TWO-ARC-TRANSITIVE GRAPHS OF ODD ORDER – I

CAI HENG LI, JING JIAN LI, AND ZAI PING LU

Abstract. This is one of a series of papers that aims towards to classify finite connected graphs of odd order admitting a 2-arc-transitive almost simple group of automorphisms. This one presents such a classification for an automorphism group that has soluble vertex stabilisers or is an exceptional group of Lie type.

1. Introduction

All graphs in this paper are assumed to be finite, simple and undirected.

Let $\varGamma = (V, E)$ be a graph with vertex set V and edge set E. For a positive integer s, an *s-arc* in $\varGamma$ is a sequence of $s+1$ vertices of which any two consecutive vertices are adjacent and any three consecutive vertices are distinct; in particular, a 2-arc is a triple of distinct vertices (α, β, γ) such that β is adjacent to both α and γ. A graph $\varGamma$ is called *(G, s)-arc-transitive* if a group G of automorphisms is transitive on the set of s-arcs of $\varGamma$, or simply called *s-arc-transitive*.

The study of s-arc-transitive graphs has attracted considerable attentions in the literature since Tutte [23] proved that there is no 6-arc-transitive cubic graph. In particular, Weiss [24] generalized Tutte's result by proving that there is no 8-arc-transitive graph of valency at least 3. Trofimov in 1990's determined vertex stabilisers of s-arc-transitive graphs with $s \geqslant 2$ in a series of papers, refer to [21, 22]. On the other hand, Praeger [20] developed a framework for the study of s-arc-transitive graphs. Some special families of s-arc-transitive graphs have been classified, see for example [2, 5, 6, 10, 11, 14, 18]. However, it would be infeasible to classify 2-arc-transitive graphs in the general case. The first-named author [15] proved that there is no s-arc-transitive graph of odd order and valency at least 3 with $s \geqslant 4$, and an s-arc-transitive graph of odd order with $s = 2$ or 3 is a normal cover of 2-arc-transitive graph admitting an almost simple group. (Recall that an *almost simple group* G is a group such that $S \leqslant G \leqslant \mathsf{Aut}(S)$ for some finite nonabelian simple group S; in other words, S is the *socle* of G and is denoted by $\mathsf{soc}(G)$.) This initiated a project of classifying 2-arc-transitive graphs of odd order, and the following natural problem arises:

Problem A. *Classify 2-arc-transitive graphs of odd order admitting an almost simple group.*

This is one of a series of papers which aim towards to solve this problem. The first main result of this paper is a classification of such graphs with soluble stabiliser.

The project was partially supported by the NNSF of China (11771200, 11931005, 11861012, 11971248, 11731002) and the Fundamental Research Funds for the Central Universities.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph of odd order. Assume that G is an almost simple group with socle T, and the stabiliser G_α is soluble for $\alpha \in V$. Then one of the following statements holds:*

(i) *Γ is the odd graph O_4, and $G = \mathrm{A}_7$ or S_7;*
(ii) *$G = \mathrm{J}_1$, $G_\alpha \cong \mathrm{AGL}(1, 8)$ or $\mathrm{A\Gamma L}(1, 8)$, and Γ has valency 8 and order 3135 or 1045, respectively.*
(iii) *$T = \mathrm{PSL}(2, 2^f)$, Γ is valency 2^f for $f \geqslant 2$, and $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{A\Gamma L}(1, 2^f)$;*
(iv) *$T = \mathrm{PSL}(2, p^f)$, Γ is of valency 4, and $G_\alpha = \mathrm{A}_4$ or S_4, where p is a prime;*
(v) *$T = \mathrm{Ree}(3^{2m+1})$, $T_\alpha \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7$ and Γ is of valency 8.*

Remarks on Theorem 1.1: The graphs in parts (iii)-(iv) are classified in [10], see Lemma 3.6 for details, and the graphs in part (v) are determined in [5].

The second result of this paper classifies 2-arc-transitive graphs of odd order admitting an exceptional group of Lie type.

Theorem 1.2. *Let Γ be a connected $(G, 2)$-arc-transitive graph of odd order. Assume that G is an exceptional group of Lie type. Then $T = \mathsf{soc}(G) = \mathrm{Ree}(3^{2m+1})$, $T_\alpha = \mathbb{Z}_2^3{:}\mathbb{Z}_7$ and $k = 8$.*

In a sequel, we shall solve Problem A first for alternating and symmetric groups, and then for classical groups of Lie type.

We end this introduction by introducing some notion. The group-theoretic notation used in this paper is standard (see, for example, [4] and [25]). For two groups K and H, denote by $K.H$ an extension of K by H, while $K{:}H$ stands for a split extension, and further, $K^{\cdot}H$ indicates any case of $K.H$ which is a non-split extension. The notation $K{\circ}H$ stands for a central product of K and H.

Following [4], for a positive integer n, the symbol $[n]$ sometimes denotes an (unspecified) group of order n for convenience; in particular, n denotes a cyclic group of order n, p^f with p prime denotes an elementary abelian group of order p^f, namely, a direct product of f copies of $\mathbb{Z}_p$.

As usual, for a prime factor p of n, by n_p we mean the largest power of p dividing n, sometimes written as $n_p || n$; while G_p denote a Sylow p-subgroup of a group G.

2. Stabilisers

Let $\Gamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph. Fix an edge $\{\alpha, \beta\} \in E$, and let $\Gamma(\alpha)$ be the neighbourhood of α, which is the set of vertices adjacent to α. Then G_α induces a 2-transitive permutation group on $\Gamma(\alpha)$, denoted by $G_\alpha^{\Gamma(\alpha)}$.

Let $G_\alpha^{[1]}$ be the kernel of G_α acting on $\Gamma(\alpha)$. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$. Let $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$, the kernel of $G_{\alpha\beta}$ acting on $\Gamma(\alpha) \cup \Gamma(\beta)$. Then $G_{\alpha\beta}^{[1]} \lhd G_\alpha^{[1]} G_\beta^{[1]} \lhd G_{\alpha\beta}$, and

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \cong G_\alpha^{[1]}/G_{\alpha\beta}^{[1]} \cong G_\alpha^{[1]}G_\beta^{[1]}/G_\beta^{[1]} \lhd G_{\alpha\beta}/G_\beta^{[1]} \cong G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}.$$

In particular, $G_\alpha^{[1]} = G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}$, and

$$G_\alpha = G_\alpha^{[1]}.G_\alpha^{\Gamma(\alpha)} = (G_{\alpha\beta}^{[1]}.(G_\alpha^{[1]})^{\Gamma(\beta)}).G_\alpha^{\Gamma(\alpha)}.$$

The following theorem is a fundamental result in the area of symmetric graphs, refer to [7, 24].

Theorem 2.1. *Assume $G_{\alpha\beta}^{[1]}\neq 1$. Then $G_{\alpha\beta}^{[1]}$ is a p-group for some prime p, $G_{\alpha\beta}^{\Gamma(\alpha)}$ has a nontrivial normal p-subgroup, and $G_\alpha^{\Gamma(\alpha)}\rhd \mathrm{PSL}(d,q)$ with $q=p^f$ and $|\Gamma(\alpha)|=\frac{q^d-1}{q-1}$. Furthermore, if $d=2$ then Γ is $(G,4)$-arc-transitive.*

We remark that in this theorem the prime p divides $|\Gamma(\alpha)|-1$, and hence p does not divide the valency $|\Gamma(\alpha)|$. For a group X and a prime p, let $\mathbf{O}_p(X)$ be the largest normal p-subgroup of X, which is a characteristic subgroup of X. The next lemma slightly improves part (vi) of [17, Theorem 1.1].

Lemma 2.2. *Let r be a prime divisor of $|\Gamma(\alpha)|$. Then $\mathbf{O}_r(G_\alpha^{[1]})=1$, and either*

(i) *$\mathbf{O}_r(G_\alpha)=1$, or*
(ii) *$G_\alpha^{\Gamma(\alpha)}$ is affine of degree r^e, $\mathbf{O}_r(G_\alpha)\cong\mathbb{Z}_r^e$, and $G_\alpha=\mathbf{O}_r(G_\alpha){:}G_{\alpha\beta}=(\mathbf{O}_r(G_\alpha)\times G_\alpha^{[1]}).G_{\alpha\beta}^{\Gamma(\alpha)}$.*

Proof. Since $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive, $G_{\alpha\beta}$ is transitive on $\Gamma(\alpha)\setminus\{\beta\}$. Since $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}\lhd G_{\alpha\beta}$, all orbits of $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}$ on $\Gamma(\alpha)\setminus\{\beta\}$ have the same size. As r divides $|\Gamma(\alpha)|$, r is comprime to $|\Gamma(\alpha)\setminus\{\beta\}|$. Thus $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}$ acts trivially on $\Gamma(\alpha)\setminus\{\beta\}$. Similarly, $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}$ acts trivially on $\Gamma(\beta)\setminus\{\alpha\}$. So $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}\leqslant G_{\alpha\beta}^{[1]}$.

We claim that $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}=1$. If $G_{\alpha\beta}^{[1]}=1$, then the claim is true. Suppose that $G_{\alpha\beta}^{[1]}\neq 1$. Then Theorem 2.1 says that $G_{\alpha\beta}^{[1]}$ is a p-group with p prime, and p divides $|\Gamma(\alpha)\setminus\{\beta\}|$. So p does not divides $|\Gamma(\alpha)|$, and $r\neq p$ and $\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}=1$, as claimed. Since $\mathbf{O}_r(G_\alpha^{[1]})\leqslant\mathbf{O}_r(G_\alpha)$ and $G_\alpha^{[1]}\leqslant G_{\alpha\beta}$, we conclude that $\mathbf{O}_r(G_\alpha^{[1]})\leqslant\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}=1$.

Assume that $\mathbf{O}_r(G_\alpha)\neq 1$. Since $\mathbf{O}_r(G_\alpha^{[1]})=1$, we have $1\neq\mathbf{O}_r(G_\alpha)\cong\mathbf{O}_r(G_\alpha)^{\Gamma(\alpha)}\lhd G_\alpha^{\Gamma(\alpha)}$. As $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group, it follows that $G_\alpha^{\Gamma(\alpha)}$ is affine and $\mathbf{O}_p(G_\alpha)^{\Gamma(\alpha)}\cong\mathbf{O}_r(G_\alpha)$ is the socle of $G_\alpha^{\Gamma(\alpha)}$ and regular on $\Gamma(\alpha)$. Hence $\mathbf{O}_r(G_\alpha)=\mathbb{Z}_r^e$ for some prime r, and $G_\alpha=\mathbf{O}_r(G_\alpha){:}G_{\alpha\beta}$. Since $\mathbf{O}_r(G_\alpha)\cap G_\alpha^{[1]}\leqslant\mathbf{O}_r(G_\alpha)\cap G_{\alpha\beta}=1$, we have $\mathbf{O}_r(G_\alpha)G_\alpha^{[1]}=\mathbf{O}_r(G_\alpha)\times G_\alpha^{[1]}$. Finally, $G_\alpha/\mathbf{O}_r(G_\alpha)G_\alpha^{[1]}\cong(G_\alpha/G_\alpha^{[1]})/(\mathbf{O}_r(G_\alpha)G_\alpha^{[1]}/G_\alpha^{[1]})\cong G_\alpha^{\Gamma(\alpha)}/(\mathbf{O}_r(G_\alpha))^{\Gamma(\alpha)}\cong G_{\alpha\beta}^{\Gamma(\alpha)}$, and $G_\alpha=(\mathbf{O}_r(G_\alpha)\times G_\alpha^{[1]}).G_{\alpha\beta}^{\Gamma(\alpha)}$. This completes the proof. □

In the rest of this section, assume that the vertex number $|V|$ is odd.

Lemma 2.3. *Assume that M is a subgroup of G containing G_α such that $\mathbf{O}_2(M)\neq 1$. Then $\mathbf{O}_2(M)=\mathbf{O}_2(G_\alpha)\cong\mathbf{O}_2(G_\alpha)^{\Gamma(\alpha)}$, and $G_\alpha^{\Gamma(\alpha)}$ is affine with socle $\mathbf{O}_2(G_\alpha)^{\Gamma(\alpha)}\cong\mathbf{O}_2(G_\alpha)$. Moreover, the order $|M|$ is divisible by $|\mathbf{O}_2(M)|-1$.*

Proof. Since $|M:G_\alpha|$ is a divisor of $|G:G_\alpha|=|V|$, the index $|M:G_\alpha|$ is odd. Since $G_\alpha\leqslant\mathbf{O}_2(M)G_\alpha\leqslant M$, we have

$$|M:G_\alpha|=|M:\mathbf{O}_2(M)G_\alpha||\mathbf{O}_2(M)G_\alpha:G_\alpha|,$$

and thus $|\mathbf{O}_2(M)G_\alpha : G_\alpha|$ is odd. Now

$$|\mathbf{O}_2(M)G_\alpha : G_\alpha| = \frac{|\mathbf{O}_2(M)G_\alpha|}{|G_\alpha|} = \frac{|\mathbf{O}_2(M)||G_\alpha|}{|\mathbf{O}_2(M)\cap G_\alpha||G_\alpha|} = \frac{|\mathbf{O}_2(M)|}{|\mathbf{O}_2(M)\cap G_\alpha|}.$$

The fact that $|\mathbf{O}_2(M)G_\alpha : G_\alpha|$ is odd implies $\frac{|\mathbf{O}_2(M)|}{|\mathbf{O}_2(M)\cap G_\alpha|} = 1$. It follows that $\mathbf{O}_2(M) = \mathbf{O}_2(M)\cap G_\alpha$, and $\mathbf{O}_2(M) \leqslant G_\alpha$. Further, $\mathbf{O}_2(M) \lhd G_\alpha$ and $\mathbf{O}_2(M) \leqslant \mathbf{O}_2(G_\alpha)$. By Lemma 2.2, the 2-part $\mathbf{O}_2(G_\alpha)$ is a minimal normal subgroup of G_α. Since $\mathbf{O}_2(M) \leqslant G_\alpha$ and $\mathbf{O}_2(M) \leqslant \mathbf{O}_2(G_\alpha)$, we conclude that $\mathbf{O}_2(M) = \mathbf{O}_2(G_\alpha)$. The other statements of the lemma follow from Lemma 2.2. □

By Lemma 2.3, either $\mathbf{O}_2(M) = 1$, or $\mathbf{O}_2(M) = \mathbf{O}_2(G_\alpha) \cong \mathbf{O}_2(G_\alpha^{\Gamma(\alpha)}) \cong \mathbb{Z}_2^e$ with $|\Gamma(\alpha)| = 2^e$. For the latter case, since G_α is insoluble, $|\Gamma(\alpha)| > 4$, and then $e \geqslant 3$. Since Γ is a $(G,2)$-arc-transitive graph, the stabiliser $G_\alpha^{\Gamma(\alpha)}$ is an affine 2-transitive permutation group of degree 2^e. Thus the order $|G_\alpha|$ is divisible by $|\mathbf{O}_2(M)| - 1 = 2^e - 1$, and so is $|M|$.

Lemma 2.4. *Assume that M is a subgroup of G containing G_α such that $M \cong [m].S.[l].2$, where m and l are odd, either $S = \mathrm{PSL}(3,q_0)$ with $q_0 \equiv 1 \,(\mathsf{mod}\ 4)$ or $S = \mathrm{PSU}(3,q_0)$ with $q_0 \equiv -1 \,(\mathsf{mod}\ 4)$. Then $G_\alpha^{\Gamma(\alpha)}$ is almost simple.*

Proof. Assume that $G_\alpha^{\Gamma(\alpha)}$ is affine. Note that M has 2-rank at most 3. Then, since G_α is insoluble, $G_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2^3{:}\mathrm{SL}(3,2)$ and Γ is of valency 8. By Theorem 2.1, $G_{\alpha\beta}^{[1]} = 1$. Then $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Gamma(\alpha)} \cong \mathrm{SL}(3,2)$, and so either $|G_\alpha^{[1]}| = 1$ and $|G_\alpha|_2 = 2^6$, or $|G_\alpha|_2 = 2^9$.

Checking the 2-part of $|M|$, we have $|M|_2 = 2^{2t+2}$, where t is such that $2^t \parallel (q_0-1)$ for $S = \mathrm{PSL}(3,q_0)$, or $2^t \parallel (q_0+1)$ for $S = \mathrm{PSU}(3,q_0)$. Since $|M : G_\alpha|$ is odd, the only possibility is that $t = 2$, $|G_\alpha|_2 = 2^6$ and $G_\alpha \cong \mathbb{Z}_2^3{:}\mathrm{SL}(3,2) = \mathrm{AGL}(3,2)$. Note that M has a subgroup of index 2, which intersects G_α at a subgroup of index 2 in G_α. However, $\mathrm{AGL}(3,2)$ has no subgroup of index 2, a contradiction. Thus $G_\alpha^{\Gamma(\alpha)}$ is almost simple. □

Finally, we prove a technical lemma.

Lemma 2.5. *Assume that G is an almost simple group with socle T. Then Γ is T-arc-transitive, and either G_α is soluble, or T_α is insoluble and Γ is $(T,2)$-arc-transitive. In particular, if $|\Gamma(\alpha)| = 28$ then $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \not\cong \mathrm{PSL}(2,8)$.*

Proof. Since Γ has odd order, Γ is not bipartite. Noting that T is nonabelian simple, it follows that T is transitive on the vertex set V of Γ. Thus $G = TG_\alpha$, and $G_\alpha/T_\alpha = G_\alpha/(T\cap G_\alpha) \cong TG_\alpha/T = G/T$, so G_α/T_α is soluble.

Since $T_\alpha \trianglelefteq G_\alpha$ and G_α is 2-transitive on $\Gamma(\alpha)$. Then either $T_\alpha \leqslant G_\alpha^{[1]}$ or T_α is transitive on $\Gamma(\alpha)$. Then former case implies that T is regular on V, and so $|T| = |V|$ is even, a contradiction. The latter case says that Γ is T-arc-transitive.

Assume that G_α is insoluble. Then T_α is insoluble, and hence $T_\alpha^{\Gamma(\alpha)}$ is an insoluble normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. Checking all 2-transitive permutation groups of even degree (refer to [3, Sections 7.3 and 7.4]), we conclude that either $T_\alpha^{\Gamma(\alpha)}$ is 2-transitive, or $T_\alpha^{\Gamma(\alpha)} \cong \mathrm{PSL}(2,8)$. For the latter, $G_\alpha^{\Gamma(\alpha)} = \mathrm{P\Gamma L}(2,8)$ and Γ has valency 28. Since Γ is of odd order, by [15], Γ is not $(G,4)$-arc-transitive. Then $G_{\alpha\beta}^{[1]} = 1$ by Theorem 2.1, and so $G_\alpha^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} = \mathbb{Z}_9{:}\mathbb{Z}_6$. In particular, $G_\alpha^{[1]}$ has a unique $2'$-Hall subgroup, say L, which is characteristic in $G_\alpha^{[1]}$. Thus $L \lhd G_\alpha$, and $G_\alpha/L = \ell.\mathrm{PSL}(2,8).3$, where $\ell = 1$ or 2. Since the Schur multiplier of $\mathrm{PSL}(2,8)$ is trivial, we have $G_\alpha/L = (\ell \times \mathrm{PSL}(2,8)).3$. Thus G_α/L and hence G_α has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$. By [8, Theorem 16.6], $G = T = \mathrm{PSL}(2,16)$, which does not have a subgroup $\mathrm{PSL}(2,8)$, a contradiction. □

3. Proof of Theorem 1.1

Let $\Gamma = (V,E)$ be a connected $(G,2)$-arc-transitive graph, and let (α,β) be an arc of Γ. Assume that $|V|$ is odd, G is an almost simple group, and G_α is soluble.

It follows from the assumption that the valency $|\Gamma(\alpha)|$ is even, and $G_\alpha^{\Gamma(\alpha)}$ is a soluble 2-transitive permutation group of even degree. By Huppert's classification (see [9] for example), we have

$$G_\alpha^{\Gamma(\alpha)} \leqslant 2^d{:}\Gamma\mathrm{L}(1,2^d) \text{ for some } d \geqslant 2.$$

In particular, Γ is of valency 2^d, and since $G_{\alpha\beta}^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)\setminus\{\beta\}$, the order $|G_{\alpha\beta}^{\Gamma(\alpha)}|$ is divisible by 2^d-1. Furthermore, by Lemma 2.2, we have

$$\mathbf{O}_2(G_\alpha^{[1]}) = 1,\ \mathbf{O}_2(G_\alpha) \cong \mathbb{Z}_2^d,\ G_\alpha = \mathbf{O}_2(G_\alpha){:}G_{\alpha\beta} = (\mathbf{O}_2(G_\alpha) \times G_\alpha^{[1]}).G_{\alpha\beta}^{\Gamma(\alpha)}. \quad (3.1)$$

Lemma 3.1. *Every Sylow 2-subgroup of G is of the form of $(\mathbb{Z}_{2^a} \times \mathbb{Z}_2^d).\mathbb{Z}_{2^b}$, where $0 \leqslant a \leqslant b$ and 2^b is a divisor of d.*

Proof. Note that $G_{\alpha\beta}^{\Gamma(\alpha)} \leqslant \Gamma\mathrm{L}(1,2^d) \cong \mathbb{Z}_{2^d-1}{:}\mathbb{Z}_d$, as observed above. Each Sylow 2-subgroup of $G_{\alpha\beta}^{\Gamma(\alpha)}$ is isomorphic to a subgroup of $\mathbb{Z}_d$, say $\mathbb{Z}_{2^b}$. By Theorem 2.1, $G_{\alpha\beta}^{[1]}$ is a p-group with p coprime to $|\Gamma(\alpha)| = 2^d$, and thus p is an odd prime. Now $G_\alpha^{[1]}/G_{\alpha\beta}^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}$. So a Sylow 2-subgroup of $G_\alpha^{[1]}$ is a Sylow 2-subgroup of $(G_\alpha^{[1]})^{\Gamma(\beta)}$, and is of the form $\mathbb{Z}_{2^a}$ for some $a \leqslant b$. Then our lemma follows from (3.1). □

The conclusion of this lemma enables us to apply some classical results to determine all the possibilities for G.

Lemma 3.2. *The socle T of G is one of the following groups:*

> A_7, M_{11}, J_1, $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,q)$ *(q odd)*, $\mathrm{PSU}(3,q)$ *(q odd), and* $\mathrm{Ree}(3^{2m+1})$.

Proof. Since $|T:T_\alpha|$ is odd, each Sylow 2-subgroup of T_α is a Sylow 2-subgroup of T. By Lemma 3.1, a Sylow 2-subgroup of the simple group T is abelian or ableian-by-cyclic.

Such a simple group T is classified by Gorenstein [8, Theorem 16.6] and Kondrat'ev [13, Corollary 1], which shows that T is one of the simple groups listed above. □

Thus, to complete the proof of Theorem 1.1, we only need to analyse these candidates.

For an abstract group X and subgroups $K < H < X$ such that H is core-free in X, if there exists $g \in \mathbf{N}_X(K)$ such that $g^2 \in K$ and $\langle H, g\rangle = X$, we define a graph with vertex set $[X : H]$ and edge set $\{\{Hx, Hy\} \mid yx^{-1} \in HgH\}$, denoted by $\mathsf{Cos}(X, H, HgH)$ and called a *coset graph.*

Lemma 3.3. *A coset graph* $\mathsf{Cos}(X, H, HgH)$ *is connected if and only if* $\langle H, g\rangle = X$*, and is* $(X, 2)$*-arc-transitive if and only if* H *is* 2*-transitive on* $[H : K]$*.*

Under our assumption, G is 2-arc-transitive on Γ, $H = G_\alpha$, $K = G_{\alpha\beta}$, and $g \in G_{\{\alpha,\beta\}}$; in particular,

$$G_{\alpha\beta} < G_{\{\alpha,\,\beta\}} \leqslant \mathbf{N}_G(G_{\alpha\beta}). \tag{3.2}$$

Note that $G_\alpha = \mathbb{Z}_2^d{:}G_{\alpha\beta}$ and $2^d - 1 \bigm| |G_{\alpha\beta}|$, so G_α is not a 2-group since the valency $2^d > 2$.

Now we analyse the candidates listed in Lemma 3.2 one by one.

Lemma 3.4. *Let* $G = \mathrm{A}_7$ *or* S_7*. Then* Γ *is the Odd graph* O_4 *of valency* 4*.*

Proof. Note that a Sylow 2-subgroup of G has order 2^3 or 2^4. By (3.1), we conclude that $d = 2$; in particular, Γ has valency 4. Then $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{A}_4$ or S_4. It follows that $|G : G_\alpha|$ is odd and square-free. Then by [16, Lema 6.2], Γ is the Odd graph O_4. □

Let M be a maximal subgroup of G such that $G_\alpha \leqslant M$. Then $|G : M|$ and $|M : G_\alpha|$ are odd as $|G : G_\alpha|$ is odd, and $(2^d - 1) \bigm| |M|$. By Lemma 2.3,

$$\mathbf{O}_2(M) = 1 \text{ or } \mathbf{O}_2(G_\alpha). \tag{3.3}$$

Lemma 3.5. *Let* T *be a sporadic simple group. Then* $G = \mathrm{J}_1$*,* $G_\alpha \cong 2^3{:}7{:}3$ *or* $2^3{:}7$*, and* Γ *is of valency* 8*. Further, there indeed exist such graphs.*

Proof. By Lemma 3.2, $T = \mathrm{M}_{11}$ or J_1, and so $G = T$.

Suppose that $G = \mathrm{M}_{11}$. Since $|G : M|$ is odd, by the Atlas [4], we have $M \cong \mathrm{M}_{10}$, $\mathrm{M}_9{:}2$ or $\mathrm{M}_8{:}\mathrm{S}_3$. Further since G_α is soluble and $|M : G_\alpha|$ is odd, also by Atlas [4], one can get $M \cong \mathrm{M}_{10}$ and $G_\alpha \cong \mathbb{Z}_8{:}\mathbb{Z}_2$, or $M \cong \mathrm{M}_9{:}2$ and $G_\alpha = M$ or $\mathrm{Q}_8.2$, or $M \cong \mathrm{M}_8{:}\mathrm{S}_3$ and $G_\alpha = M$ or $\mathrm{Q}_8{:}2$. However, since G_α is not a 2-group, $G_\alpha \cong \mathrm{M}_9{:}2$ or $\mathrm{M}_8{:}\mathrm{S}_3$. It implies that $\mathbf{O}_2(G_\alpha) \not\cong \mathbb{Z}_2^d$, which contradicts (3.1).

We thus have $G = \mathrm{J}_1$. Since $|G : M|$ is odd and $\mathbf{O}_2(M) = 1$ or 2^d for $d \geqslant 2$ by Lemma 3.1, we have $M \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7{:}\mathbb{Z}_3$ by the Atlas [4]. Since $|G_\alpha|$ is divisible by $2^d - 1$, we have $G_\alpha \cong 2^3{:}7{:}3$ or $2^3{:}7$, and $d = 3$. Let K be a Hall $2'$-subgroup of G_α. Then $K \cong \mathbb{Z}_7{:}\mathbb{Z}_3$ or $\mathbb{Z}_7$, respectively.

By the Atlas [4], we conclude that $\mathbf{N}_G(K) \cong \mathbb{Z}_7{:}\mathbb{Z}_6$. Let g be an involution of $\mathbf{N}_G(K)$, and let $X = \langle G_\alpha, g\rangle$. Then X is a subgroup of G which contains two subgroups, one isomorphic to $2^3{:}7{:}3$ and the other isomorphic to $\mathbb{Z}_7{:}\mathbb{Z}_6$. Thus X has odd index in G, and by [4], it follows that $X = G$. So, by Lemma 3.3, the coset graph $\mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is connected and $(G, 2)$-arc-transitive of valency 8. □

We next treat graphs associated with the groups of Lie type listed in Lemma 3.2.

Lemma 3.6. *Let $T = \mathsf{soc}(G)$ be a group of Lie type given in Lemma 3.2. Then one of the following holds:*

(i) $T = \mathrm{PSL}(2, 2^d)$, *and* $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{A\Gamma L}(1, 2^f)$;
(ii) Γ *is of valency* 4, *and one of the following occurs:*
  (a) $T = \mathrm{PSL}(2, p^f)$ *and* $T_\alpha \cong \mathbb{Z}_2^2$, *where f is odd and divisible by* 3, *and p is a prime with* $p \equiv \pm 3 \pmod 8$; *or*
  (b) $T = \mathrm{PSL}(2, p^f)$ *and* $T_\alpha \cong \mathrm{A}_4$, *where f is odd and p is a prime with* $p \equiv \pm 3 \pmod 8$; *or*
  (c) $G = \mathrm{PSL}(2, p)$ *and* $G_\alpha \cong \mathrm{S}_4$, *where p is a prime with* $p \equiv \pm 1 \pmod 8$; *or*
  (d) $G = \mathrm{PSL}(2, p^2)$ *and* $G_\alpha \cong \mathrm{S}_4$, *where p is a prime with* $p \equiv \pm 3 \pmod 8$;
(iii) $T = \mathrm{Ree}(3^{2m+1})$, $T_\alpha \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7$ *and Γ is of valency* 8.

Proof. First, 2-arc-transitive graphs admitting a group G with socle $\mathrm{PSL}(2, q)$ or $\mathrm{Ree}(q)$ are classified in [10] or [5], respectively, from which the lemma follows. Thus, we only need prove that there is no graph arising from the groups $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q)$.

Suppose that $T = \mathrm{PSL}(3, q)$ or $\mathrm{PSU}(3, q)$, where q is odd. Then T has 2-rank 2 by [1], that is, a Sylow 2-subgroup of T does not have any subgroup which is homomorphic to $\mathbb{Z}_2^f$ with $f \geqslant 3$. Recall that $G_\alpha = (G_\alpha^{[1]} \times \mathbf{O}_2(G_\alpha)).G_{\alpha\beta}^{\Gamma(\alpha)}$, $\mathbf{O}_2(G_\alpha) \cong \mathbb{Z}_2^d$ and $G_{\alpha\beta}^{\Gamma(\alpha)} \leqslant \Gamma\mathrm{L}(1, 2^d)$ for $d \geqslant 2$, by Lemmas 2.2 and 3.1. Since T_α is normal in G_α and transitive on $\Gamma(\alpha)$, we have $T_\alpha^{\Gamma(\alpha)} \rhd \mathbb{Z}_2^d$, that is, T_α has 2-rank d. Clearly, the 2-rank of T_α is not larger than the 2-rank of T. Thus $d = 2$, and Γ is of valency 4. By Lemma 3.1, G_α and hence G has a Sylow 2-subgroup isomorphic to a subgroup of $(\mathbb{Z}_2 \times \mathbb{Z}_2^2).\mathbb{Z}_2$, which has 2-rank at least 3. Since the 2-rank of T is 2, it follows that $|T|_2 < |G_2$, and so $|T|_2 \leqslant 2^3$. However, $|T|_2 = (q^2-1)_2(q^3 \pm 1)_2 \geqslant 2^4$, which is a contradiction. This completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\Gamma = (V, E)$ be a connected graph of odd order and valency at least 3, and let G be an almost simple group of automorphisms of Γ. Assume that Γ is $(G, 2)$-arc-transitive and, for $\alpha \in V$, the stabiliser G_α is soluble. Then all possible candidates for $\mathsf{soc}(G)$ are given in Lemma 3.2. We finish the proof by analysing all the candidates. For $\mathsf{soc}(G) = \mathrm{A}_7$, Lemma 3.4 says that Γ is the odd graph of valency 4, and then Theorem 1.1 (i) holds. If $\mathsf{soc}(G) = \mathrm{M}_{11}$ or J_1, then by Lemma 3.5, $\mathsf{soc}(G) = \mathrm{J}_1$ and Γ is given as in Theorem 1.1 (ii). If $\mathsf{soc}(G) = \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$ or $\mathrm{Ree}(3^{2m+1})$, then Lemma 3.6 shows that $\mathsf{soc}(G) = \mathrm{PSL}(2, q)$ or $\mathrm{Ree}(3^{2m+1})$, and thus one of (ii)-(v) of Theorem 1.1 holds. □

4. Proof of Theorem 1.2

Let $\varGamma = (V, E)$ be a connected $(G, 2)$-arc-transitive graph of odd order and valency $k \geqslant 3$. Take an edge $\{\alpha, \beta\} \in E$. Assume further that G is an exceptional group of Lie type. If G_α is soluble then our theorem holds by Lemma 3.6. In the rest of this proof, we shall exclude the case that G_α is insoluble.

Suppose next that G_α is insoluble. By Lemma 2.5, we may assume that G is an exceptional simple group of Lie type, which is defined over $\mathrm{GF}(q)$.

Lemma 4.1. *q is odd.*

Proof. Suppose that q is even. Let M be a maximal subgroup of G containing G_α. Then M has odd index in G. Such a pair (G, M) is classified in [19], which shows that M is a maximal parabolic subgroup of G, and so $\mathbf{O}_2(M) \neq 1$. Thus, by Lemma 2.3, the order $|M|$ is divisible by $|\mathbf{O}_2(M)| - 1$.

By [25, Theorem 4.1], there is no insoluble maximal parabolic subgroup of $\mathrm{Sz}(q)$ of odd index, so $G \neq \mathrm{Sz}(q)$. Maximal parabolic subgroups of $\mathrm{G}_2(q)$, ${}^3\mathrm{D}_4(q)$, ${}^2\mathrm{F}_4(q)$ or $\mathrm{F}_4(q)$ are given in Table 4.1, Theorem 4.3, Theorem 4.5, and Section 4.8.6 of [25], respectively, which show that the order $|M|$ is not divisible by $|\mathbf{O}_2(M)|-1$. So these groups are excluded.

For the 'large' groups $G = \mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, $\mathrm{E}_8(q)$, the parabolic subgroup M is determined by the Dynkin diagrams in the methods described in [25, p. 176], from which we conclude that $|\mathbf{O}_2(M)| - 1$ is not a divisor of $|M|$. Thus these groups are also excluded. □

Since G is defined over $\mathrm{GF}(q)$, we write $G = L(q)$ for convenience. Let q_0 be minimum such that $G_\alpha \leqslant L(q_0) \leqslant L(q)$, where q is a power of q_0.

Lemma 4.2. *The stabiliser G_α is not equal to $L(q_0)$ for any subfield $\mathrm{GF}(q_0)$ of $\mathrm{GF}(q)$.*

Proof. Suppose that $G_\alpha = L(q_0)$. Then $G_\alpha \cong G_\alpha^{\varGamma(\alpha)}$ is a 2-transitive permutation group. Since $L(q_0)$ is an exceptional group of Lie type of odd characteristic, we have $L(q_0) = \mathrm{Ree}(q_0)$ and $|\varGamma(\alpha)| = q_0^3 + 1$. By [5], there is no $(G, 2)$-arc-transitive graph corresponding to this case, proving the lemma. □

Let $N = L(q_0)$, and let M be a maximal subgroup of N which contains G_α. Then the index $|N : M|$ is odd, and the pair (N, M) is determined in [19].

Lemma 4.3. *The only possibilities for N are $\mathrm{G}_2(q_0)$ and ${}^3\mathrm{D}_4(q_0)$.*

Proof. Suppose that $N = \mathrm{Ree}(q_0)$, $\mathrm{F}_4(q_0)$, ${}^2\mathrm{E}_6(q_0)$, $\mathrm{E}_6(q_0)$, $\mathrm{E}_7(q_0)$ or $\mathrm{E}_8(q_0)$. Since M is a maximal subgroup of N such that $|N : M|$ is odd, by the classification in [19], either $1 < |\mathbf{O}_2(M)| \leqslant 4$, or $\mathbf{O}_2(M)$ is not elementary abelian. This is not possible by Lemma 2.3. We thus have $N = \mathrm{G}_2(q_0)$ or ${}^3\mathrm{D}_4(q_0)$. □

Noting that $|G : N|$ is odd, q is an odd power of q_0. Set $q_0 = p^e$ for an odd prime p.

Lemma 4.4. *Let $N = \mathrm{G}_2(q_0)$ or ${}^3\mathrm{D}_4(q_0)$. Then $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ is not one of the following simple groups:*

(i) $\mathrm{PSL}(3, q_1)$, *q_0 is an odd power of q_1, $q_0 \equiv 1 \pmod 4$;*
(ii) $\mathrm{PSU}(3, q_1)$, *q_0 is an odd power of q_1, $q_0 \equiv -1 \pmod 4$;*
(iii) A_5, $\mathrm{PSL}(2, r)$, *q_0 is a power of r.*

Proof. If $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) = \mathrm{PSL}(3, q_1)$, then as $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive it follows that $|\Gamma(\alpha)| = \frac{q_1^3-1}{q_1-1} = q_1^2 + q_1 + 1$ is odd, which is not possible.

Suppose that $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{A}_5$. Then Γ has valency 6, and $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{A}_5$ or S_5. By Theorem 2.1, $G_{\alpha\beta}^{[1]} = 1$, and so $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Gamma(\alpha)} \lesssim \mathbb{Z}_5{:}\mathbb{Z}_4$. Then $|G|_2 = |G_\alpha|_2 = |G_\alpha^{[1]}|_2 |G_{\alpha\beta}^{\Gamma(\alpha)}|_2 \leqslant 2^5$. On the other hand, $|G_\alpha|_2 = |G|_2 = (q^2-1)_2(q^6-1)_2 \geqslant 2^6$, a contradiction.

Suppose that $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{PSU}(3, q_1)$. Then Γ has valency $q_1^3 + 1$. By Theorem 2.1, $G_{\alpha\beta}^{[1]} = 1$. If $G_\alpha^{[1]} = 1$ then $G_{\alpha\beta} \cong (G_\alpha^{\Gamma(\alpha)})_\beta$ is p-local. Since $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Gamma(\alpha)}$, if $G_{\alpha\beta}^{[1]} \neq 1$ then $G_\alpha^{[1]}$ and hence $G_{\alpha\beta}$ is p-local. Thus $\mathbf{O}_p(G_{\alpha\beta}) \neq 1$ and $\mathbf{N}_G(G_{\alpha\beta}) \leqslant \mathbf{N}_G(\mathbf{O}_p(G_{\alpha\beta}))$. By [12, Proposition 5.2.10], $\mathbf{N}_G(\mathbf{O}_p(G_{\alpha\beta}))$ lies in a maximal parabolic subgroup of G, say P. Check the 2-part of $|\mathrm{P}|$, refer to [25, Table 4.1, Theorem 4.3] for the structure of P. We have $|\mathbf{N}_G(G_{\alpha\beta})|_2 \leqslant |\mathrm{P}|_2 = (q^2-1)_2(q-1)_2$. Note that $|G_\alpha|_2 = |G|_2 = (q^2-1)_2(q^6-1)_2$. Since $|G_\alpha : G_{\alpha\beta}| = q_1^3 + 1$ and q is an odd power of q_1, the 2-part $|G_{\alpha\beta}|_2 \geqslant (q^2-1)_2(q^3-1)_2 = (q^2-1)_2(q-1)_2$. It follows that $|\mathbf{N}_G(G_{\alpha\beta}) : G_{\alpha\beta}|$ is odd, a contradiction.

Suppose that $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathrm{PSL}(2, r)$. Then Γ has valency $r+1$. Since Γ is not $(G, 4)$-arc-transitive, again by Theorem 2.1, $G_{\alpha\beta}^{[1]} = 1$. By a similar argument as above, $G_{\alpha\beta}$ is p-local and $|\mathbf{N}_G(G_{\alpha\beta})|_2 \leqslant (q^2-1)_2(q-1)_2$. If $r \equiv 1 \pmod 4$ then, since $|G_\alpha : G_{\alpha\beta}| = r+1$, we have $|G_{\alpha\beta}|_2 \geqslant (q^2-1)_2(q^3-1)_2 = (q^2-1)_2(q-1)_2$, a contradiction. Thus $r \equiv -1 \pmod 4$; in particular, r is an odd power of p. Note that $G_\alpha^{[1]}$ is isomorphic to a normal subgroup of $G_{\alpha\beta}^{\Gamma(\alpha)} \lesssim ([r]{:}(r-1)).\ell$, where ℓ is odd. Then $(q^2-1)_2(q^6-1)_2 = |G|_2 = |G_\alpha|_2 = |G_\alpha^{[1]}|_2 |G_{\alpha\beta}^{\Gamma(\alpha)}|_2 \leqslant (r^2-1)_2(r-1)_2$, a contradiction. □

To complete the proof of Theorem 1.2, by Lemma 4.3, we only need to exclude the groups $N = \mathrm{G}_2(q_0)$ or ${}^3\mathrm{D}_4(q_0)$.

Lemma 4.5. *$N \neq \mathrm{G}_2(q_0)$.*

Proof. Suppose $N = \mathrm{G}_2(q_0)$. Recall that $\mathbf{O}_2(M) = 1$ or $\mathbb{Z}_2^e$ for $e \geqslant 3$ by Lemma 2.3. Noting the minimality of q_0, we read out all the possibilities for M from the classification in [19, Table 1] (or refer to [25, Table 4.1]):

(i) $M \cong \mathrm{G}_2(2) \cong \mathrm{PSU}(3,3).2$, with $q_0 = p \equiv \pm 3 \pmod 8$;
(ii) $M \cong 2^{3\cdot}\mathrm{SL}(3,2)$, with $q_0 = p \equiv \pm 3 \pmod 8$;
(iii) $M \cong \mathrm{SL}(3, q_0){:}2$, with $q_0 \equiv 1 \pmod 4$;
(iv) $M \cong \mathrm{SU}(3, q_0){:}2$, with $q_0 \equiv -1 \pmod 4$.

Suppose that $M \cong \mathrm{G}_2(2)$ as in part (i). Since $|M : G_\alpha|$ is odd and G_α is insoluble, checking the maximal subgroups of $\mathrm{G}_2(2)$ (see Atlas [4]), we conclude $G_\alpha = M$. It follows that $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha \cong \mathrm{PSU}(3,3).2$, and thus the arc stabiliser $G_{\alpha\beta}^{\Gamma(\alpha)} \cong G_{\alpha\beta} \cong [3^3]{:}\mathbb{Z}_8{:}\mathbb{Z}_2$. Consequently, the edge stabiliser $G_{\{\alpha,\beta\}} = G_{\alpha\beta}.2 \cong ([3^3]{:}\mathbb{Z}_8{:}\mathbb{Z}_2).\mathbb{Z}_2$. However, by [25, Table 4.1], there is no subgroup of $\mathrm{G}_2(q)$ that is isomorphic to $([3^3]{:}\mathbb{Z}_8{:}\mathbb{Z}_2).\mathbb{Z}_2$, which is a contradiction.

Now, suppose that $M \cong 2^{3\cdot}\mathrm{SL}(3,2)$ as in part (ii), which is a non-split extension of 2^3 by $\mathrm{SL}(3,2)$. By Lemma 2.2, we have that $\mathbf{O}_2(G_\alpha) = \mathbf{O}_2(M) \cong \mathbb{Z}_2^3$, which is regular on $\Gamma(\alpha)$. Obviously, M does not have an insoluble proper subgroup of odd index, and hence $G_\alpha = M$ and $G_{\alpha\beta} \cong \mathrm{SL}(3,2)$. It implies that $M = G_\alpha = \mathbf{O}_2(G_\alpha)G_{\alpha\beta} \cong 2^3{:}\mathrm{SL}(3,2)$, which contradicts the fact that M is not a split extension of $\mathbb{Z}_2^3$ by $\mathrm{SL}(3,2)$.

Next, suppose that $M \cong \mathrm{SL}(3,q_0){:}2$ with $q_0 \equiv 1 \pmod 4$, as in part (iii). Then, by Lemma 2.4, $G_\alpha^{\Gamma(\alpha)}$ is almost simple, and hence $\mathbf{O}_2(G_\alpha) = 1$ by Lemma 2.2. Take a chain of maximal subgroups from $\mathbf{Z}(L)G_\alpha/\mathbf{Z}(L)$ to $L/\mathbf{Z}(L)$. Using [19], by induction, we shows that G_α has a unique insoluble composition factor, which is isomorphic to one of $\mathrm{PSL}(3,q_1)$, $\mathrm{PSL}(2,r)$ and A_5. Thus $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$ is known, and we get a contradiction by Lemma 4.4.

Finally, assume that $M \cong \mathrm{SU}(3,q){:}2$ with $q \equiv -1 \pmod 4$, as in part (iv). Then the argument in the previous paragraph works with $\mathrm{SU}(3,q_0)$ replacing $\mathrm{SL}(3,q_0)$, and so the case where $M \cong \mathrm{SU}(3,q){:}2$ is also excluded. □

Lemma 4.6. *$N \neq {}^3\mathrm{D}_4(q_0)$.*

Proof. Suppose $N = {}^3\mathrm{D}_4(q_0)$. Note the minimality of q_0. Since $|N : M|$ is odd and $|\mathbf{O}_2(M)| = 1$ or 2^e for $e \geqslant 3$, by [19, Table 1] and [25, Theorem 4.1], we conclude that M is listed as follows:

(i) $M \cong \mathrm{G}_2(q_0)$; or
(ii) $M \cong ((q_0^2 + q_0 + 1) \circ \mathrm{SL}(3,q_0)).(q_0^2 + q_0 + 1, 3).2$, where $q_0 \equiv 1 \pmod 4$; or
(iii) $M \cong ((q_0^2 - q_0 + 1) \circ \mathrm{SU}(3,q_0)).(q_0^2 - q_0 + 1, 3).2$, where $q_0 \equiv -1 \pmod 4$.

Employing Lemmas 2.4 and 4.4, (ii) and (iii) are excluded by a similar argument as used in (iii) and (iv) in the proof of Lemma 4.5.

Assume that $M \cong \mathrm{G}_2(q_0)$ as in part (i). Since $\mathrm{G}_2(q_0)$ has no 2-transitive permutation representation of even degree (refer to [3, Sections 7.3 and 7.4]), the stabiliser G_α is a proper subgroup of M. Let L be a maximal subgroup of $M = \mathrm{G}_2(q_0)$ such that $G_\alpha \leqslant L$. Since $|M : L|$ is odd, L is one of the groups listed in (i)-(iv) in the proof of Lemma 4.5. Then the argument in the proof of Lemma 4.5 works in this case and excludes all the possibilities. □

Now we can finish the proof of Theorem 1.2 by summarising the arguments.

Proof of Theorem 1.2. Let $\Gamma = (V,E)$ be a connected $(G,2)$-arc-transitive graph of odd order and valency at least 3. Assume that G is an exceptional group of Lie type. Take $\alpha \in V$. Then, by Lemmas 4.3 and 4.4, either G_α is soluble or G has a subgroup $\mathrm{G}_2(q_0)$ or ${}^3\mathrm{D}_4(q_0)$ which contains G_α as a proper subgroup, where

q_0 is odd and q is an odd power of q_0. By Lemmas 4.5 and 4.6, the latter case does not occur. Thus G_α is soluble, and the proof follows from Theorem 1.1. □

References

[1] J.L. Alperin, R. Brauer and D. Gorenstein, Finite simple groups of 2-rank two, *Scripta Math.* **29** (1973), 191-214.

[2] R.W. Baddeley, Two-arc-transitive graphs and twisted wreath products. *J. Algebraic Combin.* **2**(1993), 215-237.

[3] P.J. Cameron, *Permutation Groups*, London Mathematical Society Student Texts, 45. Cambridge University Press, Cambridge, 1999.

[4] J.H. Conway, R.T. Curtis, S.P. Noton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985. (http://brauer.maths.qmul.ac.uk/Atlas/v3/).

[5] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group, *Comm. Algebra* **27(8)** (1999), 3755–3769.

[6] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27(8)**(1999), 3727-3754.

[7] A. Gardiner, Doubly primitive vertex stabilisers in graphs, *Math. Z.* **135** (1974), 157-166.

[8] D. Gorenstein, *Finite groups*, Chelsea publishing company, New Youk, 1980.

[9] B. Huppert, Zweifach transitive, auflöbare Permutationsgruppen. *Math. Z.*, **68**(1957), 126-150.

[10] A. Hassani, L. Nochefranca and C.E. Praeger, Two-arc-transitive graphs admitting a two-dimensional projective linear group, *J. Group Theory* **2**(1999), 335-353.

[11] A.A. Ivanov and C.E. Praeger, On finite affine 2-arc transitive graphs, *European J. Combin.* **14(5)**(1993), 421-444.

[12] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, Cambridge University Press, **1990**.

[13] A.S. Kondrat'ev, Finite simple groups whose Sylow 2-subgroup is an extension of an abelian group by a group of rank 1, *Algebra i Logika* **14** (1975), 288-303.

[14] C.H. Li, The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$, *Tran. Amer. Math. Soc.* **353** (2001), 3511-3529.

[15] C.H. Li, On finite s-transitive graphs of odd order, *J. Combin. Theory Ser. B* **81**(2001), 307-317.

[16] C.H. Li, Z.P. Lu and G.X. Wang, On vertex-transitive and edge-transitive tetravalent graphs of square-free order, *J. Algebraic Combin.* **42** (2015), 25-50.

[17] C.H. Li, A. Seress and S.J. Song, s-Arc-transitive graphs and normal subgroups, *J. Algebra* **421** (2015) 331-348.

[18] C.H. Li and Hua Zhang, The finite primitive groups with soluble stabilizers, and the edge-primitive s-arc transitive graphs. *Proc. Lond. Math. Soc. (3)* **103** (2011), 441-472.

[19] M.W. Liebeck and J. Saxl, The primitive permutation groups of odd degree, *J. London Math. Soc.* **(2) 31**(1985) 250-264.

[20] C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **47**(1992), 227-239.

[21] V.I. Trofimov, Graphs with projective suborbits, cases of small characteristics, I(in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.*, **58**(1994), 124-171.

[22] V.I. Trofimov, Vertex stabilizers of locally projective groups of automorphisms of graphs. A summary, *Groups, combinatorics and geometry* (2001), 313-334.

[23] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43**(1947), 459-474.

[24] R. Weiss, s-transitive graphs, Algebraic methods in graph theory, *Colloq. Soc. Janos Bolyai* **25**(1981), 827-847.

[25] R.A. Wilson, *The Finite Simple Groups*, Springer Berlin Heidelberg NewYork Hong Kong London Milan Paris Tokyo Press, **2007**.

Cai Heng Li, Department of Mathematics, Southern University of Science and Technology, Shenzhen, P. R. China

E-mail address: lich@sustech.edu.cn

Jing Jian Li, School of Mathematics and Information Sciences, Guangxi University, Nanning 530004, P. R. China.

Colleges and Universities Key Laboratory of Mathematics and Its Applications.

E-mail address: lijjhx@163.com

Zai Ping Lu, Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lu@nankai.edu.cn