ON EDGE-PRIMITIVE 2-ARC-TRANSITIVE GRAPHS

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ABSTRACT. A graph is edge-primitive if its automorphism group acts primitively on the edge set. In this short paper, we prove that a finite 2-arc-transitive edge-primitive graph has almost simple automorphism group if it is neither a cycle nor a complete bipartite graph. We also present two examples of such graphs, which are 3-arc-transitive and have faithful vertex-stabilizers.

KEYWORDS. Primitive group, almost simple group, edge-primitive graph, 2-arc-transitive graph.

1. INTRODUCTION

All graphs and groups considered in this paper are assumed to be finite.

A graph in this paper is a pair $\Gamma = (V, E)$ of a nonempty set V and a set E of 2-subsets of V. The elements in V and E are called the vertices and edges of Γ, respectively. The number $|V|$ of vertices is called the order of Γ. For $v \in V$, the set $\Gamma(v) = \{u \in V \mid \{u, v\} \in E\}$ is called the neighborhood of v in Γ, while $|\Gamma(v)|$ is called the valency of v. We say that Γ has valency d or Γ is d-regular if its vertices all have equal valency d. For an integer $s \geq 1$, an s-arc in Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_i, v_{i+1}\} \in E$ and $v_i \neq v_{i+2}$ for all possible i. A 1-arc is also called an arc.

Let $\Gamma = (V, E)$ be a graph. A permutation g on V is called an automorphism of Γ if $\{u^g, v^g\} \in E$ for all $\{u, v\} \in E$. Let $\mathsf{Aut}\Gamma$ denote the set of all automorphisms of Γ. Then $\mathsf{Aut}\Gamma$ is a subgroup of the symmetric group $\mathrm{Sym}(V)$, and called the automorphism group of Γ. Note that the group $\mathsf{Aut}\Gamma$ has a natural action on the edge set E (and also on the set of s-arcs). The graph Γ is called *edge-transitive* if $E \neq \emptyset$ and for each pair of edges there exists some $g \in \mathsf{Aut}\Gamma$ mapping one of these two edges to the other one. (Similarly, we may define *vertex-transitive*, *arc-transitive* or *s-arc-transitive* graphs.) An edge-transitive graph is called *edge-primitive* if some (and hence every) *edge-stabilizer*, the subgroup of its automorphism group which fixes a given edge, is a maximal subgroup of the automorphism group.

It is well-known that edge-transitive graphs and hence edge-primitive graphs are either bipartite or vertex-transitive. As a subclass of the edge-transitive graphs, edge-primitive graphs posses more restrictions on their symmetries and automorphism groups. For example, a connected edge-primitive graph is necessarily arc-transitive provided that it is not a star graph. In [9], appealing to the O'Nan-Scott Theorem for (quasi)primitive

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groups [22], Giudici and Li investigated the structural properties of edge-primitive graphs, particularly, on their automorphism groups. Let $\Gamma = (V, E)$ be an edge-primitive graph which is neither a cycle nor a complete bipartite graph. If Γ is bipartite then let $\mathsf{Aut}^+\Gamma$ be the subgroup of $\mathsf{Aut}\Gamma$ preserving the bipartition. By [9], as a primitive group on E, only four of the eight O'Nan-Scott types for (quasi)primitive groups may occur for $\mathsf{Aut}\Gamma$, namely, SD, CD, PA and AS. For the first two types, Γ is bipartite and $\mathsf{Aut}^+\Gamma$ is quasiprimitive of type CD on each bipartite half. For the last two types, with one exceptional case, $\mathsf{Aut}\Gamma$ or $\mathsf{Aut}^+\Gamma$ is quasiprimitive on V or on each bipartite half respectively of the same type for $\mathsf{Aut}\Gamma$ on E. In this paper, we will work on the types of $\mathsf{Aut}\Gamma$ on E and on V under the further assumption that Γ is 2-arc-transitive.

The interest for edge-primitive graphs arises partially from the fact that many (almost) simple groups may be represented as the automorphism groups of edge-primitive graphs. Consulting the Atlas [3], one may get first-hand such examples. For example, the sporadic Higman-Sims group HS is a group of automorphisms of a rank 3 graph (i.e., HS acts on the vertex set as a transitive permutation group of rank 3) with order 100 and valency 22, which is in fact a 2-arc-transitive and edge-primitive graph with automorphism group HS.2; the sporadic Rudvalis group Ru is the automorphism group of a rank 3 graph with order 4060 and valency 2304, which is edge-primitive but not 2-arc-transitive. Besides, the almost simple groups $\mathrm{PSU}(3,5).2$, $\mathrm{M}_{22}.2$, $\mathrm{J}_2.2$ and McL.2 all have representations on edge-primitive graphs. The reader may refer to [11, 12, 18, 21, 26] for more examples of edge-primitive graphs which have almost simple automorphism groups. Of course, using the constructions given in [9], one can easily construct examples of edge-primitive graphs with automorphism groups not almost simple.

From the known examples of edge-primitive graphs in the literature, we get the impression that a 2-arc-transitive edge-primitive graph has almost simple automorphism group unless it is a cycle or a complete bipartite graph. In Section 3, we shall prove the following result.

Theorem 1.1. *Let $\Gamma = (V, E)$ be an edge-primitive d-regular graph for some $d \geq 3$. If Γ is 2-arc-transitive, then either Γ is a complete bipartite graph, or $\mathsf{Aut}\Gamma$ is almost simple.*

Remarks on Theorem 1.1. (1) Li and Zhang [18] proved that 4-arc-transitive and edge-primitive graphs have almost simple automorphism groups. Further, as a consequence of their classification on almost simple primitive groups with soluble point-stabilizers, they give a complete list for 4-arc-transitive and edge-primitive graphs.

(2) By Theorem 1.1, appealing to the classification of almost simple groups with soluble maximal subgroups, it might be feasible to classify 2-arc-transitive and edge-primitive graphs with soluble edge-stabilizers.

2. Preliminaries

For the subgroups of (almost) simple groups, we sometimes follow the notation used in the Atlas [3], while we also use $\mathbb{Z}_l$ and $\mathbb{Z}_p^k$ to denote respectively the cyclic group of order l and the elementary abelian group of order p^k.

2.1. **Primitive groups.** In this subsection, Ω is a nonempty finite set, and G is a transitive subgroup of the symmetric group $\mathrm{Sym}(\Omega)$. Let $\mathrm{soc}(G)$ be the socle of G, that is, $\mathrm{soc}(G)$ is generated by all minimal normal subgroups of G.

Consider the point-stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$, where $\alpha \in \Omega$. Then

(1) G is primitive if G_α is a maximal subgroup of G;
(2) G is $\frac{3}{2}$-transitive if G_α is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$, that is, all G_α-orbits on $\Omega \setminus \{\alpha\}$ have equal length > 1;
(3) G is a Frobenius group if G_α is semiregular on $\Omega \setminus \{\alpha\}$;
(4) G is 2-transitive if G_α is transitive on $\Omega \setminus \{\alpha\}$.

Note that a 2-transitive group is also primitive and $\frac{3}{2}$-transitive, and a $\frac{3}{2}$-transitive group is either a primitive group or a Frobenius group (refer to [29, Theorem 10.4]).

Let $1 \neq N \trianglelefteq G$. Then N is $\frac{1}{2}$-transitive, and $N_\alpha = N \cap G_\alpha \trianglelefteq G_\alpha$, and so G_α is contained in the normalizer $\mathbf{N}_G(N_\alpha)$ of N_α in G. Thus, if G_α is maximal then either $N_\alpha \trianglelefteq G$ or $\mathbf{N}_G(N_\alpha) = G_\alpha$. The former case yields $N_\alpha = 1$, while the latter case gives

$$\mathbf{N}_N(N_\alpha) = N \cap \mathbf{N}_G(N_\alpha) = N \cap G_\alpha = N_\alpha.$$

Then we have following simple fact for primitive groups.

Lemma 2.1. *Assume that G is primitive on Ω and N is a normal subgroup of G with $N \neq 1$. Then either N is regular on Ω or N_α is self-normalizing. If further G is 2-transitive then N is either regular or $\frac{3}{2}$-transitive on Ω.*

For an almost simple 2-transitive group G, each non-trivial normal subgroup N of G is primitive, and in fact 2-transitive except for the case where $N = \mathrm{soc}(G) = \mathrm{PSL}(2, 8)$ acting on 28 points, refer to [1, page 197, Table 7.4]. Next we consider the normal subgroups of affine 2-transitive groups. Refer to [1, page 195, Table 7.3] for a complete list of affine 2-transitive groups. We consider the affine 2-transitive groups in their natural actions.

Lemma 2.2. *Let G be an affine 2-transitive group and $1 \neq N \trianglelefteq G$. If N is imprimitive on Ω, then N is a soluble Frobenius group, N_0 is cyclic, and either $G_0 \leq \Gamma\mathrm{L}(1, q)$ or $N_0 \leq \mathbf{Z}(G_0)$, where q is not a prime.*

Proof. Assume that N is imprimitive. Then $N \neq G$, and so $N_0 \neq G_0$. Further, by Lemma 2.1 and [29, Theorem 10.4], N is a Frobenius group. Let $|\Omega| = p^k$ for a prime p. We may write $G_0 \leq \mathrm{GL}(k, p)$, $G = \mathbb{Z}_p^k{:}G_0$ and $N = \mathbb{Z}_p^k{:}N_0$. Since N is imprimitive, N_0 is not maximal in N, and thus N_0 is a normal reducible subgroup of G_0. Then, by [13, Lemma 5.1], N_0 is cyclic and $|N_0|$ is a divisor of $p^l - 1$, where $l < k$ and $l \,\big|\, k$. Finally, the lemma follows from checking all affine 2-transitive groups one by one. □

If every minimal normal subgroup of G is transitive on Ω, then G is called a quasiprimitive group. Praeger [22, 24] generalized the O'Nan-Scott Theorem for primitive groups to quasiprimitive groups, which says that a quasiprimitive group has one of the following eight types: HA, HS, HC, TW, AS, SD, CD and PA. In particular, if G is quasiprimitive then G has at most two minimal normal subgroups, and if two (for HS and HC) then they are isomorphic and regular.

Suppose that G has a transitive insoluble minimal normal subgroup N. Then $G = NG_\alpha$ for $\alpha \in \Omega$. Write $N = T_1 \times \cdots \times T_k$ for isomorphic nonabelian simple groups T_i

and integer $k \geq 1$. Then G_α acts transitively on $\{T_i \mid 1 \leq i \leq k\}$ by conjugation. Note that, for $g \in G_\alpha$ and $1 \leq i \leq k$,

$$((T_i)_\alpha)^g = (T_i \cap G_\alpha)^g = T_i^g \cap G_\alpha^g = (T_i)_\alpha^g = (T_j)_\alpha \text{ for some } j.$$

Thus G_α acts transitively on $\{(T_i)_\alpha \mid 1 \leq i \leq k\}$ by conjugation. Clearly, $(T_1)_\alpha \times \cdots \times (T_k)_\alpha \leq N_\alpha$; however, the equality does not necessarily hold even if G is quasiprimitive. A sufficient condition for this equality is that G is primitive and of type AS or PA, refer to [4, Theorem 4.6] and its proof. In conclusion, we have the simple fact as follows.

Lemma 2.3. *Assume that G has a transitive minimal normal subgroup $N = T_1 \times \cdots \times T_k$, where T_i are isomorphic nonabelian simple groups. Let $\alpha \in \Omega$. Then G_α acts transitively on $\{(T_i)_\alpha \mid 1 \leq i \leq k\}$ by conjugation. If further G is primitive and of type* AS *or* PA*, then $N_\alpha = (T_1)_\alpha \times \cdots \times (T_k)_\alpha$.*

2.2. **Locally-primitive graphs.** In this subsection, $\Gamma = (V, E)$ is a connected d-regular graph for some $d \geq 3$, and $G \leq \mathsf{Aut}\Gamma$. Assume further that the graph Γ is G-locally primitive, that is, G_v acts primitively on $\Gamma(v)$ for all $v \in V$.

Fix an edge $\{u, v\} \in E$. Note that G_v induces a primitive permutation group $G_v^{\Gamma(v)}$ (on $\Gamma(v)$). Let $G_v^{[1]}$ be the kernel of G_v acting on $\Gamma(v)$. Then $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$. Set $G_{uv}^{[1]} = G_u^{[1]} \cap G_v^{[1]}$. Then $G_v^{[1]}$ induces a normal subgroup of $(G_u^{\Gamma(u)})_v$ with the kernel $G_{uv}^{[1]}$, and so $(G_v^{[1]})^{\Gamma(u)} \cong G_v^{[1]}/G_{uv}^{[1]}$.

Assume that G is transitive on V. Then $G_{uv}^{[1]}$ is a p-group for some prime p, refer to [6]. Note that G is transitive on the arc set of Γ. There is some element in G interchanging u and v. This implies that $(G_v^{[1]})^{\Gamma(u)} \trianglelefteq (G_u^{\Gamma(u)})_v \cong (G_v^{\Gamma(v)})_u$. Suppose that $G_v^{\Gamma(v)}$ is soluble. Then $(G_v^{\Gamma(v)})_u$ is soluble, and hence $(G_u^{\Gamma(u)})_v$ is soluble. Thus $(G_v^{[1]})^{\Gamma(u)}$ is soluble. Recalling that $(G_v^{[1]})^{\Gamma(u)} \cong G_v^{[1]}/G_{uv}^{[1]}$ and $G_v^{\Gamma(v)} \cong G_v/G_v^{[1]}$, it follows that G_v is soluble. Thus we have the following lemma.

Lemma 2.4. *Assume that G is transitive on V, and $\{u, v\} \in E$. Then $G_{uv}^{[1]}$ is a p-group, and $(G_v^{[1]})^{\Gamma(u)}$ is isomorphic to a normal subgroup of a point-stabilizer in $G_v^{\Gamma(v)}$. In particular, G_v is soluble if and only if $G_v^{\Gamma(v)}$ is soluble.*

The graph $\Gamma = (V, E)$ is said to be (G, s)-arc-transitive if Γ has an s-arc and G acts transitively on the set of s-arcs of Γ, where $s \geq 1$. Note that Γ is $(G, 2)$-arc-transitive if and only if G is transitive on V, and $G_v^{\Gamma(v)}$ is a 2-transitive group for some (and hence every) $v \in V$. By [7, 27, 28], we have the following result.

Theorem 2.5. *Assume that $\Gamma = (V, E)$ is $(G, 2)$-arc-transitive. Then Γ is not $(G, 8)$-arc-transitive. Further,*

(1) *if $G_{uv}^{[1]} = 1$ then Γ is not $(G, 4)$-arc-transitive.*
(2) *if $G_{uv}^{[1]} \neq 1$ then $G_{uv}^{[1]}$ is a nontrivial p-group for some prime p, $\mathbf{O}_p(G_{uv}^{\Gamma(v)}) \neq 1$, $\mathrm{PSL}(n, q) \trianglelefteq G_v^{\Gamma(v)}$, and $|\Gamma(v)| = \frac{q^n-1}{q-1}$, where $n \geq 2$ and q is a power of p; in this case, Γ is $(G, 4)$-arc-transitive if and only if $n = 2$.*

3. The proof of Theorem 1.1

In this section, we let $\Gamma = (V, E)$ be a connected graph of valency $d \geq 3$, and $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is G-edge-primitive, that is, G acts primitively on E. Then, by [9, Lemma 3.4], G acts transitively on the arc set of Γ. Thus, for an edge $\{u, v\} \in E$, $d = |G_v : G_{uv}|$ and $|G_{\{u,v\}} : G_{uv}| = 2$.

Let $1 \neq N \trianglelefteq G$. Then N is transitive on E, and so either N is transitive on V or N has two orbits on V; for the latter case, N_v is transitive on $\Gamma(v)$. This implies that either $G = NG_v$, or $|G : (NG_v)| = 2$ and $N_{uv} = N_{\{u,v\}}$. Note that $G = NG_{\{u,v\}}$ by the maximality of $G_{\{u,v\}}$ or the transitivity of N on E. We have

$$\begin{aligned} |G| &= \tfrac{|N||G_{\{u,v\}}|}{|N\cap G_{\{u,v\}}|} = \tfrac{|N||G_{\{u,v\}}|}{|N_{\{u,v\}}|} = \tfrac{2|N||G_{uv}|}{|N_{\{u,v\}}|} = \tfrac{2|N||G_v|}{d|N_{\{u,v\}}|} \\ &= \tfrac{|N||G_v|}{|N_v|}\cdot\tfrac{2|N_v|}{d|N_{\{u,v\}}|} = |NG_v|\tfrac{2|N_v|}{d|N_{\{u,v\}}|}. \end{aligned}$$

Then the next lemma follows.

Lemma 3.1. *Let $1 \neq N \trianglelefteq G$. If N is transitive on V then $2|N_v| = d|N_{\{u,v\}}|$; if N is intransitive on V then $|N_v| = d|N_{\{u,v\}}| = d|N_{uv}|$. In particular, $N_v \neq 1$ and $N_{uv} \neq N_v \neq N_{\{u,v\}}$.*

Let $\mathsf{K}_{d,d}$ and K_{d+1} be the complete bipartite graph and complete graph of valency d, respectively.

Lemma 3.2. *Let $1 \neq N \trianglelefteq G$. Then either $\Gamma \cong \mathsf{K}_{d,d}$, or $N_{uv} \neq 1$ and $N_{\{u,v\}}$ is self-normalizing in N, where $\{u, v\} \in E$.*

Proof. Assume that $\Gamma \not\cong \mathsf{K}_{d,d}$. Then, by the O'Nan-Scott Theorem and [9, Lemmas 6.1, 6.2 and Propersition 6.13], G has no normal subgroup acting regularly on E. Thus $N_{\{u,v\}} \neq 1$, and so $N_{\{u,v\}}$ is self-normalizing in N by Lemma 2.1.

Suppose that $N_{uv} = 1$. Then $N_{\{u,v\}}$ has order 2, and so $N_{\{u,v\}} \leq \mathbf{C}_N(N_{\{u,v\}}) \leq \mathbf{N}_N(N_{\{u,v\}}) = N_{\{u,v\}}$. This implies that $\mathbf{C}_N(N_{\{u,v\}}) = \mathbf{N}_N(N_{\{u,v\}})$, and then $N_{\{u,v\}}$ is a Sylow 2-subgroup of N. By Burnside's transfer theorem (refer to [14, IV.2.6]), N has a normal Hall $2'$-subgroup, say M. Then this M is normal in G and regular on E, a contradiction. □

Suppose that $\Gamma \not\cong \mathsf{K}_{d,d}$. By [9], as a primitive group on E, the O'Nan-Scott type of G is one of SD, CD, AS and PA. Then G has a unique minimal normal subgroup, which is insoluble, refer to [22, 24]. In particular, G is insoluble, and so $G_{\{u,v\}}$ is not abelian by [14, IV.7.4]. For the case where the arc-stabilizer G_{uv} is abelian, the following result says that Γ is a complete graph.

Theorem 3.3. *Assume that $\Gamma \not\cong \mathsf{K}_{d,d}$. Let $1 \neq N \trianglelefteq G$.*

1. *If $N_{\{u,v\}}$ has a normal Sylow subgroup $P \neq 1$ then P is also a Sylow subgroup of N; in particular, $N_{\{u,v\}}$ is not abelian.*
2. *If N_{uv} is abelian then N is transitive on the arc set of Γ.*
3. *If N_{uv} is an abelian 2-group then $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ and $\Gamma \cong \mathsf{K}_{q+1}$, where q is a power of some prime with $q - 1$ a power of 2 greater than 8.*
4. *If G_{uv} is an abelian group then $d = q$ and either $\mathrm{soc}(G) \cong \mathrm{PSL}(2, q)$ and $\Gamma \cong \mathsf{K}_{q+1}$, or $\mathrm{soc}(G) = \mathrm{Sz}(q)$, $\mathsf{Aut}\Gamma = \mathsf{Aut}(\mathrm{Sz}(q))$ and Γ is $(\mathrm{Sz}(q), 2)$-arc-transitive, where q is a power of some prime.*

Proof. (1) Assume that $P \neq 1$ is a normal Sylow p-subgroup of $N_{\{u,v\}}$. Then P is a characteristic subgroup of $N_{\{u,v\}}$, and so $P \trianglelefteq G_{\{u,v\}}$ as $N_{\{u,v\}} \trianglelefteq G_{\{u,v\}}$. Thus $\mathbf{N}_G(P) \geq G_{\{u,v\}}$, and then $\mathbf{N}_G(P) = G_{\{u,v\}}$ by the maximality of $G_{\{u,v\}}$. This gives $\mathbf{N}_N(P) = N \cap \mathbf{N}_G(P) = N \cap G_{\{u,v\}} = N_{\{u,v\}}$. Choose a Sylow p-subgroup Q of N with $P \leq Q$. Then $\mathbf{N}_Q(P) \leq Q \cap \mathbf{N}_G(P) = Q \cap N_{\{u,v\}} = P$. This yields $P = Q$, so P is a Sylow p-subgroup of N.

Suppose that $N_{\{u,v\}}$ is abelian. Let $R \neq 1$ be a Sylow subgroup of $N_{\{u,v\}}$. Then R is a Sylow subgroup of N, and $N_{\{u,v\}} \leq \mathbf{C}_N(Q) \leq \mathbf{N}_N(R) = N_{\{u,v\}}$, yielding $\mathbf{C}_N(R) = \mathbf{N}_N(R)$. By Burnside's transfer theorem, R has a normal complement H in N, that is $N = RH$ with $R \cap H = 1$ and $H \trianglelefteq N$. Note that H is a Hall subgroup of N. It follows that H is characteristic in N, and hence $H \trianglelefteq G$. Let R runs over the Sylow subgroups of $N_{\{u,v\}}$. Then the resulting normal complements intersect at a normal complement of $N_{\{u,v\}}$ in N, which is normal in G and regular on E. This contradicts Lemma 3.2. Therefore, $N_{\{u,v\}}$ is nonabelian, and (1) of this theorem follows.

(2) Assume that N_{uv} is abelian. Then $N_{uv} \neq N_{\{u,v\}}$ by (1), and thus $(u, v) = (v, u)^x$ for some $x \in N_{\{u,v\}}$. Since $\mathit{\Gamma}$ is N-edge-transitive, $\mathit{\Gamma}$ is N-arc-transitive.

(3) Assume that N_{uv} is an abelian 2-group. Recall that G has a unique minimal normal subgroup, say M. Then $M \leq N$, and (1) and (2) hold for M. Then, since M_{uv} is an abelian 2-group, $M_{\{u,v\}}$ is a Sylow 2-subgroup of M, and $M_{\{u,v\}}$ is not abelian.

Write $M = T_1 \times \cdots \times T_k$, where T_i are isomorphic nonabelian simple groups. Recall that $M_{\{u,v\}}$ is a Sylow 2-subgroup of M. For each i, choose a Sylow 2-subgroup Q_i of T_i with $Q_i \leq M_{\{u,v\}}$. Then $M_{\{u,v\}} = Q_1 \times \cdots \times Q_k$. Noting that Q_i are all isomorphic, every Q_i is nonabelian; otherwise, $M_{\{u,v\}}$ is abelian, a contradiction. In particular, $Q_1 \not\leq M_{uv}$. Then $M_{\{u,v\}} = M_{uv}Q_1$, and so

$$Q_2 \times \cdots \times Q_k \cong M_{\{u,v\}}/Q_1 = M_{uv}Q_1/Q_1 \cong M_{uv}/(M_{uv} \cap Q_1).$$

Since M_{uv} is abelian, the only possibility is $k = 1$. Thus $M = \mathrm{soc}(G)$ is simple.

By [10, Corollary 5], $M_{\{u,v\}}$ has cyclic commutator subgroup. Since $M_{\{u,v\}}$ is nonabelian, by [2], M is isomorphic to one of the groups M_{11}, $\mathrm{PSL}(2, q)$ (with q^2-1 divisible by 16), $\mathrm{PSL}(3, q)$ (with q odd) and $\mathrm{PSU}(3, q)$ (with q odd). If $M \cong \mathrm{M}_{11}$, then $G = M$, and so $M_{\{u,v\}}$ is maximal in M; however, by the Atlas [3], a Sylow 2-subgroup of M_{11} is not a maximal subgroup, a contradiction. Thus we next let $M \cong \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$ or $\mathrm{PSU}(3, q)$.

Since M is transitive on E, we know that $|E| = |M : M_{\{u,v\}}|$ is odd. Thus G is an almost simple primitive group (on E) of odd degree. Noting that $M_{\{u,v\}} = M \cap G_{\{u,v\}}$, by [20], $M_{\{u,v\}}$ is known. Noting the isomorphisms among simple groups (refer to [15, Proposition 2.9.1 and Theorem 5.1.1]), since $M_{\{u,v\}}$ is a Sylow 2-subgroup of M, the only possibility is that $M \cong \mathrm{PSL}(2, q)$, and $M_{\{u,v\}}$ is the stabilizer of some orthogonal decomposition of a natural projective module associated with M into 1-dimensional subspaces. It follows that $M_{\{u,v\}} \cong \mathrm{D}_{q-1}$ or D_{q+1}, and so $M_{uv} \cong \mathbb{Z}_{\frac{q-1}{2}}$ or $\mathbb{Z}_{\frac{q+1}{2}}$, respectively. Since M is transitive on the arc set of $\mathit{\Gamma}$, we have $|M_v : M_{uv}| = d \geq 3$. Checking the subgroups of $\mathrm{PSL}(2, q)$ (refer to [14, II.8.27]), we conclude that $M_{uv} \cong \mathbb{Z}_{\frac{q-1}{2}}$, $d = q$, $V = |M : M_v| = q + 1$ and M is 2-transitive on V. Thus $\mathit{\Gamma} \cong \mathsf{K}_{q+1}$.

(4) Assume that G_{uv} is abelian. Let M be the unique minimal normal subgroup of G. If M_{uv} is a 2-group, then (4) of this theorem follows from (3).

We next assume that $|M_{uv}|$ has an odd prime divisor p. By (1), the unique Sylow p-subgroup of M_{uv} is also a Sylow p-subgroup of M. Write $M = T_1 \times \cdots \times T_k$, where T_i are isomorphic nonabelian simple groups. By (1) of this theorem, $\mathrm{M}_{\{u,v\}}$ is not abelian, so $M_{\{u,v\}} \not\leq G_{uv}$, and then $G_{\{u,v\}} = M_{\{u,v\}}G_{uv}$. Thus $G = MG_{uv}$, and hence G_{uv} acts transitively on $\{T_1, \ldots, T_k\}$ by conjugation. Choose, for each i, a Sylow p-subgroup P_i of T_i such that $P_1 \times \cdots \times P_k$ is the unique Sylow subgroup of M_{uv}. Since G_{uv} is abelian, we have $P_1 = P_1^x \leq T_1^x$ for $x \in G_{uv}$. It follows that $P_1 \leq T_i$ for all i. The only possibility is that $k = 1$, and so M is simple.

Note that G is an almost simple group with a soluble maximal subgroup $G_{\{u,v\}}$. Then, by [18], both $M = \mathrm{soc}(G)$ and $M_{\{u,v\}} = M \cap G_{\{u,v\}}$ are known. Since $M_{\{u,v\}}$ has an abelian subgroup of index 2, it follows that either $M \cong \mathrm{PSL}(2, q)$ and $M_{\{u,v\}} \cong \mathrm{D}_{\frac{2(q\pm1)}{(2,q-1)}}$, or $M = \mathrm{Sz}(q)$ and $M_{\{u,v\}} \cong \mathrm{D}_{2(q-1)}$. Recalling that $G = MG_{uv}$, we know that M is transitive on V. By Lemma 3.1, $|M_v| = \frac{d}{2}|M_{\{u,v\}}|$. Check the subgroups of M, refer to [25] for $\mathrm{Sz}(q)$. For $M \cong \mathrm{PSL}(2, q)$, we have $M_v \cong [q]{:}\mathbb{Z}_{\frac{q-1}{(2,q-1)}}$, and then $\Gamma \cong \mathsf{K}_{q+1}$. Assume that $M = \mathrm{Sz}(q)$ and $M_{\{u,v\}} \cong \mathrm{D}_{2(q-1)}$. Then $M_v \cong [q]{:}\mathbb{Z}_{q-1}$ and $d = q$; in this case, Γ is $(M, 2)$-arc-transitive. By [5], we have that $\mathsf{Aut}\Gamma = \mathsf{Aut}(\mathrm{Sz}(q))$ and Γ is unique up to isomorphism. Thus (4) of this theorem follows. □

Lemma 3.4. *Assume that G has type* PA *on E. Let* $\mathrm{soc}(G) = T_1 \times \cdots \times T_k$. *Then* $(T_i)_{uv} \neq 1$ *for each i and $\{u, v\} \in E$; in particular, every T_i is neither semiregular on V nor semiregular on E.*

Proof. Let $M = \mathrm{soc}(G)$. By Lemma 2.3, $M_{\{u,v\}} = (T_1)_{\{u,v\}} \times \cdots \times (T_k)_{\{u,v\}}$, and $(T_i)_{\{u,v\}}$ all have equal order. By Theorem 3.3, $M_{\{u,v\}}$ is nonabelian. Thus $(T_i)_{\{u,v\}}$ is nonabelian for all i. Then the lemma follows. □

For the case where Γ is a bipartite graph, we let G^+ be the subgroup of G preserving the bipartition of Γ. Then $|G : G^+| = 2$, and each bipartite half of Γ is a G^+-orbit on V.

Lemma 3.5. *Assume that the graph $\Gamma = (V, E)$ is $(G, 2)$-arc-transitive, and G has type* PA *on E. Then either $\Gamma \cong \mathsf{K}_{d,d}$, or one of the following holds:*

(1) *G is quasiprimitive on V;*
(2) *Γ is bipartite, and G^+ is faithful and quasiprimitive on each bipartite half of Γ.*

Proof. Since G is primitive on E, every minimal normal subgroup of G is transitive on E, and so has at most two orbits on V. If Γ is not bipartite then G is quasiprimitive on V.

Now let Γ be bipartite with bipartition, say, $V = V_1 \cup V_2$. Note that $G_v \leq G^+$ for each $v \in V$. Then G^+ is locally-primitive on Γ. Suppose that $\Gamma \not\cong \mathsf{K}_{d,d}$. Then, by [23], G^+ is faithful on both V_1 and V_2, and either (2) of this lemma holds, or the unique minimal normal subgroup of G is a direct product $M_1 \times M_2$, where M_1 and M_2 are normal in G^+ and conjugate in G, and M_i is intransitive on V_i for $i = 1, 2$. For the latter case, if M_1 is intransitive on V_2 then M_1 is semiregular on V by [8, Lemma 5.1]; if M_1 is transitive on V_2 then M_2 is semiregular on V_2. These two cases all contradict Lemma 3.4. Thus G^+ is quasiprimitive on both V_1 and V_2. □

As permutation groups on V and on E, the types of G (and G^+) have been determined in [9]. Then by Lemma 3.5 and combined with the reduction theorems for 2-arc-transitive graphs given by Praeger [22, 23], we get the following result.

Lemma 3.6. *Assume that the graph $\Gamma = (V, E)$ is $(G, 2)$-arc-transitive. Suppose that $\Gamma \not\cong \mathsf{K}_{d,d}$. If G is not almost simple, then G has type* PA *on E and either*
(1) *G is quasiprimitive and of type* PA *on V; or*
(2) *Γ is bipartite, G^+ is faithful and quasiprimitive on each bipartite half of Γ with type* PA*.*

Now we are ready to give a proof of Theorem 1.1.

Theorem 3.7. *Let $\Gamma = (V, E)$ be a connected d-regular graph for some $d \geq 3$, and let $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is both G-edge-primitive and $(G, 2)$-arc-transitive. Then either $\Gamma \cong \mathsf{K}_{d,d}$, or G is almost simple.*

Proof. Assume that $\Gamma \not\cong \mathsf{K}_{d,d}$, and let $\{u, v\} \in E$. By the 2-arc-transitivity of G on Γ, we know that $G_v^{\Gamma(v)}$ is a 2-transitive permutation group of degree d.

Let $M = \mathrm{soc}(G) = T_1 \times \cdots \times T_k$, where T_i are isomorphic nonabelian simple groups. Then $M_v \trianglelefteq G_v$, and $1 \neq M_v \neq M_{uv}$ by Lemma 3.1; in particular, $M_v \not\leq G_v^{[1]}$. Thus $M_v^{\Gamma(v)}$ is a transitive normal subgroup of $G_v^{\Gamma(v)}$.

Assume that $M_v^{\Gamma(v)}$ is primitive on $\Gamma(v)$. Noting that G is transitive on V, we conclude that $M_w^{\Gamma(w)}$ is primitive for every $w \in V$. Thus Γ is M-locally primitive. Then, by Lemma 3.4 and [8, Lemma 5.1], we conclude that $k = 1$, and so G is almost simple.

Next assume that $M_v^{\Gamma(v)}$ is imprimitive on $\Gamma(v)$.

Note that every non-trivial normal subgroup of an almost simple 2-transitive group is primitive. Then $G_v^{\Gamma(v)}$ is an affine 2-transitive group, and by Lemma 2.2, $M_v^{\Gamma(v)}$ is a soluble Frobenius group and $(M_v^{\Gamma(v)})_u$ is cyclic. Set $(M_v^{\Gamma(v)})_u \cong \mathbb{Z}_e$ and $\mathrm{soc}(G_v^{\Gamma(v)}) \cong \mathbb{Z}_r^l$ for a prime r and integer $l \geq 1$ with $d = r^l$. Then e is a divisor of $r^l - 1$, and $e < r^l - 1$.

Assume that $e = 1$. Then $M_v^{\Gamma(v)} = \mathrm{soc}(G_v^{\Gamma(v)}) \cong \mathbb{Z}_r^l$, and so $M_v^{\Gamma(v)}$ is regular on $\Gamma(v)$. By [17, Lemma 2.3], M_v is faithful and hence regular on $\Gamma(v)$, and thus $M_{uv} = 1$, which contradicts Lemma 3.2. Thus $e \neq 1$.

If $l = 1$ then $|\Gamma(v)| = d = r$ and $M_v^{\Gamma(v)}$ is primitive on $\Gamma(v)$, a contradiction. Thus $l > 1$. Note that e is a proper divisor of $d - 1 = r^l - 1$. Then $d - 1$ is a not a prime. It follows that $d = r^l \geq 9$. Since $G_v^{\Gamma(v)}$ is an affine 2-transitive group of degree d, $G_v^{\Gamma(v)}$ has no normal subgroup isomorphic to a projective special linear group of dimension ≥ 2. By Theorem 2.5, $G_{uv}^{[1]} = 1$, and so $M_{uv}^{[1]} = 1$.

Let $x \in G_{\{u,v\}} \setminus G_{uv}$. Then $(u, v)^x = (v, u)$, this implies that $M_v^{\Gamma(v)}$ and $M_u^{\Gamma(u)}$ are permutation isomorphic. In particular, $(M_u^{\Gamma(u)})_v \cong (M_v^{\Gamma(v)})_u = \mathbb{Z}_e$. Since $M_v^{[1]} \cap M_u^{[1]} = M_{uv}^{[1]} = 1$, we know that M_{uv} is isomorphic to a subgroup of $(M_{uv}/M_u^{[1]}) \times (M_{uv}/M_v^{[1]})$. Note that $M_{uv}/M_v^{[1]} \cong (M_v^{\Gamma(v)})_u$ and $M_{uv}/M_u^{[1]} \cong (M_u^{\Gamma(u)})_v$. Then M_{uv} is isomorphic to a subgroup of $\mathbb{Z}_e \times \mathbb{Z}_e$. In particular, M_{uv} is abelian. Then, by Theorem 3.3, M is transitive on the arc set of Γ, and so $M_{\{u,v\}} = M_{uv}.2$.

If e is a power of 2 then, by Theorem 3.3, $M \cong \mathrm{PSL}(2, r^l)$, $\Gamma \cong \mathsf{K}_{r^l+1}$; however, in this case, M is locally primitive on Γ, a contradiction. Thus e has odd prime divisors.

Let s be an odd prime divisor of e, and S be a Sylow s-subgroup of M_{uv}. Then, noting that $M_{\{u,v\}} = M_{uv}.2$, we know that S is also a Sylow s-subgroup of M by Theorem 3.3. Thus $S = S_1 \times \cdots \times S_k$, where S_i is a Sylow s-subgroup of T_i for $1 \le i \le k$. Since M_{uv} is isomorphic to a subgroup of $\mathbb{Z}_e \times \mathbb{Z}_e$, we know that M_{uv} has no subgroup isomorphic to $\mathbb{Z}_s^3$. It follows that $k \le 2$.

Now we deduce a contradiction by supposing that $k = 2$.

Let $k = 2$. Since $G \le (\mathsf{Aut}(T_1) \times \mathsf{Aut}(T_1))$:2, we have

$$G_{\{u,v\}}/M_{\{u,v\}} = G_{\{u,v\}}/(M \cap G_{\{u,v\}}) \cong MG_{\{u,v\}}/M = G/M \le (\mathsf{Out}(T_1) \times \mathsf{Out}(T_1)){:}2.$$

It follows that $G_{\{u,v\}}/M_{\{u,v\}}$ is soluble, and so $G_{\{u,v\}}$ is soluble as $M_{\{u,v\}}$ is soluble. Thus $(G_v^{\Gamma(v)})_u$ is soluble, and $G_v^{\Gamma(v)} = \mathrm{soc}(G_v^{\Gamma(v)}){:}(G_v^{\Gamma(v)})_u$ is also soluble. Checking the soluble affine 2-transitive groups, by Lemma 2.2, $(G_v^{\Gamma(v)})_u \le \Gamma\mathrm{L}(1, r^l)$ or $\mathbb{Z}_e \cong (M_v^{\Gamma(v)})_u \le \mathbf{Z}((G_v^{\Gamma(v)})_u) \cong \mathbb{Z}_2$. Note that $(M_v^{\Gamma(v)})_u$ is a reducible subgroup of $(G_v^{\Gamma(v)})_u$. Recalling that e is not a power of 2, the latter case does not occur.

Since $|M_{\{u,v\}} : M_{uv}| = 2$, we have $M_{\{u,v\}} \not\le G_{uv}$, and so $G_{uv} \ne M_{\{u,v\}}G_{uv} \le G_{\{u,v\}}$. Then $M_{\{u,v\}}G_{uv} = G_{\{u,v\}}$, and $G = MG_{\{u,v\}} = MG_{uv}$. Recalling that $M = T_1 \times T_2$, it follows that G_{uv} acts transitively on $\{T_1, T_2\}$ by conjugation. Let H be the kernel of this action. Then $|G_{uv} : H| = 2$, and each T_i is normalized by H. For $h \in H$,

$$((T_i)_v)^h = (T_i \cap G_v)^h = T_i^h \cap (G_v)^h = T_i \cap G_v = (T_i)_v,\ i = 1, 2.$$

This implies that H normalizes each $(T_i)_v$. Then $(T_i)_v^{\Gamma(v)}$ is normalized by $H^{\Gamma(v)}$. Note that $(T_i)_v^{\Gamma(v)}$ is a normal subgroup of $M_v^{\Gamma(v)} = \mathrm{soc}(G_v^{\Gamma(v)}){:}(M_v^{\Gamma(v)})_u$, and $e = |(M_v^{\Gamma(v)})_u|$ is a proper divisor of $r^l - 1$. Let K_i be the Sylow r-subgroup of $(T_i)_v^{\Gamma(v)}$. Then K_i is normalized by $H^{\Gamma(v)}$, and $K_i \le \mathrm{soc}(G_v^{\Gamma(v)})$ and $K_1 \cap K_2 = 1$.

Recalling that $|G_{uv} : H| = 2$, we have $|(G_v^{\Gamma(v)})_u : H^{\Gamma(v)}| \le 2$. Since $G_v^{\Gamma(v)}$ is 2-transitive, $|(G_v^{\Gamma(v)})_u|$ is divisible by $r^l - 1$, and so $|H^{\Gamma(v)}|$ is divisible by $\frac{r^l-1}{2}$. Note that $\frac{r^l-1}{2} > \frac{r^l}{2} - 1 \ge r^{l-1} - 1$. Then $|H^{\Gamma(v)}|$ is not a divisor of $r^b - 1$ for any $1 \le b < l$. Then, by [13, Lemma 5.1], $H^{\Gamma(v)}$ is irreducible on $\mathrm{soc}(G_v^{\Gamma(v)})$. It implies that $K_1 = K_2 = 1$, and thus $(T_i)_v^{\Gamma(v)} \le (M_v^{\Gamma(v)})_u$ for $i = 1, 2$. Let u run over $\Gamma(v)$. It follows that $(T_i)_v^{\Gamma(v)} = 1$, and hence $(T_i)_v \le M_v^{[1]}$, $i = 1, 2$. Since M is transitive on V, by [17, Lemma 2.3], we have $(T_1)_v = (T_2)_v = 1$, which contradicts Lemma 3.4. This completes the proof. □

As a consequence of Theorems 3.3 and 3.7, an edge-primitive graph of prime valency is 2-arc-transitive, and then it has almost simple automorphism group if it is not a complete bipartite graph. See also [21].

Corollary 3.8. *Assume that d is a prime and $\Gamma \not\cong \mathsf{K}_{d,d}$. Then G is almost simple, and either $G = \mathrm{PSL}(2, d)$ with $d > 11$ and $\Gamma \cong \mathsf{K}_{d+1}$ or G is transitive on the set of 2-arcs of Γ.*

Proof. Note that G is transitive on the arc set of Γ. Let $\{u, v\} \in E$. By Theorem 3.7, it suffices to deal with the case where $G_v^{\Gamma(v)}$ is not 2-transitive.

Suppose that $G_v^{\Gamma(v)}$ is not 2-transitive. Then $G_v^{\Gamma(v)} \cong \mathbb{Z}_d{:}\mathbb{Z}_l$ with $l < d - 1$ and l a divisor of $d - 1$. If $l = 1$ then $G_v \cong \mathbb{Z}_d$ by [17, Lemma 2.3], and so $G_{uv} = 1$, which contradicts Lemma 3.2. Then $l > 1$, and so $d \ge 5$. By Theorem 2.5, $G_{uv}^{[1]} = 1$.

Then G_{uv} is isomorphic to a subgroup of $(G_u^{\Gamma(u)})_v \times (G_v^{\Gamma(v)})_u \cong \mathbb{Z}_l \times \mathbb{Z}_l$. Thus G_{uv} is abelian. By Theorem 3.3, $\Gamma \cong \mathsf{K}_{d+1}$, $\mathrm{soc}(G) \cong \mathrm{PSL}(2,d)$, $\mathrm{soc}(G)_v \cong \mathbb{Z}_d{:}\mathbb{Z}_{\frac{d-1}{2}}$ and $\mathrm{soc}(G)_{\{u,v\}} \cong \mathrm{D}_{d-1}$. If $G \cong \mathrm{PGL}(2,d)$ then G is transitive on the set of 2-arcs of Γ, which is not the case. Thus $G \cong \mathrm{PSL}(2,d)$, and so $d > 11$ by the maximality of $G_{\{u,v\}}$. $\Box$

4. Examples

Let $\Gamma = (V,E)$ be a connected d-regular graph, where $d \geq 3$. Let $v \in V$ and $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is $(G,2)$-arc-transitive. Choose an integer $s \geq 2$ such that Γ is (G,s)-arc-transitive but not $(G,s+1)$-arc-transitive; in this case, we call Γ a (G,s)-transitive graph. Then $s \leq 7$ by [28]. If G_v is faithful on $\Gamma(v)$ then $s \leq 3$ by Theorem 2.5, and $s = 3$ yields that $d = 7$ and $G_v \cong \mathrm{A}_7$ or S_7, see [16, Proposition 2.6]. This leads to the following interesting problem: *Do there exist* 3*-arc-transitive graphs with faithful stabilizers?* We next answer this problem by giving several examples of edge-primitive graphs which are 3-arc-transitive and have faithful stabilizers.

The first example is the Hoffman-Singleton graph, which has valency 7, order 50 and automorphism group $G = T.2$, where $T = \mathrm{PSU}(3,5)$. Let $X = T$ or G. For an edge $\{u,v\}$ of this graph, $X_v \cong \mathrm{A}_7$ or S_7 and $X_{\{u,v\}} \cong \mathrm{M}_{10}$ or $\mathrm{P\Gamma L}(2,9)$, which are maximal subgroups of X. Thus the Hoffman-Singleton graph is both X-edge-primitive and $(X,2)$-arc-transitive. To see the 3-arc-transitivity, we fix an edge $\{u,v\}$ and consider the action of the arc-stabilizer X_{uv} ($\cong \mathrm{A}_6$ or S_6) on $\Gamma(u) \cup \Gamma(v)$. By the 2-arc-transitivity of X, we have two faithful transitive actions of X_{uv} on $\Gamma(u)$ and $\Gamma(v)$, respectively. Let $v_1 \in \Gamma(v) \setminus \{u\}$ and $x \in X_{\{u,v\}} \setminus X_{uv}$. Then $u_1 := v_1^x \in \Gamma(u) \setminus \{v\}$, and

$$(X_{uv})_{u_1} = (X_{\{u,v\}})_{u_1} = (X_{\{u,v\}})_{v_1^x} = ((X_{\{u,v\}})_{v_1})^x = ((X_{uv})_{v_1})^x.$$

By the choice of x, we know that $(X_{uv})_{v_1}$ and $((X_{uv})_{v_1})^x$ are not conjugate in X_{uv}, and so do for $(X_{uv})_{v_1}$ and $(X_{uv})_{u_1}$. This implies that the actions of X_{uv} on $\Gamma(u)$ and $\Gamma(v)$ are not equivalent. Thus $(X_{uv})_{v_1}$ acts on $\Gamma(u) \setminus \{v\}$ without fixed-points, this yields that $(X_{uv})_{v_1}$ is transitive on $\Gamma(u) \setminus \{v\}$. It follows that the Hoffman-Singleton graph is $(X,3)$-arc-transitive.

In general, combined with [16, Proposition 2.6], a similar argument as above yields the following result.

Lemma 4.1. *Let $\Gamma = (V,E)$ be a connected d-regular graph for $d \geq 3$, $\{u,v\} \in E$ and $G \leq \mathsf{Aut}\Gamma$. If Γ is $(G,2)$-arc-transitive and G_v is faithful on $\Gamma(v)$, then Γ is $(G,3)$-arc-transitive if and only if $d = 7$, $\mathrm{soc}(G_v) \cong \mathrm{A}_7$ and $G_{\{u,v\}} \not\cong \mathrm{S}_6$, i.e. $G_{\{u,v\}} \cong \mathrm{PGL}(2,9)$, M_{10} or $\mathsf{Aut}(\mathrm{A}_6)$.*

We next give another example.

Example 4.2. By the information given in the Atlas [3] for the O'Nan simple group $\mathrm{O'N}$, there are exactly two conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of (maximal) subgroups isomorphic to A_7, which are merged into one class in $\mathrm{O'N}.2$. Further, there are $H \in \mathcal{C}_1$ and involutions $x_1, x_2 \in \mathrm{O'N}.2 \setminus \mathrm{O'N}$ such that $(H \cap H^{x_i}){:}\langle x_i \rangle$ both are maximal subgroups of $\mathrm{O'N}.2$ with $(H \cap H^{x_1}){:}\langle x_1 \rangle \cong \mathrm{PGL}(2,9)$ and $(H \cap H^{x_2}){:}\langle x_2 \rangle \cong \mathrm{PSL}(2,7){:}2$. Define two bipartite graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ with vertex set $V = \mathcal{C}_1 \cup \mathcal{C}_2$ and edge sets

$E_1 = \{\{H_1, H_2\} \mid H_1 \in \mathcal{C}_1, H_2 \in \mathcal{C}_2, H_1 \cap H_2 \cong \mathrm{A}_6\}$;
$E_2 = \{\{H_1, H_2\} \mid H_1 \in \mathcal{C}_1, H_2 \in \mathcal{C}_2, H_1 \cap H_2 \cong \mathrm{PSL}(2,7)\}$.

Then $\varGamma_1$ and $\varGamma_2$ are both $\mathrm{O'N}.2$-edge-primitive and $(\mathrm{O'N}.2, 2)$-arc-transitive, which have valency 7 and 15 respectively. By Lemma 4.1, only $\varGamma_1$ is $(\mathrm{O'N}.2, 3)$-arc-transitive. □

Lemma 4.3. *Let* $\varGamma_1$ *be as in Example* 4.2. *Then* $\mathsf{Aut}\varGamma_1 = \mathrm{O'N}.2$.

Proof. Let $G = \mathsf{Aut}\varGamma_1$. Then $G \geq \mathrm{O'N}.2$. By Theorem 1.1, G is almost simple, and so $\mathrm{O'N} \leq \mathrm{soc}(G)$. Let $\{u, v\}$ be an edge of $\varGamma_1$. Then $G_v^{\varGamma(v)} \cong \mathrm{A}_7$ or S_7, and $G_{uv}^{[1]} = 1$ by Theorem 2.5. Thus, by the group extensions ($\circledast$) in Section 2, we conclude that $|G_v|$ has no prime divisor other than 2, 3, 5 and 7. Since $\mathrm{O'N}.2$ is transitive on the vertices of $\varGamma_1$, we have $G = (\mathrm{O'N}.2)G_v$. It follows that $|\mathrm{O'N}|$ and $|\mathrm{soc}(G)|$ have the same prime divisors. Using [19, Corollary 5], we get $\mathrm{soc}(G) = \mathrm{O'N}$, and so $G = \mathrm{O'N}.2$. □

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