Extremal problems on saturation for the family of k-edge-connected graphs

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Abstract

Let $\mathcal{F}$ be a family of graphs. A graph G is *$\mathcal{F}$-saturated* if G contains no member of $\mathcal{F}$ as a subgraph but $G+e$ contains some member of $\mathcal{F}$ whenever $e \in E(\overline{G})$. The *saturation number* and *extremal number* of $\mathcal{F}$, denoted $\mathrm{sat}(n,\mathcal{F})$ and $\mathrm{ex}(n,\mathcal{F})$ respectively, are the minimum and maximum numbers of edges among n-vertex $\mathcal{F}$-saturated graphs. For $k \in \mathbb{N}$, let $\mathcal{F}_k$ and $\mathcal{F}'_k$ be the families of k-connected and k-edge-connected graphs, respectively. Wenger proved $\mathrm{sat}(n,\mathcal{F}_k) = (k-1)n - \binom{k}{2}$; we prove $\mathrm{sat}(n,\mathcal{F}'_k) = (k-1)(n-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k-1}{2}$. We also prove $\mathrm{ex}(n,\mathcal{F}'_k) = (k-1)n - \binom{k}{2}$ and characterize when equality holds. Finally, we give a lower bound on the spectral radius for $\mathcal{F}_k$-saturated and $\mathcal{F}'_k$-saturated graphs.

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1 Introduction

When $\mathcal{F}$ is a family of graphs, a graph G is *$\mathcal{F}$-saturated* if (1) no subgraph of G belongs to $\mathcal{F}$, and (2) for any edge e in the complement $\overline{G}$ of G, the graph obtained by adding e to G contains a subgraph that belongs to $\mathcal{F}$ (our definition of "graph" prohibits loops and multiedges). The *saturation number* of $\mathcal{F}$, denoted $\mathrm{sat}(n,\mathcal{F})$, is the minimum number of

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edges in an n-vertex $\mathcal{F}$-saturated graph. The *extremal number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n-vertex $\mathcal{F}$-saturated graph. When $\mathcal{F}$ has only one graph F, we instead write $\mathrm{sat}(n, F)$ and $\mathrm{ex}(n, F)$, such as when F is K_t, the complete graph with t vertices.

Initiating the study of extremal graph theory, Turán [10] determined the extremal number $\mathrm{ex}(n, K_{r+1})$; the unique extremal graph is the n-vertex complete r-partite graph whose part-sizes differ by at most 1. Saturation numbers were first studied by Erdős, Hajnal, and Moon [3]; they proved $\mathrm{sat}(n, K_{k+1}) = (k-1)n - \binom{k}{2}$. They also proved that equality holds only for the graph formed from a copy of K_{k-1} with vertex set S by adding $n-k+1$ vertices that each have neighborhood S. We call this the *complete split graph* $S_{n,k}$; note that $S_{n,k}$ has clique number k and no k-connected subgraph, and $S_{n,2}$ is a star. For an excellent survey on saturation numbers, we refer the reader to Faudree, Faudree, and Schmitt [4].

In this paper, we study the relationship between saturation and edge-connectivity. For a given positive integer k, let $\mathcal{F}_k$ be the family of k-connected graphs, and let $\mathcal{F}'_k$ be the family of k-edge-connected graphs. Wenger [11] determined $\mathrm{sat}(n, \mathcal{F}_k)$. Since K_{k+1} is a minimal k-connected graph, it is not surprising that $S_{n,k}$ is also an $\mathcal{F}_k$-saturated graph with fewest edges, but in fact the family of extremal graphs is much larger. A *k-tree* is any graph obtained from K_k by iteratively introducing a new vertex whose neighborhood in the previous graph consists of k pairwise adjacent vertices. Note that $S_{n,k}$ is a $(k-1)$-tree.

Theorem 1.1 (Wenger [11])**.** $\mathrm{sat}(n, \mathcal{F}_k) = (k-1)n - \binom{k}{2}$ *when* $n \geq k$*. Furthermore, every* $(k-1)$*-tree with* n *vertices has this many edges and is* $\mathcal{F}_k$*-saturated.*

For $n \geq k+1$, we determine $\mathrm{sat}(\mathcal{F}'_k)$ and $\mathrm{ex}(\mathcal{F}'_k)$. An $\mathcal{F}'_1$-saturated graph has no edges, so henceforth we may assume $k \geq 2$. Let $\rho_k(n) = (k-1)(n-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k-1}{2}$. In Section 2, we construct for $n \geq k+1$ an $\mathcal{F}'_k$-saturated graph with n vertices having $\rho_k(n)$ edges, proving $\mathrm{sat}(n, \mathcal{F}'_k) \leq \rho_k(n)$. Using induction on n, in Section 3 we prove that if G is $\mathcal{F}'_k$-saturated, then $\rho_k(n) \leq |E(G)| \leq (k-1)n - \binom{k}{2}$, where $E(G)$ denotes the edge set of a graph G. Since $S_{n,k}$ is also $\mathcal{F}'_k$-saturated, the upper bound is sharp. Thus $\mathrm{sat}(n, \mathcal{F}'_k) = \rho_k(n)$ and $\mathrm{ex}(n, \mathcal{F}'_k) = (k-1)n - \binom{k}{2}$.

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. In Section 4, we give a lower bound on the spectral radius for $\mathcal{F}_k$-saturated and $\mathcal{F}'_k$-saturated graphs. There is a long history of studying the relationship between eigenvalues and connectivity. The Laplacian matrix of G is the diagonal matrix of degrees minus the adjacency matrix. Fiedler [5] proved that the (vertex)-connectivity of a graph is at least the second smallest eigenvalue of its Laplacian matrix. This fundamental result has stimulated much additional research, such as [2, 8, 9]. It appears that for saturation problems the spectral radius is more relevant than the second smallest Laplacian eigenvalue.

Additional notation is as follows. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the neighborhood of v in G, respectively. For $A, B \subseteq V(G)$, let $\overline{A} = V(G) - A$, let $[A, B]$ be the set of edges with endpoints in A and B, and let $G[A]$ to denote the subgraph of G induced by A. Let $[k] = \{1, 2, \ldots, k\}$.

Let K_{k+1}^- denote the graph obtained from K_{k+1} by deleting one edge; this graph is the unique smallest $\mathcal{F}_k'$-saturated graph that is not a complete graph. The complete graphs with at most k vertices are trivially $\mathcal{F}_k'$-saturated, since there are no edges to add. We therefore use *nontrivial $\mathcal{F}_k'$-saturated graph* to mean an $\mathcal{F}_k'$-saturated graph with at least $k+1$ vertices.

2 Construction

Recall that $\rho_k(n) = (k-1)(n-1) - \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k-1}{2}$ and that we restrict to $k \geq 2$ since $\mathcal{F}_1'$-saturated graphs have no edges. In this section, for $n \geq k+1$, we construct an n-vertex $\mathcal{F}_k'$-saturated graph with $\rho_k(n)$ edges. Since every $\mathcal{F}_2'$-saturated graph is a tree (and $\rho_2(n) = n-1$), we need only consider $k \geq 3$.

Definition 2.1. Fix $k \in \mathbb{N}$ with $k \geq 3$. For $n \in \mathbb{N}$ with $n > k$, let $t = \left\lfloor \frac{n}{k+1} \right\rfloor$ and $r = n - t(k+1)$. Let H_i be a copy of K_{k+1}^- using vertices $u_{i,1}, \ldots, u_{i,k+1}$, with $u_{i,1}$ and $u_{i,k+1}$ nonadjacent. Let $U_i = V(H_i)$ for $i \in [t]$. Let F_t be the graph obtained from the disjoint union $H_1 + \cdots + H_t$ by adding the edge $u_{i,j}u_{i+1,j}$ for all i and j such that $i \in [t-1]$ and $j \in [k+1] - \{2, k\}$. Let $G_{k,n}$ be the graph obtained from F_t by adding new vertices $w_1, \ldots, w_r$, each having neighborhood $V(H_t) - \{u_{t,1}, u_{t,k+1}\}$.

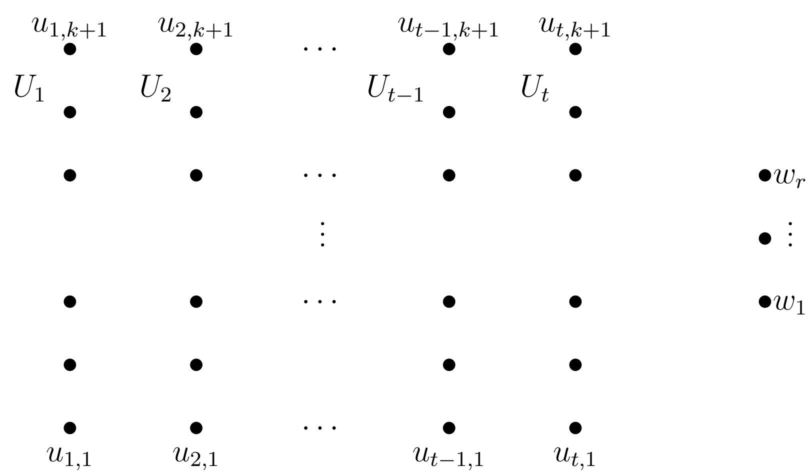

Figure 1: The graph $G_{k,n}$.

Proposition 2.2. *For $n > k \geq 3$, the graph $G_{k,n}$ of Definition 2.1 is $\mathcal{F}_k'$-saturated and has n vertices and $\rho_k(n)$ edges.*

Proof. Since $n = t(k+1) + r$, the graph $G_{k,n}$ has n vertices.

In $G_{k,n}$, the vertices $w_1, \ldots, w_r$ have degree $k-1$ and hence cannot lie in a k-edge-connected subgraph. In F_t, the edges joining U_i and U_{i+1} form a cut of size $k-1$, so any k-edge-connected subgraph of $G_{k,n}$ is contained in just one copy of K_{k+1}^-. However, K_{k+1}^- has two vertices of degree $k-1$, leaving only $k-1$ other vertices. Hence $G_{k,n}$ has no k-edge-connected subgraph.

In F_t, there are $t\left[\binom{k+1}{2} - 1\right] + (k-1)(t-1)$ edges. The added vertices $w_1, \ldots, w_r$ contribute $r(k-1)$ more edges. Since $n = t(k+1) + r$, we compute

$$\begin{aligned} |E(G_{k,n})| &= t\left[\binom{k+1}{2} - 1\right] + (k-1)(t+r-1) = t\frac{k^2+3k-4}{2} + (k-1)(r-1) \\ &= t\frac{(k-1)(k+4)}{2} + (k-1)(r-1) = (k-1)[t(k+1)+r-1] - t\binom{k-1}{2} \\ &= (k-1)(n-1) - t\binom{k-1}{2} = \rho_k(n). \end{aligned}$$

Let xy be an edge in the complement of $G_{k,n}$. It remains to show that the graph G' obtained by adding xy to $G_{k,n}$ has a k-edge-connected subgraph. Note that the subgraph of $G_{k,n}$ induced by $U_t \cup \{w_1, \ldots, w_r\}$ is the K_{k+1}-saturated graph $S_{k+r+1,k}$ of [3], so G' contains K_{k+1} when x and y lie in this set. Similarly, if xy is the one missing edge of H_i, then G' again contains K_{k+1}. Hence we may assume that $x \in U_i$ with $1 \le i < t$ and that $y \in \{w_1, \ldots, w_r\}$ or $y \in U_j$ with $i < j \le t$. If $y \in \{w_1, \ldots, w_r\}$, then let $j = t+1$ and $U_j = \{y\}$, in order to combine cases. Let H' be the subgraph of G' induced by $\bigcup_{l=i}^{j} U_l$. To prove that H' is k-edge-connected, we show that $H' - S$ is connected, where S is a set of $k-1$ edges in H'.

Suppose first that $H'[U_l] - S$ is disconnected for some l with $i \le l \le j$ (this can only occur with $l \le t$). Since $\kappa'(H_l) = k-1$ for $l \in [t]$, this case requires $S \subseteq E(H'[U_l])$. In $H' - S$, every vertex of U_l except $u_{l,2}$ and $u_{l,k}$ has a neighbor in U_{l-1} when $l > i$ and in U_{l+1} when $l < j$. Also $u_{l,2}$ and $u_{l,k}$ have degree k in H' (or degree $k+1$ if in $\{x, y\}$), so in $H' - S$ each has a neighbor in U_l. If one of them is the only neighbor of the other in $H' - S$, then in $H' - S$ it has an additional neighbor in U_l. Thus in $H' - S$ each component of the subgraph induced by U_l has a neighbor in U_{l+1} if $l < j$ and a neighbor in U_{l-1} if $l > i$, so paths can reach U_j and U_i, at least one of which is connected.

Hence we may assume that $H'[U_l] - S$ is connected for each l with $i \le l \le j$. With this reduction, for $i \le l < j$ the subgraph induced by $U_l \cup U_{l+1}$ is also connected unless S consists of all $k-1$ edges joining U_l and U_{l+1}. If S is not any of these sets, then altogether $H' - S$ is connected. If S consists of the $k-1$ edges joining U_l and U_{l+1}, then the subgraphs of $H' - S$ induced by $U_i \cup \cdots \cup U_l$ and by $U_{l+1} \cup \cdots \cup U_j$ are connected, and the presence of xy connects these two subgraphs. □

By Proposition 2.2, $\mathrm{sat}(n, \mathcal{F}'_k) \le \rho_k(n)$. Thus $\mathrm{sat}(n, \mathcal{F}'_k)$ is much smaller than $\mathrm{sat}(n, \mathcal{F}_k)$ when $n \ge 2(k+1)$. Indeed, $G_{k,n}$ is not $\mathcal{F}_k$-saturated. In particular, adding an edge joining $u_{1,1}$ to a vertex v outside U_1 does not create a k-connected subgraph. Since $G_{k,n}$ has no k-edge-connected subgraph, it has no k-connected subgraph, so a k-connected subgraph H' of the new graph G' must contain the edge $u_{1,1}v$. Let $S = U_1 - \{u_{1,2}, u_{1,k}\}$; note that $|S| = k - 1$. Since H' must have $k-1$ internally disjoint paths from v to $u_{1,1}$ in addition to the edge $vu_{1,1}$, and S is the set of vertices in U_1 with neighbors outside U_1, all of S must also lie in $V(H')$. Since $d_G(u_{1,k+1}) = k$, we must also include $u_{1,2}$ and $u_{1,k}$ in $V(H')$. Now $H' - S$ has $u_{1,2}u_{1,k}$ as an isolated edge.

3 Saturation and extremal number of $\mathcal{F}'_k$

In this section, we show that if G is an $\mathcal{F}'_k$-saturated n-vertex graph with $n \ge k+1$, then $|E(G)| \ge \rho_k(n)$. First, we investigate the properties of an $\mathcal{F}'_k$-saturated graph.

Lemma 3.1. *If G is $\mathcal{F}'_k$-saturated and has more than k vertices, then $\kappa'(G) = k-1$.*

Proof. Since G has no k-edge-connected subgraph, $\kappa'(G) \le k-1$. If $\kappa'(G) < k-1$, then G has an edge cut $[S, \overline{S}]$ of size less than $k-1$. Since $|V(G)| > k$, there are at least k pairs (x, y) with $x \in S$ and $y \in \overline{S}$. Hence there is such a pair (x, y) with $xy \notin E(G)$. Let G' be the graph obtained by adding the edge xy to G.

Since G has no k-edge-connected subgraph, any such subgraph of G' must contain the edge xy. Hence it contains k edge-disjoint paths with endpoints x and y, by Menger's Theorem. Besides the edge xy, there must be at least $k-1$ with endpoints x and y that use edges of $[S, \overline{S}]$. This contradicts $|[S, \overline{S}]| < k-1$. Hence G' has no k-edge-connected subgraph, and G cannot be $\mathcal{F}'_k$-saturated. □

Lemma 3.2. *Assume $k \ge 3$, and let G be a $\mathcal{F}'_k$-saturated graph with at least $k+2$ vertices. If S is a vertex subset in $V(G)$ such that $|[S, \overline{S}]| = k-1$ and $|S| \ge |\overline{S}|$, then $G[S]$ is a nontrivial $\mathcal{F}'_k$-saturated graph, and $G[\overline{S}]$ is K_1 or is a nontrivial $\mathcal{F}'_k$-saturated graph.*

Proof. First, we prove for $T \in \{S, \overline{S}\}$ that the induced subgraph $G[T]$ is a complete subgraph or is $\mathcal{F}'_k$-saturated with at least $k+1$ vertices. When $G[T]$ is not complete, take $e \in E(\overline{G[T]})$, and let G' be the graph obtained from G by adding e. Since G is $\mathcal{F}'_k$-saturated, G' contains a k-edge-connected subgraph H, and $e \in E(H)$. Since $|[T, \overline{T}]| = k-1$, no vertex of H lies in $\overline{T}$. Hence $H \subseteq G[T]$, which implies that $G[T]$ is $\mathcal{F}'_k$-saturated. Since $G[T]$ is not complete, that requires $|T| \ge k+1$.

If $G[\overline{S}]$ is a nontrivial $\mathcal{F}'_k$-saturated graph, then $G[S]$ is also, by $|S| \ge |\overline{S}|$ and the preceding paragraph. If $G[\overline{S}] = K_1$, then $|V(G)| \ge k+2$ and the preceding paragraph again yields that $G[S]$ is a nontrivial $\mathcal{F}'_k$-saturated graph. Hence it suffices to show that $G[\overline{S}]$ cannot be a complete graph with $|\overline{S}| \geq 2$.

By Lemma 3.1, $\delta(G) \geq k-1$. The vertex of $\overline{S}$ incident to the fewest edges of $[S, \overline{S}]$ has degree at most $\left\lfloor \frac{k-1}{j} \right\rfloor + j - 1$, where $j = |\overline{S}|$. Since $j \geq 2$, we thus have $j \geq k-1$.

If $j = k-1$, then $\delta(G) \geq k-1$ requires each vertex of $\overline{S}$ to be incident to exactly one edge of the cut. Adding an edge across the cut then increases the degree of only one vertex of $\overline{S}$ to k. Hence only that vertex can lie in H, which restricts its degree in H to 1.

We may therefore assume $|\overline{S}| = k$, since $K_{k+1} \not\subseteq G$, and $|S| \geq k$. Since $|[S, \overline{S}]| = k-1$, some $v \in \overline{S}$ has degree only $k-1$ in G, and every vertex of $\overline{S}$ has a nonneighbor in S. Choose $y \in \overline{S}$ with $y \neq v$, and choose $x \in S$ with $xy \notin E(G)$. Let G' be the graph obtained by adding xy to G. A k-edge-connected subgraph H of G' must contain y but cannot contain v. If H has $i+1$ vertices in $\overline{S} - \{v\}$, then a vertex among these with least degree in H has degree at most $\lfloor \frac{k}{i+1} \rfloor + i$ in H. Since $i \leq k-2$ and $\delta(H) \geq k$, we have $i = 0$.

Hence $V(H) \cap \overline{S} = \{y\}$ and all edges of $[S, \overline{S}]$ are incident to y. All vertices of $\overline{S}$ other than y have degree $k-1$ in G. In this case let G' be the graph obtained by adding xv to G. Since vertices in the resulting k-edge-connected subgraph H must have degree at least k, the only vertices from $\overline{S}$ that can be included are y and v. However, now $d_H(v) = 2$, which prohibits such a subgraph H since $k \geq 3$. □

Lemma 3.3. *If G is an n-vertex $\mathcal{F}'_k$-saturated graph with $n \geq k+1$, then G contains K^-_{k+1}.*

Proof. We use induction on n, the number of vertices. The claim holds when $n = k+1$, since K^-_{k+1} is the only $\mathcal{F}'_k$-saturated graph with $k+1$-vertices.

Now consider $n \geq k+2$. Since $\kappa'(G) = k-1$ by Lemma 3.1, there exists $S \subseteq V(G)$ such that $|[S, \overline{S}]| = k-1$ and $|S| \geq |\overline{S}|$. By Lemma 3.2, $|S| \geq k+1$ and $G[S]$ is $\mathcal{F}'_k$-saturated. By the induction hypothesis, $G[S]$ (and hence also G) contains K^-_{k+1}. □

The lemmas allow us to prove the main result of this section.

Theorem 3.4. *For $n \in \mathbb{N}$, with $t = \lfloor \frac{n}{k+1} \rfloor$,*

$$\mathrm{sat}(n, \mathcal{F}'_k) = (k-1)(n-1) - t\binom{k-1}{2},$$

with equality achieved for $k = 1$ by $\overline{K}_n$, for $k = 2$ by trees, and for $k \geq 3$ by $G_{k,n}$.

Proof. The claims for $k \leq 2$ are immediate. For $k \geq 3$, Proposition 2.2 yields the upper bound. For the lower bound, we use induction on n. When $n = k+1$, so $t = 1$, the only $\mathcal{F}'_k$-saturated n-vertex graph is K^-_{k+1}, which indeed has $(k-1)k - \binom{k-1}{2}$ edges.

For $n > k+1$, let G be a $\mathcal{F}'_k$-saturated n-vertex graph. Since $\kappa'(G) = k-1$ by Lemma 3.1, there exists $S \subseteq V(G)$ such that $|[S,\overline{S}]| = k-1$ and $|S| \geq |\overline{S}|$. By Lemma 3.2, $G[S]$ is a nontrivial $\mathcal{F}'_k$-saturated graph and $G[\overline{S}]$ is K_1 or is a nontrivial $\mathcal{F}'_k$-saturated graph. Let $t' = \left\lfloor \frac{|S|}{k+1} \right\rfloor$. By the induction hypothesis, $|E(G[S])| \geq (k-1)(|S|-1) - t'\binom{k-1}{2}$.

If $G[\overline{S}] = K_1$, then $|S| = n-1$ and exactly $k-1$ edges lie outside $G[S]$. Hence $|E(G) \geq (k-1)(n-1) - t'\binom{k-1}{2}$. Since $t' \in \{t, t-1\}$, the desired inequality holds.

Therefore, we may assume that $G[S]$ and $G[\overline{S}]$ are both nontrivial $\mathcal{F}'_k$-saturated graphs. Let $t'' = \left\lfloor \frac{|\overline{S}|}{k+1} \right\rfloor$. Note that $t' + t'' \leq t$. Using the induction hypothesis and adding the $k-1$ edges of the cut,

$$|E(G)| \geq (k-1)(|S| + |\overline{S}| - 2) + (k-1) - (t'+t'')\binom{k-1}{2} \geq (k-1)(n-1) - t\binom{k-1}{2}.$$

Hence $|E(G)| \geq \rho_k(n)$. □

Next we determine the maximum number of edges in $\mathcal{F}'_k$-saturated n-vertex graphs.

Theorem 3.5. *If $n \geq k+1$, then $\mathrm{ex}(n, \mathcal{F}'_k) = (k-1)n - \binom{k}{2}$. Furthermore, the n-vertex $\mathcal{F}'_k$-saturated graphs with the most edges arise from $(n-1)$-vertex $\mathcal{F}'_k$-saturated graphs with the most edges by adding one vertex with $k-1$ neighbors.*

Proof. As we have noted, $\mathcal{F}'_1$-saturated graphs have no edges and $\mathcal{F}'_2$-saturated graphs are trees, so we may assume $k \geq 3$. We use induction on n; when $n = k+1$, the only $\mathcal{F}'_k$-saturated n-vertex graph is K^-_{k+1}.

For $n > k+1$, let G be an $\mathcal{F}'_k$-saturated n-vertex graph. As in Theorem 3.4, there exists $S \subseteq V(G)$ such that $|[S,\overline{S}]| = k-1$ and $|S| \geq |\overline{S}|$. By Lemma 3.2, $G[S]$ is a nontrivial $\mathcal{F}'_k$-saturated graph and $G[\overline{S}]$ is K_1 or is a nontrivial $\mathcal{F}'_k$-saturated graph.

Applying the induction hypothesis, if $G[\overline{S}] = K_1$, then $|E(G)| \leq (k-1)(n-1) + (k-1) - \binom{k}{2} = (k-1)n - \binom{k}{2}$, with equality only under the claimed condition. On the other hand, if $[\overline{S}]$ is a nontrivial $\mathcal{F}'_k$-saturated graph, then

$$|E(G)| \leq (k-1)|S| - \binom{k}{2} + (k-1)|\overline{S}| - \binom{k}{2} + (k-1) = (k-1)n - (k+1)(k-1).$$

Since $k+1 > k/2$ when $k > 0$, the upper bound in this case is strictly smaller than the claimed upper bound. □

4 Spectral radius and $\mathcal{F}'_k$-saturated graphs

In this section, we give sharp lower bounds on the spectral radius for $\mathcal{F}'_k$-saturated graphs and for $\mathcal{F}_k$-saturated graphs. The spectral radius of a graph G, denoted $\lambda_1(G)$, is the largest eigenvalue of the adjacency matrix of G. The following two lemmas are well-known in spectral graph theory.

Lemma 4.1 ([6]). *If H is a subgraph of G, then $\lambda_1(H) \le \lambda_1(G)$.*

Lemma 4.2 ([1]). *For any graph G,*

$$\frac{2|E(G)|}{|V(G)|} \le \lambda_1(G) \le \Delta(G)$$

with equality if and only if G is regular.

For a vertex partition P of a graph G, with parts $V_1, \dots, V_t$, the *quotient matrix* Q has (i,j)-entry $\frac{|[V_i,V_j]|}{|V_i|}$ when $i \ne j$ and $\frac{2|E(G[V_i])|}{|V_i|}$ when $i = j$. Let $q_{i,j}$ denote the (i,j)-entry in Q. A vertex partition P with t parts is *equitable* if whenever $i,j \in [t]$ and $v \in V_i$, the number of neighbors of v in V_j is $q_{i,j}$.

Lemma 4.3 ([6]). *If $\{V_1, \dots, V_t\}$ is an equitable partition of $V(G)$, then $\lambda_1(G) = \lambda_1(Q)$, where Q is the quotient matrix for the partition.*

Theorem 4.4. *If G is a nontrivial $\mathcal{F}'_k$-saturated graph, then $\lambda_1(G) \ge (k-2+\sqrt{k^2+4k-4})/2$, with equality for K^-_{k+1}.*

Proof. First we prove $\lambda_1(K^-_{k+1}) = (k-2+\sqrt{k^2+4k-4})/2$. Let $V(K^-_{k+1}) = \{x_1, \dots, x_{k+1}\}$, with $d(x_1) = d(x_{k+1}) = k-1$. The vertex partition of K^-_{k+1} given by $V_1 = \{x_1, x_{k+1}\}$ and $V_2 = \{x_2, \dots, x_k\}$ is equitable. The corresponding quotient matrix Q is $\binom{0 \quad 2}{k-1 \quad k-2}$. By Lemma 4.3, $\lambda_1(K^-_{k+1}) = \lambda_1(Q) = (k-2+\sqrt{k^2+4k-4})/2$.

For any nontrivial $\mathcal{F}'_k$-saturated graph G, Lemma 3.3 yields $K^-_{k+1} \subseteq G$. By Lemma 4.1, $\lambda_1(G) \ge \lambda_1(K^-_{k+1})$, as desired. □

Theorem 4.5. *If G is $\mathcal{F}_k$-saturated with n vertices, where $n \ge k+1$, then $\lambda_1(G) \ge (k-2+\sqrt{k^2+4k-4})/2$.*

Proof. For $k=1$, the bound is 0. Since $\mathcal{F}_1$-saturated graphs have no edges and hence all eigenvalues 0, we may assume $k \ge 2$. When $n = k+1$, the only $\mathcal{F}_k$-saturated graph is K^-_{k+1}, whose spectral radius as computed in Theorem 4.4 is the claimed bound. Hence we may assume $n \ge k+2 \ge 4$.

By Theorem 1.1, $|E(G)| \ge (k-1)n - \binom{k}{2}$. By Lemma 4.2,

$$\lambda_1(G) \ge \frac{2|E(G)|}{n} \ge \frac{2(k-1)n - 2\binom{k}{2}}{n} = 2(k-1) - \frac{k(k-1)}{n}.$$

Thus it suffices to prove $2(k-1) - k(k-1)/n \ge (k-2+\sqrt{k^2+4k-4})/2$.

For $k = 2$, this reduces to $2 - 2/n \geq \sqrt{2}$, which holds when $n \geq 4$. For $k = 3$, it reduces to $4 - 6/n \geq (1 + \sqrt{17})/2$, which holds when $n \geq 5$.

For $k \geq 4$, since $k > (k - 2 + \sqrt{k^2 + 4k - 4})/2$, it suffices to prove $2(k-1) - \frac{k(k-1)}{n} \geq k$. We compute

$$2(k-1) - \frac{k(k-1)}{n} - k \geq k - 2 - \frac{k(k-1)}{k+2} = \frac{k-4}{k+2} \geq 0.$$

This completes the proof. □

For $t \geq 3$, let $\mathcal{F}_{d,t}$ be the family of d-regular simple graphs H with $\kappa'(H) \leq t$. In [7], it was proved that the minimum of the second largest eigenvalue over graphs in $\mathcal{F}_{d,t}$ is the second largest eigenvalue of a smallest graph in $\mathcal{F}_{d,t}$. Theorems 4.4 and 4.5 similarly say that the minima of the spectral radius over $\mathcal{F}$-saturated graphs and over $\mathcal{F}'$-saturated graphs are the spectral radii of the smallest graph in these families.

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