ON ARC-TRANSITIVE CAYLEY DIGRAPHS OF OUT-VALENCY 3

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Abstract. In this paper, based on the calculation using GAP, we give a classification result on arc-transitive Cayley digraphs of finite simple groups. Let G be a finite simple group and $S \subset G \setminus \{1\}$ with $|S| = 3$, $S \neq S^{-1}$ and $G = \langle SS^{-1} \rangle$. If the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ is arc-transitive, then Γ is either normal or isomorphic to one of 382 Cayley digraphs of the alternating group A_{47}. Further, we consider the underlying graphs and standard double covers of these 382 Cayley digraphs, and then we get 172 (non-isomorphic) half-transitive Cayley graphs of valency 6, and 144 semisymmetric cubic graphs.

Keywords. Cayley digraph, arc-transitive digraph, alternatingly connected digraph, simple group.

1. introduction

In this paper, all graphs and digraphs are assumed to be finite and simple.

Let V be a nonempty set. A digraph (directed graph) Γ on V is a pair (V, A) with A a set of ordered pairs of distinct elements in V, where the elements in V and A are called vertices and arcs, respectively. For the case where A is self-paired, that is, $A = A^* := \{(v, u) \mid (u, v) \in A\}$, the digraph Γ gives an (undirected) graph on V with edge set $\{\{u, v\} \mid (u, v) \in A\}$, and vice versa.

Let $\Gamma = (V, A)$ be a digraph. For vertices $u, v \in V$, an *alternating walk* between u and v means a sequence $u = v_0\, v_1\, \ldots\, v_{2l} = v$ of odd number of vertices such that $(v_{2i}, v_{2i+1}), (v_{2i+2}, v_{2i+1}) \in A$ for $0 \leq i \leq l-1$. The digraph Γ is said to be *alternatingly connected* if there exists an alternating walk between every pair of vertices. Note that an alternatingly connected digraph must be connected, and that a graph (self-paired digraph) is alternatingly connected if and only if it is connected but not bipartite. Let $\mathsf{Aut}\Gamma$ be the automorphism group of Γ. For a subgroup $X \leq \mathsf{Aut}\Gamma$, the digraph Γ is said to be X-*arc-transitive* if X acts transitively on both V and A. In this paper we deal with alternatingly connected arc-transitive Cayley digraphs.

Let G be a finite group, and S a subset of G which does not contain the identity 1 of G. The Cayley digraph $\mathrm{Cay}(G, S)$ on G with respect to S is the digraph with vertex set $u(G)$, the underlying set of G, and arc set $\{(u(g), u(sg)) \mid g \in G,\ s \in S\}$. Clearly, all vertices of $\mathrm{Cay}(G, S)$ have the same out-valency $|S|$. If S is inverse-closed, that is,

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$S = S^{-1} := \{s^{-1} \mid s \in S\}$, then $\mathrm{Cay}(G, S)$ can be viewed as an (undirected) graph; in this case, we call it a Cayley graph. By the definition of $\mathrm{Cay}(G, S)$, each $g \in G$ induces an automorphism of $\mathrm{Cay}(G, S)$ by right multiplication, say

$$g : u(G) \to u(G);\ u(x) \mapsto u(xg).$$

Then the group G can be viewed as a regular subgroup of $\mathsf{Aut}\mathrm{Cay}(G, S)$. In particular, $\mathrm{Cay}(G, S)$ is vertex-transitive, that is, $\mathsf{Aut}\mathrm{Cay}(G, S)$ acts transitively on the vertex set of $\mathrm{Cay}(G, S)$.

Another obvious subgroup of $\mathsf{Aut}\mathrm{Cay}(G, S)$ consists of automorphisms of the group G which fix S set-wise, say

$$\mathsf{Aut}(G, S) := \{\alpha \in \mathsf{Aut}(G) \mid S = S^{\alpha}\},$$

which acts on $u(G)$ by

$$u(g)^{\alpha} = u(g^{\alpha});\ g \in G,\ \alpha \in \mathsf{Aut}(G, S).$$

By [11, Lemma 2.1], the normalizer of G in $\mathsf{Aut}\mathrm{Cay}(G, S)$ is

$$\mathbf{N}_{\mathsf{Aut}\mathrm{Cay}(G,S)}(G) = G{:}\mathsf{Aut}(G, S),$$

a semidirect product of G by $\mathsf{Aut}(G, S)$. In general, $\mathbf{N}_{\mathsf{Aut}\mathrm{Cay}(G,S)}(G)$ is not equal to $\mathsf{Aut}\mathrm{Cay}(G, S)$. For example, considering the complete graph K_n as a Cayley graph of the cyclic group $\mathbb{Z}_n$, we have an example with $\mathbf{N}_{\mathsf{Aut}\mathrm{Cay}(G,S)}(G) \neq \mathsf{Aut}\mathrm{Cay}(G, S)$.

A Cayley digraph $\mathrm{Cay}(G, S)$ is said to be *normal* with respect to G if G is a normal subgroup group of $\mathsf{Aut}\mathrm{Cay}(G, S)$, i.e., $\mathbf{N}_{\mathsf{Aut}\mathrm{Cay}(G,S)}(G) = \mathsf{Aut}\mathrm{Cay}(G, S)$. In 1998, Xu [27] conjectured that almost Cayley digraphs are normal. This stimulates the study of the following natural problem: *Describe the pairs* (G, S) *such that* $\mathrm{Cay}(G, S)$ *is not normal*, particularly, do this under certain restrictions on G or S such as restrictions on the structure of G or on the size of S, etc.. This problem has received considerable attention in the literature, see for example [5, 6, 8, 9, 10, 15, 17, 20, 22, 28, 30].

Let G be a finite nonabelian simple group and S a generating subset of G with $|S| = 3$. For the case where S is inverse-closed, Li [16] proved that if $\mathrm{Cay}(G, S)$ is arc-transitive then it is normal unless G is one of the seven exceptions: A_5, $\mathrm{PSL}(2, 11)$, M_{11}, A_{11}, M_{23}, A_{23} and A_{47}. Xu et al. [28, 29] improved Li's result by showing that the only exception is A_{47} and, up to isomorphism, there are two non-normal arc-transitive Cayley graphs on A_{47}. In 2009, Li and Lu [22] gave a complete classification of core-free arc-transitive Cayley graphs of valency 3, which covers the main results in [28, 29]. For the directed case, that is, $S \neq S^{-1}$, Fang at al. [8] proved that if G is generated by $\{st^{-1} \mid s, t \in S\}$ (equivalently, $\mathrm{Cay}(G, S)$ is alternatingly connected) and $\mathrm{Cay}(G, S)$ is arc-transitive then $\mathrm{Cay}(G, S)$ is normal unless G is one of a finite number of exceptions. In Section 3 of the present paper, we improve the above result by proving that the only exception is A_{47}, and then we give a classification of non-normal arc-transitive Cayley digraph of out-valency 3 on finite simple groups.

Recall that, for a group X and a subgroup $Y \leq X$, the core $\mathsf{Core}_X(Y)$ of Y in X is the intersection of all conjugations of Y in X, i.e., $\mathsf{Core}_X(Y) = \cap_{x \in X} Y^x$. The subgroup Y is said to be core-free in X if $\mathsf{Core}_X(Y) = 1$. For a subset D of X, write $DD^{-1} = \{xy^{-1} \mid x, y \in D\}$, and let $\langle DD^{-1}\rangle$ be the subgroup of X generated by DD^{-1}.

Theorem 1.1. *Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$, $S^{-1} \neq S$ and $\langle SS^{-1}\rangle = G$. Let $\Gamma = \mathrm{Cay}(G, S)$ and $G \leq X \leq \mathsf{Aut}\Gamma$. Assume that Γ is X-arc-transitive. If G is core-free in X, then either*

(1) *$G \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$, $\mathsf{Aut}\Gamma = X \cong \mathrm{PSL}(2, 11)$, and Γ is unique up to isomorphism; or*
(2) *$G = \mathrm{A}_{47}$, $\mathsf{Aut}\Gamma = X = \mathrm{A}_{48}$ and Γ is isomorphic to one of 382 Cayley digraphs of A_{47}.*

Remark. Assume that Γ is a connected arc-transitive digraph of out-valency 3, and H is the stabilizer in $\mathsf{Aut}\Gamma$ of some vertex. Then $|H| = 2^e3^f$ for some integers $e \geq 0$ and $f \geq 1$. For the case where Γ is self-paired, $e \leq 3$ and $f = 1$, refer to [1, 18c]. If Γ is a normal Cayley digraph then it is easily shown that $e \leq 1$ and $f = 1$. *In general, are there upper bounds for e and f*? This may be an interesting problem.

Assume further that Γ is a Cayley digraph $\mathrm{Cay}(G, S)$ with $S^{-1} \neq S$. Under the assumption that $G = \langle SS^{-1}\rangle$, the order of H is a divisor of 48. Thus, if further G is core-free in $\mathsf{Aut}\Gamma$ then we can consider $\mathsf{Aut}\Gamma$ as a subgroup of the symmetric group of some small degree, which has a point-stabilizer G and a regular subgroup H. This allows us get the classification in Theorem 1.1 successfully, see Section 3 for the details. $\square$

Note that, for a finite simple group G and $S \subseteq G \setminus \{1\}$, either the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ is normal or G is core-free in $\mathsf{Aut}\Gamma$. The following result is a direct consequence of Theorem 1.1.

Corollary 1.2. *Let G be a finite simple group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$, $S^{-1} \neq S$ and $\langle SS^{-1}\rangle = G$. Assume that $\Gamma = \mathrm{Cay}(G, S)$ is arc-transitive. If Γ is not normal then $G = \mathrm{A}_{47}$, $\mathsf{Aut}\Gamma = \mathrm{A}_{48}$ and Γ is isomorphic to one of 382 Cayley digraphs of A_{47}.*

Recall that a regular graph is semisymmetric if it is edge-transitive but not vertex-transitive, and half-transitive if it is both vertex-transitive and edge-transitive but not arc-transitive. In Sections 4 and 5, we investigate the underlying graphs and the standard double covers of the 382 Cayley digraphs on A_{47}. As a result, we get some examples of half-transitive graphs and semisymmetric graphs.

Theorem 1.3. *Let $\mathcal{G}$ be the set of 382 digraphs involved in Corollary 1.2. For $\Gamma = \mathrm{Cay}(G, S) \in \mathcal{G}$, we let $\underline{\Gamma} = \mathrm{Cay}(G, S \cup S^{-1})$, and let $\Gamma^{(2)}$ be the bipartite graph on $u(G) \times \mathbb{Z}_2$ such that $\{(u(h)), 0), (u(g), 1)\}$ is an edge if and only if $gh^{-1} \in S$.*

(1) *If $\Gamma \in \mathcal{G}$ then $\mathsf{Aut}\underline{\Gamma} = \mathrm{A}_{48}$ or $\mathbb{Z}_2 \times \mathrm{A}_{48}$, and $\underline{\Gamma}$ is arc-transitive if and only if $\mathsf{Aut}\underline{\Gamma} \cong \mathbb{Z}_2 \times \mathrm{A}_{48}$. In particular, when Γ runs over $\mathcal{G}$, there are 38 arc-transitive graphs $\underline{\Gamma}$, and 172 half-transitive graphs $\underline{\Gamma}$.*
(2) *If $\Gamma \in \mathcal{G}$ then $\mathsf{Aut}\Gamma^{(2)} = \mathrm{A}_{48}$ or $\mathbb{Z}_2 \times \mathrm{A}_{48}$, and $\Gamma^{(2)}$ is arc-transitive if and only if $\mathsf{Aut}\Gamma^{(2)} \cong \mathbb{Z}_2 \times \mathrm{A}_{48}$. In particular, when Γ runs over $\mathcal{G}$, there are 66 arc-transitive cubic graphs $\Gamma^{(2)}$ and 144 semisymmetric cubic graphs $\Gamma^{(2)}$.*

Remark. *In Theorem 1.3 (1), the* 210 *graphs are not isomorphic to every other. According to vertex-transitivity, the* 210 *graphs in Theorem 1.3 (2) are divided into two subdivisions; however, we do not know the isomorphisms among the members in a same subdivision.*

2. Preliminaries

Let $\Gamma = (V, A)$ be a digraph. For $u \in V$, set $\Gamma^+(u) = \{v \mid v \in V, (u, v) \in A\}$ and $\Gamma^-(u) = \{v \mid v \in V, (v, u) \in A\}$. The sizes $|\Gamma^+(u)|$ and $|\Gamma^-(u)|$ are the out-valency and in-valency of u in Γ, respectively. Note that, for a graph (self-paired digraph) Γ, we have $\Gamma^+(u) = \Gamma^-(u) := \Gamma(u)$; in this case, $\Gamma(u)$ is called the neighborhood of u, and $|\Gamma(u)|$ called the valency of u. For $X \leq \mathsf{Aut}\Gamma$ and $u \in V$, set $X_u = \{x \in X \mid u^x = u\}$, called the stabilizer of u in X.

2.1. **Alternatingly connected digraphs.** Let Γ be a digraph. If every vertex of Γ has positive out-valency and in-valency, then it is easily shown that the following two statements are equivalent:

- (a1) Γ is alternatingly connected;
- (a2) For every pair of vertices u and v, there is a sequence $u = v_0\, v_1\, \ldots\, v_{2l} = v$ of odd number of vertices such that $(v_{2i+1}, v_{2i}), (v_{2i}, v_{2i+2}) \in A$ for $0 \leq i \leq l - 1$.

We next assume that $\Gamma = (V, A)$ is a digraph with $A \neq \emptyset$, and let $X \leq \mathsf{Aut}\Gamma$ such that Γ is X-vertex-transitive, that is, X is transitive on V. Then the above (a1) and (a2) are equivalent for Γ. Fix a vertex $u \in V$. Then $V = \{u^x \mid x \in X\}$.

Let $X_u = \{x \in X \mid u^x = u\}$, the stabilizer of u in X, and $D = \{x \mid x \in X, u^x \in \Gamma^+(u)\}$. Then it is easily shown that D is a union of some double cosets $X_u x X_u$ of X_u in X, and Γ is X-arc-transitive if and only if D is a single double coset. Clearly, since we consider only simple digraphs, D does not contain the identity of X. It is well-known and easily shown that Γ is connected if and only if $X = \langle H, D\rangle$.

Consider the subgroups of X generated by DD^{-1} and $D^{-1}D$. Then, for $x, y \in D$, we have $x\langle D^{-1}D\rangle y^{-1} \subseteq \langle DD^{-1}\rangle$ and $\langle D^{-1}D\rangle \supseteq x^{-1}\langle DD^{-1}\rangle y$. This implies that $\langle D^{-1}D\rangle$ and $\langle DD^{-1}\rangle$ have the same order. Thus $X = \langle D^{-1}D\rangle$ if and only if and $X = \langle DD^{-1}\rangle$.

Lemma 2.1. *Let $\Gamma = (V, A)$ be an X-vertex-transitive digraph, $u \in V$, $H = X_u$ and $D = \{x \mid x \in X, u^x \in \Gamma^+(u)\}$. Then Γ is alternatingly connected if and only if $X = \langle DD^{-1}\rangle = \langle D^{-1}D\rangle$. In particular, if $D = HxH$ for some $x \in G \setminus H$, then Γ is alternatingly connected if and only if $X = \langle H, H^x\rangle$.*

Proof. Assume that Γ is alternatingly connected. For each $x \in X$, take an alternating walk $u = v_0\, v_1\, \ldots\, v_{2l} = u^x$. Set $v_i = u^{y_i}$ for $y_i \in X$ with $y_0 = 1$ and $y_{2l+1} = x$. Then $(u^{y_{2i}}, u^{y_{2i+1}}), (u^{y_{2i+2}}, u^{y_{2i+1}}) \in A$, i.e., $y_{2i+1}y_{2i}^{-1}, y_{2i+1}y_{2i+2}^{-1} \in D$, where $0 \leq i \leq l - 1$. Thus

$$x = y_{2l} = (y_{2l}y_{2l-1}^{-1})(y_{2l-1}y_{2l-2}^{-1}) \cdots (y_{2i+2}y_{2i+1}^{-1})(y_{2i+1}y_{2i}^{-1}) \cdots (y_2y_1^{-1})(y_1y_0^{-1}) \in \langle D^{-1}D\rangle.$$

Then we have $X = \langle D^{-1}D\rangle = \langle DD^{-1}\rangle$.

Let $X = \langle D^{-1}D\rangle$. For $v \in V$, take $x \in X$ with $u^x = v$. Write $x = c_{l-1}^{-1}d_{l-1}\cdots c_0^{-1}d_0$, where $c_i, d_i \in D$. For $1 \le i \le l$, let

$$v_{2i-1} = u^{d_{i-1}c_{i-2}^{-1}d_{i-2}\cdots c_0^{-1}d_0},\ v_{2i} = u^{c_{i-1}^{-1}d_{i-1}\cdots c_0^{-1}d_0}.$$

Then, $v_{2l} = u^x = v$, and we get an alternating walk $u = v_0\, v_1\, \ldots\, v_{2l} = v$.

For $D = HxH$, we have $\langle DD^{-1}\rangle = \langle Hx^{-1}HxH\rangle = \langle H, H^x\rangle$. Then the second part of this lemma follows. $\square$

If X has a regular subgroup then we have the following simple fact.

Lemma 2.2. *Let Γ be an X-vertex-transitive digraph, $u \in V$, $H = X_u$ and $D = \{x \mid x \in X,\ u^x \in \Gamma^+(u)\}$. If $G \le X$ with $G \cap H = 1$ and $X = GH$, then $\Gamma \cong \mathrm{Cay}(G, S)$ with $S = G \cap D$, and Γ is alternatingly connected if and only if $G = \langle SS^{-1}\rangle$.*

By [18, Lemma 2.3], we have the next result.

Lemma 2.3. *Let $\Gamma = (V, A)$ and $\Sigma = (V, A')$ be two X-vertex-transitive digraphs on V, $u \in V$, $H = X_u$, $D = \{x \mid x \in X,\ u^x \in \Gamma^+(u)\}$ and $D' = \{x \mid x \in X,\ u^x \in \Sigma^+(u)\}$. Assume that $\mathsf{Aut}\Gamma = X = \mathsf{Aut}\Sigma$. Then $\Gamma \cong \Sigma$ if and only if $D' = D^\sigma$ for some $\sigma \in \mathsf{Aut}(X)$ with $H^\sigma = H$.*

Let $\Gamma = (V, A)$ be an alternatingly connected X-arc-transitive digraph of out valency 3. Consider the standard double cover $\Gamma^{(2)}$ of Γ, which is the graph on $V \times \mathbb{Z}_2$ such that $\{(u, 0), (v, 1)\}$ is an edge if and only if $(u, v) \in A$. Then $\Gamma^{(2)}$ is a connected bipartite cubic graph, and X has an action on the vertex set of $\Gamma^{(2)}$ by

$$(u, i)^x = (u^x, i);\ x \in X, u \in V, i \in \mathbb{Z}_2.$$

It is easily shown that X acts transitively on the edge set but not on the vertex set of $\Gamma^{(2)}$. Noting that the stabilizers $X_{(u,0)} = X_u$ and $X_{(v,1)} = X_v$ are conjugate in X, by [12], we can determine the structures of X_u and X_v.

Lemma 2.4. *Let $\Gamma = (V, A)$ be an alternatingly connected X-arc-transitive digraph of out valency 3. If $(u, v) \in A$ then $X_u \cong X_v \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathbb{Z}_2 \times \mathrm{S}_4$.*

Lemma 2.5. *Let $\Gamma = (V, A)$ be an alternatingly connected X-arc-transitive digraph of out valency 3, $(u, v) \in A$ with $(v, u) \notin A$, $X_{uv} = X_u \cap X_v$, and $v = u^y$ for some $y \in X$. Then $\mathbf{N}_X(X_{uv})/X_{uv}$ is not an elementary 2-group, and $\mathbf{N}_X(X_{uv})$ contains an element z such that $X_u y X_u = X_u z X_u$ and $X = \langle X_u, X_u^z\rangle$.*

Proof. Let $w = u^{y^{-1}}$. Then $w \in \Gamma^-(u)$. Since Γ is X-arc-transitive, X_u is transitive on both $\Gamma^+(u)$ and $\Gamma^-(u)$. This implies that $|X_u : X_{uv}| = |X_u : X_{uw}| = 3$, and so both X_{uv} and X_{uw} are Sylow 2-subgroups of X_u, and then $X_{uv} = X_{uw}^x$ for some $x \in X_u$. Thus

$$X_{uv} = X_{uw}^x = (X_u \cap X_u^{y^{-1}})^x = (X_u^y \cap X_u)^{y^{-1}x} = X_{uv}^{y^{-1}x},$$

yielding $y^{-1}x \in \mathbf{N}_X(X_{uv})$. Let $z = x^{-1}y$. Then $z \in \mathbf{N}_X(X_{uv})$, $X_u y X_u = X_u z X_u$, and $X = \langle X_u, X_u^y\rangle = \langle X_u, X_u^z\rangle$. Noting that $v = u^y = u^z$, if $z^2 \in X_{uv}$ then $(v, u) =$

$(u^z, u^{z^2}) = (u, u^z)^z = (u, v)^z \in A$, a contradiction. Thus $\mathbf{N}_X(X_{uv})/X_{uv}$ contains an element of order greater than 2, and so $\mathbf{N}_X(X_{uv})/X_{uv}$ is not an elementary 2-group. $\square$

2.2. **Vertex-stabilizers.** Let $\Gamma = (V, E)$ be a graph and $X \leq \mathsf{Aut}\Gamma$. Then, for $u \in V$, the stabilizer X_u induces a permutation group $X_u^{\Gamma(u)}$ (on $\Gamma(u)$). The next lemma is easily shown, see [3] for example.

Lemma 2.6. *Let $\Gamma = (V, E)$ be a connected regular graph, $X \leq \mathsf{Aut}\Gamma$ and $u \in V$. Assume that $X_u \neq 1$. Let p be a prime divisor of $|X_u|$. Then $p \leq |\Gamma(u)|$. If further Γ is X-vertex-transitive, then p divides $|X_u^{\Gamma(u)}|$ and, for $v \in \Gamma(u)$, each prime divisor of $|X_{uv}|$ is less than $|\Gamma(u)|$.*

For a positive integer s, an *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of vertices such that $v_{i-1} \in \Gamma(v_i)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. For $X \leq \mathsf{Aut}\Gamma$, the graph Γ is said to be *(X, s)-arc-transitive* if it contains at least one s-arc and X acts transitively on both the vertex set and the set of s-arcs, and said to be *(X, s)-transitive* if it is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive.

It is well known that an X-vertex-transitive graph Γ is $(X, 2)$-arc-transitive if and only if X_v induces a 2-transitive permutation group $X_u^{\Gamma(u)}$ for some vertex u. By Lemma 2.6, if Γ is a connected X-arc-transitive cubic graph then either X_u has order 3 or Γ is $(X, 2)$-arc-transitive. The well-known result of Tutte determines X_u, refer to [1, 18c].

Theorem 2.7. *If $\Gamma = (V, E)$ is a connected X-arc-transitive cubic graph, then $X_u \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathrm{S}_4 \times \mathrm{S}_2$, where $u \in V$.*

Let $\Gamma = (V, E)$ be a connected graph of valency 6, $X \leq \mathsf{Aut}\Gamma$ and $u \in V$. Then, by Lemma 2.6, X_u is a $\{2, 3, 5\}$-group. Further, if $|X_u|$ is divisible by 5, then either $X_u^{\Gamma(u)}$ is a 2-transitive permutation group or $X_u = X_v$ for some $v \in \Gamma(u)$. Assume that X is transitive on both V and E. Then X_u either has two orbits of size 3 or acts transitively on $\Gamma(u)$. It follows that either X_u is a $\{2, 3\}$-group, or $X_u^{\Gamma(u)}$ is 2-transitive. Then, by Lemma 2.6 and [19], we have the following result.

Theorem 2.8. *Let $\Gamma = (V, E)$ be a connected graph of valency 6, $X \leq \mathsf{Aut}\Gamma$ and $u \in V$. Assume that X is transitive on both V and E. Then either X_u is a $\{2, 3\}$-group, or Γ is (X, s)-transitive with s and X_u listed in the following table.*

s	2	3	4
X_u	A_6, S_6 A_5, S_5	$\mathrm{A}_5 \times \mathrm{A}_6$, $(\mathrm{A}_5 \times \mathrm{A}_6).2$, $\mathrm{S}_5 \times \mathrm{S}_6$ $\mathrm{D}_{10} \times \mathrm{PSL}(2,5)$, $(5 \times \mathrm{PSL}(2,5)).2$ $\mathrm{D}_{10} \times \mathrm{PGL}(2,5)$, $(5{:}4) \times \mathrm{PGL}(2,5)$	$5^2{:}\mathrm{GL}(2,5)$

2.3. **Quotients.** In this subsection, we assume that $\Gamma = (V, E)$ is a connected graph, and $X \leq \mathsf{Aut}\Gamma$.

Suppose that X has a normal subgroup N which is intransitive on every X-orbit on V. Denote by $\mathcal{B}$ the set of N-orbits. The *quotient* graph Γ_N is defined on $\mathcal{B}$ such that distinct $B_1 \in \mathcal{B}$ and $\mathrm{B}_2 \in \mathcal{B}$ are adjacent if and only if some $u_1 \in B_1$ and some $u_2 \in B_2$ are adjacent in $\varGamma$. The graph $\varGamma$ is called a *normal cover* of $\varGamma_N$ if for every edge $\{B_1, B_2\}$ of $\varGamma_N$ the subgraph $[B_1, B_2]$ of $\varGamma$ induced by $B_1 \cup B_2$ is a complete matching. We collect in the following lemma some well-known facts on $\varGamma_N$.

Lemma 2.9. *Let $\varGamma = (V, E)$, X, N, $\mathcal{B}$ and $\varGamma_N$ be as above. Assume that $\varGamma$ is X-edge-transitive, that is, X is transitive on E. Then X has at most two orbits on V, every $B \in \mathcal{B}$ is an independent set of $\varGamma$, and $\varGamma_N$ is $X^{\mathcal{B}}$-edge-transitive. Moreover,*

(1) *if $u, v \in B \in \mathcal{B}$ then $|\varGamma(u)| = |\varGamma(v)|$ and $|\varGamma_N(B)|$ is a divisor of $|\varGamma(u)|$;*
(2) *$\varGamma$ is a normal cover of $\varGamma_N$ if and only if there is an edge $\{B_1, B_2\}$ of $\varGamma_N$ such that $|\varGamma_N(B_i)| = |\varGamma(u_i)|$ for $u_i \in B_i$ and $i = 1, 2$;*
(3) *if $\varGamma$ is a normal cover of $\varGamma_N$ then $X^{\mathcal{B}} \cong X/N$, N is regular on every $B \in \mathcal{B}$, and*
   (i) *$\varGamma$ is X-vertex-transitive if and only if $\varGamma_N$ is $X^{\mathcal{B}}$-vertex-transitive;*
   (ii) *$\varGamma$ is (X, s)-arc-transitive if and only if $\varGamma_N$ is $(X^{\mathcal{B}}, s)$-arc-transitive.*

Suppose that $\varGamma = (V, E)$ is X-half-transitive, that is, X is transitive on both V and E but not transitive on the arcs of $\varGamma$. Then X has two orbits on the arc set of $\varGamma$, and the following lemma holds, see [21, Theorem 4].

Lemma 2.10. *Assume that $\varGamma = (V, E)$ is a connected X-half-transitive graph. Let Δ be the digraph on V with arc set an X-orbit on the arcs of $\varGamma$. Suppose that Δ is alternatingly connected, and for $u \in V$, the stabilizer X_u acts primitively on both $\Delta^+(u)$ and $\Delta^-(u)$. Suppose that N is an intransitive normal subgroup of X (acting on V). Let $\mathcal{B}$ be the set of N-orbits on V. Then $X^{\mathcal{B}} \cong X/N$, every $B \in \mathcal{B}$ is an independent set of $\varGamma$, N is regular on every $B \in \mathcal{B}$, and either*

(1) *$\varGamma$ is a normal cover of $\varGamma_N$; or*
(2) *$\varGamma_N$ is $X^{\mathcal{B}}$-arc-transitive, non-bipartite and of valency $\frac{|\varGamma(u)|}{2}$.*

3. Core-free Cayley digraphs of out-valency 3

Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$, $S^{-1} \neq S$ and $G = \langle SS^{-1} \rangle$. Let $\varGamma = \mathrm{Cay}(G, S)$ and $G \leq X \leq \mathsf{Aut}\varGamma$. Assume that $\varGamma$ is X-arc-transitive.

3.1. **Conclusions based on GAP calculation.** In this subsection, we assume that G is core-free in X, and determine the triple (X, G, S).

Denoted by $u(g)$ the vertex of $\varGamma$ corresponding to $g \in G$. Let $u = u(1)$ and let $H = X_u$. Then $X = GH = HG$, $H \cap G = 1$, and H is core-free in X. For any given $x \in X$ and $g \in G$, the product gx can be written uniquely as hg' with $h \in H$ and $g' \in G$, and thus

$$u(g)^x = u(1)^{gh} = u(g'). \tag{3.1}$$

Consider the action of X on $\Omega = [X, G]$, the set of right cosets of G in X, by right multiplication. Then the kernel of this action is $\mathsf{Core}_X(G)$. Recall that G is core-free in X. Thus we may identify X with a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$. Noting that $|\Omega| = H$ and $|G| > 6$, we have $|H| > 3$. By Lemma 2.4,

$$H \cong \mathrm{S}_3,\ \mathrm{D}_{12},\ \mathrm{S}_4 \text{ or } \mathbb{Z}_2 \times \mathrm{S}_4. \tag{3.2}$$

Further, we may identify Ω with the set $\{1, 2, 3, \ldots, n\}$, where $n = |H|$. Then we have $X \leq \mathrm{S}_n$ and, without loss of generality, we let $G = X_n$, the stabilizer of the point n in X. Note that, for each $v \in \mathit{\Gamma}^+(u)$, the arc-stabilizer X_{uv} is a Sylow 2-subgroup of H. By Lemmas 2.1, 2.2 and 2.5, $S = G \cap HxH$, where x normalizes a Sylow 2-subgroup of H such that $HxH \neq Hx^{-1}H$ and $X = \langle H, H^x \rangle$.

By the above argument, our task is to find all possible $x \in \mathrm{S}_n$ which satisfies the following conditions:

(C1) x normalizes a Sylow 2-subgroup of H, and $HxH \neq Hx^{-1}H$;
(C2) $x \in \langle H, H^x \rangle$ and H is core-free in $\langle H, H^x \rangle$.

The following lemma is easily shown.

Lemma 3.1. *Let H be a regular subgroup of S_n with $H \cong \mathrm{S}_3, \mathrm{D}_{12}, \mathrm{S}_4$ or $\mathbb{Z}_2 \times \mathrm{S}_4$. If a permutation $x \in \mathrm{S}_n$ satisfies the conditions (C1) and (C2), setting $X = \langle H, H^x \rangle$, $G = X_n$ and $S = G \cap HxH$, then $|S| = 3$, $S = \{y \in HxH \mid n^y = n\} \neq S^{-1}$ and $\mathrm{Cay}(G, S)$ is alternatingly connected and X-arc-transitive, where X acts on $u(G)$ as in (3.1).*

Let $x \in \mathrm{S}_n$ satisfy (C1) and (C2). Assume that $x \in \mathbf{N}_{\mathrm{S}_n}(P)$ for a Sylow 2-subgroup P of H. For $y \in \mathrm{S}_n$ with $n^y = n$, we have $x^y \in \mathbf{N}_{\mathrm{S}_n}(P)^y = \mathbf{N}_{\mathrm{S}_n}(P^y)$, and

$$\mathrm{Cay}(G, G \cap HxH) \cong \mathrm{Cay}(G^y, G^y \cap H^y x^y H^y).$$

Let $y \in \mathbf{N}_{\mathrm{S}_n}(H)$ with $n^y = n$. Noting that P and P^y are Sylow 2-subgroups of H, we may let $P^y = P^h$ for some $h \in H$. Then $x^y = (x')^h$ for some $x' \in \mathbf{N}_{\mathrm{S}_n}(P)$. Thus

$$\mathrm{Cay}(G, G \cap HxH) \cong \mathrm{Cay}(G^y, G^y \cap H^y x^y H^y) = \mathrm{Cay}(G^y, G^y \cap Hx'H),$$

where $G^y \cap Hx'H = \{z \in Hx'H \mid n^z = n\}$. It is easily shown that any two Sylow 2-subgroups of H are conjugate under $(\mathbf{N}_{\mathrm{S}_n}(H))_n$. Thus when we work on x, up to isomorphism of digraphs, we may fix a regular subgroup H of S_n and a Sylow 2-subgroup P of H, and locate x in a set I_t satisfying the following conditions:

(C3) I_t is a set of representatives of right cosets of P in $\mathbf{N}_{\mathrm{S}_n}(P)$, and $HxH \neq Hx^{-1}H$ for each $x \in I_t$, where t is such that $n = 2^t 3$;
(C4) Distinct elements $x_1, x_2 \in I_t$ are not conjugate under $\mathbf{N}_{\mathrm{S}_n}(H)$, and $Hx_1H \neq (Hx_2H)^y$ for all $y \in \mathbf{N}_{\mathrm{S}_n}(H)$.
(C5) $\langle H, x \rangle = \langle H, H^x \rangle$, and H is core-free in $\langle H, H^x \rangle$ for each $x \in I_t$;

Further, by [22], we choose H and P as follows:

(1) For $n = 6$, $H = \langle a, b \rangle$ and $P = \langle b \rangle$, where a=(1 2 3)(4 5 6) and b=(1 5)(2 4)(3 6).

(2) For $n = 12$, $H = \langle a, b\rangle$ and $P = \langle a^3, b\rangle$, where $a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$ and $b = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7)$.

(3) For $n = 24$, $H = \langle a, b, c\rangle$ and $P = \langle a, c\rangle$, where

$$\begin{aligned} a &= (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)(13\,14\,15\,16)(17\,18\,19\,20)(21\,22\,23\,24),\\ b &= (1\,18)(2\,11)(3\,6)(4\,15)(5\,16)(7\,10)(8\,21)(9\,22)(12\,17)(13\,24)(14\,19)(20\,23),\\ c &= (1\,23)(2\,22)(3\,21)(4\,24)(5\,19)(6\,18)(7\,17)(8\,20)(9\,13)(10\,16)(11\,15)(12\,14). \end{aligned}$$

(4) For $n = 48$, $H = \langle a, b, c, d\rangle$ and $P = \langle a, b, d\rangle$, where

$$\begin{aligned} a= \ & (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10\ 11\ 12)(13\ 14\ 15\ 16)(17\ 18\ 19\ 20)(21\ 22\ 23\ 24)\\ & (25\ 26\ 27\ 28)(29\ 30\ 31\ 32)(33\ 34\ 35\ 36)(37\ 38\ 39\ 40)\\ & (41\ 42\ 43\ 44)(45\ 46\ 47\ 48),\\ b= \ & (1\ 8)(2\ 7)(3\ 6)(4\ 5)(9\ 16)(10\ 15)(11\ 14)(12\ 13)(17\ 24)(18\ 23)(19\ 22)\\ & (20\ 21)(25\ 32)(26\ 31)(27\ 30)(28\ 29)(33\ 40)(34\ 39)(35\ 38)\\ & (36\ 37)(41\ 48)(42\ 47)(43\ 46)(44\ 45),\\ c= \ & (1\ 17\ 33)(2\ 39\ 20)(3\ 24\ 38)(4\ 34\ 23)(5\ 37\ 21)(6\ 19\ 40)(7\ 36\ 18)\\ & (8\ 22\ 35)(9\ 25\ 41)(10\ 47\ 28)(11\ 32\ 46)(12\ 42\ 31)\\ & (13\ 45\ 29)(14\ 27\ 48)(15\ 44\ 26)(16\ 30\ 43),\\ d= \ & (1\ 9)(2\ 10)(3\ 11)(4\ 12)(5\ 13)(6\ 14)(7\ 15)(8\ 16)(17\ 25)(18\ 26)(19\ 27)\\ & (20\ 28)(21\ 29)(22\ 30)(23\ 31)(24\ 32)(33\ 41)(34\ 42)(35\ 43)\\ & (36\ 44)(37\ 45)(38\ 46)(39\ 47)(40\ 48). \end{aligned}$$

The following conclusions are based on the calculation using GAP [26].

Lemma 3.2. *Let I_t be defined as above. Let I_4' be the set of elements $x \in I_4$ with $x^{-1} \in I_4$, let I_4'' be the set of elements $x \in I_4$ with $Hx^{-1}H = (HxH)^y$ for some $y \in \mathbf{N}_{\mathrm{S}_{48}}(H)$, and let I_4''' be the set of elements $x \in I_4$ with $t(\mathrm{A}_{47} \cap Hx^{-1}H)t = \mathrm{A}_{47} \cap HxH$ for some $t \in \mathbf{N}_{\mathrm{A}_{47}}(H)$. Then*

(1) $I_1 = \emptyset$;

(2) *I_2 is a singleton, say $\{x\}$, $\langle H, x\rangle = \langle H, H^x\rangle \cong \mathrm{PSL}(2, 11)$, and the stabilizer of point 12 in $\langle H, H^x\rangle$ is isomorphic to $\mathbb{Z}_{11}{:}\mathbb{Z}_5$;*

(3) $I_3 = \emptyset$;

(4) *$|I_4| = 382$, $\langle H, H^x\rangle = \mathrm{A}_{48}$ for each $x \in I_4$, the stabilizer of point 48 in $\langle H, H^x\rangle$ is equal to A_{47}; $|I_4'| = 18$, $|I_4''| = 38$, $I_4' \cap I_4'' = \emptyset$, $|I_4'''| = 94$, $I_4'' \subset I_4'''$, and $I_4' \cap I_4'''$ consists of 4 inverse-paired elements. Moreover, $\mathbf{N}_{\mathrm{S}_{48}}(H)$ is a subgroup of A_{48}, and if $x \in I_4''$ then $HxH = (Hx^{-1}H)^y$ and $Hx^{-1}H = (HxH)^y$ for some $y \in \mathbf{N}_{\mathrm{S}_{48}}(H)$.*

Remark. *For (2) of Lemma 3.2, we may choose $x = (1\,7\,11)(2\,4\,3)(5\,12\,10)(6\,8\,9)$. Then $S = \{(1\,5\,9\,6\,8)(2\,3\,4\,7\,10), (1\,5\,10\,7\,3\,9\,4\,11\,8\,2\,6), (1\,9\,2\,6\,10)(3\,4\,5\,8\,11)\}$. Let $y = (1\,6)(2\,5)(3\,4)(7\,11)(8\,10)$. Then $(HxH)^y = Hx^{-1}H$, $\langle y, x, H\rangle \cong \mathrm{PGL}(2, 11)$, and y normalizes both H and the stabilizer of point 12 in $\langle H, H^x\rangle$.* □

For $x \in I_t$, $t = 2$ or 4, we set

$$S_x = G \cap HxH,\ \Gamma_x = \mathrm{Cay}(G, S_x),$$

where either $G = \mathrm{A}_{47}$ if $t = 4$ or, otherwise, G is the stabilizer of the point 12 in $\langle x, H\rangle$. By Lemma 3.2 and the choice of I_t, we have the following result.

Corollary 3.3. *If $x \in I_2$ then $\mathsf{Aut}\Gamma_x \cong \mathrm{PSL}(2, 11)$; if $x \in I_4$ then $\mathsf{Aut}\Gamma_x = \mathrm{A}_{48}$.*

It is easily shown that two paired digraphs (V, A) and (V, A^*) have the same automorphism group. By Corollary 3.3, Lemmas 2.3 and 3.2, we have the following result.

Corollary 3.4. *$I'_4 \cap I''_4 = \emptyset$, and for distinct $x, y \in I_4$,*

$\Gamma_x \ncong \Gamma_y$;
$\Gamma_x \cong \mathrm{Cay}(\mathrm{A}_{47}, S_x^{-1})$ *if and only if* $x \in I''_4$.

Remark. *By the above corollary and the choices of I'_4 and I''_4, there are examples of digraphs which are not isomorphic to their paired digraphs, and there also are examples of digraphs which are isomorphic to their paired digraphs.*

3.2. **The proof of Theorem 1.1.** Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$, $S^{-1} \neq S$ and $G = \langle SS^{-1}\rangle$. Let $\Gamma = \mathrm{Cay}(G, S)$ and $G \leq X \leq \mathsf{Aut}\Gamma$. Assume that Γ is X-arc-transitive, and $\mathsf{Core}_X(G) = 1$. By the foregoing discussion, we conclude that Γ is isomorphic to one of the Cayley digraphs arising from the elements in I_2 and I_4. In particular, either $G \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ and $\mathsf{Aut}\Gamma = X \cong \mathrm{PSL}(2, 11)$, or $G = \mathrm{A}_{47}$ and $\mathsf{Aut}\Gamma = X = \mathrm{A}_{48}$. For I_2, we have a unique digraph. For the digraphs arising from I_4, by the choice of I_4 and Lemma 2.3, we know that they are not isomorphic to every other. Then our result follows.

3.3. **Consequences.** By Theorem 1.1, we may give a description on the automorphism groups of alternatingly connected arc-transitive Cayley digraph with out-valency 3.

Corollary 3.5. *Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$, $S^{-1} \neq S$ and $\langle SS^{-1}\rangle = G$. Let $\Gamma = \mathrm{Cay}(G, S)$, $G \leq X \leq \mathsf{Aut}\Gamma$ and $N = \mathsf{Core}_X(G)$. Assume that Γ is X-arc-transitive. Then either $N = G$, or $N \cap S = \emptyset$ and one of the following holds.*

(1) $NS = NS^{-1}$, $G/N \cong \mathbb{Z}_4$ *and* $X/N \cong \mathrm{S}_4$;
(2) $NS \neq NS^{-1}$, $G/N \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ *and* $X/N \cong \mathrm{PSL}(2, 11)$;
(3) $G/N \cong \mathrm{A}_{47}$ *and* $X/N \cong \mathrm{A}_{48}$.

Proof. By Theorem 1.1, we let $N \neq 1$. Assume further that $N \neq G$. Then N is intransitive on $u(G)$. Let $\overline{X}$ be the permutation group induced by X on the set of N-orbits. Consider the Cayley graph $\underline{\Gamma} := \mathrm{Cay}(G, S \cup S^{-1})$. Clearly, $\underline{\Gamma}$ is X-half-transitive and of valency 6. By Lemma 2.10, $N \cap S = \emptyset$, $\overline{X} \cong X/N$, and either $\underline{\Gamma}$ is a normal cover of $\underline{\Gamma}_N$ or Γ_N is $\overline{X}$-arc-transitive and of valency 3. Note that $\underline{\Gamma}_N \cong \mathrm{Cay}(G/N, \overline{S} \cup \overline{S}^{-1})$, where $\overline{S} = \{Ns \mid s \in S\}$.

Assume that $\underline{\Gamma}$ is a normal cover of $\underline{\Gamma}_N$. Then $\underline{\Gamma}$ is of valency 6; in particular, $\overline{S} \neq \overline{S}^{-1}$, and so $NS \neq NS^{-1}$. By Lemma 2.9, $\underline{\Gamma}_N$ is $\overline{X}$ is half-transitive. Then it is easily shown that $\mathrm{Cay}(G/N, \overline{S})$ is isomorphic to the digraph arising from one of the $\overline{X}$-orbits on the arcs of $\underline{\Gamma}_N$. Clearly, $\mathrm{Cay}(G/N, \overline{S})$ is X/N-arc-transitive and G/N is core-free in X/N. By Theorem 1.1, one of (2) and (3) of this corollary occurs.

Assume that $\underline{\Gamma}_N$ is cubic and $\overline{X}$-arc-transitive. Then $\overline{S} = \overline{S}^{-1}$, yielding $NS = NS^{-1}$, and $\underline{\Gamma}_N$ is an arc-transitive core-free Cayley graph of valency 3. By Lemma 2.10, $\underline{\Gamma}_N$ is not bipartite. Checking the 13 graphs given in [22], we conclude that either $\underline{\Gamma}_N$ is the complete graph of order 4, or $G/N \cong \mathrm{A}_{47}$ and $X/N \cong \mathrm{A}_{48}$. Then (1) or (3) of Corollary 3.5 occurs. $\square$

Corollary 3.6. *Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $|S| = 3$ and $S \neq S^{-1}$. If $\langle S\rangle = G$ and $|\mathsf{Aut}(G, S)|$ is divisible by 3, then either $G \neq \langle SS^{-1}\rangle$ or $\mathrm{Cay}(G, S)$ is a normal Cayley digraph.*

Proof. Assume that $|\mathsf{Aut}(G, S)|$ is divisible by 3. Let $X = \mathsf{Aut}\Gamma$. Since $\langle S\rangle = G$, the action of $\mathsf{Aut}(G, S)$ on S is faithful. Then Γ is X-arc-transitive. In the following, we prove that G is normal in X if $G = \langle SS^{-1}\rangle$.

Suppose that $G = \langle SS^{-1}\rangle$ and $N = \mathsf{Core}_X(G) \neq G$. Since Γ is X-arc-transitive and alternatingly connected, Corollary 1.3 holds for Γ, X and N. Note that $\mathbf{N}_X(G) = G{:}\mathsf{Aut}(G, S)$ and $\mathbf{N}_X(G)/N = \mathbf{N}_{X/N}(G/N)$. Then

$$|\mathbf{N}_{X/N}(G/N) : (G/N)| = |\mathbf{N}_X(G) : G| = |\mathsf{Aut}(G, S)|.$$

For (1) of Corollary 3.5, we have $G/N \cong \mathbb{Z}_4$ and $\mathbf{N}_{X/N}(G/N) \cong \mathrm{D}_8$, and so $|\mathsf{Aut}(G, S)| = |\mathbf{N}_{X/N}(G/N) : (G/N)| = 2$, a contradiction. For (2) of Corollary 3.5, we have $G/N \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ and $\mathbf{N}_{X/N}(G/N) \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$, yielding $|\mathsf{Aut}(G, S)| = 2$, a contradiction. For (3) of Corollary 3.5, we have $G/N \cong \mathrm{A}_{47}$ and $X/N \cong \mathrm{A}_{48}$, and so $\mathbf{N}_{X/N}(G/N) = G/N$, yielding $\mathsf{Aut}(G, S) = 1$, again a contradiction. This completes the proof. $\square$

4. The underlying graphs of digraphs Γ_x

In this section, we consider some graphs arising from the Cayley digraphs given in Section 3. For $x \in I_t$, $t = 2$ or 4, we set

$$S_x = G \cap HxH,\ \Gamma_x = \mathrm{Cay}(G, S_x),\ \underline{\Gamma_x} = \mathrm{Cay}(G, S_x \cup S_x^{-1}),$$

where either $G = \mathrm{A}_{47}$ if $t = 4$ or, otherwise, G is the stabilizer of the point 12 in $\langle x, H\rangle$. Then each $\underline{\Gamma_x}$ is a Cayley graph of valency 6. By Lemma 3.2 and Corollary 3.4, there are exactly 383 Cayley digraphs Γ_x up to isomorphism, and

(I2) if $x \in I_2$ then $\mathsf{Aut}\Gamma_x \cong \mathrm{PSL}(2, 11)$ and $G \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$;
(I4) if $x \in I_4$ then $\mathsf{Aut}\Gamma_x = \mathrm{A}_{48}$ and $G = \mathrm{A}_{47}$.

Clearly, $\mathsf{Aut}\Gamma_x \leq \mathsf{Aut}\underline{\Gamma_x}$, and $\underline{\Gamma_x}$ is $\mathsf{Aut}\Gamma_x$-half-transitive. Then $\underline{\Gamma_x}$ is arc-transitive if and only if $\mathsf{Aut}\Gamma_x \neq \mathsf{Aut}\underline{\Gamma_x}$.

Lemma 4.1. *If $x \in I_4''$ then $\mathbf{C}_{\mathsf{Aut}\underline{\Gamma_x}}(\mathsf{Aut}\Gamma_x) \cong \mathbb{Z}_2$ and $\underline{\Gamma_x}$ is arc-transitive.*

Proof. Assume that $x \in I_4''$. Then, by (4) of Lemma 3.2, there is some $y \in \mathbf{N}_{\mathrm{S}_{48}}(H)$ with $HxH = (Hx^{-1}H)^y$ and $Hx^{-1}H = (HxH)^y$. Note that $\mathbf{N}_{\mathrm{S}_{48}}(H) = LH$, where L is the point-stabilizer of 48 in $\mathbf{N}_{\mathrm{S}_{48}}(H)$. By (4) of Lemma 3.2, L is a subgroup of $G = \mathrm{A}_{47}$. Thus, we write $y = ab$ with $a \in L$ and $b \in H$. Then we have $HxH = (Hx^{-1}H)^a$ and $Hx^{-1}H = (HxH)^a$, and hence $(S_x)^a = (G \cap HxH)^a = G \cap Hx^{-1}H = S_x^{-1}$, and $(S_x^{-1})^a = (G \cap Hx^{-1}H)^a = G \cap HxH = S_x$. Let $\widehat{a}$ be the (inner) automorphism of G induced by the conjugation of a. Then $\widehat{a} \in \mathsf{Aut}\underline{\mathit{\Gamma}_x}$, and so $\widehat{a}a^{-1} \in \mathsf{Aut}\underline{\mathit{\Gamma}_x}$. Noting that $u(g)^{\widehat{a}a^{-1}} = u(a^{-1}g)$ for all $g \in G$, we know that $z \neq 1$ and z centralizes G. For $h \in H$ and $g \in G$, noting that $\mathsf{Aut}\mathit{\Gamma}_x = GH = HG$, we may write $gh = h'g'$ for $g' \in G$ and $h' \in H$. Then

$$u(g)^{h\widehat{a}a^{-1}} = u(1)^{gh\widehat{a}a^{-1}} = u(1)^{h'g'\widehat{a}a^{-1}} = u(1)^{g'\widehat{a}a^{-1}} = u(a^{-1}g')$$

and

$$u(g)^{\widehat{a}a^{-1}h} = u(a^{-1}g)^h = u(1)^{a^{-1}gh} = u(1)^{a^{-1}h'g'} = u(1)^{a^{-1}h'aa^{-1}g'} = u(1)^{a^{-1}g'} = u(a^{-1}g').$$

It follows that z centralizes H, and so z centralizes $\mathsf{Aut}\mathit{\Gamma}_x = GH$. Since $\mathsf{Aut}\mathit{\Gamma}_x = \mathrm{A}_{48}$, we have $z \notin \mathsf{Aut}\mathit{\Gamma}_x$. Thus $\mathsf{Aut}\mathit{\Gamma}_x \neq \mathsf{Aut}\underline{\mathit{\Gamma}_x}$, and then $\underline{\mathit{\Gamma}_x}$ is arc-transitive.

Let $C = \mathbf{C}_{\mathsf{Aut}\underline{\mathit{\Gamma}_x}}(\mathsf{Aut}\mathit{\Gamma}_x)$, and $Y = \langle \mathsf{Aut}\mathit{\Gamma}_x, C\rangle$. Then $Y = \mathsf{Aut}\mathit{\Gamma}_x \times C$, and $\underline{\mathit{\Gamma}_x}$ is Y-arc-transitive. Note that $\mathsf{Aut}\mathit{\Gamma}_x$ has two orbits on the arc set of $\underline{\mathit{\Gamma}_x}$, say A and A^*, which are exactly the arc sets of $\mathit{\Gamma}_x$ and $\mathrm{Cay}(G, S_x^{-1})$. Since $\mathsf{Aut}\mathit{\Gamma}_x$ is normal in Y, we know that C interchanges A and A^*. In particular, C has a subgroup N of index 2, which fixes both two A and A^* set-wise, and then $N \leq \mathsf{Aut}\mathit{\Gamma}_x$. Since $\mathsf{Aut}\mathit{\Gamma}_x \cap C = 1$, we have $N = 1$, and so $C \cong \mathbb{Z}_2$. □

Lemma 4.2. *If* $\mathbf{C}_{\mathsf{Aut}\underline{\mathit{\Gamma}_x}}(\mathsf{Aut}\mathit{\Gamma}_x) \neq 1$ *then* $x \in I_4''$.

Proof. Let $C = \mathbf{C}_{\mathsf{Aut}\underline{\mathit{\Gamma}_x}}(\mathsf{Aut}\mathit{\Gamma}_x) \neq 1$. Considering the action of $C\mathsf{Aut}\mathit{\Gamma}_x$, by a similar argument as in the proof of Lemma 4.1, we have C has order 2. Let $C = \langle z\rangle$, and let A and A^* be the orbits of $\mathsf{Aut}\mathit{\Gamma}_x$ on the arc set of $\underline{\mathit{\Gamma}_x}$. Then z interchanges A and A^*, and so z gives an isomorphism from $\mathit{\Gamma}_x$ to $\mathrm{Cay}(G, S_x^{-1})$, and also an isomorphism from $\mathrm{Cay}(G, S_x^{-1})$ to $\mathit{\Gamma}_x$. Noting that $\mathsf{Aut}\mathit{\Gamma}_x = \mathsf{Aut}\mathrm{Cay}(G, S_x^{-1})$, by Lemma 2.3, there is an automorphism σ of $\mathsf{Aut}\mathit{\Gamma}_x$ which fixes H and interchanges HxH and $Hx^{-1}H$. Noting that $\mathsf{Aut}\mathit{\Gamma}_x = \mathrm{A}_{48}$, such a σ is induced by some $y \in \mathbf{N}_{\mathrm{S}_{48}}(H)$ by the conjugation on A_{48}. Thus the lemma follows. □

Lemma 4.3. *Let* $x \in I_2$ *or* $x \in I_4$*, and let* N *be a minimal normal subgroup of* $\mathsf{Aut}\underline{\mathit{\Gamma}_x}$*. Then either*

(1) $\mathsf{Aut}\mathit{\Gamma}_x \leq N$*; or*
(2) $x \in I_4''$*,* $N = \mathbf{C}_{\mathsf{Aut}\underline{\mathit{\Gamma}_x}}(\mathsf{Aut}\mathit{\Gamma}_x)$ *and* ${\underline{\mathit{\Gamma}_x}}_N$ *has valency* 6*.*

Proof. Let $X = \mathsf{Aut}\mathit{\Gamma}_x$ and $Y = \mathsf{Aut}\underline{\mathit{\Gamma}_x}$. Then, since X is simple and $N \cap X \trianglelefteq X$, either $X \leq N$ or $X \cap N = 1$. Next we assume that $X \cap N = 1$ and show (2) occurs.

Let $X \cap N = 1$. Then $X = \mathsf{Aut}\mathit{\Gamma}_x \neq XN$, it implies that $\mathsf{Aut}\underline{\mathit{\Gamma}_x}$ is $(XN$-)arc-transitive. Since $G \leq X \leq XN \leq Y$, we know that $|N| = |XN : X|$ is a divisor of $|Y : G|$, which equals the order of Y_u for $u \in u(G)$. By Lemma 2.6, $|Y_u|$ has no prime divisor other that 2, 3 and 5. Thus $|N|$ is not divisible by 11, and so N is intransitive on $u(G)$ as $|G|$ has a divisor 11. Consider the quotient graph $\underline{\Gamma_x}_N$. Then X induces a insoluble subgroup of $\mathsf{Aut}\underline{\Gamma_x}_N$; in particular, $\underline{\Gamma_x}_N$ is neither a cycle nor the complete graph on two vertices. It follows from Lemma 2.9 that $\underline{\Gamma_x}_N$ is of valency 3 or 6.

Suppose that $\underline{\Gamma_x}_N$ is cubic. Then $\underline{\Gamma_x}_N$ is $\overline{X}$-arc-transitive, where $\overline{X}$ is the group induced by X on $\underline{\Gamma_x}_N$. Since X is simple, $\overline{X} \cong X$. Take an N-orbit B on $u(G)$. Then $\overline{X}_B$ and hence X_B has order a divisor of 48, see Theorem 2.7. Noting that X is transitive on $u(G)$, it follows that B is an X_B-orbit. Thus $|B| = |X_B : X_u|$ for $u \in B$. Since $\underline{\Gamma_x}_N$ has odd valency, it has even order, and so $\underline{\Gamma_x}$ is of even order. Then $x \in I_4$, and so $|X_u| = 48$. Thus $|B| = |X_B : X_u| = 1$. Noting that all N-orbits on $u(G)$ have the same size, it follows that N acts trivially $u(G)$, yielding $N = 1$, a contradiction.

Assume that $\underline{\Gamma_x}_N$ is of valency 6. Then, by Lemma 2.9, $\underline{\Gamma_x}$ is a normal cover of $\underline{\Gamma_x}_N$, and then N is semiregular on $u(G)$. In particular, $|N|$ is a divisor of $|G|$, and hence $|N|$ is a common divisor of $|G|$ and $|Y_u|$. Write $N \cong T^l$ for a positive integer l and a simple group T. Then every prime divisor of $|T|$ is contained in $\{2, 3, 5\}$. Suppose that $x \in I_2$. Then $|G| = 55$, and so $\underline{\Gamma_x}_N$ has order 5 or 11; however, by [2], such a $\underline{\Gamma_x}_N$ does not exist, a contradiction. Thus $x \in I_4$, and $G = \mathrm{A}_{47}$.

Suppose that T is insoluble, then $T \cong \mathrm{A}_5$ or A_6, see [13, pp. 12-14] for example. In this case, $|Y_u|$ is divisible by 5, and so $\underline{\Gamma_x}$ is 2-arc-transitive. Since $|N| = |T|^l$ is a divisor of $|Y_u|$, by Lemma 2.8, $l \leq 2$. Note that X induces by conjugation a subgroup of $\mathsf{Aut}(N) \cong \mathsf{Aut}(T)^l{:}\mathrm{S}_l$. It follows that X centralizes N, and so $|N| \leq 2$ by Lemmas 4.1 and 4.2, a contradiction.

Now let N is soluble. Then $|N| = p^l$ for some $p \in \{2, 3, 5\}$. Recalling that $|N|$ is a divisor of $|G|$, we have $l < 37$. By [14, Proposition 5.3.7], A_{48} can not be embedded in a classical group of Lie type with dimension less than 46. It follows that X acts trivially on N by conjugation. Thus $N \leq \mathbf{C}_Y(X)$. Then part (2) of this lemma follows from Lemmas Lemmas 4.1 and 4.2. $\square$

Corollary 4.4. *If $x \notin I_4''$ then $\mathsf{Aut}\underline{\Gamma_x}$ is almost simple.*

Proof. Let $x \in I_2 \cup I_4 \setminus I_4''$. By Lemma 4.3, $\mathsf{Aut}\Gamma_x$ is contained in every minimal normal subgroup of $\mathsf{Aut}\underline{\Gamma_x}$. Thus $Y := \mathsf{Aut}\underline{\Gamma_x}$ has a unique minimal normal subgroup, say $N \cong T^l$ with T simple. Since $\mathsf{Aut}\Gamma_x \leq N$, we know that $|T|$ is divisible by the largest prime divisor p of $|G|$. Then $|N|$ is divisible by p^l, and so $|N : \mathsf{Aut}\Gamma_x|$ is divisible by p^{l-1}. Noting that $|N : \mathsf{Aut}\Gamma_x|$ is a divisor of $|Y_u|$ for $u \in u(G)$, by Theorem 2.8, we have $l = 1$, and so N is simple. Then the result follows. $\square$

Theorem 4.5. *Let $x \in I_2$. Then $\mathsf{Aut}\underline{\Gamma_x} \cong \mathrm{PGL}(2, 11)$; in particular, $\underline{\Gamma_x}$ is an arc-transitive graph of valency 6.*

Proof. By the remak after Lemma 3.2, we know that there is an involution $y \in \mathrm{S}_{12}$ such that $S_x^y = S_x^{-1}$ and $\langle x, y, H\rangle \cong \mathrm{PGL}(2, 11)$. Then this y induces an automorphism of

$\underline{\mathit{\Gamma}_x}$. Then $Y := \mathsf{Aut}\underline{\mathit{\Gamma}_x}$ has a subgroup X with $T_1 := \mathsf{Aut}\mathit{\Gamma}_x \le X \cong \mathrm{PGL}(2,11)$, and $\underline{\mathit{\Gamma}_x}$ is X-arc-transitive. By [19], there is no 2-arc-transitive graph of order 55 and valency 6. Thus for $v \in u(G)$, the stabilizer Y_v is a $\{2,3\}$-group, see Lemma 2.6. Let T_2 be the socle $\mathrm{soc}(Y)$. By Corollary 4.4, T_2 is simple, and then $T_1 \le T_2$. Since $Y = GY_v = T_1Y_v$, we have $T_2 = T_2 \cap T_1Y_v = T_1(T_2 \cap Y_v)$. Then the index $|T_2 : T_1|$ is a $\{2,3\}$ number. By [23], if $T_1 \ne T_2$ then $T_2 = \mathrm{M}_{11}$ or M_{12}. Noting that Y contains a subgroup isomorphic to $\mathrm{PGL}(2,11)$, by the Atlas [4], we concluder either $T_1 = T_2$, or $T_2 = \mathrm{M}_{12}$ and $Y = \mathrm{M}_{12}.2$. Note that $T_2 = T_2 \cap GY_v = G(T_2 \cap Y_v)$, $|G| = 55$ and $G \cap Y_v = 1$. Then T_2 has a subgroup of index 55, it follows that $T_1 = T_2 = \mathrm{PSL}(2,11)$. This completes the proof. □

Theorem 4.6. *Let $x \in I_4$. If $\mathsf{Aut}\underline{\mathit{\Gamma}_x}$ is almost simple then $\mathsf{Aut}\underline{\mathit{\Gamma}_x} = \mathrm{A}_{48}$, $x \notin I_4''$ and $\underline{\mathit{\Gamma}_x}$ is half-transitive. In particular, $\mathsf{Aut}\underline{\mathit{\Gamma}_x} = \mathrm{A}_{48}$ if and only if $x \notin I_4''$.*

Proof. Let $G = \mathrm{A}_{47}$, $T = \mathsf{Aut}\mathit{\Gamma}_x = \mathrm{A}_{48}$ and $X = \mathrm{soc}(\mathsf{Aut}\underline{\mathit{\Gamma}_x})$. Then $\underline{\mathit{\Gamma}_x}$ is T-half-transitive and, for $v \in u(G)$, either X_v is transitive on $\underline{\mathit{\Gamma}_x}(v)$ or X_v has 2-orbits of size three on $\underline{\mathit{\Gamma}_x}(v)$. Since $\underline{\mathit{\Gamma}_x}$ has valency 6, either X_v is a $\{2,3\}$-group or $|X_v|$ is divisible by 5, see Lemma 2.6. Assume further that X is simple. Note that $X = GX_v = TX_v$ and $T = GT_v \le X$, and so $|X : T| = |X_v : T_v|$.

Suppose that $|X_v|$ is divisible by 5. By Lemma 2.6, we conclude that X_v acts 2-transitively on $\underline{\mathit{\Gamma}_x}(v)$. Then $\underline{\mathit{\Gamma}_x}$ is 2-arc-transitive, and thus X_v is known as in Theorem 2.8. In particular, either $|X_v|$ is a divisor of $2^7 \cdot 3^3 \cdot 5^2$ or $|X_v| = 2^5 \cdot 3 \cdot 5^3$. Recalling that T_v has order $2^4 \cdot 3$, either $|X : T| = 2 \cdot 5^3$ or $|X : T|$ is a divisor of $2^3 \cdot 3^2 \cdot 5^2$. Take a maximal subgroup M of X with $T \le M$. Then $|X : M| < 1000$ or $M = T$ and $|X : M| = 1800$. Then the simple group X has a primitive representation of degree 1800 or less than 1000. All primitive permutation groups of almost type with degree less than 2500 are explicitly classified in [7, 25]. By this classification, we conclude that there is no simple group X which contains A_{48} as a proper subgroup of index less than 2500, a contradiction. Thus X_v is a $\{2,3\}$-group.

Suppose that $X \ne T = \mathrm{A}_{48}$. Take a maximal subgroup M of X with $T \le M$. Then $|X : M| = 2^a \cdot 3^b$ and $|M : T| = 2^c \cdot 3^d$ for some integers a, b, c and d. By [23], the triple X, M and $|X : M|$ are known. The only possibility is that $X = \mathrm{A}_n$ and $M \cong \mathrm{A}_{n-1}$ with $n = 2^a \cdot 3^b > 48$. Then $|X : T|$ has a prime divisor no less than 7, which is impossible. Thus we have $X = T = \mathrm{A}_{48}$, and so $\mathsf{Aut}\underline{\mathit{\Gamma}_x} = \mathrm{A}_{48}$ or S_{48}.

Suppose that $\mathsf{Aut}\underline{\mathit{\Gamma}_x} = \mathrm{S}_{48}$. Take an permutation $y \in \mathrm{S}_{48} \setminus \mathrm{A}_{48}$. Note that A_{48} has two orbits on the arc set and $\underline{\mathit{\Gamma}_x}$, which are in fact the arc set A of $\mathit{\Gamma}_x$ and the arc set A^* of $\mathit{\Gamma}_{x^{-1}} := \mathrm{Cay}(G, S_{x^{-1}})$. If y fixes both A and A^* set-wise, then y is an automorphism of $\mathit{\Gamma}_x$, which contradicts Corollary 3.3. Thus we have $A^y = A^*$ and $(A^*)^y = A$, and then y is an isomorphism between $\mathit{\Gamma}_x$ and $\mathit{\Gamma}_{x^{-1}}$. Since both $\mathit{\Gamma}_x$ and $\mathit{\Gamma}_{x^{-1}}$ are vertex transitive, we may let $u(1)^y = u(1)$ without loss of generality. This implies that y normalizes the stabilizer $T_{u(1)} = H$, i.e., $y \in \mathbf{N}_{\mathrm{S}_{48}}(H)$. However, by (4) of Lemma 3.2, $y \in \mathbf{N}_{\mathrm{S}_{48}}(H) \le \mathrm{A}_{48} = \mathsf{Aut}\mathit{\Gamma}_x$, and so $A^y = A$, a contradiction. Thus the theorem follows from Lemma 4.1. □

Theorem 4.7. *Let $x \in I_4''$. Then* $\mathsf{Aut}\underline{\Gamma_x} = C \times \mathrm{A}_{48}$, *where* $C \cong \mathbb{Z}_2$.

Proof. Let $Y = \mathsf{Aut}\underline{\Gamma_x}$ and $X = \mathsf{Aut}\Gamma_x$. By Lemma 4.3 and Theorem 4.6, we conclude that Y has exactly two minimal normal subgroups, one of them is $C = \mathbf{C}_Y(X)$. By Lemma 4.2, $C \cong \mathbb{Z}_2$. Let T be the other minimal normal subgroup. Then $X \leq T$. By a similar argument as in the proof of Corollary 4.4, we know that T is simple. Considering the maximal subgroups of T, a similar argument as in the proof Theorem 4.6, we conclude that $T = \mathrm{A}_{48} = X$. It follows from the normality of X that each element in Y either fixes the arc set of Γ_x or interchanges the arc sets of Γ_x and $\mathrm{Cay}(G, S_x^{-1})$. Then $X = \mathsf{Aut}\Gamma_x$ contains a subgroup of Y of index 2, yielding $|Y : X| = 2$, and then $\mathsf{Aut}\underline{\Gamma_x} = Y = C \times \mathrm{A}_{48}$. □

By Theorems 4.6 and 4.7, we get the following corollaries, which complete the proof of Theorem 1.3 (1).

Corollary 4.8. *Let $x, y \in I_4$ with $x \neq y$. Then $\underline{\Gamma_x} \cong \underline{\Gamma_y}$ if and only if $\Gamma_x \cong \Gamma_{y^{-1}}$.*

Proof. If $\sigma : \Gamma_x \to \Gamma_{y^{-1}}$ is an isomorphism then $\sigma : \Gamma_{x^{-1}} \to \Gamma_y$ is also an isomorphism, and then $\sigma : \underline{\Gamma_x} \to \underline{\Gamma_y}$ is an isomorphism.

Suppose that there is an isomorphism $\sigma : \underline{\Gamma_x} \to \underline{\Gamma_y}$. Without loss of generality, we may let $u(1)^\sigma = u(1)$. Then $H^\sigma = H$. Note that $\underline{\Gamma_x}$ and $\underline{\Gamma_y}$ have the same automorphism group A_{48} or $\mathbb{Z}_2 \times \mathrm{A}_{48}$. Thus $\mathrm{A}_{48}^\sigma = \mathrm{A}_{48}$. Let A_x and A_x^* be the orbits of A_{48} on the arc set of $\underline{\Gamma_x}$, and let A_y and A_y^* be the orbits of A_{48} on the arc set of $\underline{\Gamma_y}$. Then we have $\{A_x, A_x^*\}^\sigma = \{A_y, A_y^*\}$. Thus σ is in fact an isomorphism from Γ_x to Γ_y or $\Gamma_{y^{-1}}$. By Corollary 3.4, σ is an isomorphism from Γ_x to $\Gamma_{y^{-1}}$. □

Corollary 4.9. *Up to isomorphism, there are* 210 *graphs* $\underline{\Gamma_x}$ *when x runs over I_4, involving* 38 *arc-transitive graphs and* 172 *half-transitive graphs.*

Proof. For distinct $x, y \in I_4'$, by Corollaries 3.4 and 4.8, $\underline{\Gamma_x} \not\cong \underline{\Gamma_y}$ unless $y = x^{-1}$. Thus we get 9 non-isomorphic graphs $\underline{\Gamma_x}$ when x runs over I_4'. Similarly, we have 38 non-isomorphic graphs $\underline{\Gamma_x}$ when x runs over I_4''. By the choices of I_4' and I_4'', if $x \in I_4'$ and $y \in I_4''$ then $\Gamma_x \not\cong \Gamma_{y^{-1}}$, and so $\underline{\Gamma_x} \not\cong \underline{\Gamma_y}$ by Corollary 4.8. Thus, when x runs over $I_4' \cup I_4''$, we get 47 non-isomorphic graphs $\underline{\Gamma_x}$. For $x \notin I_4' \cup I_4''$ and $y \in I_4' \cup I_4''$, by Corollaries 3.4 and 4.8, $\underline{\Gamma_x} \not\cong \underline{\Gamma_y}$. Thus it suffices to enumerate the graphs Γ_x up to isomorphism for $x \in I_4 \setminus (I_4' \cup I_4'')$. Note that $|I_4 \setminus (I_4' \cup I_4'')| = 326$.

Let $x \in I_4 \setminus (I_4' \cup I_4'')$. By the choice of I_4, there is a unique $y \in I_4 \setminus (I_4' \cup I_4'')$ such that $\Gamma_{x^{-1}} \cong \Gamma_y$, and then $\underline{\Gamma_x} \cong \underline{\Gamma_y}$. Then, by Corollary 4.8, all graphs $\underline{\Gamma_x}$ with $x \in I_4 \setminus (I_4' \cup I_4'')$ are isomorphic in pairs, and distinct pairs of graphs are not isomorphic. Thus, when x runs over $I_4 \setminus (I_4' \cup I_4'')$, we get 163 non-isomorphic graphs $\underline{\Gamma_x}$. Then the corollary follows from Lemma 4.1 and Theorem 4.6. □

5. Standard double covers of digraphs Γ_x

Let $\Gamma = \mathrm{Cay}(R, S)$ be a Cayley (di)graph. We may identify $\mathsf{Aut}\Gamma$ with a subgroup of $\mathsf{Aut}\Gamma^{(2)}$ by

$$(u(g), i)^y = (u(g)^y, i);\ g \in R,\ y \in \mathsf{Aut}\Gamma,\ i \in \mathbb{Z}_2.$$

Thus, if Γ is arc-transitive, then $\Gamma^{(2)}$ is $\mathsf{Aut}\Gamma$-semisymmetric. Set $\iota : u(R) \times \mathbb{Z}_2 \to u(R) \times \mathbb{Z}_2$, $(u(g), i) \mapsto (u(g), i)$. It is easily shown that ι is an isomorphism between $\Gamma^{(2)}$ and $\mathrm{Cay}(R, S^{-1})^{(2)}$. In particular, ι is an automorphism if and only if $S = S^{-1}$. In general, we have the following result.

Lemma 5.1. *Let $\Gamma = \mathrm{Cay}(R, S)$ be a Cayley (di)graph, and $R \le Y \le \mathsf{Aut}\Gamma$. Then $tS^{-1}t = S$ for some $t \in \mathbf{N}_R(Y_{u(1)}) \setminus Y_{u(1)}$ if and only if there is an involution $\sigma \in \mathsf{Aut}\Gamma^{(2)} \setminus Y$ such that σ centralizes Y and $Y \times \langle\sigma\rangle$ is a transitive subgroup of $\mathsf{Aut}\Gamma^{(2)}$.*

Proof. Assume first that $tS^{-1}t = S$ for some $t \in \mathbf{N}_R(Y_{u(1)}) \setminus Y_{u(1)}$. Define σ by

$$(u(g), 0)^\sigma = (u(tg), 1),\ (u(g), 1)^\sigma = (u(t^{-1}g), 0),$$

where $g \in R$. It is easily shown that σ is a permutation on $u(G) \times \mathbb{Z}_2$ of order 2, and σ interchanges $u(G) \times \{0\}$ and $u(G) \times \{1\}$. For $g, h \in R$, the vertices $(u(h), 0)$ and $(u(g), 1)$ are adjacent if and only if $gh^{-1} \in S$, i.e., $hg^{-1} \in S^{-1}$. Note that $thg^{-1}t = (th)(t^{-1}g)^{-1}$ and $tS^{-1}t = S$. Then $gh^{-1} \in S$ if and only if $(th)(t^{-1}g)^{-1} \in S$, that is, $(u(t^{-1}g), 0)$ and $(u(th), 1)$ are adjacent. Thus $\{(u(h), 0), (u(g), 1)\}$ is an edge of $\Gamma^{(2)}$ if and only of $\{(u(h), 0)^\sigma, (u(g), 1)^\sigma\}$ is an edge of $\Gamma^{(2)}$. It follows that $\sigma \in \mathsf{Aut}\Gamma^{(2)}$, and then $\langle\sigma, Y\rangle$ is a transitive subgroup of $\mathsf{Aut}\Gamma^{(2)}$. For any $g \in R$ and $y \in Y$, since $\mathsf{Aut}\Gamma = R(\mathsf{Aut}\Gamma)_{u(1)} = (\mathsf{Aut}\Gamma)_{u(1)}R$, we write $gy = y_1g_1$ with $y_1 \in (\mathsf{Aut}\Gamma)_{u(1)}$ and $g_1 \in R$. Then

$$\begin{aligned}&(u(g), 1)^{y\sigma} = (u(1), 1)^{gy\sigma} = (u(1), 1)^{y_1g_1\sigma} = (u(tg_1), 2) = (u(1), 2)^{tg_1}\\&= (u(1), 2)^{(ty_1^{-1}t^{-1})(ty_1g_1)} = (u(1), 2)^{tgy} = (u(tg), 2)^y = (u(g), 1)^{\sigma y}.\end{aligned}$$

Similarly, $(u(g), 2)^{y\sigma} = (u(g), 2)^{\sigma y}$. Then $\sigma y = y\sigma$ for every $y \in Y$ and so $\langle\sigma, Y\rangle = Y \times \langle\sigma\rangle$.

Now let σ be an involution $\sigma \in \mathsf{Aut}\Gamma^{(2)}$ such that $\langle\sigma, Y\rangle = Y \times \langle\sigma\rangle$ and σ interchanges $u(G) \times \{0\}$ and $u(G) \times \{1\}$. Set $(u(1), 0)^\sigma = (u(t), 1)$ for some $t \in R$. Then $Y_{u(1)}^t = Y_{u(t)} = Y_{(u(t),1)} = Y_{(u(1),0)}^\sigma = Y_{(u(1),0)} = Y_{u(1)}$, and so $t \in \mathbf{N}_R(Y_{u(1)})$. Note that σ maps the neighborhood $u(S) \times \{1\}$ of $(u(1), 0)$ to the neighborhood $u(S^{-1}t) \times \{0\}$ of $(u(t), 1)$. For $s \in S$, we have $(u(s), 1)^\sigma = (u(t^{-1}s), 0) \in u(S^{-1}t) \times \{0\}$, yielding $t^{-1}s \in S^{-1}t$, and so $s \in tS^{-1}t$. It follows that $S \subseteq tS^{-1}t$, and then $S = tS^{-1}t$ as $|S| = |tS^{-1}t|$ and S is finite. This completes the proof. □

Clearly, for each $x \in I_4$, the standard double cover $\Gamma_x^{(2)}$ is connected, cubic and A_{48}-semisymmetric.

Theorem 5.2. *Let $x \in I_4$. Then $\Gamma_x^{(2)}$ is arc-transitive if and only if there is some $t \in \mathbf{N}_{\mathrm{A}_{47}}(H)$ with $tS_x^{-1}t = S_x$; in this case, $\mathsf{Aut}\Gamma^{(2)} = \mathsf{Aut}\Gamma_x \times \mathbb{Z}_2$.*

Proof. If $tS_x^{-1}t = S_x$ for some $t \in \mathbf{N}_{\mathrm{A}_{47}}(H)$ then, by Lemma 5.1, $\Gamma_x^{(2)}$ has an automorphism interchanging two parts of $\Gamma_x^{(2)}$, yielding the arc-transitivity of $\Gamma_x^{(2)}$.

Assume that $\Gamma_x^{(2)}$ is arc-transitive. Let $X = \mathsf{Aut}\Gamma_x^{(2)}$. By Theorem 2.7, for a vertex v of $\Gamma_x^{(2)}$, the stabilizer X_v of v has order a divisor of 48. Note that $T := \mathsf{Aut}\Gamma_x \le X$ and $|T_v| = 48$. It follows that $X_v = T_v$, and $|X| = 2|G||T_v| = 2|T|$. Thus $X = T.2$. Assume $\mathbf{C}_X(T) = 1$. Then we have $X \lesssim \mathsf{Aut}(T) \cong \mathrm{S}_{48}$, and so $X \cong \mathrm{S}_{48}$. If $\mathbf{C}_X(T) \neq 1$ then, since T is simple, we have $X = T \times \mathbf{C}_X(T)$ and $\mathbf{C}_X(T) \cong \mathbb{Z}_2$. All in all, there is an involution $\tau \in X \setminus T$ which normalizes G and interchanges $u(G) \times \{0\}$ and $u(G) \times \{1\}$. Thus $\Gamma_x^{(2)}$ is an arc-transitive cubic Cayley graph of $R := G{:}\langle t\rangle$. By [22], we conclude that R is not core-free in X. Note that $R \cap T = G = \mathrm{A}_{47}$, which is not normal in T. Then R is not normal in X. Thus $1 \neq \mathrm{Core}_X(R) \neq R$, and hence $R = G \times \mathrm{Core}_X(R)$, $X = T \times \mathrm{Core}_X(R)$, and $|\mathrm{Core}_X(R)| = 2$. By Lemma 5.1, $tS_x^{-1}t = S_x$ for some $t \in \mathbf{N}_{\mathrm{A}_{47}}(H)$. $\square$

Theorem 5.3. *Let $x \in I_4$. If $x \in I_4'''$ then $\mathsf{Aut}\Gamma^{(2)} = \mathsf{Aut}\Gamma_x \times \mathbb{Z}_2$; if $x \notin I_4'''$ then $\mathsf{Aut}\Gamma^{(2)} = \mathsf{Aut}\Gamma_x$.*

Proof. By Theorem 5.2, the first part of our result holds. Thus we next let $x \notin I_4'''$. Then $\Gamma_x^{(2)}$ is semisymmetric. Let $Y = \mathsf{Aut}\Gamma_x^{(2)}$ and $X = \mathsf{Aut}\Gamma_x$.

By Lemma 2.9, noting that $\Gamma_x^{(2)}$ is of valency 3, if N is normal in Y and neither transitive on $u(G) \times \{0\}$ nor transitive on $u(G) \times \{1\}$ then $\Gamma_x^{(2)}$ is a normal cover of $(\Gamma_x^{(2)})_N$. Let K be the kernel of Y acting on $u(G) \times \{0\}$. If K is transitive on $u(G) \times \{1\}$ then $\Gamma_x^{(2)}$ is the complete bipartite graph on 6 vertices, which is impossible. Thus K is intransitive on $u(G) \times \{1\}$. Then $K = 1$, and so Y is faithful on $u(G) \times \{0\}$. Similarly, Y is faithful on $u(G) \times \{1\}$.

Suppose that Y is not almost simple. Then, considering X as a (simple) subgroup of Y, there is a minimal normal subgroup N of Y such that $X \cap N{=}1$. Since X is transitive on $u(G) \times \{0\}$, we have $Y = XY_{(u(1),0)}$, and then $|Y : X| = |Y_{(u(1),0)} : X_{(u(1),0)}|$. Since $XN \le Y$, we know that $|N| = |XN : X|$ is a divisor of $|Y_{(u(1),0)} : X_{(u(1),0)}|$. Note that $X_{(u(1),0)} = X_{u(1)}$ has order 48 and, by [12], $|Y_{(u(1),0)}|$ is a divisor of $2^7 \cdot 3$. Then $|Y : X| = |Y_{(u(1),0)} : X_{(u(1),0)}| = 2^l$ for some $l \le 3$. It follows that $|N| = 2$, 4 or 8. Considering the conjugation of X on N, we have $XN = X \times N$. Take an involution $\sigma \in N$, and set $(u(1), i)^\sigma = (u(t_i), i)$ for $i \in \mathbb{Z}_2$ and $t_i \in G = \mathrm{A}_{47}$. Then, for $g \in G$, we have

$$(u(g), i)^\sigma = (u(1), i)^{g\sigma} = (u(1), i)^{\sigma g} = (u(t_i g), i),\ i \in \mathbb{Z}_2.$$

In particular, since σ has order 2, we have that $t_i^2 = 1$, and $t_0 \neq 1$ or $t_1 \neq 1$. Since σ centralizes X, the vertices $(u(1), i)$ and $(u(t_i), i)$ have the same stabilizer in X. Then $H = X_{u(1)} = X_{u(t_i)} = X_{u(1)}^{t_i} = H^{t_i}$, and so $t_i \in \mathbf{N}_{\mathrm{S}_{48}}(H) \cap G$. Note that σ maps the neighborhood of $(u(1), 0)$ onto the neighborhood of $(u(t_0), 0)$. It follows that $St_0 = t_1 S$. However, checking by GAP, there is no such a pair (t_0, t_1), a contradiction.

By the above argument, we may assume that Y is almost simple. Then $X \le \mathrm{soc}(Y)$ and $|\mathrm{soc}(Y) : X| \le |Y : X| \le 8$, yielding $\mathrm{soc}(Y) = X$. Thus $Y = \mathrm{A}_{48}$ or S_{48}, and hence

$2 \geq |Y : X| = |Y_{(u(1),0)} : X_{(u(1),0)}|$. Then $Y_{(u(1),0)} \leq \mathbf{N}_Y(X_{(u(1),0)}) = \mathbf{N}_{\mathrm{A}_{48}}(H) \leq \mathrm{A}_{48}$, yielding $\mathrm{A}_{48} \leq Y = GY_{(u(1),0)} \leq \mathrm{A}_{48}$, and the lemma follows. □

Note that a Cayley digraph and its paired digraph give isomorphic standard double covers. Then by the above theorem and Lemma 3.2 (4), we have the following result, which fulfils the proof of Theorem 1.3 (2).

Corollary 5.4. *If $x \in I_4$, then $\Gamma_x^{(2)}$ is isomorphic to one of are* 210 *cubic graphs which consist of* 66 *arc-transitive graphs and* 144 *semisymmetric graphs.*

Proof. By Theorem 5.2, $\Gamma_x^{(2)}$ is arc-transitive if and only if $x \in I_4'''$. Note that $|I_4'''| = 94$, $|I_4''' \cap I_4'| = 4$, $|I_4''' \cap (I_4 \setminus (I_4' \cup I_4''))| = 52$ and $I_4'' \subset I_4'''$. Then elements in $I_4''' \cap I_4'$ give 2 arc-transitive graphs $\Gamma_x^{(2)}$, the 14 elements in $I_4' \setminus I_4'''$ give 7 semisymmetric graphs $\Gamma_x^{(2)}$, elements in I_4'' give 38 arc-transitive graphs $\Gamma_x^{(2)}$. Recall that $x \in I_4 \setminus (I_4' \cup I_4'')$ there is a unique $y \in I_4 \setminus (I_4' \cup I_4'')$ such that $\Gamma_{x^{-1}} \cong \Gamma_y$, and then $\Gamma_x^{(2)} \cong \Gamma_y^{(2)}$. Thus elements in $I_4''' \cap (I_4 \setminus (I_4' \cup I_4''))$ give 26 arc-transitive graphs, and the other 274 elements in I_4 result in 137 semisymmetric graphs. Thus we get 66 arc-transitive cubic graphs and 144 half-transitive graphs. Then the corollary follows. □

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