Planar Turán numbers of short paths

Yongxin Lan[1] and Yongtang Shi[2*]

[1] School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin, 300387, China

[2] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China

Email: yxlan0@126.com, shi@nankai.edu.cn

Abstract Given a graph H, the planar Turán number of H, denoted $ex_{\mathcal{P}}(n,H)$, is the maximum number of edges in an H-free planar graph on n vertices. The idea of determining $ex_{\mathcal{P}}(n,P_k)$ was promoted by Lan, Song and Shi, in which they obtained that the planar Turán number of paths P_k with $k\in\{8,9\}$. In this paper, we determine the planar Turán number of paths P_k with $k\in\{6,7,10,11\}$.

Keywords Turán number, planar Turán number, path

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1 Introduction

Graphs considered below will always be simple and finite. Our notation in this paper is standard and refers to [3]. Given a graph G, let $|G|$ and $e(G)$ denote the size of the vertex set $V(G)$ and edge set $E(G)$, respectively. For a vertex $v\in V(G)$, we will use $N_G(v)$ to denote the set of vertices which are adjacent to v in G and its size, denoted $d_G(v)$, is the degree of vertex v. Let $\delta(G)$ denote the minimum degree in a graph G. Given two graphs G and H, the *union* $G\cup H$ is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$; the *join* $G+H$ is the graph obtained from $G\cup H$ by adding all edges with one endpoint in G and the other in H; and let kG denote the disjoint union of k copies of G, where k is a positive integer. For a vertex set $S\subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S and $G\backslash S$ the subgraph of G induced by $V(G)\backslash S$ (i.e., the set $V(G)-S$). For $A\subseteq E(G)$, let G/A denote the simple graph obtained from G by contracting each component of $G[A]$ into a single vertex. If $A=\{uv\}$, then we simple write G/uv. Moreover, a graph is a minor of a given graph G if it can be obtained from a subgraph of G by contracting edges. Denote by P_k a path and C_k a cycle on k vertices. Let K_k^- denote the complete graph on k vertices minus one edge.

Given a graph H, we say that a graph is H-free if it does not contain H as a subgraph. One of the fundamental questions in extremal graph theory is to study the maximum number

*Corresponding author

of edges in an H-free graph on n vertices. The maximum, denoted $ex(n, H)$, is called the Turán number of H. Turán Theorem [15] gave a precise answer to this question for complete graphs by determining the balanced complete $(r-1)$-partite graph (called *Turán graph*) with the maximum number of edges in a K_r-free graph on n vertices. This was extended by Erdős and Stone [6], who proved that every H-free graph has at most $(1+o(1))(1-\frac{1}{\chi(H)-1})\binom{n}{2}$ edges for given arbitrary graph H, where $\chi(H)$ denotes the chromatic number of H. This means that the asymptotics of $ex(n, H)$ was determined for all non-bipartite graphs H. For bipartite graphs H, the problem of determining $ex(n, H)$ is still largely open. The Turán problem for even cycles is of particular interest. Erdős [5] conjectured that $ex(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}})$. The upper bound on $ex(n, C_{2k})$ was showed by Bondy and Simonovits [2], but the corresponding lower bound is only known for $k \in \{2, 3, 5\}$. The Turán number of paths was completely determined by Faudree and Schelp [7].

When host graphs are hypergraphs, the Turán number of k-uniform linear paths and cycles was also investigated and we refer to [9, 10, 11]. More results for Turán problem of hypergraphs see surveys [8, 12].

When host graphs are planar graphs, the Turán problem was introduced by Dowden [4] (under the name of "extremal" planar graphs). The *planar Turán number* of H, denoted $ex_{\mathcal{P}}(n, H)$, is the maximum number of edges in an H-free planar graph on n vertices. Euler's formula implies that the maximum number of edges in a planar graph on $n \geq 3$ vertices equals $3n-6$. It is trivial that $ex_{\mathcal{P}}(n, H) = 3n-6$ for every non-planar graph H. The planar Turán number of K_r can be obtained easily as K_5 is not planar. Dowden first observed the results for K_r with $3 \leq r \leq 4$ and also determined the tight upper bounds of $ex_{\mathcal{P}}(n, C_k)$ for $k \in \{4, 5\}$. Actually, $ex_{\mathcal{P}}(n, K_4) = 3n-6$, since the triangulation $\overline{K_2} + C_{n-2}$ is K_4-free. In [14], the authors completely determine $ex_{\mathcal{P}}(n, H)$ when H is a wheel or a star, and obtain several sufficient conditions on H which yield $ex_{\mathcal{P}}(n, H) = 3n-6$ for all $n \geq |V(H)|$, which partially answers a question of Dowden [4]. In [13], the upper bound of $ex_{\mathcal{P}}(n, C_6)$ was determined and they promoted the idea of determining $ex_{\mathcal{P}}(n, P_k)$. In addition, they determined that the planar Turán number for paths P_k with $k = 8, 9$. Let $\mathcal{T}_t$ denote the family of all planar triangulations on t vertices and let $\mathcal{T}_t^* \subseteq \mathcal{T}_t$ denote the family of planar triangulations with a spanning path. Now we will construct a family of graphs containing a copy of P_{k-1} but no P_k. Let $n = \lfloor k/3 \rfloor - 1 + \varepsilon + t(\lfloor k/3 \rfloor - 1) + r + 2$, $t \geq 2$ and $0 \leq r < \lfloor k/3 \rfloor - 1$, where $\varepsilon = k \pmod 3$. Given a positive integer $k \geq 9$, let $(a_0, b_0), \ldots, (a_{t+1}, b_{t+1})$ be the two ends of one fixed spanning path of $T_0, T_1, \ldots, T_{t+1}$, respectively, and let $\mathcal{G}_{\lfloor k/3 \rfloor+1+\varepsilon,\, n}$ be the family of graphs obtained from $T_0, T_1, \ldots, T_{t+1}$ by identifying all a_i as a and identifying all b_i as b, where

$$T_0 \in \mathcal{T}^*_{\lfloor k/3 \rfloor+1+\varepsilon},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor+1} \text{ for any } i \in [t] \text{ when } \varepsilon \in \{0, 1\};$$

$$T_0, T_1 \in \mathcal{T}^*_{\lfloor k/3 \rfloor+2},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor+1} \text{ for any } 2 \leq i \leq t, \text{ or}$$

$$T_0 \in \mathcal{T}^*_{\lfloor k/3 \rfloor+3},\ T_{t+1} \in \mathcal{T}^*_{r+2},\ T_i \in \mathcal{T}^*_{\lfloor k/3 \rfloor+1} \text{ for any } i \in [t] \text{ when } \varepsilon = 2.$$

For $n \geq k-1$, it is easy to see that the longest path of $\mathcal{G}_{\lfloor k/3\rfloor+1+\varepsilon,\, n}$ has $(|T_0|-2)+(|T_1|-2)+(|T_2|-2)+2 = k-1$ vertices and so $\mathcal{G}_{\lfloor k/3\rfloor+1+\varepsilon,\, n}$ is P_k-free, where $\varepsilon = k \pmod 3$.

Theorem 1.1 ([13]) *Let $n \geq 3$ be an integer. Let G be a P_8-free planar graph on n vertices. Then $e(G) \leq 15n/7$, with equality when $n = 7t$ for any positive integer t and $G = T_1 \cup \cdots \cup T_t$, where $T_i \in \mathcal{T}_7$ for all $i \in [t]$.*

Theorem 1.2 ([13]) *Let $n \geq 3$ be an integer. Let G be a P_9-free planar graph on n vertices. Then $e(G) \leq \max\{\frac{9n}{4}, \frac{5n}{2}-4\}$, with equality when $G \in \mathcal{T}_8$ or when $G = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{T}_8$ or when $n \geq 16$ is even and $G \in \mathcal{G}_{4,n}$.*

In this paper, we continue to determine the planar Turán number of paths P_k with $k \in \{6, 7, 10, 11\}$. Clearly, $ex_{\mathcal{P}}(n, P_6) = 3n-6$ when $n \in \{3, 4, 5\}$. We first introduce more notation. We say that U is *complete to* W if for every $u \in U$ and every $w \in W$, $uw \in E(G)$. If $U = \{u\}$, we simply say u is complete to W. Let $e_G(S)$ denote the number of edges in G meeting the vertex set $S \subseteq V(G)$.

Theorem 1.3 *Let $n \geq 6$ be an integer and let G be a P_6-free planar graph on n vertices. Then*

$$e(G) \leq \begin{cases} 2n-3 & \text{if} \quad n \in \{6,9\}, \text{ with equality when } G \in \{K_2 + \overline{K_{n-2}},\, K_5^- \cup K_{n-5}\}; \\ 2n-3 & \text{if} \quad n \in \{7,8\}, \text{ with equality when } G = K_2 + \overline{K_{n-2}}; \\ 2n-2 & \text{if} \quad n = 10, \text{ with equality when } G = 2K_5^-; \\ 2n-3 & \text{if} \quad n \geq 11, \text{ with equality when } G \in \{K_2 + \overline{K_{n-2}},\, 3K_5^-\}. \end{cases}$$

Theorem 1.4 *Let $n \geq 7$ be an integer. Let G be a P_7-free planar graph on n vertices. Then*

$$e(G) \leq \begin{cases} 2n & \text{if} \quad n = 6t, \text{ with equality when } G = T_1 \cup \cdots \cup T_t; \\ 2n-1 & \text{if} \quad n = 6t+5, \text{ with equality when } G = T_1 \cup \cdots \cup T_t \cup K_5^-; \\ 2n-2 & \text{if} \quad n = 6t+4 \text{ with equality when } G \in \{K_2 + (\overline{K_{n-4}} + K_2), T_1 \cup \cdots \cup T_t \cup K_4, \\ & \qquad T_1 \cup \cdots \cup T_{t-1} \cup 2K_5^-, T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_6} \cup K_2))\}; \\ 2n-2 & \text{if} \quad n = 6t+1 \text{ with equality when } G \in \{K_2 + (\overline{K_{n-4}} + K_2), T_1 \cup \cdots \cup T_t \cup K_1, \\ & \qquad T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_3} \cup K_2))\}; \\ 2n-2 & \text{if} \quad n = 6t+r \text{ with equality when } G \in \{K_2 + (\overline{K_{n-4}} + K_2), \\ & \qquad T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_{2+r}} \cup K_2))\}, \end{cases}$$

where $T_i \in \mathcal{T}_6$ for all $i \in [t]$ and $r \in \{2, 3\}$.

Theorem 1.5 *Let $n \geq 3$ be an integer. Let G be a P_{10}-free planar graph on n vertices. Then $e(G) \leq \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$, with equality when $G \in \mathcal{T}_9$, or when $G = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{T}_9$, or when $n \geq 21$ is odd and $G \in \mathcal{G}_{5,n}$.*

Theorem 1.6 *Let $n \geq 3$ be an integer. Let G be a P_{11}-free planar graph on n vertices. Then $e(G) \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$, with equality when $G \in \mathcal{T}_{10}$, or when $G \in \{T_1 \cup T_2, T_1 \cup T_2 \cup T_3$ with $T_1, T_2, T_3 \in \mathcal{T}_{10}$, or when $n \geq 30$ is even and $G \in \mathcal{G}_{6,n}$.*

2 Preliminary Results

To study planar Turán numbers of paths, we shall make use of the following results.

Lemma 2.1 ([7]) *Let t, k, r be integers satisfying $t \geq 0$ and $0 \leq r < k$. If G is a P_{k+1}-free graph on $tk+r$ vertices, then $e(G) \leq t\binom{k}{2} + \binom{r}{2}$, with equality when $G = tK_k \cup K_r$ or when k is odd, $t > 0$, and $r \in \{(k+1)/2, (k-1)/2\}$, $G = (t-s-1)K_k \cup (K_{(k-1)/2} + \overline{K}_{(k+1)/2+sk+r})$ for some $s \in \{0, 1, \ldots, t-1\}$.*

Lemma 2.2 ([1]) *Let n, k with $n > k \geq 3$ be integers. If G is a connected, P_{k+1}-free graph on n vertices, then*

$$e(G) \leq \max\left\{\binom{k-1}{2} + (n-k+1), \binom{\lceil (k+1)/2 \rceil}{2} + \left\lfloor \frac{k-1}{2} \right\rfloor \left(n - \left\lceil \frac{k+1}{2} \right\rceil\right)\right\},$$

with equality when $G = K_s + (K_{k-2s} \cup \overline{K}_{n-k+s})$ for some $s \in \{1, \lfloor (k-1)/2 \rfloor\}$.

Lemma 2.3 ([13]) *Let G be a connected graph and let P be a longest path in G with vertices $v_1, v_2, \ldots, v_\ell$ in order, where $\ell = |P|$ and $|G| > \ell \geq 3$. Then*

(a) $G[V(P)]$ has no spanning cycle. In particular, $v_1 v_\ell \notin E(G)$, and if $v_1 v_s \in E(G)$ for some $s \in \{2, \ldots, \ell-1\}$, then $v_{s-1} v_\ell \notin E(G)$. Similarly, if $v_\ell v_s \in E(G)$ for some $s \in \{2, \ldots, \ell-1\}$, then $v_1 v_{s+1} \notin E(G)$.

(b) $v_{s-1} v_{t+1} \notin E(G)$ if $v_1 v_s \in E(G)$ and $v_\ell v_t \in E(G)$, where $s, t \in [\ell]$ with $2 \leq s \leq t \leq \ell-1$. Similarly, v_{t-1} has no edges to $\{v_{s-1}, v_{s+1}\}$ if $v_1 v_s \in E(G)$ and $v_\ell v_t \in E(G)$, where $s, t \in [\ell]$ with $4 \leq t+2 \leq s \leq \ell-1$.

(c) $2\delta(G) \leq d_G(v_1) + d_G(v_\ell) \leq \ell - 1$.

(d) v_ℓ (resp. v_1) is not adjacent to any two consecutive vertices in $\{v_2, v_3, \ldots, v_{\ell-1}\}$ if $v_1 v_{\ell-1} \in E(G)$ (resp. $v_\ell v_2 \in E(G)$).

It is worth noting that for all $k \in \{2, 3, 4, 5\}$, every P_k-free graph must be planar. Hence the values of $ex_{\mathcal{P}}(n, P_k)$ when $k \in \{2, 3, 4, 5\}$ and the extremal graphs are determined by Lemma 2.1.

3 Proof of Theorem 1.3

Let G and n be given as in the statement. By Lemma 2.2, the components of an extremal P_6-free graph are either K_r when $r \leq 4$ and K_5^- or $K_2 + \overline{K_{r-2}}$ when $r \geq 6$. When G is connected, by Lemma 2.2, $e(G) \leq 2n - 3$ with equality when $G = K_2 + \overline{K_{n-2}}$. So we may assume that G is disconnected. Let $H_1, H_2, \ldots, H_s$ be components of G. We see that $e(H_i) \leq 2n - 3$ when $|H_i| \in \{2, 3\}$ or $|H_i| \geq 6$, $e(H_i) \leq 2n - 2$ when $|H_i| \in \{1, 4\}$ and $e(H_i) \leq 2n - 1$ when $|H_i| = 5$. If $s \geq 3$, then $e(G) = e(H_1) + \cdots + e(H_s) \leq 2n - 3$ with equality when $G = 3K_5^-$. So we assume $s = 2$. If $H_1 = H_2 = K_5^-$, then $e(G) = 2n - 2$. If either $H_1 \neq K_5^-$ or $H_2 \neq K_5^-$, then $e(G) = e(H_1) + e(H_2) \leq 2n - 3$ with equality when $G = K_5^- \cup K_1$ or $G = K_5^- \cup K_4$. □

4 Proof of Theorem 1.4

Let G and n be given as in the statement. By Lemma 2.2, the components of an extremal P_7-free graph are either K_r when $r \leq 4$, K_5^-, $K_2 + P_4$ and $K_{2,2,2}$, or $K_2 + (\overline{K_{r-4}} + K_2)$ when $r \geq 7$. When G is connected, by Lemma 2.2, $e(G) \leq 2n - 2$ with equality when $G = K_2 + (\overline{K_{r-4}} + K_2)$. So we may assume that G is disconnected and let $H_1, H_2, \ldots, H_s$ be the components of G. We see that $e(H_i) \leq 2n - 3$ when $|H_i| \in \{2, 3\}$, $e(H_i) \leq 2n - 2$ when $|H_i| \in \{1, 4\}$ or $|H_i| \geq 7$, $e(H_i) \leq 2n-1$ when $|H_i| = 5$ and $e(H_i) \leq 2n$ when $|H_i| = 6$. Then $e(G) \leq 2n$ with equality when $n = 6t$ and $G = T_1 \cup \cdots \cup T_t$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$. Let $t \geq 1$ be any positive integer. If $n = 6t + 5$, then $e(H_j) \leq 2|H_j| - 1$ for some $j \in [s]$, and so $e(G) \leq 2n - 1$ with equality when $G = T_1 \cup \cdots \cup T_t \cup K_5^-$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$. If $n = 6t + 4$, then $e(H_{j_i}) \leq 2|H_{j_i}| - 1$ for $i \in [2]$ and some $j_1, j_2 \in [s]$ or $e(H_\ell) \leq 2|H_\ell| - 2$ for some $\ell \in [s]$, and so $e(G) \leq 2n-2$ with equality when $G \in \{K_2 + (\overline{K_{n-4}} + K_2), T_1 \cup \cdots \cup T_t \cup K_4, T_1 \cup \cdots \cup T_{t-1} \cup 2K_5^-, T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_6} + K_2))\}$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$. If $n = 6t+1$, then $e(H_\ell) \leq 2|H_\ell| - 2$ for some $\ell \in [s]$, and so $e(G) \leq 2n-2$ with equality when $G \in \{K_2 + (\overline{K_{n-4}} + K_2), T_1 \cup \cdots \cup T_t \cup K_1, T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_3} + K_2))\}$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$. Finally, if $n = 6t+r$ for $r \in \{2, 3\}$, then $e(H_k) \leq 2|H_k| - 2$ for some $k \in [s]$ and so $e(G) \leq 2n-2$ with equality when $G \in \{K_2 + (\overline{K_{n-4}} + K_2), T_1 \cup \cdots \cup T_{t-1} \cup (K_2 + (\overline{K_{2+r}} + K_2))\}$, where $T_i \in \mathcal{T}_6$ for all $i \in [t]$, as desired. □

5 Proof of Theorem 1.5

Let G and n be given as in the statement. Note that $\max\{\frac{7n}{3}, \frac{5n-7}{2}\} = \frac{5n-7}{2}$ when $n \geq 21$ and $\max\{\frac{7n}{3}, \frac{5n-7}{2}\} = \frac{7n}{3}$ when $n \leq 21$. By induction on n. Since any graph on at most 9 vertices is P_{10}-free and $|G| \geq 3$, we see that $e(G) \leq 3n-6 \leq \frac{7n}{3}$, with equality when $n = 9$ and $G \in \mathcal{T}_9$. So we may assume that $n \geq 10$. We next show that $e(G) \leq \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$. Let $x \in V(G)$ be a vertex with $d_G(x) = \delta(G)$. Then $G - x$ is a P_{10}-free planar graph on $n-1$ vertices. By the induction hypothesis, $e(G-x) \le \max\{\frac{7}{3}(n-1), \frac{5}{2}(n-1)-\frac{7}{2}\}$ and so $e(G) = e(G-x) + d_G(x) \le \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$ when $d_G(x) \le 2$. So we may assume that $d_G(x) \ge 3$. Assume next that G is disconnected and H is one of its components. Then $|H| \ge 4$ and $|G \backslash V(H)| \ge 4$ since $\delta(G) \ge 3$. By the induction hypothesis, $e(H) \le \max\{\frac{7}{3}|H|, \frac{5}{2}|H| - \frac{7}{2}\}$ and $e(G\backslash V(H)) \le \max\{\frac{7}{3}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)| - \frac{7}{2}\}$. Hence, $e(G) = e(H) + e(G\backslash V(H)) \le \max\{\frac{7}{3}|H|, \frac{5}{2}|H| - \frac{7}{2}\} + \max\{\frac{7}{3}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)| - \frac{7}{2}\} \le \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$, with equality when both H and $G\backslash V(H)$ are planar triangulations on 9 vertices. Hence, $e(G) \le \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$, with equality when $n = 18$ and $G = T_1 \cup T_2$, where $T_1, T_2 \in \mathcal{T}_9$. Now suppose G is connected. Let P be a longest path in G with vertices $v_1, v_2, \dots, v_t$ in order. We may assume that $d_G(v_1) \le d_G(v_t)$. Then $t \le 9$ because G is P_{10}-free. By Lemma 2.3(c), $6 \le 2\delta(G) \le d_G(v_1) + d_G(v_t) \le t-1 \le 8$. Then $7 \le t \le 9$. Assume that $t \in \{7, 8\}$. Then by Theorem 1.1 (when $t = 7$) and Theorem 1.2 (when $t = 8$), $e(G) < \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$, as desired. So we may assume that $t = 9$.

Let F be an induced subgraph of G on $V(P)$. Let ℓ_v denote the number of vertices of the longest path in F starting at v and $S = \{v \in V(F) | \ell_v = 9\}$. Notice that $S \ne \emptyset$. Since G is connected, it follows that $S \ne V(F)$. Observe that $e_F(S') = e_G(S')$ for any $S' \subseteq S$. Assume that there exists some $R \subseteq S$ with $e_F(R) < \frac{7}{3}|R|$. By the induction hypothesis, $e(G\backslash R) \le \max\{\frac{7}{3}|G\backslash R|, \frac{5}{2}|G\backslash R| - \frac{7}{2}\}$. Hence, $e(G) = e_G(R) + e(G\backslash R) < \frac{7}{3}|R| + \max\{\frac{7}{3}|G\backslash R|, \frac{5}{2}|G\backslash R| - \frac{7}{2}\} \le \max\{\frac{7n}{3}, \frac{5n-7}{2}\}$. So we may assume that $e_F(S') \ge \frac{7}{3}|S'|$ for any $S' \subseteq S$. Thus we have the following claim.

Claim. The graph G has an induced subgraph F on 9 vertices satisfying the properties: (1) F is planar; (2) $S \ne \emptyset$ and $S \ne V(F)$; (3) $e_F(S') \ge \frac{7}{3}|S'|$ for any $S' \subseteq S$.

It can be shown by computer that there are only 18 graphs with the above three properties, as depicted in **Appendix 1**. It can be observed that $(\ell_v)_{v \notin S} = (\ell_{u_1}, \ell_{u_2}) = (7, 7)$ when $F = F_i$ for any $i \in [10]$, $(\ell_v)_{v \notin S} = (\ell_{w_1}, \ell_{w_2}, \ell_{w_3}) = (7, 7, 8)$ when $F = F_{11}$ or $F = F_{12}$, and $(\ell_v)_{v \notin S} = (\ell_{w_1}, \ell_{w_2}, \ell_{w_3}) = (8, 8, 8)$ when $F = F_i$ for any $13 \le i \le 18$. Assume that $F = F_i$ for any $11 \le i \le 18$. Since $\delta(G) \ge 3$ and G is P_{10}-free, it follows that $N_G(v) = \{w_1, w_2, w_3\}$ for some $v \in V(G)\backslash V(F)$. But then G contains a copy of $K_{3,3}$ because $n \ge 10$ and F contains $K_{2,3}$ as a subgraph with one part $\{w_1, w_2, w_3\}$, a contradiction. Assume then that $F = F_i$ for any $i \in [10]$. Since $\delta(G) \ge 3$ and G is P_{10}-free, it follows that for any $w \in V(G)\backslash V(F)$, $d_G(w) = 3$, w is complete to $\{u_1, u_2\}$ in G and every component of $G\backslash V(F)$ is isomorphic to K_2. This is only possible when n is odd. We see that $G\backslash V(F) = \frac{n-9}{2}K_2$. Hence, when n is odd, we see that $e(G) \le e(F) + \frac{5(n-9)}{2} \le 19 + \frac{5n-45}{2} \le \frac{5n-7}{2}$. With equality when $F = F_6$ or $F = F_{10}$, that is, when $G \in \mathcal{G}_{5,n}$. □

6 Proof of Theorem 1.6

Let G and n be given as in the statement. Note that $\max\{\frac{12n}{5}, \frac{5n-6}{2}\} = \frac{5n-6}{2}$ when $n \ge 30$ and $\max\{\frac{12n}{5}, \frac{5n-6}{2}\} = \frac{12n}{5}$ when $n \le 30$. By induction on n. Since any graph on at most 10 vertices is P_{11}-free and $|G| \geq 3$, we see that $e(G) \leq 3n-6 \leq \frac{12n}{5}$, with equality when $n=10$ and $G \in \mathcal{T}_{10}$. So we may assume that $n \geq 11$. We next show that $e(G) \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$. Let $x \in V(G)$ be a vertex with $d_G(x) = \delta(G)$. Then $G-x$ is a P_{11}-free planar graph on $n-1$ vertices. By the induction hypothesis, $e(G-x) \leq \max\{\frac{12}{5}(n-1), \frac{5}{2}(n-1)-3\}$ and so $e(G) = e(G-x)+d_G(x) \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$ when $d_G(x) \leq 2$. So we may assume that $d_G(x) \geq 3$. Assume next that G is disconnected and H is one of its components. Then $|H| \geq 4$ and $|G\backslash V(H)| \geq 4$ since $\delta(G) \geq 3$. By the induction hypothesis, $e(H) \leq \max\{\frac{12}{5}|H|, \frac{5}{2}|H|-3\}$ and $e(G\backslash V(H)) \leq \max\{\frac{12}{5}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)|-3\}$. Hence, $e(G) = e(H)+e(G\backslash V(H)) \leq \max\{\frac{12}{5}|H|, \frac{5}{2}|H|-3\}+\max\{\frac{12}{5}|G\backslash V(H)|, \frac{5}{2}|G\backslash V(H)|-3\} \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$, with equality when H is planar triangulation on 10 vertices and $G\backslash V(H)$ is either planar triangulation on 10 vertices or the disjoint union of two planar triangulations on 10 vertices. Hence, $e(G) \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$, with equality when $n=20$ and $G = T_1 \cup T_2$, or when $n=30$ and $G = T_1 \cup T_2 \cup T_3$, where $T_1, T_2, T_3 \in \mathcal{T}_{10}$. Now suppose G is connected. Let P be a longest path in G with vertices $v_1, v_2, \ldots, v_t$ in order. Assume $d_G(v_1) \leq d_G(v_t)$ and $t \leq 10$ since G is P_{11}-free. By Lemma 2.3(c), $6 \leq 2\delta(G) \leq d_G(v_1)+d_G(v_t) \leq t-1 \leq 9$. Then $7 \leq t \leq 10$. If $t \in \{7,8,9\}$, then by Theorem 1.1 (when $t=7$), Theorem 1.2 (when $t=8$) and Theorem 1.5 (when $t=9$), we have $e(G) < \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$, as desired. So now suppose $t=10$.

Let H be an induced subgraph of G on $V(P)$. Let ℓ_v denote the number of vertices of the longest path in H starting at v and $S = \{v \in V(H) | \ell_v = 10\}$. Notice that $S \neq \emptyset$. Since G is connected, it follows that $S \neq V(H)$. Observe that $e_H(S') = e_G(S')$ for any $S' \subseteq S$. Assume that there exists some $R \subseteq S$ with $e_H(R) < \frac{12}{5}|R|$. By the induction hypothesis, $e(G\backslash R) \leq \max\{\frac{12}{5}|G\backslash R|, \frac{5}{2}|G\backslash R|-3\}$. Hence, $e(G) = e_G(R)+e(G\backslash R) < \frac{12}{5}|R| + \max\{\frac{12}{5}|G\backslash R|, \frac{5}{2}|G\backslash R|-3\} \leq \max\{\frac{12n}{5}, \frac{5n-6}{2}\}$. So we may assume that $e_H(S') \geq \frac{12}{5}|S'|$ for any $S' \subseteq S$. Thus we have the following claim.

Claim. The graph G has an induced subgraph H on 10 vertices satisfying the properties: (1) H is planar; (2) $S \neq \emptyset$ and $S \neq V(H)$; (3) $e_H(S') \geq \frac{12}{5}|S'|$ for any $S' \subseteq S$.

It could be shown by computer that there are only 200 graphs with the above three properties, as depicted in **Appendix 2**. It can be observed that $(\ell_w)_{w \notin S} = (\ell_x, \ell_y, \ell_u, \ell_v) = (8,8,9,9)$ when $H = H_i$ for any $i \in [28]$, $(\ell_w)_{w \notin S} = (\ell_x, \ell_y, \ell_u, \ell_v) = (9,9,9,9)$ when $H = H_i$ for any $29 \leq i \leq 88$, $(\ell_w)_{w \notin S} = (\ell_x, \ell_y, \ell_u) = (8,8,9)$ when $H = H_i$ for any $89 \leq i \leq 118$, $(\ell_w)_{w \notin S} = (\ell_x, \ell_y, \ell_u) = (9,9,9)$ when $H = H_i$ for any $119 \leq i \leq 166$, and $(\ell_w)_{w \notin S} = (\ell_x, \ell_y) = (8,8)$ when $H = H_i$ for any $167 \leq i \leq 200$.

Case 1. $H = H_i$ for any $i \in [28]$. Since $V(H)\backslash S = \{x,y,u,v\}$, $N_H(w) \subseteq \{x,y,u,v\}$ and $d_H(w) \leq 4$ for any $w \in V(G)\backslash V(H)$. We claim that $d_H(w) \leq 2$ for any $w \in V(G)\backslash V(H)$. Suppose $d_H(w) \geq 3$ for some $w \in V(G)\backslash V(H)$, then either $\{x,y\} \subseteq N_H(w)$ or $\{u,v\} \subseteq N_H(w)$. If $\{u,v\} \subseteq N_H(w)$, then G contains P_{11} as a subgraph when $uv \in E(H)$, and G contains $K_{3,3}$-minor as a subgraph with one part $\{x,y,\{u,w,v\}\}$ when $uv \notin E(H)$. If $\{x,y\} \subseteq N_H(w)$, then G contains $K_{3,3}$-minor as a subgraph with one part $\{x,y,\{u,v\}\}$ or

$\{x, y, u\}$ or $\{x, y, v\}$. Now suppose $d_H(w) \leq 2$ for any $w \in V(G)\backslash V(H)$. Since $\delta(G) \geq 3$ and G is P_{11}-free, it follows that for any $w \in V(G)\backslash V(H)$, $d_G(w) = 3$, w is complete to $\{x, y\}$ in G, and so each component is isomorphic to K_2. We see that $G\backslash V(H) = \frac{n-10}{2}K_2$. This is impossible when n is even. Hence, $e(G) = e(H) + \frac{5(n-10)}{2} \leq 22 + \frac{5n-50}{2} \leq \frac{5n-6}{2}$. With equality when $H = H_{20}$ or $H = H_{28}$, that is, when $G \in \mathcal{G}_{6,n}$.

Case 2. $H = H_i$ for any $29 \leq i \leq 88$. Notice that $N_H(w) = N_G(w)$ for any $w \in V(G)\backslash V(H)$. Since $V(H)\backslash S = \{x, y, u, v\}$, $N_G(w) \subseteq \{x, y, u, v\}$ and $d_G(w) \leq 4$ for any $w \in V(G)\backslash V(H)$. Since $\delta(G) \geq 3$, it follows that either $\{x, y\} \subseteq N_H(w)$ or $\{u, v\} \subseteq N_H(w)$ for any $w \in V(G)\backslash V(H)$. If $\{u, v\} \subseteq N_H(w)$, then G contains $K_{3,3}$-minor as a subgraph with one part $\{x, y, \{u, w, v\}\}$ or $\{u, v, \{x, y, w'\}\}$, where $w' \in N_H(x) \cap N_H(y)$. If $\{x, y\} \subseteq N_H(w)$, then G contains $K_{3,3}$-minor as a subgraph with one part $\{x, y, u\}$ or $\{x, y, v\}$ or $\{x, y, \{u, w', v\}\}$, where $w' \in N_H(u) \cap N_H(v)$.

Case 3. $H = H_i$ for any $89 \leq i \leq 166$. Since $V(H)\backslash S = \{x, y, u\}$, $N_H(w) \subseteq \{x, y, u\}$ and $d_H(w) \leq 3$ for any $w \in V(G)\backslash V(H)$. Notice that $G\backslash V(H)$ contains at most one vertex w with $d_H(w) = 3$ since H contains $K_{1,3}$ as a subgraph with one part $\{x, y, u\}$. Hence, $e(G) \leq e(H) + 3 + 2(n-11) = 2n + 3 < \frac{12n}{5}$ when $H = H_i$ for any $119 \leq i \leq 166$. So next consider $H = H_i$ for any $97 \leq i \leq 118$. We see that $d_H(w) = 2$ for any $w \in V(G)\backslash V(H)$ since G contains $K_{2,3}$-minor as a subgraph with one part $\{x, y, u\}$. Since $\delta(G) \geq 3$ and G is P_{11}-free, it follows that for any $w \in V(G)\backslash V(H)$, $d_G(w) = 3$, w is complete to $\{x, y\}$ in G, and so each component is isomorphic to K_2. We see that $G\backslash V(H) = \frac{n-10}{2}K_2$. This is impossible when n is even. Hence, $e(G) \leq e(H) + \frac{5(n-10)}{2} \leq 22 + \frac{5n-50}{2} \leq \frac{5n-6}{2}$. With equality when $H = H_{112}$ or $H = H_{118}$, that is, when $G \in \mathcal{G}_{6,n}$. Finally, consider $H = H_i$ for $89 \leq i \leq 96$. Since $\delta(G) \geq 3$ and G is P_{11}-free, it follows that for any $w \in V(G)\backslash V(H)$, $d_G(w) = 3$, w is complete to $\{x, y\}$ ($\{x, y, u\}$ when $d_H(w) = 3$) in G, and so each component is isomorphic to K_2 (isolated vertex when $d_H(w) = 3$). Hence, $e(G) \leq e(H) + 3 + \frac{5(n-11)}{2} \leq 21 + 3 + \frac{5n-55}{2} = \frac{5n-7}{2}$.

Case 4. $H = H_i$ for any $167 \leq i \leq 200$. Since $V(H)\backslash S = \{x, y\}$, $N_H(w) \subseteq \{x, y\}$ and $d_H(w) \leq 2$ for any $w \in V(G)\backslash V(H)$. Since $\delta(G) \geq 3$ and G is P_{11}-free, it follows that for any $w \in V(G)\backslash V(H)$, $d_G(w) = 3$, w is complete to $\{x, y\}$ in G, and so each component is isomorphic to K_2. We see that $G\backslash V(H) = \frac{n-10}{2}K_2$. This is impossible when n is even. Hence, $e(G) \leq e(H) + \frac{5(n-10)}{2} \leq 22 + \frac{5n-50}{2} \leq \frac{5n-6}{2}$. With equality when $H = H_{184}$ or $H = H_{186}$ or $H = H_{188}$ or $H = H_{190}$ or $H = H_{192}$ or $H = H_{200}$, that is, when $G \in \mathcal{G}_{6,n}$. □

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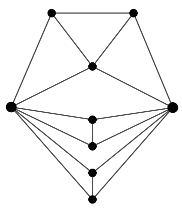
Appendix 1:

List of 18 graphs satisfying the three properties in Claim used in the proof of Theorem 1.5.



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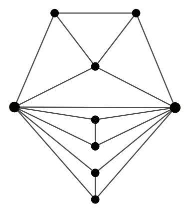

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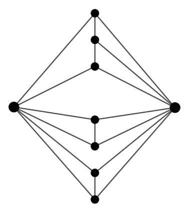

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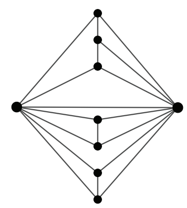

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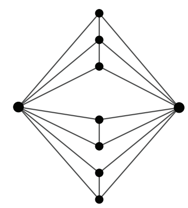

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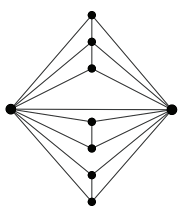

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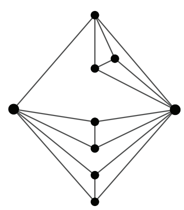

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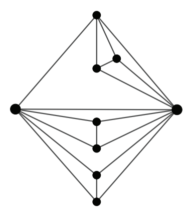

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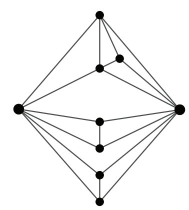

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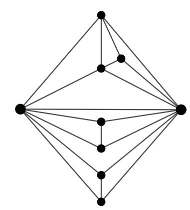
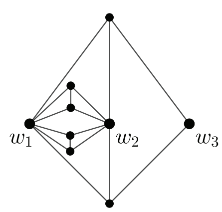
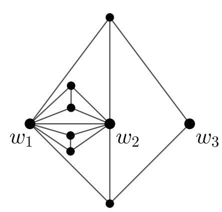


F_{10}



F_{11}



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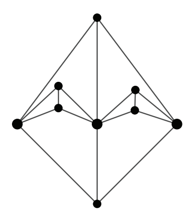

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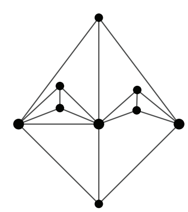

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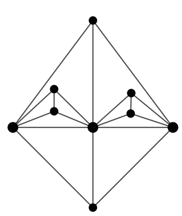

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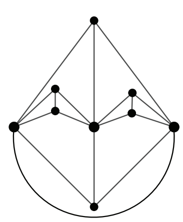

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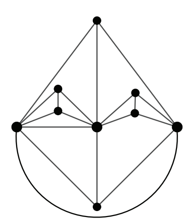

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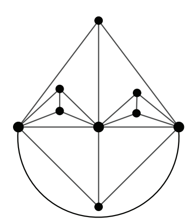
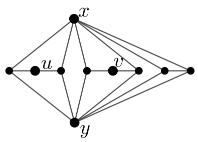


F_{18}

Appendix 2:

List of 200 graphs satisfying the three properties in Claim used in the proof of Theorem 1.6.



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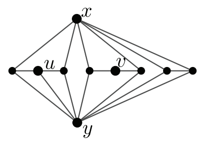
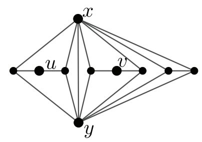


H_2



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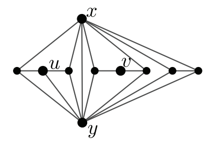

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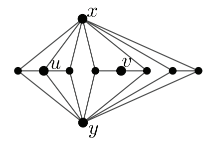

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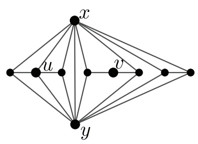

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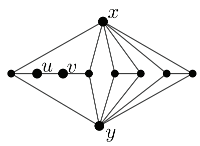

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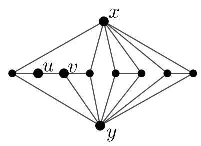
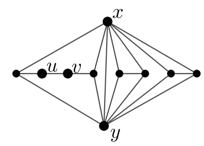


H_8



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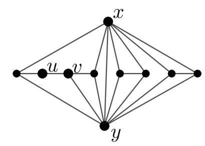
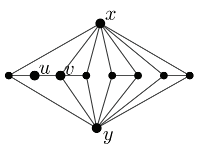
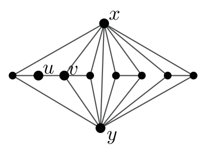


H_{10}



H_{11}



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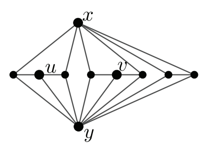

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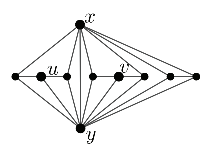

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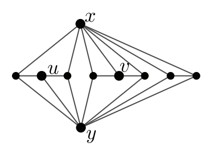

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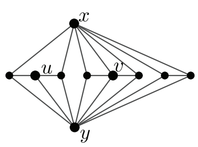

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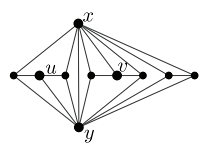

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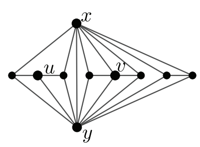

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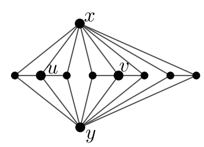

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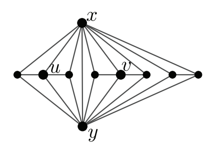

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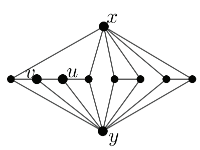

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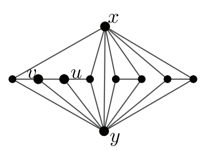

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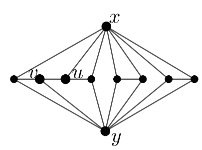

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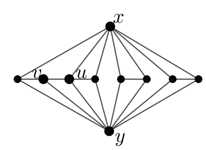

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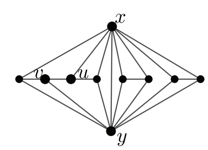

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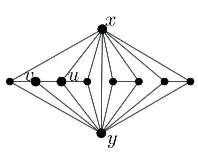

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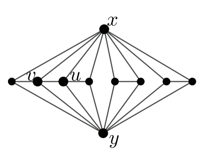

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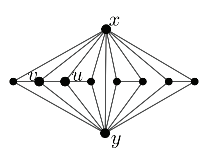

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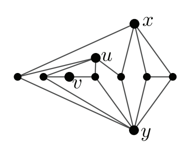

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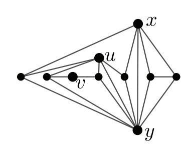

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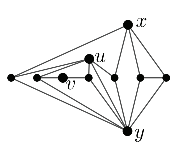

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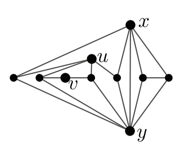

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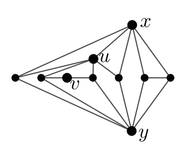

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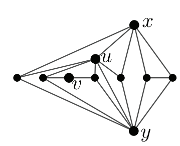

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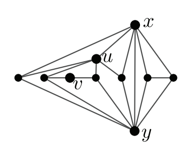

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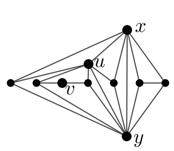

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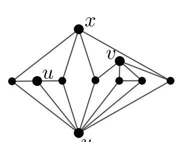

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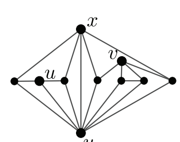

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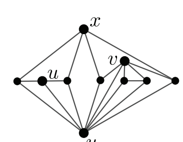

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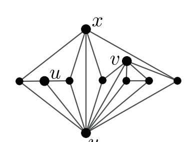

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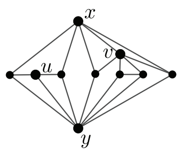

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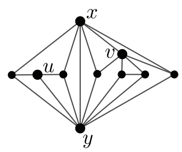

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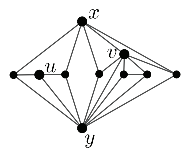

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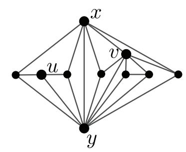

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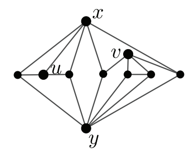

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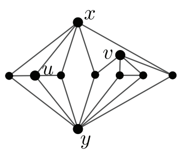

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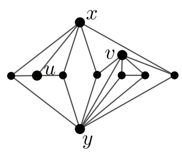

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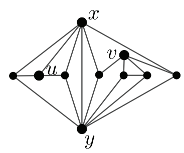

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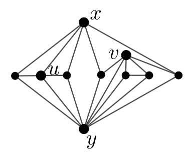

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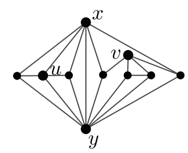

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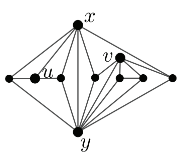

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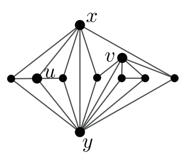

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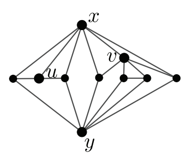

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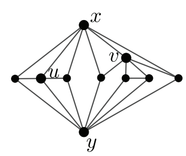

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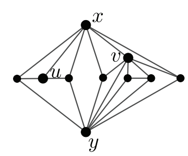

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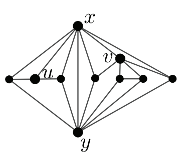

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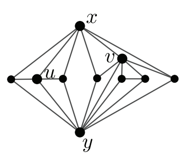

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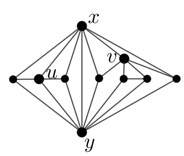

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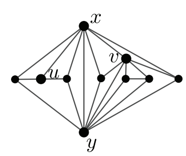

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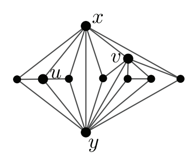

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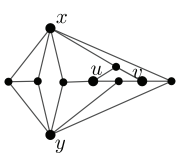

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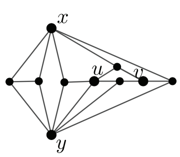

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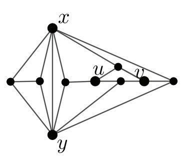

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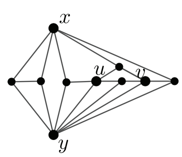

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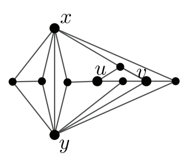

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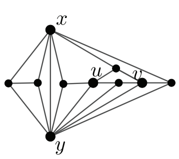

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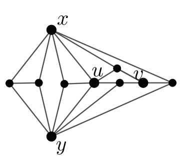

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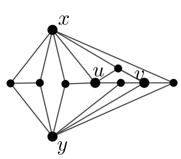

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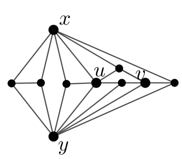

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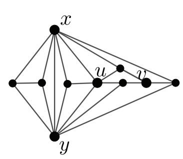

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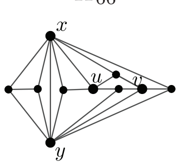

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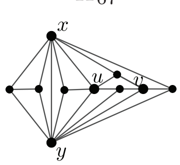

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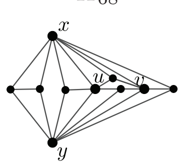

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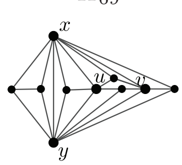

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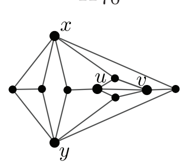

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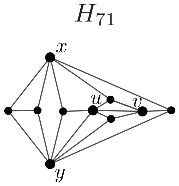

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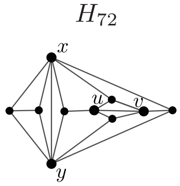

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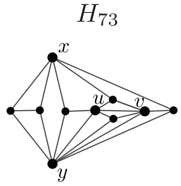

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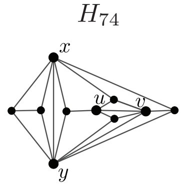

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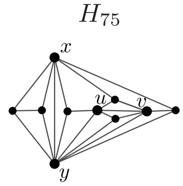

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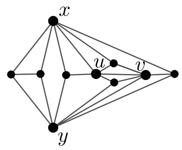

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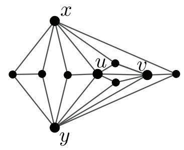

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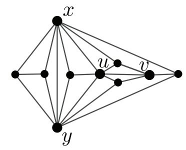

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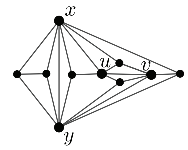

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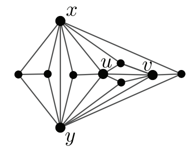

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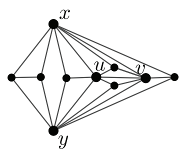

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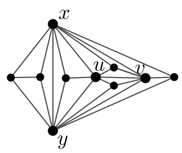

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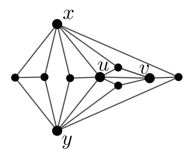

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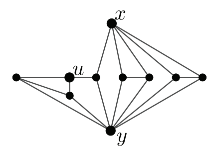

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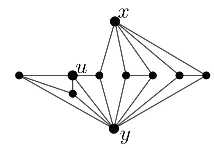

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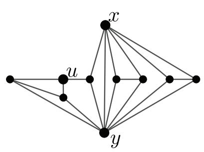

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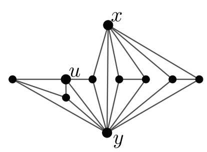

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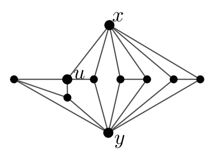

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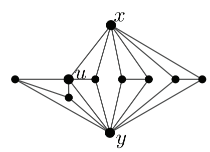

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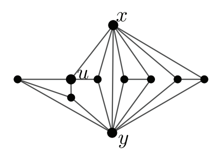

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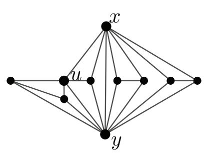

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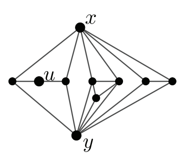

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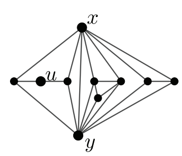

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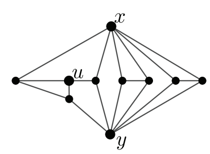

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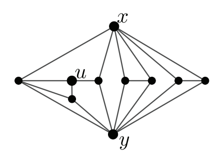

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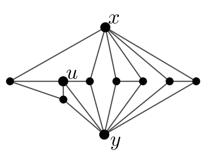

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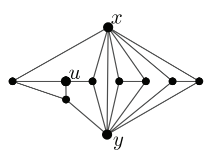

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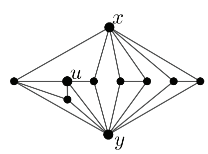

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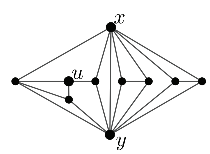

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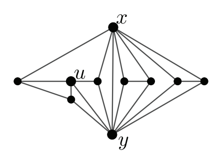

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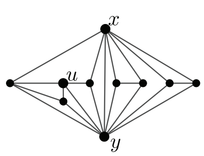

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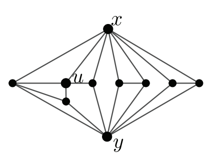

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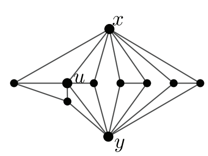

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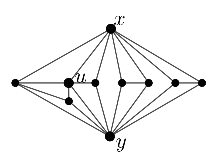

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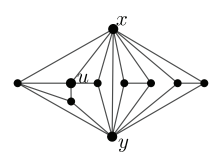

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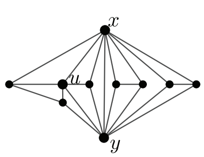

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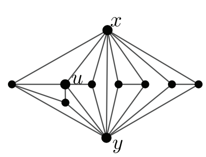

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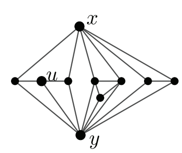

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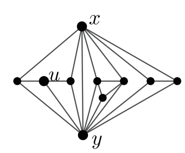

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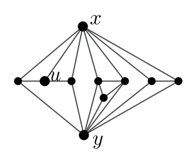

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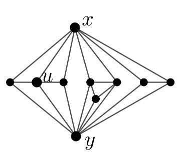

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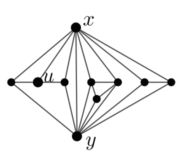

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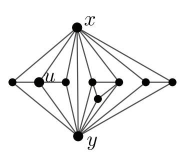

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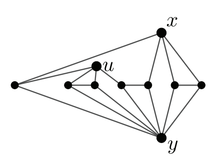

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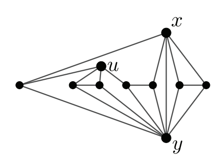

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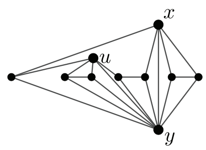

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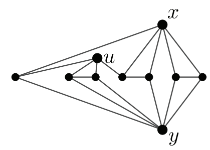

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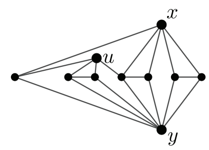

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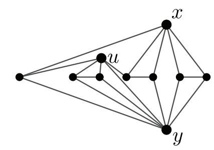

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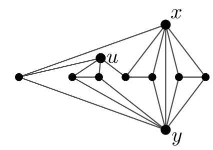

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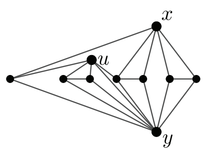

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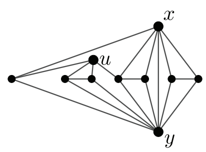

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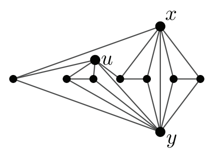

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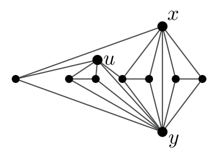

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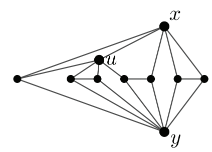

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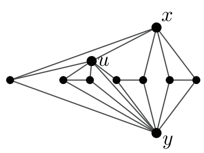

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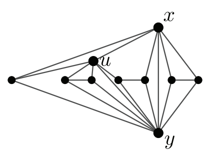

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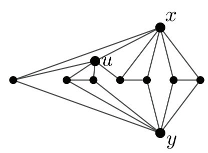

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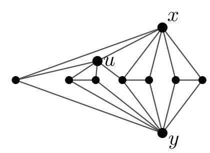

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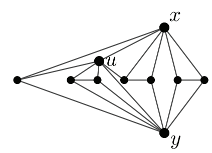

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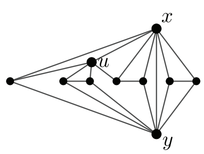

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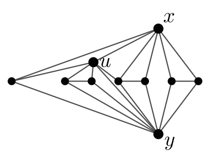

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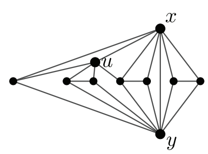

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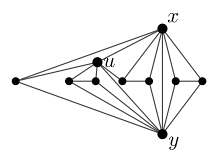

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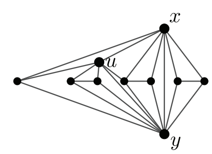

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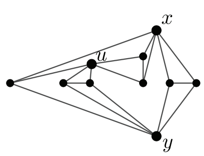

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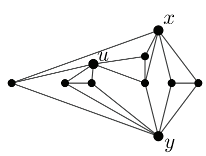

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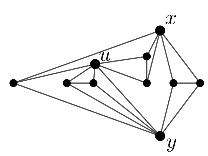

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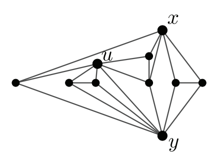

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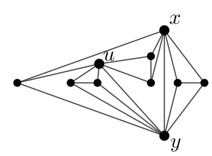

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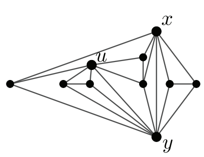

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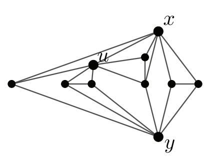

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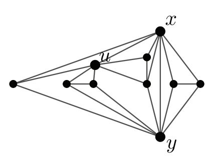

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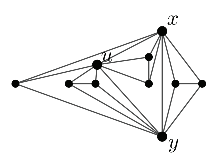

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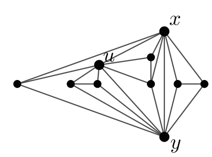

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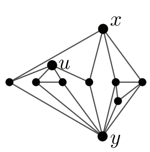

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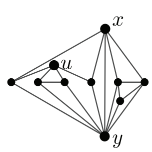

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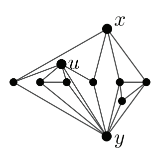

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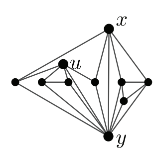

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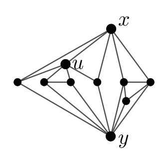

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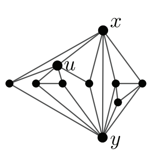

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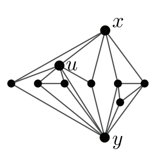

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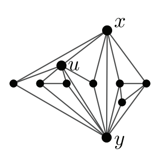

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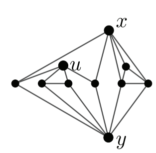

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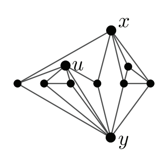

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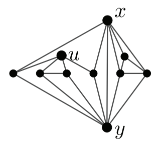

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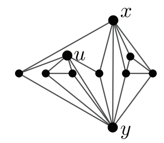

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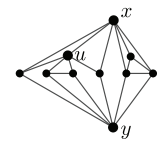

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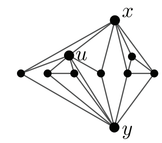

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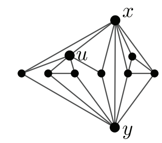

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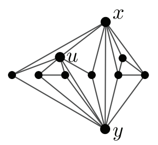

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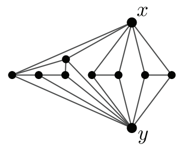

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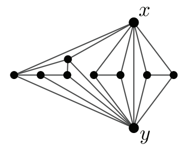

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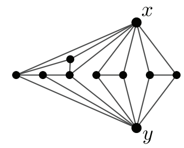

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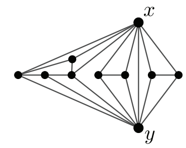

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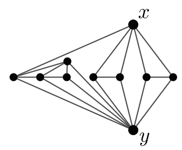

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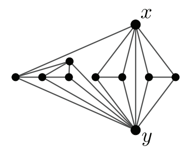

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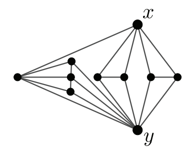

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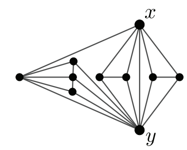

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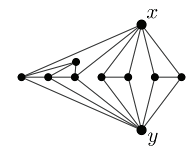

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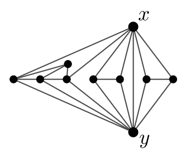

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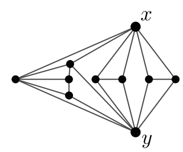

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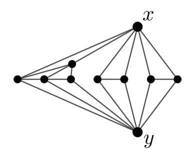

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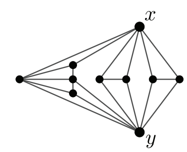

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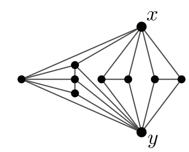

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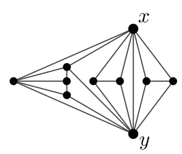

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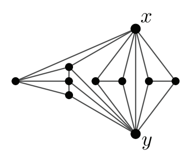

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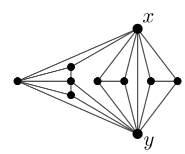

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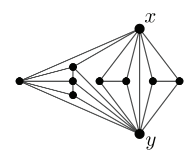

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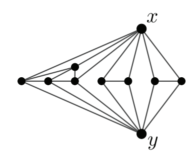

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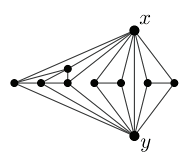

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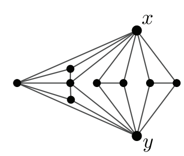

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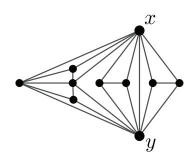

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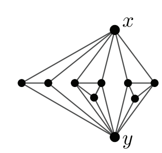

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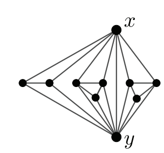

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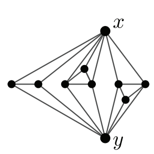

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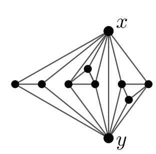

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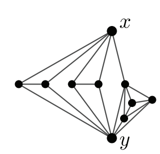

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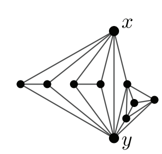

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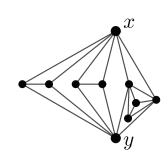

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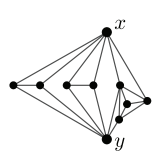

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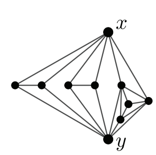

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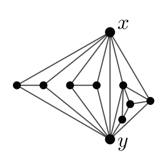

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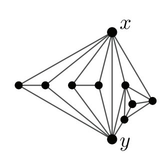

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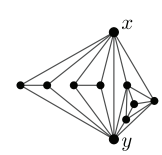

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