $scfc(M_{2k}) \leq 2$ for $3 \leq k \leq 7$. For the graph M_{2k} with $k \in \{4, 6\}$, we define a 2-edge-coloring c: for $i \in \{1, 3, 5\}$, $c(s_is_{i+1}) = c(t_it_{i+1}) = c(s_it_i) = 1$; for the remaining edges e, $c(e) = 2$. For the graph M_{2k} with $k \in \{3, 5, 7\}$, we define a 2-edge-coloring c: for $i \in \{1, 3, 5\}$, $c(s_is_{i+1}) = c(t_{i+1}t_{i+2}) = 1$; for $i \in \{1, 2 \cdots, k\}$, $c(s_it_i) = c(s_kt_1) = 1$; for the remaining edges e, $c(e) = 2$. It is easy to check that every pair of vertices are connected by a strong conflict-free path under the above 2-edge-colorings. □

Before the proof of Lemma 5.6, we first illustrate a cubic graph U (*see* Fig. 5) with the property that $spc(U) > 2$ and $scfc(U) = 2$. We first illustrate that $spc(U) > 2$. Since each 2-path of $\{v_1v_7v_6, v_8v_7v_6, v_7v_6v_{10}, v_7v_6v_5, v_{10}v_4v_3,$

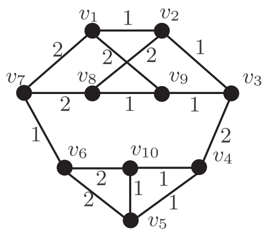

Fig. 5. The graph U.

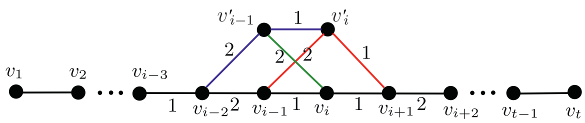

Fig. 6. The path P with an attachment W. (The path $v_{i-1}v'_iv_{i+1}$ is the replacement for $v_{i-1}v_iv_{i+1}$; the path $v_{i-2}v'_{i-1}v'_i$ is the replacement for $v_{i-2}v_{i-1}v'_i$; the path $v'_{i-1}v_iv_{i+1}$ is the replacement for $v'_{i-1}v'_iv_{i+1}$.).

$v_5v_4v_3v_4v_3v_2, v_4v_3v_9\}$ is a forced 2-path, every two adjacent edges are needed to be colored by distinct colors from {1, 2}. Thus, there is not a strong proper path between v_7 and v_3, therefore, $spc(U) > 2$. We have $scfc(U) \geq 2$ by Theorem 4.1, and together with the 2-edge-coloring in Fig. 5, it follows that $scfc(U) = 2$.

Now we will show Lemma 5.6. In order to be more convenient to handle with Lemma 5.6, in the very beginning, we give some explanations. Let G be a cubic graph and let c: $E(G) \mapsto \{1, 2\}$ be a strong conflict-free connection coloring of G. Let $P = (u =)v_1v_2 \cdots v_{t-1}v_t(= v)$ be a strong conflict-free path between u and v. Suppose that there exists a 2-path $v_iv_{i+1}v_{i+2}$ with $c(v_iv_{i+1}) = c(v_{i+1}v_{i+2})$ in P. Then there must exist another 2-path v_ixv_{i+2} $(x \notin V(P))$ with $c(v_ix) \neq c(xv_{i+2})$ since c is a strong conflict-free connection coloring of G. Then v_ixv_{i+2} is called *a replacement* for $v_iv_{i+1}v_{i+2}$. Suppose that $c(v_{i-1}v_i) = c(v_ix)$. Then there must also exist a replacement $v_{i-1}yx$ for $v_{i-1}v_ix$. Furthermore, suppose the path yxv_{i+2} contains the coloring $c(yx) = c(xv_{i+2})$. If $yv_{i+1}v_{i+2}$ is a replacement for yxv_{i+2}, then $G[V']$, where $V' = \{v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, y\}$, is called an *attachment* of P. Then, clearly, there is not a strong proper path between v_{i-1} and v_{i+2}. If there does not exist a replacement sharing the same edges with P, then there is a strong proper path between v_{i-1} and v_{i+2}. Thus, we call the replacements *noncyclic replacements* of P. Otherwise, it is called a *cyclic replacement* of P.

Lemma 5.6. *Let G be a cubic graph with $G \not\cong U$. If $scfc(G) = 2$, then $spc(G) = 2$.*

Proof. Let c: $E(G) \mapsto [2]$ be an arbitrary strong conflict-free connection coloring of G. Let $P = (u =)v_1v_2 \cdots v_{t-1}v_t(= v)$ be an arbitrary strong conflict-free path between u and v. For every pair of v_i and v_{i+2} $(i \in [t-2])$, if $c(v_iv_{i+1}) \neq c(v_{i+1}v_{i+2})$, then P is a strong proper path. Thus, we have $spc(G) = 2$. Suppose that there exists a 2-path $v_iv_{i+1}v_{i+2}$ $(i \in [t-2])$ with $c(v_iv_{i+1}) = c(v_{i+1}v_{i+2})$ in P. When each replacement is a noncyclic replacement for P, then there exists a strong proper path Q between u and v in G. We also have $spc(G) = 2$. Suppose that there exist a cyclic attachment for P. We denote $G[V']$ by W, where $V' = \{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v'_i, v'_{i-1}\}$, clearly, W is an attachment for P *(see* Fig. 6*)*, in which there is not a strong proper path between v_{i-2} and v_{i+1}.

Claim 1. Let W be an attachment of path P. Then $c(v_{i-3}v_{i-2}) \neq c(v_{i-2}v_{i-1}) = c(v_{i-2}v'_{i-1})$ and $c(v_{i+1}v_{i+2}) \neq c(v_iv_{i+1}) = c(v'_iv_{i+1})$.

Proof of Claim 1: Without loss of generality, suppose that $c(v_{i-3}v_{i-2}) = c(v_{i-2}v_{i-1})$. Then, clearly, $v_{i-3}v_{i-2}v'_{i-1}$ is a unique shortest path between v_{i-3} and v'_{i-1} since G is a cubic graph. It contradicts to $c(v_{i-3}v_{i-2}) = c(v_{i-2}v_{i-1})$ under the coloring c. This completes the proof of Claim 1.

Suppose P contains an attachment W. We first show there is at most one attachment for P. Assume, to the contrary, that there are two attachments in P. Let $E' = \{v_jv_{j+1}, v_{j+1}v_{j+2}, v_{j+2}v_{j+3}, z_1v_j, z_1v_{j+2}, z_1z_2, z_2v_{j+1}, z_2v_{j+3}\}$, for $j \geq i+2$. Without loss of generality, let W and $G[E']$ be two attachments for P. Since both $v_{i-3}v_{i-2}v_{i-1}$ and $v_{i-3}v_{i-2}v'_{i-1}$ are forced 2-paths, then we have $c(v_{i-3}v_{i-2}) \neq c(v_{i-2}v_{i-1}) = c(v_{i-2}v'_{i-1})$. Similarly, we have $c(v_{j-1}v_j) \neq c(v_jz_1) = c(v_jv_{j+1})$. Clearly, there is not a strong conflict-free path between v_{i-2} and v_{j+1}. A contradiction. Hence, there is at most one attachment for P. Suppose that the path P is not contained in a cycle. Then we are concerned about the path between v_{i-2} and v_{i+3}. Clearly, the paths $v_iv_{i+1}v_{i+2}$ and $v_{i+1}v_{i+2}v_{i+3}$ are forced 2-paths. Thus, we have $c(v_iv_{i+1}) = c(v_{i+2}v_{i+3}) \neq c(v_{i+1}v_{i+2}) = c(v_{i-2}v_{i-1})$. Clearly, there is not a strong conflict-free path between v_{i-2} and v_{i+3}. A contradiction. Suppose that the path P is contained in a cycle. If we identify v_{i-3} with v_{i+1}, then $G = M_6$ with $spc(M_6) = 2$ by Theorem 5.7. Now we consider that a shortest cycle C contains P. Clearly, $|C| \geq 6$, otherwise, P does not contain an attachment. Suppose $|C| = 6$. Then there are two vertices u_1 and u_2 except the vertices of the attachment in C. If u_1 and u_2 are not adjacent to the same neighbor, then every pair of edges incident with u_1 is a forced 2-path. Hence, there need at least three colors. A contradiction. Let x be a common neighbor of u_1 and u_2, where u_2 is adjacent to v_{i+1}. Let y be a neighbor of x. Let z be another neighbor of y except x. Thus, $v_{i+1}u_2xyz$ is a unique forced path for the pair v_{i+1} and z. Then it is not a strong conflict-free path by Lemma 5.2. Suppose $|C| = 7$. Let

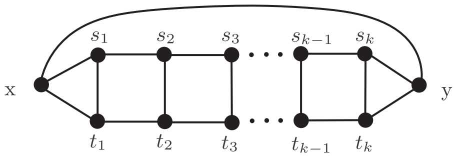

Fig. 7. The graph $F_0(k)$.

u_1, u_2 and u_3 be three vertices except the vertices of the attachment in C. If each of $\{u_1, u_2, u_3\}$ is contained in a triangle, then $G \cong U$ (*see* Fig. 5). If one of u_1, u_2 and u_3 is in a triangle, then there exists a unique forced 4-path for a pair of vertices in C, a contradiction. Let $C = v_1v_2v_3v_4u_1u_2u_3u_4v_1$. Suppose that there are two attachments in C. Then $G \cong L_2$ (*see* Fig. 10) with an edge-coloring such that $scfc(G) = spc(G) = 2$. Suppose that u_1, u_2, u_3 and u_4 are contained in triangles. Then $G \cong L_3$ (*see* Fig. 10) such that $scfc(G) = spc(G) = 2$. Otherwise, there will exist a unique forced 4-path for a pair of vertices in C, a contradiction. Suppose that at most one triangle contains two of the vertices u_1, u_2 u_3 and u_4, without loss of generality, say u_1, u_2. Suppose further that u_3u_4 is a forced edge. Then $c(v_1u_4) \neq c(u_4u_3) \neq c(u_4x)$, where x is a neighbor of u_4 except v_1, u_3, a contradiction. Then suppose that both u_3 and u_4 are contained a 4-cycle C'. Clearly, there induces a unique forced 4-path for the pair of v_2 and one vertex in C' except u_3, u_4, a contradiction. Suppose $9 \leq |C| \leq 10$. Then there is a unique forced 4-path for some pair of vertices in G. Hence, $scfc(G) \geq 3$. Assume $|C| \geq 11$. There exists a unique shortest path of length 5 between v_{i-3} and v_{i+2}, thus, $scfc(G) \geq 3$. A contradiction by Claim 1. □

Theorem 5.7 [10]**.** *Let G be a cubic graph without forced edges. Suppose further that $G \neq K_4$. Then $spc(G) = 2$ if and only if $G \in \{C_3\Box K_2, C_{2k}\Box K_2, M_{2k}\}$ for some $k \geq 2$.*

Theorem 5.8. *Let G be a cubic graph without forced edges. Then $scfc(G) = 2$ if and only if $G \in \{C_l\Box K_2, M_{2k}\}$ for $l \in \{3, 4, 6\}$ and for k with $3 \leq k \leq 7$.*

Proof. *Sufficiency.* By Lemma 5.4 and 5.5, Clearly, $scfc(G) = 2$.

Necessity. Suppose $G \ncong U$. If $scfc(G) = 2$, then we have $spc(G) = 2$ by Lemma 5.6. Furthermore, we have $G \in \{C_3\Box K_2, C_{2k}\Box K_2, M_{2k}\}$ for some $k \geq 2$ from Theorem 5.8. Then $G \in \{C_l\Box K_2, M_{2k}\}$ for $l \in \{3, 4, 6\}$ and for k with $3 \leq k \leq 7$ by Lemma 5.4 and 5.5. □

Let $F_0(k)$ be the cubic graph which is obtained from L_k by adding two new vertices x and y and adding five new edges xy, xs_1, xt_1, ys_k, yt_k (*see* Fig. 7).

Lemma 5.9. *$scfc(F_0(k)) = 2$ with $k \geq 2$ if and only if $k \in \{2, 4\}$.*

Proof. When $k \geq 3$, the cycle $xs_1s_2\cdots s_kyx$, say C, is a forced one in $F_0(k)$. Then we have that $k = 4$ by Lemma 5.3. When $k = 2$, we define an edge-coloring c for $F_0(k)$: $c(xy) = 2$; $c(xs_1) = c(xt_1) = c(ys_k) = c(yt_k) = 1$; $c(s_is_{i+1}) = c(t_it_{i+1}) = c(s_it_i) = 1$ for even $i \in [k]$; for all the remaining edges, $c(s_is_{i+1}) = c(t_it_{i+1}) = c(s_it_i) = 2$ for odd $i \in [k]$. We can easily check that every pair of vertices have a strong conflict-free path connecting them. Since $scfc(F_0(k)) > 1$, we have that $scfc(F_0(k)) = 2$ for $k = 2$ or 4. □

We now introduce a family $\mathcal{H}$ of graphs which will be used in the latter proof (*see* Fig. 8).

$$\mathcal{H} = \{F_0^*(k), \hat{K_4}, \hat{D_3}, \tilde{K_{3,3}}, \tilde{Q_3}, F_1(k)\}(k \in \mathbb{N})$$

Theorem 5.10 [10]**.** *Let G be a cubic graph with exactly one forced edge. Then $spc(G) = 2$ if and only if $G = F_0(k)$ for some even $k \geq 4$, or G is obtained from H_1 and H_2 by identifying the pendent edges to a single edge, where $H_i \in \{\hat{K_4}, \hat{D_3}\}$ for $i = 1, 2$.*

Lemma 5.11. *Let G be a cubic graph. If G is obtained from H_1 and H_2 by identifying the pendent edges to a single edge, where $H_i \in \{\hat{K_4}, \hat{D_3}\}$ for $i = 1, 2$, then scfc(G)=2 if and only if $H_i = \hat{K_4}$ for $i = 1, 2$.*

Proof. *Sufficiency.* If G is obtained from two graphs $\hat{K_4}$ by identifying the pendent edges to a single edge, then we know that $scfc(G) = 2$ by the coloring of Fig. 9.

Necessity. Suppose $scfc(G)$=2. If G is constructed by identifying the pendent edge of H_1 with $H_2 \in \{\hat{K_4}, \hat{D_3}\}$ (*see* Fig. 8), then, clearly, there is a forced 4-path which needs three colors to make it strong conflict-free connected. Thus we have $scfc(G) \geq 3$. A contradiction by Lemma 5.2. when $H_1 = H_2 = \hat{K_4}$, it is clear that $scfc(G) > 1$. On the contrary, we have $scfc(G) \leq 2$ under the edge-coloring in Fig. 9. □

Theorem 5.12. *Let G be a cubic graph with exactly one forced edge. Then $scfc(G) = 2$ if and only if $G \cong F_0(k)$ for $k \in \{2, 4\}$ or $G \cong N$.*

Proof. *Sufficiency.* From Lemma 5.9 and 5.11, we have $scfc(G) = 2$ for $G \cong F_0(k)$ for $k \in \{2, 4\}$ or $G \cong N$.

Necessity. Suppose that $scfc(G) = 2$. From Lemma 5.6, it follows that $spc(G) = 2$. Furthermore, we then have $G \cong F_0(k)$ for $k \in \{2, 4\}$ or $G \cong N$ by Theorem 5.10 and by 5.11. □

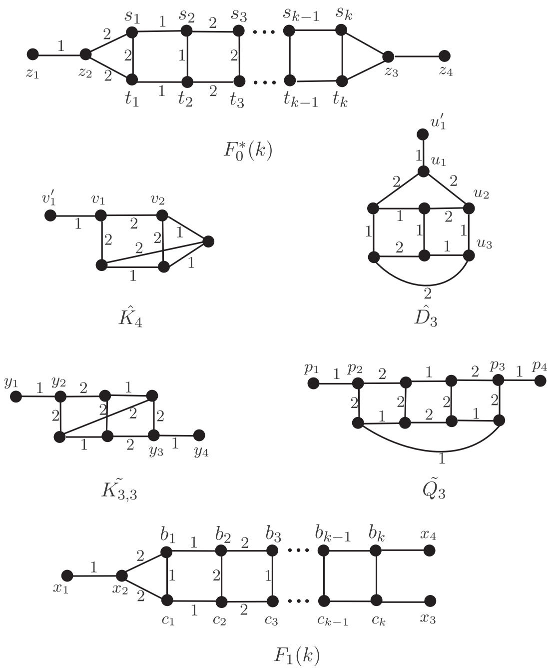

Fig. 8. The graph family $\mathcal{H}$.

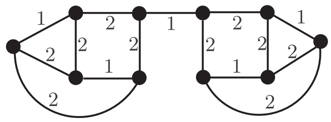

Fig. 9. The graph N.

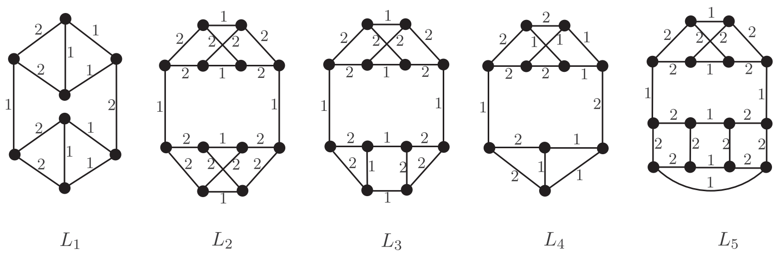

Fig. 10. The graph class $\mathcal{L}$.

Before proceeding, we need one more definition.

Definition 5.13 [10]**.** Let G be a connected graph. The *forced graph* of G is obtained from G by replacing each forced edge uv (if any) by two pendant edges uu' and vv', where u' and v' are two new vertices with respect to the forced edge uv. Each component of the forced graph of G is called a *forced branch* of G, and the new pendant edge uu' in the forced branch is called a *forced link* of G. For each forced edge uv of G, we call uu' and vv' the *twin links* corresponding to the forced edge uv. In the case that a forced link uu' and its twin link vv' are contained in a common forced branch of G, we say that uu' is a *selfish link*.

Theorem 5.14 [10]**.** *Let G be a cubic graph containing at least two forced edges, and let $H_1, H_2, \cdots, H_r$ be the forced branches of G. Then $spc(G) = 2$ if and only if $H_i \in \mathcal{H}$ for $i = 1, 2, \cdots, r$, and there are 2-SPC (strong proper connection number being 2) patterns $p_1, p_2, \cdots, p_r$ of $H_1, H_2, \cdots, H_r$, respectively, such that each pair of twin links receive the same color.*

Remark. The definition of *pattern* in Theorem 5.14 can be referred to [10].

Theorem 5.15. *Let G be a cubic graph containing at least two forced edges, and let $H_1, H_2, \cdots, H_r \in \mathcal{H}$ be the forced branches of G. Then scfc(G)=2 if and only if $G \in \mathcal{L}$, demonstrated in Fig. 10.*

Proof. *Necessity.* Since every graph $L_i \in \mathcal{L}$ is not a complete graph, then $scfc(L_i) \geq 2$. Also, we have $scfc(L_i) \leq 2$ by the edge coloring depicted in Fig. 10. Hence, $scfc(L_i) = 2$ for each $L_i \in \mathcal{L}$.

Sufficiency. Let G^* be the *forced graph* of G. If $F_0^*(k) \subset G^*$ or $F_1(k) \subset G^*$, then $k \leq 2$. Otherwise, there is a forced 4-path which needs at least three colors to make it strong conflict-free connected. Clearly, G^* contains at least two forced branches since G contains at least two forced edges.

Suppose that there exists a forced edge which is a cut-edge in G. It is clear that $\hat{D_3} \not\subseteq G^*$ or $\hat{K_4} \not\subseteq G^*$. Suppose $\hat{D_3} \subset G^*$. Since there is a forced 3-path $u_1' u_1 u_2 u_3$, then identify the pendent edge of any one graph in $\mathcal{H}$ with the pendent edge $u_1' u_1 \in E(\hat{D_3})$ will induce a forced 4-path which needs at least three colors. Hence, $\hat{D_3} \not\subseteq G^*$. Suppose $\hat{K_4} \subset G^*$. Clearly, there are at least two $\hat{K_4}$ since G contains at least two forced edges. For each graph $H' \in \{F_0^*(k), \tilde{K_{3,3}}, \tilde{Q_3}, F_1(k)\}$ $(k \in \mathbb{N})$, if identify the pendent edges $e_1, e_2 \in E(H')$ with each pendent edge of two $\hat{K_4}$, then, clearly, there are two forced 2-paths between v_2 and the copy of v_2, which needs at least three colors to make the path (which contains the two forced 2-paths) strong conflict-free connected. Hence, there does not exist a forced edge which is a cut-edge in G.

Suppose that every forced edge is not one cut-edge in G. Clearly, we have $\hat{D_3} \not\subseteq G^*$ and $\hat{K_4} \not\subseteq G^*$. There is no selfish link in G^* since G contains at least two forced edges.

Claim 1: If each connected component of G^* belongs to $\{\tilde{Q_3}, \tilde{K_{3,3}}, F_0^*(k)\}$ for $k \leq 2$, then there are at most two connected components in G^*.

Proof of Claim 1: Assume, to the contrary, that there are three connected components in G^*. Since each component of G^* contains exactly two pendant edges, then the forced edges are contained in the same cycle. Clearly, both each pendant edge of the connected components and its each adjacent edge form a forced 2-path. It means that there does not exist a strong conflict-free path containing two forced 2-paths, between two forced edges. Hence, $scfc(G) \geq 3$. A contradiction. Completing the proof of Claim 1. □

Claim 2: There is at most one copy of $\tilde{Q_3}$ in G^*.

Proof of Claim 2: Assume, to the contrary, that there are two copies of $\tilde{Q_3}$ in G^*. Clearly, the forced edges are contained in a cycle of length at least 8. Thus, there is exactly a forced 4-path between p_2 and p_4. Clearly, $scfc(G) \geq 3$. Completing the proof of Claim 2. □

Claim 3: There is no the copy of $F_1(k)$ in G^*.

Proof of Claim 3: Assume that there is a connected component $F_1(k)$ in G^*. Clearly, there are at least two connected components in G^*. Then there must exist also another one copy of $F_1(k)$ in G^* since there are three pendant edges in $F_1(k)$. Since both each pendant edge of the connected components and its each adjacent edge form a forced 2-path, there does not exist a strong conflict-free path containing two forced 2-paths, between two forced edges. This completes the proof of Claim 3. □

Then from Claim 1, Claim 2 and Claim 3, we can check by enumeration that $G \in \mathcal{L}$. □

Finally, Combining Theorems 5.8, 5.12 and 5.15, we have our main theorem of this section.

Theorem 5.16. *Let G be a cubic graph. Then $scfc(G) = 2$ if and only if*

$$G \in \mathcal{L} \text{ or } G \in \{N, C_l \Box K_2, M_{2r}, F_0(k)\},$$

where $l \in \{3, 4, 6\}$, $3 \leq r \leq 7$ *and* $k \in \{2, 4\}$.

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