# Monochromatic $k$-edge-connection colorings of graphs[1]

Ping Li[1], Xueliang Li[1,2]
[1]Center for Combinatorics and LPMC, Nankai University
Tianjin 300071, China
[2]School of Mathematics and Statistics, Qinghai Normal University
Xining, Qinghai 810008, China
qdli_ping@163.com, lxl@nankai.edu.cn

**Abstract**

A path in an edge-colored graph $G$ is called monochromatic if any two edges on the path have the same color. For $k \geq 2$, an edge-colored graph $G$ is said to be monochromatic $k$-edge-connected if every two distinct vertices of $G$ are connected by at least $k$ edge-disjoint monochromatic paths, and $G$ is said to be uniformly monochromatic $k$-edge-connected if every two distinct vertices are connected by at least $k$ edge-disjoint monochromatic paths such that all edges of these $k$ paths colored with a same color. We use $mc_k(G)$ and $umc_k(G)$ to denote the maximum number of colors that ensures $G$ to be monochromatic $k$-edge-connected and, respectively, $G$ to be uniformly monochromatic $k$-edge-connected. In this paper, we first conjecture that for any $k$-edge-connected graph $G$, $mc_k(G) = e(G) - e(H) + \lfloor \frac{k}{2} \rfloor$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$. We verify the conjecture for $k = 2$. We also prove the conjecture for $G = K_{k+1}$ when $k \geq 4$ is even, and for $G = K_{k,n}$ when $k \geq 4$ is even, or when $k = 3$ and $n \geq k$. When $G$ is a minimal $k$-edge-connected graph, we give an upper bound of $mc_k(G)$, i.e., $mc_k(G) \leq k-1$, and $mc_k(G) \leq \lfloor \frac{k}{2} \rfloor$ when $G = K_{k,n}$. For the uniformly monochromatic $k$-edge-connectivity, we prove that for all $k$, $umc_k(G) = e(G) - e(H) + 1$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.

**Keywords:** edge-coloring, monochromatic path, edge-connectivity, monochromatic $k$-edge connection number.

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# 1 Introduction

All graphs in this paper are simple and undirected. For a graph $G$, we use $V(G)$, $E(G)$ to denote the vertex set and edge set of $G$, respectively, and $e(G)$ the number of edges of $G$. For all other terminology and notation not defined here we follow Bondy and Murty [1].

For a natural number $r$, we use $[r]$ to denote the set $\{1, 2, \cdots, r\}$ of integers. Let $\Gamma : E(G) \to [r]$ be an edge-coloring of $G$ that allows a same color to be assigned to adjacent edges. For two vertices $u$ and $v$ of $G$, a *monochromatic uv-path* is a $uv$-path of $G$ whose edges are colored with a same color, and $G$ is *monochromatic connected* if any two distinct vertices of $G$ are connected by a monochromatic path. An edge-coloring $\Gamma$ of $G$ is a *monochromatic connection coloring (MC-coloring)* if it makes $G$ monochromatic connected. The *monochromatic connection number* of a connected graph $G$, denoted by $mc(G)$, is the maximum number of colors that are needed in order to make $G$ monochromatic connected. An *extremal MC-coloring* of $G$ is an $MC$-coloring that uses $mc(G)$ colors.

The notion monochromatic connection coloring was introduced by Caro and Yuster in [4]. Many results have been obtained; see [3, 6, 10, 14]. For more knowledge on the monochromatic connections of graphs we refer to a survey paper [12]. Gonzlez-Moreno, Guevara, and Montellano-Ballesteros in [5] generalized the above concept to digraphs. Now we introduce the concept of *monochromatic $k$-edge-connectivity* of graphs. An edge-colored graph $G$ is *monochromatic $k$-edge-connected* if every two distinct vertices are connected by at least $k$ edge-disjoint monochromatic paths (allow some of the paths to have different colors). An edge-coloring $\Gamma$ of $G$ is a *monochromatic $k$-edge-connection coloring ($MC_k$-coloring)* if it makes $G$ monochromatic $k$-edge-connected. The *monochromatic $k$-edge-connection number*, denoted by $mc_k(G)$, of a connected graph $G$ is the maximum number of colors that are needed in order to make $G$ monochromatic $k$-edge-connected. Since we can color all the edges of a $k$-edge-connected graph by distinct colors, $mc_k(G)$ is well-defined. An *extremal $MC_k$-coloring* of $G$ is an $MC_k$-coloring that uses $mc_k(G)$ colors.

In an edge-colored graph $G$, we say that a subgraph $H$ of $G$ is induced by color $i$ if $H$ is induced by all the edges with a same color $i$ of $G$. If a color $i$ only color one edge of $E(G)$, then we call the color $i$ is a *trivial color*, and the edge is a *trivial edge*; otherwise, we call the colors (edges) *non-trivial*. We call an extremal $MC_k$-coloring a *good $MC_k$-coloring* of $G$ if the coloring has the maximum number of trivial edges.

Suppose that $X$ is a proper vertex subset of $G$. We use $E(X)$ to denote the set of edges with both ends in $X$. For a graph $G$ and $X \subset V(G)$, to shrink $X$ is to delete all edges in

$E(X)$ and then merge the vertices of $X$ into a single vertex. A partition of a vertex set $V$ is to divide $V$ into some mutual disjoint nonempty sets. Suppose $\mathcal{P} = \{V_1, \cdots, V_s\}$ is a partition of $V(G)$. Then $G/\mathcal{P}$ is a graph obtained from $G$ by shrinking every $V_i$ into a single vertex.

An edge $e$ of a $k$-edge-connected graph $G$ is *deletable* if $G \backslash e$ is also a $k$-edge-connected graph. A $k$-edge-connected graph $G$ is *minimally $k$-edge-connected* if none of its edges is deletable. A *minimal $k$-edge-connected spanning subgraph* of $G$ is a $k$-edge-connected spanning graph of $G$ that does not have any deletable edges. *A minimum $k$-edge-connected spanning subgraph* of $G$ is a minimal $k$-edge-connected spanning subgraph of $G$ that has minimum number of edges. The next result was obtained by Mader.

**Theorem 1.1** (Mader [13]). *Let $G$ be a minimally $k$-edge-connected graph of order $n$. Then*

1. *$e(G) \leq k(n-1)$.*

2. *every edge $e$ of $G$ is contained in a $k$-edge cut of $G$.*

3. *$G$ has a vertex of degree $k$.*

The following theorem was proved by Nash-Williams and Tutte independently.

**Theorem 1.2** ([15] [16]). *A graph $G$ has at least $k$ edge-disjoint spanning trees if and only if $e(G/\mathcal{P}) \geq k(|G/\mathcal{P}|-1)$ for any vertex partition $\mathcal{P}$ of $V(G)$.*

We denote $\psi(G) = min_{|\mathcal{P}|\geq 2}\frac{e(G/\mathcal{P})}{|G/\mathcal{P}|-1}$, and $\Psi(G) = \lfloor \psi(G) \rfloor$. Then the Nash-Williams-Tutte theorem can be restated as follows.

**Theorem 1.3.** *A graph $G$ has exactly $k$ edge-disjoint spanning trees if and only if $\Psi(G) = k$.*

If $\Gamma$ is an extremal $MC_k$-coloring of $G$, then each color-induced subgraph is connected; otherwise we can recolor the edges of one of its components by a fresh color, and then the new coloring is also an $MC_k$-coloring of $G$, but then the number of colors is increased by one, which contradicts that $\Gamma$ is extremal.

For the monochromatic $k$-edge-connection number of graphs, we conjecture that the following statement is true.

**Conjecture 1.4.** *For a $k$-edge-connected graph $G$ with $k \geq 2$, $mc_k(G) = e(G) - e(H) + \lfloor \frac{k}{2} \rfloor$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.*

In Section 2, we will prove that the conjecture is true for $k = 2$, and that it is also true for some special graph classes. We also give a lower bound of $mc_k(G)$ for $2 \leq k \leq \Psi(G)$, and an upper bound of $mc_k(G)$ for minimally $k$-edge-connected graphs with $k \geq 2$.

The following lemma seems easy, but it is useful for some proofs in Section 2.

**Lemma 1.5.** *Suppose that $G$ is a 2-edge-connected graph and $H$ is a 2-edge-connected subgraph of $G$. Let $S$ be subset of $E(G)$ whose ends are contained in $V(H)$ such that $S \cap E(H) = \emptyset$. Then $G \backslash S$ is also a 2-edge-connected graph.*

*Proof.* We need to show that for any $u, v$ in $G \backslash S$ there are at least two edge-disjoint paths connecting them. From the condition, there are two edge-disjoint $uv$-path $P_1, P_2$ in $G$. Suppose $a_1$ is the first vertex of $V(P_1)$ from $u$ to $v$ contained in $V(H)$, and $a_2$ is the first vertex of $V(P_2)$ from $u$ to $v$ contained in $V(H)$ (if $u \in V(H)$, then $u = a_1 = a_2$); suppose $b_1$ is the last vertex from $u$ to $v$ contained in $V(H)$, and $b_2$ is the last vertex of $V(P_2)$ from $u$ to $v$ contained in $V(H)$ (if $v \in V(H)$, then $v = b_1 = b_2$). Let $L_i = uP_ia_i$ and $L_{i+2} = b_iP_iv$, $i = 1, 2$. Because each of $L_i$ does not contain any edge of $S$ and $H$ is a 2-edge-connected graph, we have that $H \cup \bigcup_{i\in[4]} L_i$ is also a 2-edge-connected graph of $G \backslash S$. Therefore, there are two edge-disjoint $uv$-paths in $G \backslash S$. ∎

In Section 3, we introduce other version of monochromatic $k$-edge-connection of graphs, i.e., uniformly monochromatic $k$-edge-connection of graphs, and get some results. For details we will state them there.

# 2 Results on the monochromatic $k$-edge-connection number

**Theorem 2.1.** *Conjecture 1.4 is true when $G$ and $k$ satisfy one of the following conditions:*

1. *$k = 2$, i.e., $G$ is a 2-edge-connected graph.*
2. *$G = K_{k+1}$ where $k \geq 4$ is even.*
3. *$G = K_{k,n}$ where $k \geq 4$ is even, and $k = 3$ and $n \geq k$.*

We restate the first result of Theorem 2.1 as follows.

**Theorem 2.2.** *Let $G$ be a 2-edge-connected graph. Then $mc_2(G) = e(G) - e(H) + 1$, where $H$ is a minimum 2-edge-connected spanning subgraph of $G$.*

The following is the proof of Theorem 2.2. For convenience, we abbreviate the term "monochromatic path" as "path" in the proof.

Let $\Gamma$ be a good $MC_2$-coloring of $G$. Then we denote the set of non-trivial colors of $\Gamma$ by $[r]$, and denote $G_i$ as a subgraph induced by the color $i$; subject to above, let $p(\Gamma) = \Sigma_{i\in[r]}p(G_i)$ be maximum, where $p(G_i)$ is the number of non-cut edges of $G_i$. It is obvious that each of these edges is contained in some cycles of $G_i$.

**Claim 2.3.** *Each $G_i$ is either a 2-edge-connected graph or a tree.*

*Proof.* Suppose that $G_i$ is neither a 2-edge-connected graph nor a tree, i.e., $G_i$ contains both non-trivial blocks and cut edges. Therefore we can choose a cut edge $e = uv \in E(G_i)$ such that $v$ belongs to a maximal 2-edge-connected subgraph $B$ of $G_i$ (actually, $B$ is the union of some non-trivial blocks). Because $B$ is a 2-edge-connected subgraph of $G_i$, each of its vertices belongs to a cycle. Let $v$ be contained in a cycle $C$ of $B$ and $e' = vw$ be an edge of $C$. Because $e$ is a cut edge of $G_i$, there is just one $uw$-path in $G_i$ (the $uw$-path is $P$). Therefore, there exists another $uw$-path $P'$, which is colored differently from $i$.

If $P'$ is a path colored by $j$, then we can obtain a new coloring $\Gamma'$ of $G$ from $\Gamma$ by recoloring all edges of $G_i - e'$ with $j$. We first prove that $\Gamma'$ is an $MC_2$-coloring of $G$, i.e., we need to prove that for any two vertices $a, b$ of $V(G)$, there are at least two $ab$-paths under $\Gamma'$. If at least one vertex of $a, b$ does not belong to $V(G_i)$, then the two $ab$-paths are colored differently from $i$. Because we just change the color $i$, the two $ab$-paths are not affected; if both of $a, b$ belong to $V(G_i)$ and at least one of them does not belong to $V(B)$, then we can choose a right $ab$-path such that it does not contain $e'$ (under $\Gamma$), and so there are at least two $ab$-paths under $\Gamma'$; if both $a, b \in V(B)$, then the two $ab$-paths under $\Gamma$ (call them $L_1, L_2$) belong to $B$. If $e'$ is not an edge of any $L_1, L_2$, then the two $ab$-paths are not affected. Otherwise, let $e' \in E(L_1)$, and then $L = L_1 - e' + e + P'$ is a trial connecting $a, b$. Because $E(L) \cap E(L_2) = \emptyset$, there are two $ab$-paths under $\Gamma'$.

According to the above, $\Gamma'$ is an $MC_2$-coloring of $G$. If $j \in [r]$ is a non-trivial color, then the number of colors has not changed, but the number of trivial edges is increased by one, which contradicts that $\Gamma$ is good; otherwise, if $j$ is a trivial color, i.e., $uw$ is a trivial edge, then the new coloring $\Gamma'$ is a good $MC_2$-coloring (the number of colors and non-trivial edges have not changed), but compared to $p(\Gamma)$, $p(\Gamma')$ is increased by one, which contradicts that $p(\Gamma)$ is maximum. Therefore, we have proved that $G_i$ is either a 2-edge-connected graph or a tree.

∎

By Claim 2.3, each $G_i$ is either a 2-edge-connected graph or a tree. Suppose there are $h$ trees and $s = k - h$ 2-edge-connected graphs. W.l.o.g., suppose that $G_1, \cdots, G_s$ are $s$

2-edge-connected graphs and $G_{s+1} = T_1, \cdots, G_k = T_h$ are $h$ trees. $G_i$ colored by $i$ and $F_j$ colored by $s + j$. For convenience, we also call the color of $F_j$ $j$ when there is no confusion.

**Claim 2.4.** *For each $G_i$ and $T_j$, let $e = uv \in E(G_i)$ and $e' = xy \in E(T_j)$. Then at most one of $u, v$ belongs to $V(T_j)$, and at most one of $x, y$ belongs to $V(G_i)$.*

*Proof.* We prove it by contradiction, i.e., suppose that there exist $G_i$ and $T_j$, and there exist $e = uv \in E(G_i)$ and $e' = xy \in E(T_j)$, such that either $u, v \in V(T_j)$ or $x, y \in V(G_i)$.

*Case 1:* Suppose $u, v \in V(T_j)$. Then we recolor $E(G_i) - e$ by $j$ and keep the color of $e$. We now prove that the new coloring (call it $\Gamma'$) is an extremal $MC_2$-coloring of $G$.

We denote the segment of $uT_jv$ by $L$. For any pair of vertices $a, b$ of $V(G)$, if at least one vertex does not belong to $V(G_i)$, then the two $ab$-paths colored differently from $i$ under $\Gamma$. Because we just change the color $i$, the two $ab$-paths are not affected; if $a, b \in V(G_i)$, because $G_i + L - e$ is also 2-edge-connected, then there are two $ab$-paths (with the same color $j$) under $\Gamma'$. Therefore, $\Gamma'$ is an $MC_2$-coloring, and because the number of colors are not changed, $\Gamma'$ is still an extremal $MC_2$-coloring. However, the number of non-trivial edges is increased ($e$ becomes a trivial edge), which contradicts that $\Gamma$ is good.

*Case 2:* Suppose $x, y \in V(G_i)$. Then we recolor $E(T_j) - e'$ with $i$ and keep the color of $e'$. We now prove that the new coloring (call it $\Gamma'$) is an extremal $MC_2$-coloring of $G$.

For any vertices pair $a, b$ of $V(G)$, if at least one of $a, b$ does not belong to $V(T_j)$, then the two $ab$-paths colored differently from $j$. Because we just change the color $j$, the two $ab$-paths are not affected; if $a, b \in V(T_j)$ and at leat one of $a, b$ does not belong $V(G_i)$, then there is just one $ab$-path of $T_j$ and the other $ab$-paths colored differently from $i$ under $\Gamma$. Because $G_i \cup (T_j \backslash e')$ is connected and all of them colored by $i$ under $\Gamma'$, there are two $ab$-paths under $\Gamma'$; if both $a, b \in V(G_i)$, then there are two $ab$-paths (with the same color $i$) under $\Gamma'$. Above all, $\Gamma'$ is an $MC_2$-coloring of $G$. Because the number of colors are not changed, $\Gamma'$ is an extremal $MC_2$-coloring of $G$. However, the number of non-trivial edges is increased ($e'$ becomes a trivial edge), which contradicts that $\Gamma$ is good. ∎

By Claim 2.4, for each edge $e' = xy$ of a $T_j$, the other $xy$-paths belong to some $T_q$; for each edge $e = uv$ of a $G_i$, the other $uv$-paths belong to some $G_l$.

**Claim 2.5.** *$h = 0$, i.e., $G_i$ is a 2-edge-connected graph for any $i \in [r]$.*

*Proof.* If $h \neq 0$, for an edge $e_1 = v_1u_1 \in E(T_1)$, because $P_1 = e_1 = v_1u_1$ is the only $v_1u_1$-path of $T_1$, there exists another $v_1u_1$-path $P_2$, then $|P_2| \geq 2$ (because $G$ is simple), and therefore the color of $P_2$ is non-trivial. By Claim 2.4, $P_2$ belongs to some $T_j$, w.l.o.g., suppose $j = 2$. Then $e_1 + T_2$ contains a unique cycle $C_1$. Let $f_1 = v_1u_2$ is a pendent edge of $P_2$, and $e_2 = v_2u_2$ is the edge adjacent to $f_1$ in $P_2$. Then there exists a $v_2u_2$-path $P_3$ in $T_3$ and $e_2 + T_3$ contains a unique cycle $C_2$. Let $f_2 = v_2u_3$ is a pendent edge of $P_3$, and $e_3 = v_3u_3$ is the edge adjacent to $f_2$ in $P_3$. By repeating the process, we get a series of trees $T_1, T_2, \cdots$, paths $P_1, P_2, \cdots$ and edges $f_1 = v_1u_2, f_2 = v_2u_3, \cdots$, etc. Because there are at most $h < \infty$ trees, there is a $T_d$ which is the first tree appearing before (w.l.o.g., suppose $T_d = T_1$), and the $v_{d-1}u_{d-1}$-path $P_d$ is contained in $T_d = T_1$. Because there are at least two trees in this sequence, we have $d - 1 \geq 2$. Then $f_1 \in T_2, f_2 \in T_3, \cdots, f_{d-2} \in T_{d-1}$; $P_2 \in T_2, P_3 \in T_3, \cdots, P_d \in T_d = T_1$, etc. $T_1, \cdots, T_{d-1}$ are different trees. Let $H = \bigcup_{i\in[d-1]} T_i$.

In order to complete the proof, we need to construct a 2-edge-connected subgraph $T$ of $H$, a connected graph $H'$, and an edge set $B$ of $H$ with $|B| = d - 2$ below.

*Case 1*: $e_1 \notin E(P_d)$.

We have already discussed above that $C_1 = P_2+e_1, C_2 = P_3+e_2, \cdots, C_{d-1} = P_d+e_{d-1}$. So, $T = C_1 + C_2 - e_2 + C_3 - e_3 + \cdots + C_{d-1} - e_{d-1} = \bigcup_{i=1}^{d-1} C_i - B$ is a closed trail, where $B = \bigcup_{i=2}^{d-1} e_i$, see Fig.1(1). Therefore, $T$ is a 2-edge-connected graph. Because the ends of every edge in $B$ belong to $V(T)$, we have that $H' = \bigcup_{i\in[d-1]} T_i \backslash B$ is a connected graph.

*Case 2*: $e_1 \in E(P_d)$.

Suppose $F_1, F_2$ are two small trees of $T_1 \backslash e_1$ and let $v_1 \in V(F_1)$, $u_1 \in V(F_2)$. Then there is a $u_{d-1}v_1$-path $L_1$ and a $v_{d-1}u_1$-path $L_2$ (if $u_{d-1}$ connects $u_1$ and $v_{d-1}$ connects $v_1$, the situation is similar). Let

$$T' = v_1e_1u_1P_2u_2P_3u_3 \cdots P_{d-2}u_{d-2}P_{d-1}u_{d-1}L_1v_1$$

and

$$T'' = u_1P_2u_2P_3u_3 \cdots P_{d-2}u_{d-2}P_{d-1}v_{d-1}L_2u_1.$$

It is obvious that both of $T'$ and $T''$ are closed trails and

$$T' \cap T'' = u_1P_2u_2 \cdots P_{d-2}u_{d-2}P_{d-1}v_{d-1}$$

is a trail. Therefore, $T = T' \cup T'' = \bigcup_{i=1}^{d-1} C_i - B$ is a 2-edge-connected graph, where $B = \bigcup_{i=1}^{d-2} f_i$, see Fig.1(2). Because the ends of each edge in $B$ belong to $V(T)$, $H' = \bigcup_{i\in[d-1]} T_i \backslash B$ is a connected graph.

In above two cases, $T$ is a 2-edge-connected subgraph of $H$, and $B$ is an edge set of $H$ with $|B| = d-2$. We recolor each edges of $H - B$ by 1 and recolor each edge of $B$ by different new colors, denote the new coloring of $G$ by $\Gamma'$. Then the total number of colors

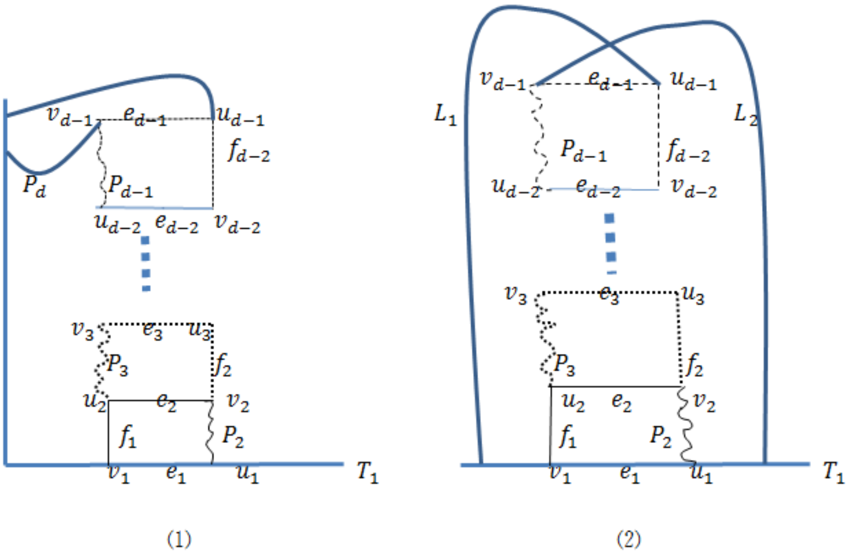


Figure 1

is not changed, but the number of trivial colors is increased by $|B| = d-2 \geq 1$. In order to complete the proof by contradiction, we need to prove that $\Gamma'$ is an $MC_2$-coloring, i.e., we need to prove that for two distinct vertices $x, y$ of $G$, there are 2 edge-disjoint $xy$-paths under $\Gamma'$. There are three cases to discuss.

(I) At least one of $x, y$ does not belong to $V(H)$. Then the two $xy$-paths do not belong to any $T_1, \cdots, T_{d-1}$. Because we just change the colors of $T_1, \cdots, T_{d-1}$, the two $xy$-paths are not affected from $\Gamma$ to $\Gamma'$.

(II) Both of $x, y$ belong to $V(H)$, but at least one of them does not belong to $V(T)$.

If there is just one $xy$-path in $H$ under $\Gamma$, then another $xy$-path will not be affected. Because $H'$ is connected, there are also two edge-disjoint $xy$-paths under $\Gamma'$.

If there are two $xy$-paths $L_1, L_2$ in $H$ under $\Gamma$. Suppose $a_i$ is the first vertex of $L_i$ contained in $V(T)$ from $x$ to $y$, and $b_i$ is the last vertex of $L_i$ contained in $V(T)$ from $x$ to $y$, $i = 1, 2$. Let $Q_i = xL_ia_i$ and $Q_{i+2} = b_iL_iy$, $i = 1, 2$. Because $T$ is a 2-edge-connected graph, $T \cup \bigcup_{i \in [4]} Q_i$ is also a 2-edge-connected graph, i.e., there are two edge-disjoint $xy$-paths under $\Gamma'$.

(III) Both of $x, y$ belong to $V(T)$. Then because $T$ is a 2-edge-connected graph, there are two edge-disjoint $xy$-path under $\Gamma'$.

∎

**Claim 2.6.** *$s = 1$, i.e., all the non-trivial edges belong to $G_1$.*

*Proof.* The proof is done by contradiction. If $s \geq 2$, by Claim 2.3, each $G_i$ is a 2-

edge-connected graph. Thus, $V(G_1)\backslash V(G_2) \neq \emptyset$ and $V(G_2)\backslash V(G_1) \neq \emptyset$; for otherwise, w.l.o,g, suppose $V(G_1) \subseteq V(G_2)$. Recoloring all the edges of $G_1$ by different new colors, then the new coloring is an $MC_2$-coloring of $G$ but it has more colors than $\Gamma$, which contradicts that $\Gamma$ is extremal.

Let $a \in V(G_1)\backslash V(G_2)$ and $b \in V(G_2)\backslash V(G_1)$. Suppose $G_a = \bigcup_{i\in c_a} G_i$ where $c_a = \{i : a \in V(G_i)\}$. Let $t$ be the minimum integer such that $V(G_2) \subseteq V(\bigcup_{j\in[t]} G_{i_j})$ where $i_j \in c_a$. Then $t \leq |G_2|$. Recoloring the edges of each $G_{i_j}$ by $i_1$, and recoloring the edges of $G_2$ by different new colors. Then the new coloring is an $MC_2$-coloring of $G$. Because $e(G_2) \geq |G_2| \geq t$, the number of colors is not decreased. However, the number of trivial colors is increased, which contradicts that $\Gamma$ is good. ∎

**Claim 2.7.** *$G_1$ is a minimum 2-edge-connected spanning subgraph of $G$.*

*Proof.* Because $s = 1$ and $h = 0$, there is just one non-trivial color (call it 1). Then $G_1$ is a 2-edge-connected spanning subgraph of $G$; for otherwise, there is a vertex $w \notin V(G_1)$, and then there is just one $uw$-path (which is a trivial path) for any $u \in V(G_1)$, a contradiction.

If $G_1$ is not minimum, we can choose a minimum 2-edge-connected spanning subgraph $H$ of $G$ with $e(G_1) > e(H)$. Coloring each edge of $H$ by a same color and coloring the other edges by trivial colors. Then the new coloring is an $MC_2$-coloring of $G$, but there are more colors than $\Gamma$, which contradicts that $\Gamma$ is extremal. ∎

**Proof of Theorem 2.2**: Actually, the theorem can be proved directly by Claims 2.5, 2.6 and 2.7. Because $\Gamma$ is an extremal $MC_2$-coloring of $G$, and the non-trivial color-inducted subgraph is just $G_1$, which is a minimum 2-edge-connected spanning subgraph of $G$. So, $mc_2(G) = e(G) - e(H) + 1$ where $H$ is a minimum 2-edge-connected spanning subgraph of $G$. ∎

We have proved that if $\Gamma$ is a coloring of $G$ in Theorem 2.2, then there is just one non-trivial color 1 and $H = G_1$ is a minimum 2-edge-connected spanning subgraph of $G$. If $G$ has $t$ blocks, then $H$ also has $t$ blocks, and each block is a minimum 2-edge-connected spanning subgraph of the corresponding block of $G$. Furthermore, the number of edges of $H$ is greater than or equal to $n + t - 1$ (equality holds if each block of $H$ is a cycle). So, the following result is obvious.

**Corollary 2.8.** *If $G$ is a 2-edge-connected graph with $t$ blocks $B_1, \cdots, B_t$, then $mc_2(G) = \sum_{i\in[t]} mc_2(B_i) - t + 1$, and $mc_2(G) \leq e(G) - n - t + 2$.*

A *cactus* is a connected graph where every edge lies in at most one cycle. If $G$ is a cactus without cut edges, then every edge lies in exactly one cycle. It is obvious that

$G$ will have cut edge when deleting any edge, and so $G$ is a minimal 2-edge-connected graph. A minimal $k$-edge-connected graph is also the minimum $k$-edge-connected spanning subgraph of itself, and this fact will not be declared again later.

**Corollary 2.9.** *If $G$ is a cactus without cut edge, then $mc_2(G) = 1$.*

We have proved the first result of Theorem 2.1. Next we will prove the remaining two results. Before this, we give an upper bound of $mc_k(G)$ for $G$ being a minimal $k$-edge-connected graph. The following lemma is necessary for our later proof.

**Lemma 2.10.** *Let $G$ be a minimal $k$-edge-connected graph and $\Gamma$ be an extremal $MC_k$-coloring of $G$ (suppose $mc_k(G) = t$), and let $G_i$ be the subgraph induced by the edges of color $i$, $1 \leq i \leq t$. Then each $G_i$ is a spanning subgraph of $G$.*

*Proof.* We prove it by contradiction. Suppose $G_i$ is not a spanning subgraph of $G$. Let $v \notin V(G_i)$. Then for any $u \neq v$, none of the $k$ edge-disjoint monochromatic $uv$-paths is colored by $i$. Let $e$ be an edge colored by $i$. By Theorem 1.1, there exists an edge cut $C(G)$ such that $e \in C(G)$ and $|C(G)| = k$. Then $G\backslash C(G)$ has two components $M_1, M_2$ (in fact, $C(G)$ is a bond of $G$). Let $v \in V(M_1)$ and some $w \in V(M_2)$. Then the $k$ edge-disjoint monochromatic $vw$-paths are retained in $G\backslash e$. However, $C(G)\backslash e$ is an edge cut of $G\backslash e$ that separates $v$ and $w$, and $|C(G)\backslash e| = k - 1$, which contradicts that there are $k$ edge-disjoint monochromatic $vw$-paths in $G\backslash e$. ∎

**Theorem 2.11.** *If $G$ is a minimal $k$-edge-connected graph with $k \geq 2$, then $mc_k(G) \leq k - 1$.*

*Proof.* We prove it by contradiction. Suppose $mc_k(H) \geq k$. Let $\Gamma$ be an extremal $MC_k$-coloring of $G$. Then by Lemma 2.10, there are at least $k$ edge-disjoint spanning subgraphs of $G$. Because there exists a vertex of $G$ with degree $k$, there are exactly $k$ edge-disjoint spanning subgraphs of $G$, denoted by $G_1, \cdots, G_k$. Because $G$ is a minimal $k$-edge-connected graph, by Theorem 1.1, $e(G) \leq k(n-1)$, which allows all of $G_1, \cdots, G_k$ to be spanning trees of $G$.

Because $k \geq 2$, there are at least two spanning trees $G_1, G_2$, and so $G_1 \cup G_2$ is a 2-edge-connected spanning subgraph of $G$. Let $e = uv$ be an edge of $G_1$ and let $P_1$ be the $uv$-path of $G_2$. Suppose $e_1 = uu_1$ and $e_2 = vv_1$ are two terminal edges of $P_1$. Let $P_2$ be the $uu_1$-path of $G_1$ and let $P_3$ be the $vv_1$-path of $G_1$.

*Case 1*: If one of $P_2$ and $P_3$ does not contain $e$, w.l.o.g., suppose $P_2$ does not contain $e$. Then $T = uP_2u_1P_1veu$ is a 2-edge-connected graph (in fact, $T$ is a closed trail, see Fig.2(1)). Because $u, u_1 \in V(T)$, by Lemma 1.5, $(G_1 \cup G_2)\backslash e_1$ is a 2-edge-connected subgraph of $G$.

*Case 2*: If both $P_2$ and $P_3$ contain $e$, then $T = uevP_2u_1P_1v_1P_3u$ is a 2-edge-connected graph (in fact, $T$ is a closed trail, see Fig.2(2)). Because $u, u_1 \in V(T)$, by Lemma 1.5, $(G_1 \cup G_2)\backslash e_1$ is a 2-edge-connected subgraph of $G$.

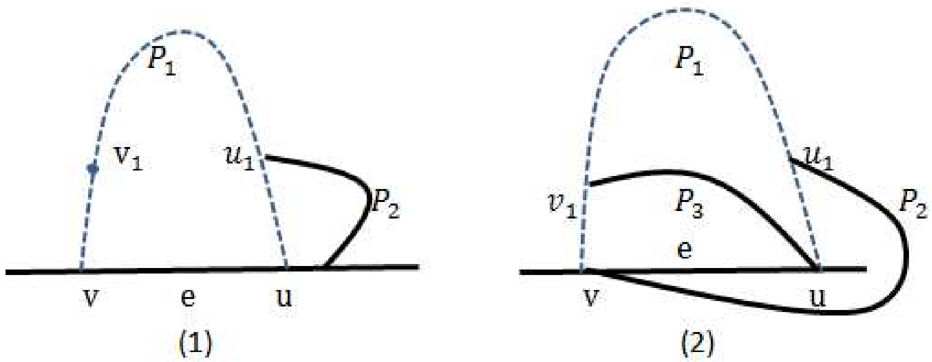


Figure 2

The coloring $\Gamma'$ obtained from $\Gamma$ by assigning 1 to the edges of $G_2\backslash e_1$ and assigning a new color to $e_1$. From above two cases, $(G_1 \cup G_2)\backslash e_1$ is a 2-edge-connected spanning subgraph of $G$ and $G_3, \cdots, G_k$ are spanning subgraph of $G$. So, every two vertices are also connected by $k$ monochromatic paths and the number of colors is not changed, i.e., $\Gamma'$ is also an extremal $MC_k$-coloring of $G$. While $e$ is a single edge, that would contradict that each induced subgraph is spanning by Lemma 2.10. ∎

Before proving the second result of Theorem 2.1, we introduce a well-known result.

**Fact 2.12.** *$K_{2n+1}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles; $K_{2n+2}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles and a perfect matching.*

**Theorem 2.13.** *$mc_{2n}(K_{2n+1}) = n$ for $n \geq 2$.*

*Proof.* By Fact 2.12, $K_{2n+1}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles $C_1, \cdots, C_n$. Color each $C_i$ by $i \in [n]$, and then the coloring is an $MC_{2n}$-coloring of $K_{2n+1}$. So, $mc_{2n}(K_{2n+1}) \geq n$.

We need to prove that $mc_{2n}(K_{2n+1}) \leq n$ to complete our proof. The proof is done by contradiction. Suppose $mc_{2n}(K_{2n+1}) = t \geq n + 1$. Let $\Gamma$ be an extremal $MC_{2n}$-coloring of $K_{2n+1}$ and let $G_i$ be the subgraph induced by all the edges with color $i$, $1 \leq i \leq t$.

Because $K_{2n+1}$ is a minimal $2n$-edge-connected graph, by Lemma 2.10 we have that each $G_i$ is a spanning subgraph of $G$. If $t \geq 2n$, then

$$n(2n+1) = e(K_{2n+1}) = e(\bigcup_{i\in[t]} G_i) \geq 2tn \geq 4n^2,$$

which is a contradiction. Otherwise, if $t < 2n$, then not every $G_i$ is a spanning tree (for otherwise, every two vertices are just connected by $t < 2n$ monochromatic paths). To ensure that every two vertices are connected by at least $2n$ monochromatic paths, there are at least $2n - t$ $G_i$ that are 2-edge-connected. Therefore, the number of edges of $\bigcup_{i\in[t]} G_i$ satisfies

$$e(\bigcup_{i\in[t]} G_i) \geq (2n+1)(2n-t) + 2(t-n)\cdot 2n = t(2n-1) + 2n \geq 2n^2 + 3n - 1.$$

This contradicts that $\bigcup_{i\in[t]} G_i = K_{2n+1}$ and $e(K_{2n+1}) = n(2n+1)$. ∎

Before prove the third result of Theorem 2.1, we introduce another well-known result.

**Fact 2.14.** *$K_{2n,2n}$ can be decomposed into $n$ Hamiltonian cycles and $K_{2n+1,2n+1}$ can be decomposed into $n$ Hamiltonian cycles and a perfect matching.*

**Theorem 2.15.** *If $n \geq k \geq 3$, then $mc_k(K_{k,n}) \leq \lfloor \frac{k}{2} \rfloor$.*

*Proof.* Let $\Gamma$ be an extremal $MC_k$-coloring with $t$ colors and let $G_i$ be the subgraph of $G$ induced by the edges with color $i$. Because $K_{k,n}$ is a minimal $k$-edge-connected graph, by Lemma 2.10 each $G_i$ is a spanning subgraph of $G$. Let $A, B$ be the bipartition (independent sets) of $G$ with $|A| = n$ and $|B| = k$. Then each vertex in $A$ has degree $k$.

We prove that $mc_k(K_{k,n}) \leq \lfloor \frac{k}{2} \rfloor$ by contradiction. Suppose $mc_k(K_{k,n}) = t \geq \lfloor \frac{k}{2} \rfloor + 1$. For a vertex $u$ of $A$, let $d_{G_i}(u) = r_i$. Then $\sum_{i\in[t]} r_i = k$ and each $r_i \geq 1$. Because every two vertices of $A$ are connected by $k$ edge-disjoint monochromatic paths, and the degree of every vertex in $A$ is $k$, we have that for each $u \in A$, $d_{G_i}(u) = r_i$. Because $t \geq \lfloor \frac{k}{2} \rfloor + 1$, there is a color $i$ such that $d_{G_i}(u) = 1$, i.e., all vertices of $A$ are leaves of $G_i$. Because $K_{k,n}$ is a bipartite graph with bipartition $A$ and $B$, $G_i$ is a perfect matching if $n = k$, and $G_i$ is the union of $k$ stars if $n > k$, both of which contradict that $G_i$ is a connected spanning subgraph of $G$. Therefore, $mc_k(K_{k,n}) \leq \lfloor \frac{k}{2} \rfloor$. ∎

**Corollary 2.16.** *Conjecture 1.4 is true for $G = K_{k,n}$, where $k$ is even and $n \geq k \geq 4$; it is also true for $G = K_{3,n}$, where $k = 3 \leq n$.*

*Proof.* If $k = 2l$ is even, then we prove that $mc_k(K_{k,n}) = \lfloor \frac{k}{2} \rfloor = l$. Actually, we only need to construct an $MC_k$-coloring of $K_{k,n}$ with $l$ colors. Let $A_1$ be a subset of $A$ with

$k$ vertices and $A_2 = A - A_1$, and let $H$ be the subgraph of $K_{k,n}$ whose vertex set is $A_1 \cup B$. Then $H = K_{k,k}$, and by Fact 2.14 $H$ can be decomposed into $l$ Hamiltonian cycles $\{C_1, \cdots, C_l\}$. Because the degree of each vertex in $A_2$ is $k = 2l$, we mark each two edges incident with $v \in A_2$ with $i$, $1 \leq i \leq l$. Let $E_i$ be the edge set with mark $i$, and let $G_i = C_i \cup E_i$. It is obvious that $G_i$ is a 2-edge-connected spanning graph of $K_{k,n}$. We color every edge of $G_i$ by $i$, and then we find an $MC_k$-coloring of $K_{k,n}$ with $l$ colors.

Because $K_{3,n}$ is a minimal 3-edge-connected graph for $n \geq 3$, and an $MC_3$-coloring of $K_{3,n}$ assigns color 1 to all its edges, we have $mc_3(K_{3,n}) \geq 1$. By Theorem 2.15, $mc_3(K_{3,n}) \leq 1$, and thus $mc_3(K_{3,n}) = 1$. ∎

If $k \leq \Psi(G)$, then $G$ is $k$-edge-connected. By Theorem 1.3, there are $k$ edge-disjoint spanning trees $T_1, \cdots, T_k$ of $G$ and we color $E(G)$ such that each $T_i$ is colored by $i$. Then any two vertices $u, v$ are connected by at least $k$ monochromatic $uv$-paths with different colors. So, we have the following result.

**Corollary 2.17.** *For a graph $G$ with $\Psi(G) \geq k \geq 2$, $mc_k(G) \geq e(G) - k(n-2)$.*

## 3 Results for uniformly monochromatic $k$-edge-connection number

The monochromatic $k$-edge-connected graph allows $k$ edge-disjoint monochromatic paths between any two vertices of the graph. In this section, we generalize the concept of monochromatic $k$-edge-connection to uniformly monochromatic $k$-edge-connection, and get some results.

An edge-colored $k$-edge-connected graph $G$ is *uniformly monochromatic $k$-edge-connected* if every two distinct vertices are connected by at least $k$ edge-disjoint monochromatic paths of $G$ such that all these $k$ paths have the same color. Note that for different pairs of vertices the paths may have different colors. An edge-coloring $\Gamma$ of $G$ is a *uniformly monochromatic $k$-edge-connection coloring ($UMC_k$-coloring)* if it makes $G$ uniformly monochromatically $k$-edge-connected. The *uniformly monochromatic $k$-edge-connection number*, denoted by $umc_k(G)$, of a $k$-edge-connected graph $G$ is the maximum number of colors that are needed in order to make $G$ uniformly monochromatic $k$-edge-connected. An *extremal $UMC_k$-coloring* of $G$ is an $UMC_k$-coloring that uses $umc_k(G)$ colors. We call an extremal $UMC_k$-coloring a *good $UMC_k$-coloring* of $G$ if the coloring has the maximum number of trivial edges. A uniformly monochromatic $k$-edge-connected graph is also a monochromatic connected graph when $k = 1$.

**Theorem 3.1.** *Let $G$ be a $k$-edge-connected graph with $k \geq 2$. Then $umc_k(G) = e(G) - e(H) + 1$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.*

We prove the theorem below. For convenience, we abbreviate "monochromatic $uv$-path" as "$uv$-path". Let $\Gamma$ be a good $UMC_k$-coloring of $G$. Then, suppose that the number of non-trivial colors of $\Gamma$ is $t$ and denote the set of them by $[t]$. Let $G_i$ be the subgraph of $G$ induced by the edges with a non-trivial color $i$, $1 \leq i \leq t$. Let $G' = \bigcup_{i\in[t]} G_i$.

**Claim 3.2.** *Each $G_i$ is $k$-edge-connected.*

*Proof.* Let $\pi_i$ denote the set of pairs $(u, v)$ such that there are at least $k$ edge-disjoint $uv$-paths colored by $i \in [t]$. Therefore, any vertex pair $(u, v)$ belongs to some $\pi_i$.

We first prove it by contradiction that each $G_i$ is $k$-edge-connected.

Suppose that $G_i$ is not a $k$-edge-connected graph. Then there exists a bond $C(G_i)$ with $|C(G_i)| \leq k - 1$, and $G_i \backslash C(G_i)$ has two components $M_1$ and $M_2$. Let $e = vu$ be an edge of $C(G_i)$, $u \in V(M_1)$, $v \in V(M_2)$. Then there are at most $|C(G)| \leq k - 1$ edge-disjoint paths in $G_i$ between $u, v$. Therefore there exists a $j \neq i$ of $[t]$ such that there are at least $k$ edge-disjoint $uv$-paths of $G_j$.

Recolor edges of $G_i - e$ with $j$ and keep the color of $e$, and denote the new coloring of $G$ by $\Gamma'$.

Because any non-trivial color $r \neq i$ is not changed. So, under $\Gamma'$, any pair $(x, y) \in \pi_r$ also have at least $k$ edge-disjoint $xy$-paths colored $r$. For any pair $(x, y) = \pi_i$, if any $k$ edge-disjoint $xy$-paths (Note that $P_1, \cdots, P_k$) of $G_i$ under $\Gamma$ do not contain $e$. Then these $k$ edge-disjoint $xy$-paths are retained. Otherwise, there is a path (Note that $P_1$) contains $e$. We choose a path $P$ of $G_j$ whose terminals are $u, v$. Then $T = (P_1 \backslash e) \cup P$ is a trail between $x, y$ and $E(T) \cap \bigcup_{l\neq 1} E(P_l) = \emptyset$. Let $P'$ be a $xy$-path of $T$. Then $P', P_2, \cdots, P_k$ are $k$ edge-disjoint $xy$-paths colored by $j$ (under $\Gamma'$). Therefore, $\Gamma'$ is still an extremal $UMC_k$-coloring of $G$, but then $e$ becomes to a trivial edge, which contradicts that $\Gamma$ is good. So, each $G_i$ is $k$-edge-connected.

∎

By Claim 3.2, because $k \geq 2$, we have $e(G_i) \geq |G_i| \geq 3$. Denote $G_x = \bigcup_{x\in V(G_i)} G_i$, $F_x = G' - G_x$.

**Claim 3.3.** *Each $G_x$ is a $k$-edge-connected spanning subgraph of $G$. Furthermore, $F_x = \emptyset$.*

*Proof.* If there is an $x \in V(G)$ such that $G_x$ is not a spanning subgraph of $G$, then there is a vertex $y \in V(G)\backslash V(G_x)$. Because $G$ is a simple graph and $k \geq 2$, any two vertices are connected by at least one non-trivial path. It is obvious that there are no non-trivial $xy$-path, a contradiction. Therefore, $G_x$ is a spanning subgraph of $G$.

Because each $G_i$ is $k$-edge-connected, $G_x$ is also $k$-edge-connected. Therefore, each $G_x$ is a $k$-edge-connected spanning subgraph of $G$.

Now we prove that $F_x = \emptyset$. Otherwise, if $F_x \neq \emptyset$, then there is a $G_j \subseteq F_x$ and $|G_j| \geq 3$. Suppose that $s$ is the minimum number such that $V(G_j) \subseteq \bigcup_{r\in[s]} G_{i_r}$, where $G_{i_1}, \cdots, G_{i_s}$ are contained in $G_x$. Then, $s \leq |G_j|$. Because $k \geq 2$, we have $e(G_j) \geq |G_j| \geq s$. We have obtained a new coloring $\Gamma'$ from $\Gamma$ by recoloring each $G_{i_1}, \cdots, G_{i_s}$ by $i_1$ and recoloring each edge of $G_j$ by different new colors. Because $G^* = \bigcup_{r\in[s]} G_{i_r}$ is $k$-edge-connected graph, each pair $(a, b)$ with $(a, b) \in \{\pi_{i_1}, \cdots, \pi_{i_s}, \pi_j\}$ has $k$-edge-disjoint $ab$-paths colored $i_1$ under $\Gamma'$. It is easy to check that $\Gamma'$ is a $UMC_k$-coloring. Then, the number of colors is not decreased, but the number of trivial colors is increased by at least $e(G_j) \geq 3$, which contradicts that $\Gamma$ is good. So, $F_x = \emptyset$. ∎

**Claim 3.4.** *$t = 1$ and $G_1$ is a minimum $k$-edge-connected spanning subgraph of $G$.*

*Proof.* Suppose $t \geq 2$. Then $V(G_1)\backslash V(G_2) \neq \emptyset$. Otherwise, if $V(G_1) \subseteq V(G_2)$, then $(u, v) \in \pi_2$ when $(u, v) \in \pi_1$. We can recolor all edges of $G_1$ by fresh colors, and then the new coloring is also a $UMC_k$-coloring of $G$ but the number of colors is increased, which contradicts that $\Gamma$ is extremal. So, $V(G_1)\backslash V(G_2) \neq \emptyset$, and there is a vertex $a \in V(G_1)\backslash V(G_2)$, i.e., $G_2 \nsubseteq G_a$, $G_2 \subseteq F_a$. By Claim 3.3, we have $F_a = \emptyset$, a contradiction. Therefore, $t = 1$, and thus $G_1 = G_a$ is a spanning subgraph of $G$.

In fact, $G_1$ is a minimum $k$-edge-connected spanning subgraph of $G$; otherwise, there exists a minimum $k$-edge-connected spanning subgraph $H$ of $G$ such that $e(H) < e(G_1)$. Coloring each edge of $H$ by 1 and coloring the other edges by some different new colors. Then the coloring is a $UMC_k$-coloring of $G$ with more colors, which contradicts that $\Gamma$ is extremal. ∎

**Proof of Theorem 3.1**: We can prove Theorem 3.1 directly by Claim 3.4. ∎

Because any $k$-edge-connected graph $G$ has the minimum degree $\delta(G) \geq k$, by Theorem 1.1 we have that $\frac{1}{2}kn \leq e(H) \leq k(n-1)$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.

**Corollary 3.5.** *For a $k$-edge-connected graph $G$ with $k \geq 2$, $e(G) - k(n-1) + 1 \leq umc_k(G) \leq e(G) - \frac{1}{2}kn + 1$.*

By definition, a $k$-edge-connected graph $G$ satisfies that $umc_k(G) \leq mc_k(G)$. Therefore, $mc_k(G) \geq e(G) - e(H) + 1$, where $H$ is a $k$-edge-connected spanning subgraph of $G$. By this theorem, we also get a result: A graph contains a Hamiltonian cycle if and only if $umc_2(G) = e(G) - n + 1$.

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